Orbifolds and their quantizations as noncommutative geometries

Hessel Posthuma
(joint work with M. Pflaum, X. Tang and H-H Tseng)

Orbifolds are a natural generalization of the concept of a manifold with a rich geometric structure. In particular, its orbifold cohomology has many surprising features, most notably a ring structure [1] generalizing, in a nontrivial way, the cup product on the cohomology of a manifold.

As a manifold with singularities, orbifolds can also be viewed as examples of noncommutative spaces. This point of view yields different cohomological tools, such as Hochschild and cyclic (co)homology. The aim of this project is to study the relation between both cohomologies, in particular its ring structure. For this, the Hochschild cohomology is of particular importance, because it carries a graded product, known as the Yoneda or cup product. To actually relate this to the cohomology of the underlying obifold, we need a deformation quantization, for which we assume the existence of a symplectic structure. Such a space is known as a symplectic orbifold, they very often arise as symplectic quotients with respect to proper group actions on symplectic manifolds.

The set-up is as follows: Let X be an orbifold and G a proper étale groupoid modelling X. We denote the convolution algebra of G by A_{G}. In [2], its Hochschild cohomology was computed to be

$$H^\bullet(A_{\mathsf{G}}, A_{\mathsf{G}}) \cong \Gamma^\infty\left(\tilde{X}, \Lambda^{\bullet-\ell}T_{\tilde{X}} \otimes \Lambda^\ell N_{\tilde{X}}\right). \tag{1}$$

Here $\tilde{X}$ is the so-called inertia orbifold, a disconnected space whose connected components embed into X locally as fixed point sets for the local group actions. An atlas of this orbifold is provided by considering the space of loops

$$\mathsf{B}_0 := \{g \in \mathsf{G},\ s(t) = t(g)\},$$

on which G acts by conjugating loops. The vector bundles $T_{\tilde{X}}$ and $N_{\tilde{X}}$ are the tangent bundle and the normal bundle with respect to the local embedding into X. Finally $\ell : \tilde{X} \to \mathbb{N}$ is the locally constant function given by $\ell = \dim(N_{\tilde{X}})$.

To describe the cup-product on this space of "multivector fields", we need to introduce a third orbifold: consider the space

$$\mathsf{S} := \{(g_1, g_2) \in \mathsf{G}_1 \times \mathsf{G}_1,\ s(g_1) = t(g_1) = s(g_1) = t(g_1)\}.$$

There are three obvious maps $pr_1, pr_2, m : \mathsf{S} \to \mathsf{B}_0$ given by projection onto the first and second component, and multiplication of loops. Again, G acts by conjugating loops and the quotient orbifold is denoted by X_3. With this, the cup product on the Hochschild cohomology (1) is given by:

$$\xi \cup \eta := \int_m pr_1^*\xi \wedge pr_2^*\eta.$$

Here the integral means integration over the fiber of m, which is discrete, and the formula is ultimately understood on the level of germs; recall that G is étale.

There is an important subtlety hidden in the above formula for the cup-product. Recall that ξ and η are actually sections of exterior powers of the tangent bundle to X tensored with the determinant line of the normal bundle. This determinant has the effect that the wedge product is zero if the normal bundles $pr_1^* N_{\tilde{X}}$ and $pr_2^* N_{\tilde{X}}$ have a nontrivial intersection. In other words, the germ of the cup product $\xi \cup \eta$ at a point $x \in \tilde{X}$ is supported on the subset $y \in m^{-1}(x)$ for which

$$\ell(m(y)) = \ell(pr_1(y)) + \ell(pr_2(y)). \tag{2}$$

Next, we consider a formal deformation quantization $A^{\hbar}_{\mathsf{G}}$ of the convolution algebra A_{G} given by a G-invariant deformation quantization of G_0, which we assume to be symplectic. For this algebra, the Hochschild cohomology is given by

$$H^{\bullet}(A^{\hbar}_{\mathsf{G}}, A^{\hbar}_{\mathsf{G}}) \cong H^{\bullet - \ell}\left(\tilde{X}, \mathbb{C}((\hbar))\right).$$

This result extends [4] to the category of symplectic orbifolds. The proof uses the $\hbar$-filtration to identify the left hand side with the Poisson cohomology of $\tilde{X}$: since $\tilde{X}$ is symplectic- the symplectic form on X pulls back to a symplectic form on $\tilde{X}$- this yields the right hand side.

With this isomorphism, the cup-product is given on the level of differential forms by

$$\alpha \cup \beta := \int_{m_\ell} pr_1^* \alpha \wedge pr_2^* \beta, \tag{3}$$

where m_ℓ is the restriction of m to the sub-orbifold of points satisfying the condition (2). Notice that this condition implies that locally the spaces on which $pr_1^*\alpha$ and $pr_2^*\beta$ are supported, have a transversal intersection, so that the formula is well-defined.

The above formula for the cup product indeed does resemble the product defined in [1] on the cohomology of $\tilde{X}$, but there is an important difference: one easily checks that in the formula for that product, there is no condition on the fibers of m! Therefore one has deal with the non-transversal intersections as well.

To write down a similar formula for a product, but without the assumption (2), one can use the Thom isomorphism to take a wedge product in X to deal with the non-transversal intersections. To make the Thom form invertible, one uses equivariant cohomology with respect to the fiberwise S^1-action on $N_{\tilde{X}}$ associated to the choice of an almost complex structure. As a module over $H_{S^1}(pt.) = \mathbb{C}[t]$ we can complete the equivariant cohomology by tensoring with $\otimes_{\mathbb{C}[t]} \mathbb{C}((t))$. Because the action of S^1 is trivial on $\tilde{X}$ we have $H^{\bullet}_{S^1}(\tilde{X})((t)) = H^{\bullet}(\tilde{X}, \mathbb{C}((t)))$.

If we denote by $Th_{\tilde{N}}$ the equivariant Thom form of the normal bundle $N_{\tilde{X}} \to \tilde{X}$, there is a natural associative product on $\Omega^{\bullet}(\tilde{X})((t))$ given by

$$\alpha \wedge_t \beta := \int_m \frac{pr_1^*(\alpha \wedge Th_{\tilde{N}}) \wedge pr_2^*(\beta \wedge Th_{\tilde{N}})}{pr_m^* Th_{\tilde{N}}}.$$

With this product, the equivariant cohomology is an associated graded ring, which turns out to be isomorphic to an equivariant version of Chen–Ruan's orbifold cohomology.

The relation with the Hochschild ring is as follows: the equivariant orbifold cohomology ring has a natural filtration given by

$$\mathcal{F}^k := \{\alpha \in H^{\bullet}_{S^1}(\tilde{X}) \otimes_{\mathbb{C}[t]} \mathbb{C}((t)),\ \deg(\alpha) - \ell \geq k\}.$$

Taking the graded quotient with respect to this filtration enforces the condition (2), in which case we have $m_\ell^* N_{\tilde{X}} \cong pr_1^* N_{\tilde{X}} \oplus pr_2^* N_{\tilde{X}}$. In this case, the contributions of the equivariant Thom forms cancel, and the product reduces (3). Therefore the Hochschild cohomology ring is nothing but the graded quotient of the orbifold cohomology ring with respect to the filtration above.

For more details we refer to [3].

References

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