Colour-dependent percolation in complex networks

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Percolation is progressively shifting from being merely a theoretical concept to becoming a tool for modelling and analysis: many processes in various applied fields can be recast as essentially a percolation problem[1–4]. Yet, with this recasting comes the realisation that complex networks are more then a pattern of connections, they include an additional layer of information that may arguably be even more decisive then the connectivity between the nodes.

Here we introduce a generalisation of bond percolation on complex networks. The conventional bond percolation can be viewed as a continuous time process defined on an interval \( p(t) \in [0, 1] \), where \( p \) is the probability that an arbitrary edge is present (or alternatively not removed), all edges are present from the start \( p(0) = 1 \), and none at the end of the process \( p(1) = 0 \). Suppose the edges are supplied with an additional layer of discrete information. This information can either be coupled to the network topology, as for instance, degrees of incident to the edge nodes, community association, etc., or irrelevant, as for instance, weights, labels, and vector features. In either case, we refer to the set of all possible features as colours \( i = 1, 2, \ldots, N \), and to the network as the edge-coloured network. Note, if the features are vectors, their discrete states have to be projected on a line for indexing.

<table>
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<th>Concept</th>
<th>Conventional bond percolation</th>
<th>Colour-dependent percolation</th>
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<tbody>
<tr>
<td>Occupancy probability</td>
<td>scalar, ( p \in [0, 1] )</td>
<td>vector, ( p \in [0, 1]^N )</td>
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<tr>
<td>Configuration space</td>
<td>interval, ( [0, 1] )</td>
<td>hypercube, ( [0, 1]^N )</td>
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<td>Critical point</td>
<td>scalar, ( p_c \in [0, 1] )</td>
<td>manifold, ( P_c \subset [0, 1]^N )</td>
</tr>
<tr>
<td>Time process</td>
<td>( p(t), \ p'(t) &lt; 0 )</td>
<td>( p(t), \ p'(t) &lt; 0 ), ( p(0) = 1, \ p(1) = 0 )</td>
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<tr>
<td></td>
<td>( p(t_0) = 1, \ p(t_1) = 0 );</td>
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Table 1. Correspondence between the concepts in conventional and colour-dependent percolations

In colour-dependent percolation, the probability for an edge to be present depends on the colour of the edge. In this way, we have a vector of probabilities \( p = (p_1, p_2, \ldots, p_N) \) that associates a separate probability per colour, so that now, a vector characterises the

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process state rather than a single scalar number. Needless to say, when \( p = p_1 = p_2 = \cdots = p_N \) the problem degenerates to the conventional bond percolation. Gradually removing edges form the network in time reflects in the vector being time-dependent \( p(t) \), where \( t \in [0, 1] \) and \( p(0) = \bar{1}, \ p(1) = \bar{0} \). In this setting, we have a very rich problem: both topology of the network and the colour-labelling scheme contribute to the percolation properties. A few better studied types of percolation can be shown to be partial cases of colour-dependant percolation.

Many concepts from conventional percolation obtain new intriguing properties in the colour-dependant framework, see Table 1 for comparison. All configurations of percolation probability vector \( p \) form an \( N \)-dimensional hypercube and all critical probability vectors \( p_c \) form a manifold in this hypercube. Furthermore, a gradual removing of all edges corresponds to a continuous path that joins the vertices \( \bar{1} \) and \( \bar{0} \) in the hypercube.

Fig. 1. Critical manifolds for a few examples of networks coloured with \( N = 3 \) colours. Blue colour indicates the interior of the manifold, red colour – exterior. The yellow lines give examples of evolution paths for the percolation vector throughout time: a. trivial path, \( p_1 = p_2 = p_3 \); b,c. paths that evolve inside the critical manifold.

**Results**

Consider a coloured network that is defined by a *light-tailed* multidegree distribution. We say that a configuration of the probability vector is critical \( p_c \), if it corresponds to a *heavy tailed* component size distribution in the network. By extending the results of our previous works, Ref. [1, 5, 6], we derived a concise criterion that identifies the critical vectors [7]: edge-coloured network features the critical behaviour at \( 0 < p_c < 1 \), if and only if,

\[
\mathbf{v} \in \ker \{ \text{diag}(\mathbf{p}_c) \mathbf{M} - \mathbf{I} \}, \quad \text{and} \quad \frac{\mathbf{v}}{\sum_i v_i} \geq 0,
\]

where

\[
M_{i,j} = \frac{\mathbb{E}[k_i k_j]}{\mathbb{E}[k_i]} - \delta_{i,j}, \quad i, j = 1, \ldots, N,
\]
is a matrix composed of expectation values of the multi-degree distribution, and $\delta_{i,j}$ is the Kronecker’s delta. The latter criterion can be viewed as a parameter equation for a surface placed in the $N$-dimensional space. One may also view this equation as a generalisation of the Molloy and Reed criterion [8] to probability vectors. The precise description of how percolation happens in this system is given by the combination of the critical manifold and the path along which $p(t)$ evolves in time. In Figure 1 we illustrate a few possibilities. What is perhaps most remarkable, is that this path might pierce the critical manifold multiple times (multiple criticalities, Fig. 2a.) or even evolve inside the critical manifold (wide criticality, Fig. 2b,c). In the latter case, the network maintains infinite expected component size for a long period of time (if we are the thermodynamic limit), whereas below the thermodynamic limit, mean size of the non-giant components features macroscopic fluctuations.

Fig. 2. The average size of non-giant components as a function of time. The panels corresponds to manifolds depicted in Fig 1. Panel a. depicts multiple critical points, whereas Panels b,c depict an intricate situation when the network becomes critical for a long, continuous period of time.

References