Homework Set #5

Axiomatische Verzamelingentheorie
2013/14: 2nd Semester; block b
Universiteit van Amsterdam

http://hugonobrega.github.io/teaching/AxVT/

The preferred method for submitting homework solutions is by handing them in before the start of the werkcollege on Wednesday morning. Electronic submissions are also possible, by email to iilin.juli (at) gmail.com or hugonobrega (at) gmail.com before the deadline. These must be in a single, legible PDF file. It is also possible to hand in your homework by putting it into Julia’s or Hugo’s mailboxes at the ILLC at Science Park 107, but in this case the deadline for submission is at 10:45am on the day of the submission file. It is also possible to hand in your homework by putting it into Julia’s or Hugo’s mailboxes at the ILLC before the deadline. These must be in a single, legible PDF file.

This homework set is due on Wednesday 7 May 2014, before the morning werkcollege.

1. If \((X, <)\) is a strict linear order, we define the order topology as follows: for \(x \leq y\), we let \(\text{Interval}(x, y) := \{z \in X; x < z < y\}\) (called the open interval between \(x\) and \(y\)) and we call a set \(Z \subseteq X\) open if it is a union of open intervals (not necessarily a finite union). Then the order topology \(\tau\) is the set of open sets in \(X\). As usual in topology, a point \(x \in X\) is called a \(\tau\)-limit point if for every \(U \in \tau\) with \(x \in U\) there is some \(y \neq x\) such that \(y \in U\).

Remember from homework set #4 that we called a function \(s : X \to X\) a successor function if for all \(x \in X\), we have \(x < s(x)\) and there is no \(x'\) such that \(x < x' < s(x)\). The elements of \(\text{ran}(s)\) are called \(s\)-successors, the elements of \(X\setminus\text{ran}(s)\) are called \(s\)-limits.

(a) Prove that if \((X, <)\) is a strict wellorder with order topology \(\tau\) and a successor function \(s\), then a point \(x\) is a \(\tau\)-limit point if and only if it is an \(s\)-limit.

(b) Prove that if \((X, <)\) is a wellorder, then it has a unique successor function.

2. An ordinal \(\alpha\) is called an initial ordinal if for all \(\beta \in \alpha\), we have \(\beta \not\prec \alpha\). Show that for every initial ordinal \(\alpha\) at least one of the following three cases holds:

   Case 1. \(\alpha = \emptyset\).

   Case 2. There is an ordinal \(\beta \in \alpha\) such that \(\alpha = \aleph(\beta)\).

   Case 3. There is a set \(X\) of ordinals such that \(\alpha = \bigcup\{\aleph(\beta); \beta \in X\}\).

3. Let \(X\) be an arbitrary set. We consider functions \(H : \varphi(X) \to \varphi(X)\). Such a function is called monotone if for all \(A \subseteq B\) implies \(H(A) \subseteq H(B)\). We say that \(T \subseteq X\) is a fixed point of \(H\) if \(H(T) = T\). As usual, if \(f\) is a function and \(A \subseteq \text{dom}(f)\), we write \(f[A] := \text{ran}(f \upharpoonright A)\).

   (a) Show that every monotone \(H\) has a fixed point. [Hint: Consider the set \(\bigcup\{A \subseteq X; A \subseteq H(A)\}\).]

   (b) Consider two sets \(X\) and \(Y\) such that there are injections \(f : X \to Y\) and \(g : Y \to X\). Define the function \(H : \varphi(X) \to \varphi(X)\) by \(H(A) := X \setminus g[\text{ran}(f[A])]\).

   Show that \(H\) is monotone (and thus has a fixed point).

   (c) Prove the Cantor-Schröder-Bernstein Theorem: if \(X \preceq Y\) and \(Y \preceq X\), then \(X \sim Y\). [Hint: Let \(f : X \to Y\) and \(g : Y \to X\) be the injections. Use a fixed point \(T\) of the function \(H\) given in (b) and define the bijection on \(T\) by \(f\) and on \(X \setminus T\) by \(g^{-1}\).]