EXERCISE SET 3, TOPOLOGY IN PHYSICS

- The hand-in exercise is the exercise 2.
- Please hand it in electronically at topologyinphysics2019@gmail.com (1 pdf!)
- Deadline is Wednesday February 27, 23.59.
- Please make sure your name and the week number are present in the file name.

Exercise 1: Maxwell theory and de Rham cohomology. The advantage of formulating Maxwell’s theory in terms of differential forms is that it now makes sense on any manifold \( M \), not even 4-dimensional! For this we consider the first few terms of the de Rham complex:

\[
\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \ldots
\]

As we have seen, the electric and magnetic fields are gathered in a two-form \( F \in \Omega^2(M) \), which the homogeneous Maxwell equations require to be closed:

\[ dF = 0. \]

a) Assume that \( H^2_{dR}(M) = 0 \). Show that for any field strength \( F \) there is a “potential” \( A \in \Omega^1(M) \) such that \( F = dA \). Show that two potentials \( A \) and \( A + d\Lambda \), with \( \Lambda \in C^\infty(M) \) describe the same configuration of the electromagnetic fields, so that the “configuration space” of possible electromagnetic fields satisfying the homogeneous Maxwell equations is given by the quotient \( \Omega^1(M)/d\Omega^0(M) \). Elements in \( d\Omega^0(M) \) are called “gauge transformations”.

b) For any manifold \( M \), we write \( \Omega^k_{cl} \) for the space of closed \( k \)-forms. Show that there is a sequence of maps

\[
(*): \quad 0 \to H^1_{dR}(M) \to \Omega^1(M)/d\Omega^0(M) \to \Omega^2_{cl}(M) \to H^2_{dR}(M) \to 0.
\]

Explain what are the maps and show that the sequence is exact.

Minkowski space \( \mathbb{R}^{1,3} \) is topologically trivial, so \( H^1_{dR}(\mathbb{R}^{1,3}) = 0 = H^2_{dR}(\mathbb{R}^{1,3}) \), and the sequence \((*)\) amounts to an identification \( \Omega^2_{cl}(\mathbb{R}^{1,3}) \cong \Omega^1(\mathbb{R}^{1,3})/d\Omega^0(\mathbb{R}^{1,3}) \). In other words: we may equally well describe the electromagnetic field using the potential \( A \), as long as we make sure that we use “gauge invariant” observables, i.e., functions \( A \mapsto O(A) \) that are invariant under shifts by \( d\Omega^0(\mathbb{R}^{1,3}) \): \( O(A + d\Lambda) = O(A) \). For a topologically nontrivial manifold \( M \) (already \( M = \mathbb{R}^3 \times S^1 \) is an example) we no longer have \( \Omega^2_{cl}(M) \cong \Omega^1(M)/d\Omega^0(M) \), as the sequence \((*)\) shows. One of the lessons from Quantum Mechanics, as witnessed for example by the Aharonov–Bohm effect is that the potential \( A \), is more fundamental than the field strength \( F \)! Therefore, it is better to describe Maxwell theory as a action functional on the space of “fields” \( A \in \Omega^1(M) \).
c) Let $\gamma : S^1 \to M$ be a smooth closed curve in $M$. Show that the function

$$O_\gamma(A) := \int_\gamma A, \quad A \in \Omega^1(M)$$

is a gauge invariant observable. When the field strength $F = 0$, use de Rham’s theorem to show that these observables can detect the class in $H^1_{\text{dR}}(M)$.

d) Show that the action functional

$$S(A) = \frac{1}{2} ||dA||^2 = \frac{1}{2} \int_M dA \wedge *dA$$

is gauge invariant and variation leads to the vacuum Maxwell equation $d \wedge F = 0$. (You actually may have done this already last week...)

* Exercise 2: The Hopf fibration. We consider the 3-sphere defined as

$$S^3 := \{(z_1, z_2) \in \mathbb{C}^2, \ |z_1|^2 + |z_2|^2 = 1\}$$

and recall the definition of the complex projective line $\mathbb{P}^1$ better known as the Riemann sphere

$$\mathbb{P}^1 := (\mathbb{C}^2 \setminus \{(0,0)\}) / \mathbb{C}^\times$$

where $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ acts by scalar multiplication. Note that $S^3$ is given by all pairs of complex numbers satisfying a certain equation, while $\mathbb{P}^1$ is given by pairs of complex numbers $(z_1, z_2)$ (not both $0$) up to a certain equivalence, namely $(z_1, z_2) \sim (w_1, w_2)$ if there is $0 \neq \lambda \in \mathbb{C}$ such that $\lambda z_1 = w_1$ and $\lambda z_2 = w_2$.

i) The group $U(1) \cong S^1$ acts on $S^3$ by

$$(z_1, z_2) \cdot e^{i\theta} := (z_1 e^{i\theta}, z_2 e^{i\theta})$$

Find a smooth map $S^3/U(1) \to \mathbb{P}^1$ that allows for a smooth inverse, i.e. show that $S^3/U(1) \cong \mathbb{P}^1$.

We may compose the map of i) with the quotient map $S^3 \to S^3/U(1)$ to obtain a map $\pi : S^3 \to \mathbb{P}^1$. Recall from the lecture notes of lecture 1 that we had the atlas of $\mathbb{P}^1$ given by the charts

$$U := \{[(z_1, z_2)] \in \mathbb{P}^1 \mid z_1 \neq 0\}$$

and

$$V := \{[(z_1, z_2)] \in \mathbb{P}^1 \mid z_2 \neq 0\}.$$ 

ii) Find sections $U \to \pi^{-1}(U)$ and $V \to \pi^{-1}(V)$.

iii) Compute the transition function $\varphi_{UV} : U \cap V \to U(1)$.

(BONUS) Consider the standard (defining) representation of $U(1)$ on $\mathbb{C}$:

$$e^{i\theta} \cdot z = e^{i\theta}z,$$

and consider the line bundle associated to the Hopf fibration above. Show that this line bundle agrees with the tautological line bundle over $\mathbb{P}^1$. 
Exercise 3: The Hodge–Maxwell Theorem. In this exercise we will define the Hodge $\star$ starting from a general (pseudo-Riemannian) metric $g$ on the oriented manifold $M$ without using coordinates. Recall that $g$ allows us to define a notion of volume on the manifold $M$. The volume of the submanifold $B$ is given as the integral $\int_B \text{vol}$. In coordinates vol is given by the formula

$$\text{vol} = \sqrt{\left| g \right|} dx^1 \wedge \ldots \wedge dx^n,$$

where

$$\left| g \right| = \left| \sum_{i_1, \ldots, i_n=1}^n \varepsilon_{i_1 \ldots i_n} g_{i_1} \cdots g_{i_n} \right|$$

denotes the absolute value of the determinant of $g$ and the $dx^i$ form a positively oriented basis. Recall that the coordinate transformation $x^i \to y^j$ is called positive if $\text{Det} \frac{\partial x^i}{\partial y^j}$ (the Jacobian determinant) is positive.

i): Show that the formula for vol above defines an $n$-form $\omega$. Do this by performing a (positive) coordinate transformation.

ii): The metric $g$ is given by a symmetric non-degenerate bilinear pairing on the tangent spaces

$$(v, w) \mapsto g_{\mu\nu}(x) v^\mu w^\nu,$$

for $v, w \in T_x M$. Show that we get a $C^\infty(M)$-bilinear pairing

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M),$$

where $\mathfrak{X}(M)$ denotes vector fields.

Note that similarly the maps

$$(\alpha, \beta) \mapsto g^{\mu_1\nu_1}(x) \cdots g^{\mu_p\nu_p}(x) \alpha_{\mu_1} \cdots \beta_{\nu_1} \cdots \nu_p$$

define a $C^\infty(M)$-bilinear pairing on $\Omega^p(M)$. In fact this is the pairing $\langle \alpha, \beta \rangle$ mentioned in the lecture notes.

iii): Assume that $M$ is compact and show that

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle \omega$$

defines an $\mathbb{R}$-bilinear, symmetric pairing on $\Omega^p(M)$.

iv): Consider $\beta \in \Omega^p(M)$, define $\ast \beta$ as the $n-p$ form satisfying

$$\alpha \wedge \ast \beta = \langle \alpha, \beta \rangle \omega$$

for all $\alpha \in \Omega^p(M)$ and show that this definition coincides with the coordinate expression given in the lectures.

Hint: Show first that $\ast \beta$ is uniquely defined. To do this note that if a form is 0 around every point, then it vanishes globally.

v): Show that the adjoint $d^*$ of the exterior derivative $d$ is given by the formula $\ast d \ast$. 