Dual Views of String Impurities

Geometric Singularities and Flux Backgrounds

DUAL VIEWS OF STRING IMPURITIES

GEOMETRIC SINGULARITIES AND FLUX BACKGROUNDS

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor

aan de Universiteit van Amsterdam

op gezag van de Rector Magnificus

prof. mr P. F. van der Heijden

ten overstaan van een door het college voor promoties ingestelde

commissie, in het openbaar te verdedigen in de Aula der Universiteit

op woensdag 29 september 2004, te 10:00 uur

door

ROBERT JAN DUIVENVOORDEN

geboren te Leiderdorp.

PROMOTIECOMMISSIE

PROMOTOR

prof. dr J. de Boer

OVERIGE LEDEN

prof. dr ir F. A. Bais

prof. dr R. H. Dijkgraaf

prof. dr A. N. J. J. Schellekens

prof. dr C. J. M. Schoutens

prof. dr E. P. Verlinde

FACULTEIT DER NATUURWETENSCHAPPEN, WISKUNDE EN INFORMATICA

ISBN 90-9018474-0

Dit proefschrift is tot stand gekomen dankzij de financiële ondersteuning van de Stichting voor Fundamenteel Onderzoek der Materie (FOM).

VOOR MIJN OUDERS

G. Tartini

Contents

1

INTRODUCTION

GEOMETRY AND STRING THEORY

The language of geometry has proved remarkably adept to formulate the presently known fundamental physical theories. The general theory of relativity on the one hand, but also gauge theories such as the standard model of particle physics can be formulated in essentially differential geometric language.

String theory, as a candidate to provide a unified framework for the description of both gravitation and the other known fundamental forces should, and does contain both familiar gravitational and gauge theory in appropriate regimes. But this is not all. Crucially, in string theory, a theory in which the fundamental objects are extended, the rôle of geometry is quite different than in theories of point particles. Even the very notion of what we would mean by geometry can be very different than is familiar from 'ordinary' differential geometry.

In the perturbative approach[1] to string theory, to a large extent the rôle of geometry is taken over by worldsheet conformal field theory.

In many situations, the worldsheet conformal field theory has a target space interpretation. It is interpreted as describing the embedding of the string worldsheet in a spacetime background, which has an 'ordinary geometric' interpretation. Yet in many other cases worldsheet conformal field theories can have all the properties required of them to define a string 'background', yet no target space interpretation is apparent. This situation is possible because the definition of a conformal field theory can be made in ways very different than as a sigma model.

[1]We mean perturbation theory in the string worldsheet genus expansion. This is not to be confused with the α' expansion, a perturbation expansion in the string scale used in worldsheet conformal field theory. The term 'non-perturbative T-duality' which permeates the setup of this work, alludes to perturbation theory in α'

T-DUALITY

At least equally interesting, is the situation that two different sigma models can define isomorphic conformal field theories. Such isomorphic conformal field theories define equivalent string backgrounds which have different 'ordinary geometric' interpretations. Intuitively speaking, the reason why backgrounds that look different to a classical geometer may look indistinguishable to a string theorist, is that strings are not in general located at a 'point' in spacetime, but they trace out a curve. Thus, a closed string can wind around a closed curve.

T-duality essentially exchanges winding modes in the string worldsheet theory with momentum modes. The momentum modes, are nothing but the modes that a theory of point particles would also have. Consequently, T-duality exchanges the 'intrinsically stringy' part of geometry (probed by winding strings) in one background, with the 'ordinary' geometry (as probed by point particles) in the dual background.

The archetypal relation of T-duality is the $R \leftrightarrow 1/R$ duality of strings on a circle. On a circle of radius R, the winding modes have energy levels which are spaced with an energy difference proportional to R, while momentum modes are spaced with energy levels proportional to R^{-1}. On a circle of radius $1/R$, the level spacing of winding and momentum modes is interchanged.

IMPURITIES 1: GEOMETRIC IMPURITIES

T-duality can have more complicated implications in more complicated backgrounds. The backgrounds which we consider in this work can be called 'impurities'. There are two kinds of impurities which we distinguish.

The first kind is a 'geometric impurity'. In this case there are no background fields, other than a metric. The metric defines a singular geometry, more precisely, a geometry which preserves some spacetime supersymmetry[2] and which has an isolated singularity. Often an explicit metric of the background will not be known. Instead, we use other more implicit means to characterize the background geometry.

In chapter 2 different ways are discussed to characterize supersymmetric singular background geometries. Of the methods discussed, two play a prominent rôle later, in chapter 4. The first method is the characterization as a metric cone. In this method the differential geometry of the background is emphasized. Supersymmetry imposes a restriction on the holonomy of the background. The structure of a metric cone, together with restricted holonomy leads to differential geometric constraints on the base of the cone. In particular, it turns out that all of the bases of supersymmetric complex metric cones have a Killing vector field, which degenerates at the apex of the cone. This is interesting because when there is an isometry, it is usually possible to consider a T-dual background.

[2] We concern ourselves with complex geometry. It would be an interesting but quite separate undertaking to transform the methods discussed to a form suitable for supersymmetric singular geometries which admit no complex structure.

The second kind of characterization, describes the geometric impurity as a hypersurface. In this description, the differential geometry is less explicit. However, there are analytic, algebraic and topological properties which can be studied and have been studied by mathematicians.

Actually, the hypersurfaces which we consider, bear a similarity to metric cones. The affine hypersurfaces under consideration are defined by weighted homogeneous polynomials. These are equivariant under a $\mathbb{C}^*$ action. Compare this to the supersymmetric metric cones, they admit a scaling of the base, and the base has a Killing vector field, which in all cases that are discussed by us, has closed orbits, so defines a $U(1)$ action. So both the hypersurfaces and metric cones we use, admit a $\mathbb{R} \times U(1)$ action.

Describing a singularity as a hypersurfaces offers some advantages which a metric cone description does not have. First, deformations of the space, and more specifically desingularizations, which smooth out the space completely, are described as simple analytic deformations of the defining polynomial. Second, weighted homogeneous polynomials, used to describe the hypersurfaces, can also be used to describe conformal field theories, as Landau-Ginzburg theories. Both these properties are very important in the construction of backgrounds which are T-dual to the geometric impurities.

IMPURITIES 2: FLUX IMPURITIES

The second kind of impurities can be called 'flux impurities'. These are, as the name indicates, sources of gauge field flux. So in backgrounds with flux impurities, there are other non-trivial background fields than just the metric.

The flux impurities are sources of Kalb-Ramond field. The archetypal example is a simple Neveu-Schwarz fivebrane. Also, these impurities create a non-constant dilaton: near the impurity the effective string coupling is large.

LOCALIZED PHYSICS NEAR AN IMPURITY

Intuitively speaking, because the string coupling grows large near a flux impurity, it may be possible to decouple the physics localized near this impurity by sending the string coupling asymptotically far from the impurity to zero. Then the bulk degrees of freedom, coupling to the 'localized' degrees of freedom through gravity, can decouple, and one can restrict attention to the degrees of freedom localized near the impurity alone.

Such decoupling limits have various interesting properties. First, typically the 'localized' physics has a holographic description. That is to say, the decoupled subsector of string theory in the original background, which contains just the 'localized' physics of the impurity, is equivalently described by the full string theory in another background (think of Anti-de Sitter backgrounds, and also of linear dilaton backgrounds).

Next, it happens often, as we will see, that these 'decoupling limit backgrounds' admit an exact worldsheet conformal field theory description, while such a description is unknown for the full, unscaled backgrounds with a local impurity.

On the other hand, geometric impurities also have localized physics. Essentially, it comes from branes wrapping vanishing cycles in the singularity. The notion of vanishing cycles is also useful to understand that geometric impurities are quite generic. One may start with a smooth geometric background. Such a background is usually one in a family of connected backgrounds, parametrized by moduli. At certain perfectly fine values of the moduli, a homology cycle in the geometric background may shrink to zero size. Then a singularity, or geometric impurity, develops. A scaling limit which isolates the physics localized at the singularity, typically involves tuning the size of a vanishing cycle to zero, while also scaling other parameters, usually the string coupling, analogous to the limit for flux impurities. Especially the hypersurface singularities are suited to such a scaling limit, as blowing up certain cycles corresponds to simply deforming the defining polynomial.

Duality between Geometric and Flux Impurities

Geometric impurities and flux impurities are related by T-duality. In practice it is difficult to explicitly carry out the duality transformation. A reason for this difficulty is, that worldsheet instanton effects are crucial to the duality. These worldsheet instantons break spacetime symmetries which seem to be present if one considers only the perturbative physics.

If one performs a perturbative analysis and dualizes a geometric impurity, it appears that the dual flux impurity has an isometry, which turns out not to be present in reality, when considering non-perturbative contributions to the duality. The prime example of such a duality, is that between IIA strings in an asymptotically Euclidean space, with an A_k singularity, and IIB string theory on $\mathbb{R}^{5,1} \times \mathbb{R}^3 \times S^1$, with a stack of $k+1$ Neveu-Schwartz fivebranes localized at a point in $\mathbb{R}^3 \times S^1$.

It is easier to consider duality in the 'near impurity background', rather than the full background, before zooming in on the impurity. The 'localization' of the flux impurity is of course crucial to get the correct 'decoupling limit' or, as we just referred to this limit, a 'near impurity limit'.

We will see exact worldsheet conformal field theory descriptions in the 'near impurity limit' of both the geometric and the dual flux impurity. Actually, in certain cases the worldsheet conformal field theory of the dual flux impurity will have an explicit construction and interpretation as a sigma model. In other cases, a geometric interpretation of the 'near flux impurity' worldsheet theory remains to be discovered.

Why Impurities?

What are the motivations for the study of 'geometric' and 'flux' impurities and the T-duality relation between the two?

First, there are the intriguing relations between the descriptions of geometric and flux impurities. We will find flux impurities which can be viewed as a background of the form

$$\text{(linear dilaton)} \times \frac{G}{H}$$

the right hand factor denotes a coset conformal field theory. When this is realized as a gauged WZW model and when the level of G is large, the target space can be approximated by a one-loop calculation in the gauged WZW model. This gives a target space $G/H = \tilde{L}$, where H acts as a vector gauging, $g \sim h^{-1}gh$. Note that this target space looks very different from the coset manifold G/H, where group elements are identified as $g \sim gh$.

We shall see intriguing cases that flux impurities of this kind are related, by T-duality and adjusting the moduli, to geometric impurities, that are described as follows. These impurities are metric cones on a base space L, where L is a fiber bundle

$$\begin{array}{ccc} S^1 & \xrightarrow{i} & L \\ & & \downarrow{\scriptstyle \pi} \\ & & Z \end{array}$$

with base Z, which is a homogeneous space $G/(H \times \Gamma)$, and Γ is a discrete subgroup of G, related to modular data of the coset model.

So there are some intriguing similarities and differences going on, which must point at some stringy geometric phenomena. It seems worthwhile to try and understand such stringy geometric aspects better.

There is also a quite different motivation. This is related to holographic duality: string theory in certain backgrounds is believed to be exactly equivalent to a non-gravitational theory in a spacetime that has one dimension less. The two types of string background that are widely believed to exhibit such behavior are linear dilaton backgrounds and Anti-de Sitter backgrounds.

Very generically, the flux impurities we find have a linear dilaton. The linear dilaton backgrounds are believed to holographically describe certain exotic quantum theories in dimensions $d \leq 6$: Little Sting Theories. These theories are non-local, and little is known about them. Clearly it would be highly interesting to better understand such unfamiliar quantum theories.

Also very generically, the linear dilaton backgrounds can be 'deformed' to AdS backgrounds. Therefore the flux impurities are of interest to study AdS backgrounds, and their holographic duals, which are conformal field theories.

Then why are the geometric impurities of interest? A fruitful way to gain knowledge about AdS/CFT, is to take certain non-dilatonic brane configurations, that is, impurities of a sort, and take a scaling limit which isolates the physics near the branes. This physics can be characterized in two different looking ways, one can think in terms of open string degrees of freedom, describing the physics on the worldvolume of the branes, or closed string degrees of freedom describing the dual, gravitational physics in the background near the branes, which is deformed by the branes. A lot can be learned about AdS/CFT by realizing the AdS background and the dual field theory through a brane setup. However, simple brane configurations only give a limited number of geometries AdS $\times$ N.

A lot of geometries can be obtained by also considering 'geometric impurities', that is, singularities. In particular, many interesting AdS/CFT realizations are possible by considering D3branes in a Calabi-Yau singularity. These produce AdS geometries of the form $\mathrm{AdS}_5 \times N$, where N is an Sasaki-Einstein manifold, which can be viewed as the base of a metric Calabi-Yau cone.

Apart from considering D3 branes, there is another way to get AdS backgrounds from geometric impurities, which is an important motivation for the study of these impurities and their T-duality. One can take a geometric impurity and put fundamental strings at the singularity. This is not a non-dilatonic background as such, but by performing a T-duality it becomes non-dilatonic. The dilaton contribution of the fundamental string, in the 'near impurity limit' compensates the linear dilaton that is generated by the T-duality. In this way many backgrounds of the form $\mathrm{AdS}_3 \times N_7$ might be realized, which cannot be obtained from other simple brane configurations. Therefore, we hope that the knowledge about T-duality of these impurities will also lead to a better understanding of holographic duality.

OUTLINE

The outline of this thesis is as follows.

In chapter 2 geometric impurities are discussed. Mainly two characterizations of supersymmetric (and complex) singularities are presented: metric cones with holonomy contained in $SU(n)$ on the one hand, and weighted homogeneous affine hypersurfaces on the other.

Differential geometric aspects of the metric cones are discussed. A particular rôle is played by quasi-regular Sasaki-Einstein manifolds. Many known examples are homogeneous spaces, or related to homogeneous spaces. Sasakian(-Einstein) manifolds have a characteristic Killing vector field, which is used to relate these spaces to quasi-smooth Kähler-Einstein varieties. These are the subject of study of algebraic geometers.

Weighted homogeneous polynomials can also be used to characterize supersymmetric complex singular hypersurfaces. Aspects of such hypersurfaces are discussed. As somewhat of an aside, some topological properties of such hypersurfaces are discussed. Weighted homogeneous polynomials also define Kähler varieties in weighted projective space. These can be interpreted as base spaces of Sasaki-Einstein circle fibrations. This establishes a connection between metric cones and affine hypersurfaces.

In chapter 3 various aspects are discussed of superconformal field theories, which are put to use later, in chapter 4, to describe strings in the background of impurities. Some particularly important constructions are Landau-Ginzburg models, which are defined through a weighted homogeneous polynomial and thus make contacts with hypersurface singularities. Also coset conformal field theories play a rôle, since the best understood dualities between geometric and flux impurities involve coset conformal field theories, which are actually coset models that are closely related to Landau-Ginzburg (and Kazama-Suzuki coset-) models. Of course an important class of conformal field theories is formed by sigma models.

Finally in chapter 3 non-conformal models are discussed which interpolate between sigma models on hypersurfaces, and Landau-Ginzburg theories. Models of this kind are employed to formulate the T-duality of impurities in chapter 4.

Chapter 4 begins with a discussion of geometric and flux impurities (in particular: fivebranes) in string theory, and the ‘near impurity geometry’ and exact conformal field theories for ‘near impurity’ geometries. Next generalities of T-duality are discussed: classical T-duality rules, the rôle of degenerating isometries, and breaking of isometries in the dual model by worldsheet instantons. Finally, in section 4.4, T-duality for a large class of impurities is discussed. Agreement is found with the known result of hyper-Kähler surface singularities and ADE-throat geometries, and some further examples are discussed, and some final observations are made.

2

GEOMETRY AND SINGULARITIES

The objective of this chapter is to collect and expose several different geometric perspectives which can be used to describe supersymmetric 'compactifications' of string theory. The term 'compactification' is somewhat inappropriate, as most of the 'compactification' spaces discussed are not compact and also often singular. Such spaces are considered as local models of degenerate limits of smooth but not necessarily compact manifolds which can make up part of a string vacuum. In chapter 4 the physical motivation of such degenerate limits is discussed. Very briefly stated, it is possible that some cycles in a smooth manifold become small. Then some massive nonperturbative degrees of freedom of the compactified theory become light and make up physics which is localized at the degeneration of the manifold. This 'localized physics' can be decoupled in appropriate scaling limits. It depends on the local geometry near such a degeneration.

At present we are concerned with the geometry of such local models. Differential and algebraic geometric methods exist to characterize some of these. The various characterizations are interconnected in intriguing and insufficiently understood ways, and also connected to various descriptions of possible worldsheet conformal field theories, which are discussed in chapter 3. In this chapter the following topics are discussed.

If a space is part of a supersymmetric string vacuum, it must satisfy certain differential geometric requirements. For example, it might have to be Ricci flat and Kähler. Such requirements also hold for singular spaces. The four dimensional singular spaces which fit the bill are the hyper-Kähler surface singularities. These have various interchangeable descriptions, notably as quotient singularities, as hypersurfaces embedded in $\mathbb{R}^6 \simeq \mathbb{C}^3$ and as metric cones.

These descriptions can be used to describe many higher dimensional singularities as well, where the focus will be on complex singularities. It is however not true, that any given singularity can be described in all of the above fashions. Metric cones are interesting because the differential geometric constraints on the cone lead to constraints on the base of the cone. Typically the base, also known as the link of the cone is a Sasaki-Einstein manifold. Sasaki manifolds have a circle isometry and the corresponding orbit space is Kähler. For a Sasaki-Einstein manifold, it is Kähler-Einstein.

Some examples of Kähler-Einstein spaces are homogeneous. A considerable number can be constructed as hypersurfaces where a weighted homogeneous polynomial vanishes in an appropriate weighted homogeneous space. Such examples often have orbifold singularities. The zero locus of such a polynomial in affine space is precisely a supersymmetric singularity. A class of very interesting polynomials are not precisely of the form for which the known proof is valid. These polynomials 'define' certain (Landau-Ginzburg) conformal field theories which also have a geometric (sigma model) interpretation.

The generic presence of a circle isometry that exists for a Sasakian manifold partly motivates the study of T-duality for complex supersymmetric singularities in chapter 4. Some ingredients in the description of such singularities return in an apparently quite different context in chapter 3, where they are used to construct abstract conformal field theories which describe supersymmetric string vacua. In particular, weighted homogeneous polynomials are quite generally used to construct superconformal field theories. Some specific choices of the polynomial correspond to conformal field theories which have a known interpretation as coset conformal field theories. The corresponding symmetric spaces are Kähler-Einstein.

2.1 Supersymmetry, Spinors and Holonomy

2.1.1 Supersymmetry and Differential Geometry

We are interested in supersymmetric vacua of string theory of the form

$$\mathcal{M}_{10} = \mathbb{R}^{9-d,1} \times \mathcal{M}_d, \tag{2.1}$$

in the absence of fluxes and with a constant dilaton. If $\mathcal{M}_d$ is a smooth d-dimensional manifold, the low energy effective theory is the appropriate supergravity theory in this background. If this geometry is indeed a vacuum, the Ricci tensor of $\mathcal{M}_{10}$ must vanish. To find the number of conserved supersymmetries in this background one considers the supersymmetric variations of all the fermionic fields. In the backgrounds of this form, these variations are parametrized by a spinor field. They are proportional to the spinor or to its covariant derivative. The number of conserved supersymmetries is equal to the number of covariantly constant sections of the spinor bundle over $\mathbb{R}^{9-d,1} \times \mathcal{M}_d$, times the number of independent supersymmetry transformations that can be constructed out of one section, which is $n = 1$ for heterotic and $n = 2$ for Type II theories. The spinors can be decomposed into spinors over $\mathcal{M}_d$ and spinors over $\mathbb{R}^{9-d,1}$. The number of supersymmetry charges conserved by the background $\mathbb{R}^{9-d,1} \times \mathcal{M}_d$ is

$$s = n\, 2^{\lfloor \frac{10-d}{2} \rfloor} \ell, \tag{2.2}$$

Dimension d of $\mathcal{M}_d$ is d	Holonomy group Hol($\mathcal{M}_d$)	Name of $\mathcal{M}_d$
$d = 2n$	$U(n)$	Kähler
$d = 2n$	$SU(n)$	Calabi-Yau
$d = 4n$	$Sp(n)$	Hyper-Kähler
$d = 4n$	$Sp(n)Sp(1)$	Quaternionic Kähler
$d = 7$	G_2	G_2-Manifold
$d = 8$	$Spin(7)$	$Spin(7)$-Manifold

Table 2.1: Berger's list of possible reduced holonomy groups of simply connected irreducible non-symmetric Riemannian manifolds.

where $n = 1$ for heterotic theories and $n = 2$ for Type II theories. The number $2^{\lfloor \frac{10-d}{2} \rfloor}$ is the number of covariantly constant spinors on $\mathbb{R}^{d-1,1}$ and ℓ is the number of covariantly constant spinors on $\mathcal{M}_d$. So the condition for supersymmetry is that $\mathcal{M}_d$ has at least one covariantly constant spinor.

Manifolds which admit covariantly constant spinors are characterized by their holonomy. The holonomy group of a general $\mathcal{M}_d$ is $SO(d)$, but if its spinor bundle admits a covariantly constant section, the holonomy group has to be a proper subgroup $H \subset SO(d)$. After all, the covariantly constant spinor obviously transforms in the trivial representation of H, but this representation must be obtained by decomposing the spinor representation of $SO(d)$ into representations of H.

The possible subgroups that can appear are classified. If $\mathcal{M}_d$ is a product manifold, its holonomy group is the product group of the individual holonomy groups. If $\mathcal{M}_d$ is a simply connected Riemannian symmetric space it can be written as G/H where G is a Lie group of isometries that acts transitively and $H \subset G$ is the isotropy subgroup, which leaves a point fixed, then Hol$(\mathcal{M}_d) = H$. If $\mathcal{M}_d$ is a simply connected Riemannian symmetric space G/H, the holonomy group is H. This was shown long ago by Cartan. Finally, if $\mathcal{M}_d$ is a simply connected Riemannian manifold that is not a product manifold and non-symmetric, there is a list of possible holonomy groups, due to Berger. In addition to the generic case $SO(d)$, there are the cases listed in table 2.1.

The holonomy groups in table 2.1 imply certain parallel tensors, and hence certain geometric structures, see, for example [65]. If the holonomy is $U(n)$, it is possible to split the tangent bundle into a holomorphic and an antiholomorphic part. Such a split is effected by the complex structure $J(.,.)$ which is an endomorphism of the complexified tangent bundle of $\mathcal{M}$. To speak of holonomy, there must be a connection. It is always possible to choose a Hermitean metric g compatible with J, i.e. $g(.,.) = g(J.,J.)$. From these two structures it is possible to construct a two-form $\omega(.,.) = g(J.,.)$, using the property that $J^2 = -1$. This two-form is non-degenerate. If it is also closed, ω is symplectic and, by compatibility with J, Kähler; the Hermitian connection coincides with the Christoffel connection and it is the sum of a holomorphic one-form taking values in the endomorphisms of the holomorphic tangent bundle, in addition there is an entirely antiholomorphic equivalent. This means that under parallel transport (anti-)holomorphic tangent vectors remain (anti-)holomorphic, so the holonomy is contained in $U(n)$. Using this connection J is covariantly constant, and so is ω.

On a Kähler manifold one can construct the Ricci form from the Riemann tensor of the Kähler metric, using the complex structure: using the Dolbeault differentials ∂ and $\bar{\partial}$ it can be expressed as $\mathcal{R} = i\partial\bar{\partial}\log\sqrt{\det g}$. This is manifestly closed, but usually not exact, because $\det g$ is not a scalar. The cohomology class of the Ricci form is 2π times the first Chern class of the (tangent bundle of the) Kähler manifold. The Chern class is an analytic invariant: continuous changes of the metric do not alter the cohomology class of $\mathcal{R}$.

In addition to preserving some supersymmetry, the geometry of (2.1) should solve the equations of motion, which means that the Ricci tensor of $\mathcal{M}_d$ must vanish. The $U(1) \hookrightarrow U(n)$ part of the holonomy is generated by Ricci tensor. So if the Ricci tensor vanishes, the holonomy is $SU(n) \subset U(n)$. But given a Kähler manifold with Kähler form ω it is possible to deform this to ω' without altering the cohomology class of the Kähler form (the Kähler form cannot be exact, because that would be contradictory to it being nondegenerate). The new Kähler form ω' is such that its associated Ricci form is precisely the first Chern class. Yau's theorem implies that such a choice of ω' is always possible. So a Kähler manifold with $SU(n)$ holonomy admits a metric with vanishing Ricci tensor. The restrictions on a hyper-Kähler metric are so strong, that necessarily any such metric is Ricci flat.

The hyper-Kähler and Calabi-Yau manifolds, and singularities, will play a considerable rôle in the rest of this chapter. Some important reasons for this are the following. As complex manifolds, powerful tools from algebraic geometry are known to study such spaces. The Kähler structure of these manifolds appears naturally in $\mathcal{N} = 2$ superconformal models discussed in chapter 3. The properties of these models are used in chapter 4 to relate hyper-Kähler and Calabi-Yau singularities to other backgrounds of string theory.

2.1.2 Hyper-Kähler Surface Singularities

This section discusses the geometry of the best understood supersymmetric singularities: complex surface singularities which are hyper-Kähler. These are complex surfaces, so locally they look like $\mathbb{C}^2 \simeq \mathbb{R}^4$, have holonomy group $Sp(1) \simeq SU(2)$, with an isolated singularity. A great deal is known about these, both from a mathematical point of view and also from the perspective of string theory. Because so much is known about them, they take a special place. Some of the special properties they have are:

- They are classified;
- The classification is isomorphic to that of many other interesting objects in mathematics and string theory;

- They have a number of different descriptions which illustrate descriptions of higher dimensional singularities;
- For the hyper-Kähler surface singularities all descriptions are interchangeable, unlike for higher dimensional ones;
- The hyper-Kähler singularities are a motivation and the clearest example of the T-duality for cones discussed in chapter 4.

One way to describe the hyper-Kähler surface singularities, is as quotients of $\mathbb{C}^2$. On a space of $SU(2)$ holonomy there is a parallel holomorphic two-form. On the covering $\mathbb{C}^2$ such a two from can be taken as $\omega = \mathrm{d}z_1 \wedge \mathrm{d}z_2$. This two-form is preserved by $SU(2)$ mixing the holomorphic coordinates. This group has a fixed point at the origin. Take an discrete subgroup $\Gamma \subset SU(2)$. Then the quotient space $\mathbb{C}^2/\Gamma$ is a complex surface with a singularity at the origin and $SU(2)$ holonomy, with the constant holomorphic two-form given by projection of $\mathrm{d}z_1 \wedge \mathrm{d}z_2$ on the covering space.

The discrete subgroups of $SU(2)$ were classified in the nineteenth century by Klein and the quotient singularities $\mathbb{C}^2/\Gamma$ are also referred to as Kleinian singularities. The Kleinian singularities exhaust the hyper-Kähler surface singularities. There is a one-to-one correspondence of the subgroups $\Gamma \subset SU(2)$ and Dynkin diagrams of simply laced Lie algebras. This motivates the name 'ADE-singularities' which is also commonly used. In fact, there is a huge web of connections, containing the topology of desigularizations of these singularities, the representation theory of $\Gamma \subset SU(2)$ [60] and a lot of different areas of mathematics and physics, such as conformal field theory [17] and gauge theories [61].

From the description as quotients, one can obtain a different description. One can think of a point in $\mathbb{C}^2$ as the zero of a monomial

$$z_0 \leftrightarrow (z - z_0) = 0. \tag{2.3}$$

Such monomials are the prime divisors of polynomials with complex coefficients, and the algebraic structure of polynomials can be used to study geometry. An arbitrary divisor in the polynomial ring $\mathbb{C}[z_1, z_2]$ is of the form

$$\prod_{i=1}^{k} (z - z_i)^{\alpha_i},$$

and can be viewed as the divisor

$$\sum_{i=1}^{k} \alpha_i [z_i]$$

in the sense of algebraic geometry.

$\Gamma \subset \mathbb{C}^2$	$F_\Gamma(z_1, z_2, z_3)$
A_n	$z_1^n + z_2^2 + z_3^2$
D_n	$z_1^{n-1} + z_1 z_2^2 + z_3^2$
E_6	$z_1^4 + z_2^3 + z_3^2$
E_7	$z_1^3 + z_1 z_2^3 + z_3^2$
E_8	$z_1^5 + z_2^3 + z_3^2$

Table 2.2: The hyper-Kähler surface singularities as quotients $\mathbb{C}^2/\Gamma$ and as surfaces $F_\Gamma^{-1}(0) \subset \mathbb{C}^3$.

From the point of view of algebraic geometry it is the polynomial ring $\mathbb{C}[z_1, z_2]$, generated by z_1 and z_2 which characterizes the space. Consider the A_n singularity

$$\begin{aligned} \mathcal{A}_n &= \mathbb{C}^2/\Gamma, \\ \Gamma &: (z_1, z_2) \mapsto (e^{\frac{2\pi i}{n+1}} z_1, e^{-\frac{2\pi i}{n+1}} z_2). \end{aligned} \tag{2.4}$$

Not every polynomial in$\mathbb{C}[z_1, z_2]$ is invariant under the action of Γ. The subset of Γ invariant polynomials is generated by the three generators

$$\begin{aligned} u &= z_1^{n+1}, \\ v &= z_2^{n+1}, \\ x &= z_1 z_2, \end{aligned} \tag{2.5}$$

which clearly satisfy the relation

$$uv = z^{n+1}. \tag{2.6}$$

So the divisors on $\mathcal{A}_n$ are those polynomials in $\mathbb{C}[u, v, x]$ which vanish on the hypersurface defined by (2.6). Or, put differently, as far as algebraic geometry is concerned, the quotient singularity $\mathbb{C}^2/\mathbb{Z}_{n+1}$ is the hypersurface $z_1^{n+1} + z_2^2 + z_3^2 = 0$ in $\mathbb{C}^3$.

Similarly all the ADE-singularities[1] have a description as surfaces $F_{ADE}^{-1}(0)$ in $\mathbb{C}^3$. The polynomials $F_{ADE}(z_1, z_2, z_3)$ are collected in table 2.2. Note that all the polynomials are weighted homogeneous, i.e. for each F_Γ there exists a set of weights a_i which are (positive) integers, such that

$$F(\lambda^{a_1} z_1, \lambda^{a_2} z_2, \lambda^{a_3} z_3) = \lambda^d F(z_1, z_2, z_3). \tag{2.7}$$

The description as a quotient singularity $\mathbb{C}^2/\Gamma$ also provides a third description, which is more differential geometric in nature. The space $\mathbb{C}^2 \backslash \{0\}$ can be fibered by three-spheres.

[1]The D_{k+2} singularity can be obtained by a $\mathbb{Z}_2$ quotient of the A_k singularity. The A_k singularity is $\mathbb{C}^2/\mathbb{Z}_{k+1}$ where $\mathbb{Z}_{k+1}$ acts on the coordinates of $\mathbb{C}_2$ as $(z_1, z_2) \sim (e^{2\pi i/(k+1)} z_1, e^{-2\pi i/(k+1)} z_2)$. Quotienting further by $\mathbb{Z}_2 :\ (z_1, z_2) \sim (z_2, -z_1)$ yields a D_k singularity.

The metric $\mathrm{d}s^2 = \mathrm{d}z_1\mathrm{d}\overline{z}_1 + \mathrm{d}z_2\mathrm{d}\overline{z}_2$ is written as $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2$, i.e. a cone over the three sphere. As $SU(2)$ acts on $\mathbb{C}^2$ in a way that leaves invariant $r^2 = |z_1|^2 + |z_2|^2$, an ADE-singularity can be written as the metric cone

$$\begin{aligned} \mathbb{C}^2/\Gamma &= \mathbb{R}_+ \times S^3/\Gamma, \\ \mathrm{d}s^2 &= \mathrm{d}r^2 + r^2\mathrm{d}\Sigma^2, \end{aligned} \tag{2.8}$$

where $\mathrm{d}\Sigma^2$ is the line element on the smooth space S^3/Γ. The action of Γ on S^3 is obtained from the action in the embedding $\mathbb{C}^2$. The spaces S^3/Γ are simple examples of a more general class discussed in section 2.2, which can all be viewed as circle fibrations.

The base of each A_k metric cone, $S^3/\mathbb{Z}_{k+1}$ is a circle bundle over S^2, and in fact all circle bundles over the two-sphere are of this form (they are so-called lens spaces). One way to view the lens spaces $S^3/\mathbb{Z}_{k+1}$, is as quotient spaces $(S^3 \times S^1)/U(1)$, see for example [80]. Let S^3 be parametrized by $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$ that satisfy the condition $|z_1|^2 + |z_2|^2 = 1$. Let S^1 be parametrized by $\sigma = e^{i\theta}$. The $U(1)$ equivalence relation identifies $(z_1, z_2) \sim (e^{i\phi}z_1, e^{i\phi}z_2)$ and $\sigma \sim e^{-i(k+1)\phi}\sigma$. By an equivalence transformation one can always set $\sigma = 1$, unless $k+1 = 0$. This 'gauge choice' fixes the $U(1)$ action up to a $\mathbb{Z}_{k+1}$ subgroup. So quotient space is $S^3/\mathbb{Z}_{k+1}$. This is bundle over S^2, with the projection

$$\begin{aligned} \pi : S^3/\mathbb{Z}_{k+1} &\to S^2 \\ \mathbf{z} &\mapsto \tilde{\mathbf{v}} = \mathbf{z}^\dagger\tilde{\sigma}\mathbf{z}, \end{aligned} \tag{2.9}$$

where $\vec{\sigma}$ indicates the three Pauli matrices. The vector $\vec{v}$ has unit length, because $|z_1|^2 + |z_2|^2 = 1$, and hence parametrizes S^2. When $k+1 = 0$, the total space is the trivial bundle $S^2 \times S^1$, and when $k+1 = 1$, the fiber bundle structure is the Hopf fibration $S^1 \to S^3 \to \mathbb{P}^1 \simeq S^2$.

The bases of the D_{k+2} metric cones can be considered in a similar fashion, as quotient spaces $(S^3 \times S^1)/(U(1) \ltimes \mathbb{Z}_2)$. The $U(1)$ part acts as it does in the A_k case, the $\mathbb{Z}_2$ acts as

$$\mathbb{Z}_2 : ((z_1, z_2); s) \mapsto ((\overline{z}_2, -\overline{z}_1); \overline{s}). \tag{2.10}$$

The $\mathbb{Z}_2$ action also acts on the image of the projection π. The image is not the entire S^2, but rather $S^2/\mathbb{Z}_2$, with antipodal points identified, i.e. the bases of the D_{k+2} metric cones are circle bundles over the base $\mathbb{RP}^2$.

The different descriptions each have their advantages, emphasizing different properties of the ADE-singularities. The algebraic geometric description as surfaces in $\mathbb{C}^3$ emphasizes the complex structure of the singularity. Actually, since these are hyper-Kähler spaces, they have three independent complex structures I_1, I_2, I_3 and $a_1I_1 + a_2I_2 + a_3I_3$ is again a complex structure if the three real numbers a_i satisfy $a_1^2 + a_2^2 + a_3^2 = 1$. So it is better to say that it emphasizes one particular complex structure out of the whole S^3's worth. A deformation of the polynomial defining the hypersurface corresponds to a deformation of the complex geometry of the singularity.

Consider the example of an A_1 singularity, defined as $uv - x^2 = 0$ in $\mathbb{C}^3$. This can be deformed to $uv = (x+\epsilon)(x-\epsilon)$. The surface defined by this deformed equation no longer passes through $u = v = 0$, where the singular point was. Instead, the product of the moduli $|u|$ and $|v|$ is determined by the equation, and it vanishes at $x = \pm\epsilon$. Only the difference of phases of u and v is free. In the surface $uv = (x+\epsilon)(x-\epsilon)$ there is a two sphere which is a circle fibration over the line segment from $x = -\epsilon$ to $x = +\epsilon$. This is an example of a kind of deformation which can be applied to any polynomial which defines a hypersurface with an isolated singularity at the origin:

$$F(x_1, \ldots, x_n) \to F(x_1, \ldots, x_n) + \mu. \tag{2.11}$$

This deformation will be considered in chapter 4.

It is possible to characterize all deformations of the ADE-singularities. The number of independent deformations actually equals the rank of the corresponding ADE Lie algebra. By successive deformations a singular surface can be 'desingularized' by blowing up two-spheres. Hyper-Kähler metrics on the resulting smooth non-compact manifolds are known [62, 63, 64]. The construction of these metrics makes use of the fact that the singular spaces are quotient singularities $\mathbb{C}^2/\Gamma$ and the McKay correspondence [60] which relates the representation theory of Γ and the topology of the smoothed space. Far away from the origin, the smoothing does not change much and the smooth metrics asymptote to the metric cones $\mathbb{R}_+ \tilde{\times} (S^3/\Gamma)$.

Crucially, in one description the differential geometry of a singularity is explicit but deformations of the singularity are not at all apparent: this is the metric cone description. In another description deformations are apparent, but there is no hyper-Kähler metric apparent: the description as surfaces in $\mathbb{C}^3$. The logical connection between these two descriptions, is the realization as quotient singularities. The deformation parameters in the polynomials are related to the representation theory of the quotient group.

In higher dimensions, not all descriptions of supersymmetric singularities are interchangeable. That is to say, there are supersymmetric singularities which are not quotient singularities. Such singularities may have descriptions as Ricci flat metric cones with the right holonomy, $SU(n)$ or $Sp(n/2)$, but whose base manifolds are not S^{2n-1}/Γ. It is not so clear how to deform such a metric conical singularity to a smooth space which still admits a Calabi-Yau or hyper-Kähler metric if there is no apparent hypersurface description $F^{-1}(0) \subset \mathbb{C}^{n+1}$. Nor is it immediately clear if there might be a hypersurface description. In fact, for a lot of interesting singularities there is no hypersurface description, like for example $\mathbb{C}^3/\mathbb{Z}_3$. Approaching the matter from the other direction, starting with a hypersurface singularity, it is often difficult to find a differential geometric description of it, like an explicit metric, or the group of isometries of the space

These issues are discussed in the subsequent sections. Typical questions are the following. What are the conditions on a polynomial F so that $F^{-1}(0)$ in $\mathbb{C}^{n+1}$ is a supersymmetric singularity which can be used as a string vacuum? What can be said about the geometry of a singularity defined by such a polynomial? If the singularities are not quotient singularities, what is left of existing and conjectured correspondences in the spirit of the McKay correspondence, and what new correspondences are gained by leaving the set of quotient singularities? Some questions will be answered in the following sections, and some interconnections will be discussed. Together with the ingredients of chapter 3 these will be put to use in chapter 4.

2.2 METRIC CONES

An acceptable supersymmetric singularity of dimension $d = 2n$ which can serve as a string background must be Ricci flat and have a holonomy group which is contained in $SU(n)$. Take as such a singularity the metric cone $\mathcal{C}(L)$,

$$\begin{aligned} \mathcal{C}(L) =& \mathbb{R}_+ \tilde{\times} L \\ \mathrm{d}s_{2n}^2 =& \mathrm{d}r^2 + r^2 \mathrm{d}s_{2n-1}^2. \end{aligned} \tag{2.12}$$

That is to say, it is the warped product of the manifold L of dimension $2n-1$ with the half line $r > 0$, with the above metric. The question is: what are the properties of the base manifold L_{2n+1}?

An answer was given by Bär [7], who studied metric cones of restricted holonomy. Essentially, one uses the canonical vector field on a metric cone, $r\partial/\partial r$, called the Euler vector field. With this vector field, the different special tensor fields on the cone can be mapped to special tensor fields on L_{2n+1}.

First, if the Ricci tensor of $\mathcal{C}(L)$ vanishes, then L is a positively curved Einstein manifold. We call a manifold Einstein if there is a *constant* number λ such that the Ricci tensor Ric and the metric tensor g satisfy

$$Ric = \lambda g, \tag{2.13}$$

i.e. its scalar curvature is a constant. Only the sign of the Ricci curvature is really interesting, since the absolute value can be changed by rescaling L. Conversely, if B is an Einstein manifold of positive curvature, it can always be appropriately scaled to make $\mathcal{C}(L)$ a Ricci flat cone[2].

THE GEOMETRY OF L

Next, the restricted holonomy of $\mathcal{C}$ gives rise to various parallel tensors on the cone. The Kähler form ω on $\mathcal{C}(L)$ satisfies $\mathrm{d}\omega = 0$ and $\bigwedge^n \omega \neq 0$. Contracting the Euler vector with ω yields a one-form η on L. This one-form satisfies

$$\eta \wedge (\mathrm{d}\eta)^{n-1} \neq 0, \tag{2.14}$$

[2]The rescaling is proportional to $n-1$, with some constants of proportionality dependent on conventions, $n = d/2$ being the complex dimension of the cone.

everywhere on L. This equation states that η is a contact form on L. A symplectic metric cone $\mathcal{C}(L)$ has a base L that is a contact manifold.

In addition to the contact form, a contact manifold also has a unique vector field, dual to η: the Reeb vector field ξ. It satisfies

$$\begin{aligned} \imath_\xi \eta &= 1 \\ \imath_\xi \mathrm{d}\eta &= 0. \end{aligned} \tag{2.15}$$

The Reeb vector field on L is obtained from the complex structure J on $\mathcal{C}(L)$, by acting with J on the Euler vector field of $\mathcal{C}(L)$. The contact form η, Reeb vector field ξ and an endomorphism T of the tangent bundle TL together define an almost contact structure on L. They satisfy

$$\begin{aligned} &\imath_\xi \eta = 1 \\ &T^2 = -id + \xi \otimes \eta. \end{aligned} \tag{2.16}$$

A compatible metric g must satisfy

$$g(T.,T.) = g(.,.) - \eta(.)\eta(.), \tag{2.17}$$

analogous to an almost Hermitean metric on an almost complex manifold.

On B the endomorphism $T : TL \to TL$ is obtained as

$$T(\phi) = -\nabla_\phi \xi, \tag{2.18}$$

via the covariant derivative, where ϕ is any section of TL. The tensor fields ξ, η, T and g on L form a special kind of metric contact structure because L is the base of a metric cone $\mathcal{C}(L)$ which is Kähler, i.e. on which the complex structure, Hermitean metric and symplectic form are compatible. This special kind of metric contact structure is called a Sasaki structure, and L is a Sasaki manifold.

One *definition* of a Sasaki manifold, is precisely that the metric cone over a manifold is Kähler iff the manifold is Sasaki. An equivalent definition, see for example [8], is a Riemannian manifold (M, g) with a Killing vector field of unit length ξ, and endomorphism T defined as $T(\phi) = -\nabla_\phi \xi$ for any section ϕ of TM that satisfies

$$(\nabla_\chi T)\,\psi = g(\chi, \psi) - g(\xi, \psi)\chi,$$

for all vector fields χ, ψ.

If the cone $\mathcal{C}(L)$ is hyper-Kähler, it has three independent complex structures which form a quaternion algebra. Analogously, L inherits three related Sasakian structures and L is a tri-Sasakian manifold. A good overview of the properties of (tri-) Sasakian manifolds used in this section and the next, is [8].

The Reeb vector field ξ that any Sasaki manifold L has (often called its characteristic vector field), gives rise to some important consequences. For one thing, it means that a metric cone has a Killing vector field which degenerates at the apex $r = 0$. One might be tempted to perform a T-duality along this isometry, and we are tempted to do so in chapter 4. The vector field ξ is also very interesting from a purely geometric point of view. Note that because ξ is nonvanishing, its integral curves define a one-dimensional foliation of L.

The space of leaves of this foliation turns out to be quite interesting. We call the space of leaves Z. When the leaves are closed curves, so the Reeb vector field is a Killing vector field of a $U(1)$ isometry, L is called quasi-regular. In this case Z is a Kähler space which can have finite quotient singularities. When Z is a smooth Kähler manifold, L is called regular. If Z has finite quotient singularities, L is called non-regular (L is called irregular if the leaves do not close).

Regularity is a very strong condition and many examples of Sasaki-Einstein manifolds are non-regular. Explicit metrics are rarely known, with the exception of homogeneous spaces. As we will see shortly, methods and results from algebraic geometry have provided means to prove the existence of (quasi-regular) Sasaki-Einstein metrics on a much larger class of spaces. However, these methods are not constructive, and they give only limited information about the differential geometry of the spaces. The spaces for which these methods apply, are described as specific kinds of affine hypersurfaces. This description is compatible in a natural way with our duality prescriptions discussed in chapter 4.

Recently explicit metrics have been found for many five and seven dimensional Sasaki-Einstein manifolds, including the first irregular ones [104, 73, 74], using a supergravity/string theory approach. Our present interest will be with quasi-regular Sasaki-Einstein manifolds, but within an adapted framework, irregular ones should be of great interest as well, especially for string theory. For example, they could be related to rather exotic irrational conformal field theories, through a gauge/gravity correspondence. We will not discuss these further. Rather, we focus of the geometry of the leaf-space Z of a quasi-regular Sasaki-Einstein manifold.

THE GEOMETRY OF Z

If each point in B has a neighborhood such that any leaf of the characteristic foliation intersects the transversal at most a finite number of times k, then L is called quasi-regular. Equivalently B is quasi-regular if the leaves are compact. So all Sasaki manifolds which appear as compact bases of cones are quasi-regular. If $k = 1$, L is called regular. A quasi-regular L that is not regular, is called non-regular. Regularity is a very strong condition. The vast majority of compact Sasaki spaces is non-regular.

At this point we have seen that the particular structure of a metric cone, or the Euler vector field, led to geometric structures on the link $L \hookrightarrow \mathcal{C}(L)$. The metric Calabi-Yau cones have Sasaki-Einstein links, either regular or non-regular. The hyper-Kähler cones have tri-Sasaki links, which will be discussed in more depth later. Now focus on the Sasaki-Einstein manifolds[3], and to be more specific, on the regular Sasaki-Einstein manifolds. It is

[3]The curvature of a Sasaki-Einstein manifold is necessarily positive and hence it can always be used to construct

G/H	$\mathbb{R}$-dimension
$\frac{SU(m+n)}{SU(m)\times SU(n)\times U(1)}$	$2mn$
$\frac{SO(n+2)}{SO(n)\times SO(2)}$	$2n+1$
$\frac{SO(3)}{SO(2)}$	2
$\frac{SO(2n)}{SU(n)\times U(1)}$	n^2-n-2
$\frac{Sp(n)}{SU(n)\times U(1)}$	n^2+n+2
$\frac{E_6}{SO(10)\times U(1)}$	32
$\frac{E_7}{E_6\times U(1)}$	54

Table 2.3: Hermitean symmetric spaces.

useful to consider the leaf space Z of the foliation of L by the Reeb vector field,

$$\pi : \quad L \longrightarrow Z.$$

The regular Sasaki structure ensures that S is a smooth Kähler manifold, and the fact that L is Sasaki-Einstein results in Z being Kähler-Einstein. Moreover Z is positively curved, $c_1(Z) > 0$: Z is a Fano[4] variety with a smooth Kähler-Einstein metric.

Explicit realizations of Kähler-Einstein Fano manifolds are provided by Hermitean symmetric spaces. These are compact Kähler manifolds and Riemannian symmetric spaces, and positively curved. As an aside, as such these spaces are geometrically formal, that is to say, the wedge product of harmonic forms is again a harmonic form. It is proved in [66] that any geometrically formal Kähler manifold of non-negative Ricci curvature is Einstein. The Hermitean symmetric spaces play an important part in the construction of superconformal field theories 3.4. The harmonic forms on the Hermitean symmetric spaces are in one-to-one correspondence with (c, c) primary operators in the conformal field theory. These special fields have the property that under the naive operator product, they form a nilpotent ring.

The Hermitean symmetric spaces are classified. Only spaces of which the dimension is not too large can be used to build metric cones for a superstring compactification. The Hermitean symmetric spaces are listed in table 2.3.

In dimension $d = 2$, the only Kähler-Einstein manifold with $c_1 > 0$ is

$$\mathbb{P}^1 \simeq SU(2)/U(1).$$

In dimension $d = 4$, the manifolds with $c_1 > 0$ are known as del Pezzo surfaces, those which admit a Kähler-Einstein metric have been classified [38] and are collected in table 2.4. On the del Pezzos obtained by blowing up $\mathbb{P}^2$ at three to eight generic points, no explicit

a Calabi-Yau metric cone.

[4]A manifold with $c_1 > 0$ is called a Fano manifold.

L, del Pezzo surface	Homogeneous, G/H
$\mathbb{P}^2$	$\frac{SU(3)}{SU(2)\times U(1)}$
$\mathbb{P}^1 \times \mathbb{P}^1$	$\frac{SU(2)}{U(1)} \times \frac{SU(2)}{U(1)}$
$dP_n = \mathbb{P}^2 \# \overline{\mathbb{P}^2}$, $3 \leq n \leq 8$	no

Table 2.4: Smooth del Pezzo surfaces admitting a Kähler-Einstein metric.

metrics are known. The del Pezzo surfaces dP_1 and dP_2 do not feature in the classification [38] of Tian and Yau. It is a well known fact in the mathematics community, that the del Pezzo surfaces dP_1 and dP_2 do not admit a Kähler-Einstein metric[5].

In general there can be several Sasaki-Einstein circle bundles over a base Z

$$\begin{array}{ccc} \mathcal{C}(L) & \hookleftarrow & L \\ & & \downarrow \pi \\ & & Z \end{array} \tag{2.19}$$

The first Chern class of the circle fibration L must divide the first Chern class of Z [36, 67] in order to get a smooth total space. *In concreto* this means that the possible regular Sasaki-Einstein manifolds are[6]

i. $S^5 \to \mathbb{P}^2$,

ii. $S^5/\mathbb{Z}_3 \to \mathbb{P}^2$,

iii. $T^{1,1} \to \mathbb{P}^1 \times \mathbb{P}^1$,

iv. $T^{1,1}/\mathbb{Z}_2 \to \mathbb{P}^1 \times \mathbb{P}^1$,

v. $\mathcal{S}_n \to dP_n$.

The metric cone over S^5 is just $\mathbb{R}^6$ and therefore not interesting from the point of view of singularities. The manifold $T^{1,1} \simeq SO(4)/SO(2) \simeq (SU(2) \times SU(2))/U(1)$ is the link of the conifold. There is a natural interpretation why only the $\mathbb{Z}_3$ quotient of S^5 gives a regular Sasaki-Einstein space, from the perspective of quotienting $\mathbb{C}^3$ by a discrete subgroup $\Gamma \subset SU(3)$. $\mathbb{C}^3$ can be viewed as the total space of the tautological bundle over $\mathbb{P}^2$. The $U(1) \hookrightarrow SU(3)$ which acts only on the fiber but not on the base, acts on the homogeneous coordinates as $[z_1 : z_2 : z_3] \mapsto [\eta z_1 : \eta z_2, \eta z_3]$. The only nontrivial discrete subgroup

[5] This is because their automorphism groups are not reductive. But a theorem of Matsushima says that a Kähler-Einstein manifold with $c_1 > 0$ must have a reductive automorphism group.

[6] The spaces $T^{1,1}/\mathbb{Z}_2$ and $S^5/\mathbb{Z}_3$ are regular because the canonical class of $\mathbb{P}^1 \times \mathbb{P}^1$ is $2H$, twice the hyperplane class, and similarly $\mathcal{K}_{\mathbb{P}^2} = 3H$. The other del Pezzo surfaces in the list have $\mathcal{K} = H$.

$\Gamma \in U(1)$ that leaves the holomorphic three-form invariant is generated by $e^{2\pi i/3}$. This precisely 'shortens' the fiber by a factor of three, and so increases the Chern class of the bundle by three.

In for string theory, the case $d = 6$ is also interesting. The Kähler-Einstein Fano manifolds of dimension $d = 6$ have not been classified. The homogeneous manifolds are known,

i. $\mathbb{P}^3$,

ii. $\mathbb{P}^2 \times \mathbb{P}^1$,

iii. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$,

iv. $\widetilde{Gr}(5, 2)$,

v. $F(1, 2|3)$,

where $\widetilde{Gr}(5, 2)$ is the real Grassmannian $SO(5)/(SO(3) \times SO(2)$ and $F(1, 2|3)$ is the flag manifold $(SU(3) \times SU(2))/(SU(2) \times U(1) \times U(1))$. There are homogeneous Sasaki-Einstein manifolds that are circle bundles over these spaces[7]. These manifolds are known from the study of compactifications of eleven dimensional supergravity of the form AdS_4$\times$ $\mathcal{M}_7$ [68]. Some of these spaces are even tri-Sasakian. Examples of inhomogeneous Kähler-Einstein manifolds are $\mathbb{P}^1 \times dP_n$.

About tri-Sasakian manifolds, more stringent results can be stated. These can be found in [8]. All homogeneous tri-Sasakian manifolds in any dimension are known and constructions exist which given one tri-Sasakian space yield others. At the base of these results lies the structure of tri-Sasakian manifolds. As Sasaki-Einstein manifolds, they can be seen as circle bundles over Kähler-Einstein spaces. But the tri-Sasakian structure allows them to be seen also as $SU(2)$ fibrations over quaternionic Kähler manifolds. Also, the twistor space of the quaternionic Kähler manifold is the Kähler-Einstein manifold. A very good discussion is presented in [8].

Tri-Sasakian manifolds will not be further discussed here. Yet, they are very interesting for a number of reasons. Explicit geometric constructions of such manifolds exist, based on the hyper-Kähler quotient [62]. The hyper-Kähler cones preserve more supersymmetry than a generic Calabi-Yau cone and the structure as $Sp(1)$ bundles might provide a way to consider non-abelian duality for hyper-Kähler cones in a spirit similar to that of T-duality in chapter 4. This, however remains a subject left entirely for future study.

SUMMARY

Perhaps the main lesson from the description of supersymmetric singularities as metric cones, is that such cones generically have a $U(1)$ isometry which degenerates at the apex of

[7]A homogeneous Sasaki-Einstein manifold has a transitive group of isometries which preserve the Sasakian structure.

Metric Cone $\mathcal{C}(L)$	L	$Z \simeq L/U(1)$
symplectic	contact	symplectic
Kähler	Sasaki	Kähler
Calabi-Yau	Sasaki-Einstein	Kähler-Einstein and Fano
hyper-Kähler	tri-Sasaki	Kähler-Einstein, Fano, twistor space of quaternionic Kähler

Table 2.5: Relation of geometries of some metric cones and associated spaces

the metric cone. This isometry is generated by the characteristic (or Reeb) vector field that any Sasaki manifold has. Some particular simple, exceptionally symmetric Sasaki-Einstein manifolds are $U(1)$ bundles over Hermitean symmetric spaces. The Hermitean symmetric spaces also appear in the construction of some particularly symmetric worldsheet conformal field theories which can be used to describe supersymmetric string compactifications, which appear in section 3.4.

The largest class of Sasaki-Einstein spaces fall outside this category. They are non-regular and thus $U(1)$ bundles over Einstein-Kähler spaces with isolated quotient singularities. Recently many such spaces were found, using algebraic geometric considerations. These constructions show that an orbifold Kähler-Einstein metric must exist on a large class of varieties, but does not explicitly construct such metric, not unlike the proof that certain varieties admit a Calabi-Yau metrics, based on algebraic geometric criteria. This construction can be used to construct supersymmetric cones as well, and it does so in terms of hypersurfaces defined by complex polynomials. These matters are discussed in section 2.3.

2.3 HYPERSURFACES

The description of singularities as hypersurfaces $\mathcal{C} = F^{-1}(0) \subset \mathbb{C}^{n+2}$ provides a direct way to deform a singularity. By deforming the defining polynomial, a hypersurface may be completely smoothed. A deformation of the defining polynomial can be interpreted as a deformation of the complex structure of $\mathcal{C}$. There is no simple way to smooth a singularity in a metric cone or quotient description. A smoothing operation normally has negligible effect asymptotically far away from the singular point, but does not fit with a global description in terms of a quotient or a metric cone that is also applicable near the smoothed singularity.

An asymptotic metric cone description is useful, as it provides a differential geometric picture with a characteristic Killing vector field on a Sasaki-Einstein link, which is generic for any supersymmetric metric cone. Hypersurface descriptions turn out to be not only useful to consider deformations of singular cones, but also to characterize Sasaki-Einstein manifolds in a way unlike those used in section 2.2. In particular, projective hypersurfaces, defined as the zero locus of a single weighted homogeneous polynomial in an appropriate weighted projective space, can be an algebraic geometric way to describe varieties that admit Kähler-Einstein metrics, possibly with orbifold singularities. Such varieties can be used to construct metric cones on non-regular Sasaki-Einstein manifolds, as S^1 bundles over the Kähler-Einstein base. Additionally, the links of projective hypersurfaces can be related to fiber bundles over a S^1 base. Topological properties of these bundles are related to the analytic properties of the hypersurface singularity. It is the object of this section to introduce these two viewpoints, both for hypersurfaces in $\mathbb{C}^3$ and in higher dimensions.

2.3.1 The ADE-singularities as hypersurfaces

The ADE-singularities have descriptions as hypersurfaces $F^{-1}_{ADE}(0) \subset \mathbb{C}^3$. The polynomials F_{ADE} are listed in table 2.2. These singularities are quite special, as discussed in section 2.1.2, for many reasons. For one, they also have descriptions as quotients $\mathbb{C}^2/\Gamma$ and hence also as metric cones. As quotient singularities, the McKay correspondence relates the homology of resolutions to the representation theory of the quotient groups, a point which has a beautiful string theoretic interpretation [61]. As surface singularities, both resolutions and deformations blow up two-cycles. The distinction between complex and Kähler deformations is not an invariant notion, because of the $Sp(1)$-family of complex structures on these hyper-Kähler surfaces. In higher dimensions, not all of these properties are simultaneously present in general.

The polynomials F_{ADE} are weighted homogeneous, they satisfy (2.7),

$$F(\lambda^{a_1} z_1, \lambda^{a_2} z_2, \lambda^{a_3} z_3) = \lambda^d F(z_1, z_2, z_3).$$

So a hypersurface $\mathcal{C} = F^{-1}(0)$ admits a $\mathbb{C}^* = \mathbb{R}_+ \times U(1)$ action, like a supersymmetric metric cone does. The link L of a metric cone $\mathcal{C}(L)$ is obtained as $L = \mathcal{C}(L)/\mathbb{R}_+$. Analogously, one can fix the $\mathbb{R}_+$ scaling of $\mathcal{C} = F^{-1} \subset \mathbb{C}^{n+2}$ by intersecting the hypersurface with a small sphere,

$$\begin{aligned} S_r^{2n+3} &= \{\mathbf{z} \in \mathbb{C}^{n+2} : \sum_{i=1}^{n+2} |z_i|^2 = r^2\}, \\ \mathcal{C} &= \{\mathbf{z} \in \mathbb{C}^{n+2} : F(\mathbf{z}) = 0\}, \\ L_r &= \mathcal{C} \cap S_r^{2n+3}. \end{aligned} \tag{2.20}$$

which envelops an isolated singularity at the origin. For any hypersurface $\mathcal{C} = F^{-1}(0)$ defined by a weighted homogeneous F with an isolated singularity at the origin it makes sense to consider its link L_r in this way and write $\mathcal{C}(L)$.

One may ask to what extent this notion of a link is related to the link of a metric cone. The ADE-singularities have descriptions as metric cones, and one can compare the two notions. Let's call these the 'metric link' and the 'analytic link'. First of all, the metric links are S^3/Γ and are Sasaki-Einstein manifolds. The base space of each S^3/Γ is $S^3/U(1) \simeq (\mathbb{C}^2\backslash\{0\})/\mathbb{C}^* \simeq \mathbb{P}^1$. The analytic links can be viewed as $U(1)$ bundles over certain base spaces $Z(\Gamma)$. The space $Z(\Gamma)$ is characterized as the projective hypersurface $F^{-1}(0)$ in a weighted projective space defined by the weighted $\mathbb{C}^*$ action on the weighted homogeneous polynomial F.

The projective hypersurfaces $Z(\Gamma)$ are characterized using the adjunction formula. Recall the adjunction formula in ordinary projective space, see, for example [76]. It gives the canonical bundle of a hypersurface $\mathcal{P} = F^{-1}(0) \subset \mathbb{P}^m$. Such a hypersurface is the zero locus of a section of the line bundle $\mathcal{O}_{\mathbb{P}^m}(d)$, where d is the degree of the homogeneous polynomial F that defines the hypersurface $\mathcal{P}$. It can also be viewed as a submanifold of $\mathbb{P}^m$. There is the following short exact sequence,

$$0 \to \mathcal{TP} \overset{i}{\hookrightarrow} \mathcal{T}\mathbb{P}^m|_{\mathcal{P}} \overset{\nabla F}{\to} \mathcal{O}(d)_{\mathbb{P}^m}|_{\mathcal{P}} \to 0. \tag{2.21}$$

The meaning of this sequence is as follows, reading from left to right. The tangent bundle to $\mathcal{P}$ is a subbundle of the tangent bundle to the embedding $\mathbb{P}^m$, restricted to $\mathcal{P}$, so there is an inclusion map. The next arrow maps every tangent vector $X^i\nabla_i \in \mathcal{T}\mathbb{P}^m|_{\mathcal{P}}$ to a section of $\mathcal{O}_{\mathbb{P}^m}(d)$, i.e. to a homogeneous polynomial of degree d. Its kernel is formed by vectors tangent to $\mathcal{P}$. The map that achieves this is the covariant gradient,

$$\nabla_X F = X^i(F,_i + \Gamma_i F).$$

The second term involves a connection Γ_i on $\mathcal{O}_{\mathbb{P}^m}(d)$, but restricted to $\mathcal{P}$ it drops out, as $F = 0$ on $\mathcal{P}$ by definition. The vectors mapped to zero are the vectors tangent to $\mathcal{P}$ since by definition $\mathcal{P}$ is the surface of which F has the constant value $F = 0$. The short exact sequence (2.21) implies for the determinant line bundles

$$\det \mathcal{T}\mathbb{P}^m|_{\mathcal{P}} \simeq \det \mathcal{TP} \otimes \mathcal{O}_{\mathbb{P}^m}|_{\mathcal{P}}.$$

The determinant bundle of the cotangent bundle to a complex manifold is also called the canonical bundle $\mathcal{K}$, and its dual, the determinant bundle of the tangent bundle, is the anti-canonical bundle, denoted by $-\mathcal{K}$ or $\mathcal{K}^*$. The above expression implies that the canonical bundle of $\mathcal{P}$ is given by

$$\mathcal{K}_{\mathcal{P}} \simeq (\mathcal{K}_{\mathbb{P}^m} \otimes \mathcal{O}(d))\,|_{\mathcal{P}}. \tag{2.22}$$

This relation is the statement of the adjunction formula. As $\mathcal{K}_{\mathbb{P}^m} \simeq \mathcal{O}_{\mathbb{P}^m}(-m-1)$, the adjunction formula can be written as

$$\mathcal{K}_{\mathcal{P}_d \subset \mathbb{P}^m} \simeq \mathcal{O}_{\mathbb{P}^m}(d-m-1)|_{\mathcal{P}_d}, \tag{2.23}$$

For a degree d hypersurface in $\mathbb{P}^m$.

The adjunction formula can be generalized to weighted projective hypersurfaces (see section 2.3.3). The ordinary projective space $\mathbb{P}^m$ is a special case, with all weights

$$a_1 = \ldots = a_{m+1} = 1.$$

The adjunction formula applied to the complex curves Z_Γ, written as zero loci of the ADE polynomials F_Γ in the appropriate weighted projective space gives the first Chern class of Z_Γ. Hence gives its Euler characteristic, $\chi = -2c_1$, in terms of the first Chern classes of the embedding space and a that of the line bundle with section F_Γ. The result is

$$c_1(Z_\Gamma) = -d + \sum_{i=1}^{3} a_i = 1. \tag{2.24}$$

For all ADE-polynomials, listed in table 2.2, the relation between weights and weighted degree is as in (2.24). Such hypersurfaces are called anticanonically embedded,

$$-\mathcal{K}_{ADE} = \mathcal{O}(1).$$

Many higher dimensional hypersurfaces are not anticanonically embedded, while their defining polynomial does define a supersymmetric *affine* hypersurface. Consequently, they are of importance for string theory. But from the mathematicians' point of view the anticanonically embedded ones have received special attention. It will turn out that the distinction between anticanonically embedded hypersurfaces and others also has a (slight) consequence for the string theory duality transformation. In particular, the worldsheet field theories employed in the formulation of the duality transformation describe exactly affine hypersurfaces of the 'anticanonical' kind, and particular cyclic quotients of surfaces which are not of the 'anticanonical' kind. These worldsheet models are discussed at the end of section 3.3.2 and in section 4.4.

2.3.2 Topology of Affine Hypersurfaces

This section is relatively disconnected from the rest. We discuss some aspects of affine hypersurface singularities, defined by a weighted homogeneous polynomial, in arbitrary dimension. So these results in particular hold for six and eight dimensional singularities, which are of interest in string theory.

The description as a hypersurface obscures any differential geometric data of the space. However, there is a remarkable connection between analytic properties of the polynomial defining the affine hypersurface and topological properties. The 'topological properties' conceptually split into two sorts. First, there is the topology of a resolution of the singularity. This is related to deformations of the defining polynomial; essentially this is a statement in the context of Morse theory.

Second, there is the topology of the 'base of the cone', the analogue of L for metric cones. Topological properties of L, or rather its equivalent in the hypersurface context, are related to analytic properties of the defining polynomial as well. This may seem quite remarkable. This may seem quite remarkable, since L, regarded as the 'base' very far from the apex of a cone, is quite insensitive to small deformations of the singular apex.

Sasaki and Milnor: Circle Fiber or Circle Base?

In higher dimensions, many interesting ‘supersymmetric’ hypersurface singularities are not anticanonically embedded, but the ones that are play a special rôle, as it can be proved that some admit Kähler-Einstein metrics. This requirements seems more of a technical condition in the proof than a fundamental necessity. We will return to the higher dimensional cases in the next section. In any case, the Kähler-Einstein base manifolds Z_Γ of all ADE-hypersurfaces are $\mathbb{P}^1$, as $\chi = -2c_1 = -2$. This coincides with the base of the metric cone description $\mathbb{C}^2/\Gamma \to (\mathbb{C}^2\backslash\{0\})/\mathbb{C}^* \simeq \mathbb{P}^1$.

Can the links of the metric cones, S^3/Γ and the links of the hypersurface singularities $F_\Gamma^{-1}(0) \cap S^5$ also be identified? Given the weights a_i of F_Γ there is a natural Sasakian structure on $S^5 \subset \mathbb{C}^3$ with contact form $\eta_{\mathbf{a}}$ and characteristic vector $\xi_{\mathbf{a}}$ defined in terms of the coordinates $z_k = x_k + iy_k$ on $\mathbb{C}^3$,

$$\begin{aligned}\eta_{\mathbf{a}} &= \frac{\sum_{k=1}^{3} (x_k \mathrm{d}y_k - y_k \mathrm{d}x_k)}{a_k\,(x_k^2 + y_k^2)} \\ \xi_{\mathbf{a}} &= \sum_{k=1}^{3} a_k \left(x_k \frac{\partial}{\partial y_k} - y_k \frac{\partial}{\partial x_k} \right).\end{aligned} \tag{2.25}$$

This Sasakian structure is in general non-regular. It generalizes to S^{2n+3} spheres for any n. This restricts to a Sasakian structure on $L_\Gamma = S^5 \cap F_\Gamma^{-1}(0)$, and the question is to find a metric on L_Γ that is not only compatible with this Sasakian structure, but that is also Sasaki-Einstein, i.e. the $U(1)$ action above should be an isometry and it should be the action of a characteristic vector field on a Sasakian manifold. Analytic sufficient conditions can be found, discussed in a more general case in the next section, which are met by the ADE-hypersurfaces. Much like the proof of existence of Calabi-Yau metrics, it is not constructive. But from the hypersurface, some topological information about the analytic link can be found.

The link of a weighted homogeneous hypersurface singularity can be viewed not only as a circle bundle over a projective variety, such as $\mathbb{P}^1$ in the case of the ADE-hypersurfaces. It can also be seen as the ‘boundary’ of a fiber bundle with a relatively complicated fiber, but with S^1 for a base. The topology of the link is studied via the topology of its complement $S^{2n+3} \cap L$. This approach is essentially similar to the study of one-dimensional knots and links via their embedding in S^3, related to complex curve singularities

$$\mathbb{C}^2 \supset \mathcal{C} \to L \subset S^3.$$

Topological information about the link is related to topological information about its complement, which in turn is related to analytic information about the hypersurface.

More specifically, deformations of the defining polynomial of a hypersurface correspond to smoothings of the singular point. Such smoothings do not change the asymptotic form of the hypersurface toward infinity. The link of a weighted homogeneous hypersurface is obtained by intersecting it with a sphere that contains the singular point, and may be large. The deformations of the singularity occur inside the enveloping sphere and may not affect the asymptotic geometry near the sphere. Yet, the possibility of these analytic deformations far inside, which can smooth out the singularity, have a consequence for the topology of the link as well. The connection between singularity theory and topology is a very interesting matter and only a very small part will be discussed, in the context of not only ADE-hypersurfaces but also higher dimensional cases. A nice starting point, containing many classic references is [72].

A polynomial $F : \mathbb{C}^{n+2} \to \mathbb{C}$ defines an affine hypersurface $\mathcal{M} = F^{-1}(0)$. This hypersurface is singular where the $\mathrm{d}F = 0$, in other words, at the critical points of F, where in addition $F = 0$. We assume that F has isolated critical points. Around such a critical point F can be expanded as

$$F(z_1, \ldots, z_{n+2}) = \sum_{i=1}^{n+2} z_i^{k_i}(\alpha_{k_i}^{(i)} + \alpha_{k_i+1}^{(i)} z_i + \ldots), \tag{2.26}$$

and the multiplicity of the critical point is

$$\mu = \sum_{i=1}^{n+2}(k_i - 1). \tag{2.27}$$

For a weighted homogeneous polynomial,

$$F(\lambda^{a_1} z_1, \ldots, \lambda^{a_{n+2}} z_{n+2}) = \lambda^d F(z_1, \ldots, z_{n+2}) \tag{2.28}$$

this number is determined by the weights and the weighted degree,

$$\mu = \sum_{i=1}^{n+2} \frac{d - a_i}{a_i}. \tag{2.29}$$

The number μ is called the Milnor number of the hypersurface.

A polynomial F with a degenerate critical point can be deformed, $F \to \tilde{F}$ so that $\tilde{F}$ has μ non-degenerate critical points. The Milnor number can also be expressed as the dimension of the following quotient ring

$$\mu = \dim_{\mathbb{C}} \frac{\mathbb{C}[z_1, \ldots, z_{n+2}]}{\partial F}. \tag{2.30}$$

The 'numerator' is the polynomial ring generated by all variables in F and the 'denominator' is the ideal generated by the first derivatives of F, known as the Jacobian ideal of F. This quotient ring is also the (c, c) ring of a $\mathcal{N} = (2, 2)$ Landau-Ginzburg model, as discussed in section 3.3.1. Every (c, c) state corresponds to a critical point of F and to count

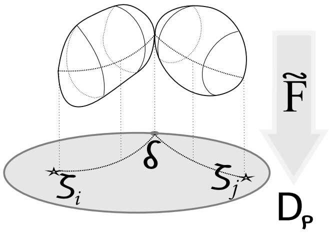


Figure 2.1: Homology cycles in $H(\Gamma, \phi; \mathbb{Z})$ for a deformed hypersurface singularity.

the multiplicity correctly, one can deform $F \to \tilde{F}$ so that the degenerate critical point of F, at the origin, splits into μ non-degenerate critical points, $\mathbf{z_i} \in \mathbb{C}^{n+2}$, of $\tilde{F}$, such that $||\mathbf{z_i}|| \leq \mathbf{r}$. These critical points are mapped to μ critical values, $\tilde{F}(\mathbf{z_i}) = \zeta_i \in \mathbb{C}$.

The function $\tilde{F}$ is continuous with non-degenerate critical points which maps the ball $B_r = \{||\mathbf{z}|| \leq \mathbf{r}\} \subset \mathbb{C}^{\mathbf{n+2}}$ containing all μ critical points into the disk $D_\rho = \{|z| \leq \rho\} \subset \mathbb{C}$ containing all critical values. Such a function is a Morse function and it can be used to extract topological information about the hypersurface, see for example [72]. Define

$$\begin{aligned} \Gamma &= \tilde{F}^{-1}(D_\rho) \cap B_r, \\ \phi &= \tilde{F}^{-1}(\zeta), \end{aligned} \tag{2.31}$$

where ζ is a generic point. The function $\tilde{F}$ can be used to find the relative homology [72]

$$H_k(\Gamma, \phi; \mathbb{Z}) \simeq \begin{cases} 0 & \text{if } k \neq n+2 \\ \mathbb{Z}^\mu & \text{if } k = n+2. \end{cases} \tag{2.32}$$

The function $\tilde{F}$ can be used to explicitly visualize a basis of $H_{n+2}(\Gamma, \phi; \mathbb{Z})$. Choose a point δ on the boundary of the disk D_ρ. Non-intersecting paths from δ to the critical values ζ_i are the images of homology cycles in the deformed hypersurface. These cycles shrink as critical points move together, see figure 2.1.

When the deformation is turned off completely, $\tilde{F} \to F$, all critical points coincide at the origin, and $F^{-1}(\zeta)$ is smooth, except when $\zeta = 0$, in which case the hypersurface has its only singularity isolated at the origin $\mathbf{z} = \mathbf{0}$.

Both the cone $\mathcal{C} = F^{-1}(0)$ and its complement $\mathbb{C}^{n+2} \backslash \mathcal{C}$ admit a $\mathbb{C}^*$ action. One can divide out the $\mathbb{R}_+$ part by intersecting with $S_r^{2n+3} = \partial B_r$. Using the fact that F has no critical points outside the origin, it can be shown that $L = F^{-1}(0) \cap S^{2n+3}$ and $M = S^{2n+3} \backslash L$ are smooth manifolds. M can be viewed as a fiber bundle with base $U(1)$. The projection map $M \to U(1)$ is given by

$$\begin{aligned} \pi : \ M &\to U(1), \\ \mathbf{z} &\mapsto \frac{F(\mathbf{z})}{|F(\mathbf{z})|}. \end{aligned} \tag{2.33}$$

The fiber is a $2n + 2$-dimensional manifold, $\Phi \simeq \Phi_\theta \simeq \pi^{-1}(e^{i\theta})$, known as the Milnor fiber. And the total space

$$\begin{array}{ccc} \Phi & \hookrightarrow & M \\ & & \downarrow \pi \\ & & S^1 \end{array} \tag{2.34}$$

is the Milnor fibration [71]. Clearly the complement of the M in the sphere, or $\partial M = \overline{M} \backslash M$, is the link L.

It was shown by Milnor [71] that $\Phi \simeq \partial M$ and also, taking a Morsification $\tilde{F}$ of F which has μ nondegenerate critical points inside a ball B_r and μ corresponding critical values inside a disk D_ρ, that

$$H_k(\Gamma, \phi; \mathbb{Z}) \simeq \begin{cases} 0 & k \neq n+1 \\ \mathbb{Z}^\mu & k = n+1, \end{cases} \tag{2.35}$$

taking $\Gamma = \tilde{F}^{-1}(D_\rho) \cap B_r$ and $\phi = \tilde{F}^{-1}(e^{i\theta}) \cap B_r$. Furthermore he showed that this Γ is contractible. Using this together with the long exact sequence for relative homology groups, $\phi \subset \Gamma$,

$$\ldots \overset{\partial}{\to} H_k(\phi) \overset{i}{\hookrightarrow} H_k(\Gamma) \overset{\sim}{\to} H_k(\Gamma, \phi) \overset{\partial}{\to} H_{k-1}(\phi) \overset{i}{\hookrightarrow} \ldots , \tag{2.36}$$

it is found that the homology of the Milnor fiber is given by

$$H_k(\phi; \mathbb{Z}) \simeq \begin{cases} 0 & k \neq n+1 \\ \mathbb{Z}^\mu & k = n+1. \end{cases} \tag{2.37}$$

This means that the Milnor fiber is homotopy equivalent to a bouquet of $(n+1)$-spheres,

$$L \simeq \underbrace{S^{n+1} \vee \ldots \vee S^{n+1}}_{\mu}. \tag{2.38}$$

The number of spheres in the bouquet is the Milnor number μ. A bouquet of spheres

$$S^{n+1} \vee S^{n+1} \vee \ldots \vee S^{n+1}$$

is the topological space obtained by taking the union of the topologies of the separate copies of S^{n+1} and identifying a marked point on each sphere to a single point, like in figure 2.2.

The total space M of the Milnor fibration is obtained by gluing the Milnor fibers over the circle in an appropriate way, using a homeomorphism

$$h : \Phi \to \Phi, \tag{2.39}$$

known as the characteristic map,

$$\begin{gathered} M = (\Phi \times [0, 2\pi]) \,/ \sim, \\ (0, \Phi) \sim (2\pi, h(\Phi)). \end{gathered} \tag{2.40}$$

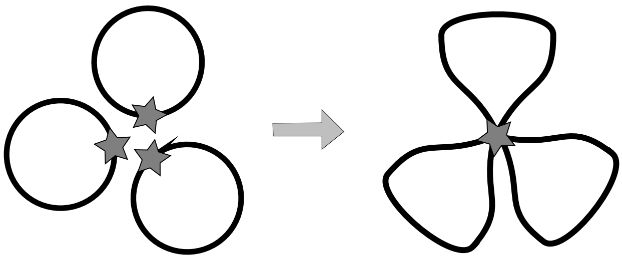


Figure 2.2: Three S^1's glued into a bouquet

Figure 2.3: Simplified version of a Milnor fibration. The link is a bouquet of three circles, a point on each of the three circles in the fiber is identified, see figure 2.2. The base space is the large circle direction. Traversing the base, the fibers are glued together in a non-trivial fashion.

An attempt to illustrate this point of view of the Milnor fibration in made in figure 2.3.

The topology of the Milnor fiber does not yet clarify the topology of the link. Note that the for a $(2n+2)$-dimensional hypersurface $\mathcal{C} = F^{-1}(0) \subset \mathbb{C}^{n+2}$, the link is a manifold of dimension $\dim(L) = 2n+1$, the complement of the Milnor fibration in S^{2n+3}, which has a $(2n+2)$-dimensional Milnor fiber. The homeomorphism $h:\ \Phi \to \Phi$ induces a linear map

$$h_*:\ H_{n+1}(\Phi;\mathbb{C}) \to H_{n+1}(\Phi;\mathbb{C}). \tag{2.41}$$

This map can be used to construct the exact sequence [70], using the fiber bundle structure[8] of M and $\partial M = L$,

$$0 \to H_{n+1}(L;\mathbb{Z}) \to H_{n+1}(\Phi;\mathbb{Z}) \overset{\mathbb{I}-h_*}{\longrightarrow} H_{n+1}(\Phi,\mathbb{Z}) \to H_n(L;\mathbb{Z}) \to 0. \tag{2.42}$$

This implies that $H_{n+1}(L,\mathbb{Z}) = \mathrm{Ker}(\mathbb{I}-h_*)$ is a free Abelian group. And $H_n(L;\mathbb{Z}) = \mathrm{Coker}(\mathbb{I}-h_*)$. This may have torsion, but its free part is isomorphic to $\mathrm{Ker}(\mathbb{I}-h_*)$ as well. The kernel of $\mathbb{I}-h_*$ is determined from the characteristic polynomial

$$\Delta(t) = \det(t\,\mathbb{I}_* - h_*). \tag{2.43}$$

There is an algorithmic way [70] to determine $\Delta(t)$ in terms of the a_i and d of a weighted homogeneous polynomial like (2.28) on page 28, and from that, the Betti numbers $b_{n+1}(L)$ and $b_n(L)$. This recipe is as follows.

For the Milnor fibration associated with a hypersurface $F^{-1}(0)$ defined by F as in (2.28), the homeomorphism h can be chosen to act on the coordinates as

$$h:\ (z_1,\ldots,z_{n+2}) \mapsto \left(e^{\frac{2\pi i a_1}{d}} z_1,\ldots,e^{\frac{2\pi i a_{n+2}}{d}} z_{n+2}\right). \tag{2.44}$$

In order to write down $\Delta(t)$, it is convenient to introduce different notation. Define $r_i = d/a_i$, and write these as fractions of relatively prime pairs $r_i = s_i/t_i$. Associate divisors to polynomials as follows,

$$\mathrm{divisor} \prod_{i=1}^{k}(t-\alpha_i) = \langle\alpha_1\rangle + \ldots + \langle\alpha_k\rangle.$$

A divisor, like the one denoted on the right hand side of the above equation, can be regarded as a formal linear combination of points in $\mathbb{C}$. More clearly, a divisor is an element of a free Abelian group[9]. Each generator $\langle\alpha_i\rangle$ of this group is in one-to-one correspondence with a point in $\mathbb{C}$, which can be regarded as the zero of a complex monomial function $t-\alpha_i$.

[8] In particular the Wang sequence is used, for fiber bundles over odd-dimensional spheres.

[9] One could even say the divisors form the group ring $\mathbb{Z}\mathbb{C}^*$, which is formally a better way to think of them. The 'special' divisors E_n are then considered not to form a subgroup, but a genuinely different group ring: $\mathbb{Q}\mathbb{C}^*$ (the coefficients of the $\langle\eta_n\rangle$ are rational numbers).

Each $\langle\alpha_i\rangle$ generates a subgroup isomorphic to $\mathbb{Z}$. The group operation in this group can be denoted as addition, and one can concisely write

$$\langle\alpha_1\rangle + \langle\alpha_1\rangle = 2\langle\alpha_1\rangle.$$

We can introduce some additional structure, multiplication, on a subgroup, if we realize that the α_i are also complex numbers, not just labels for geometric points. We restrict to a special subgroup of divisors. Define

$$E_n = \frac{1}{n}\text{divisor } (t^n - 1) = \frac{1}{n}\sum_{l=0}^{n-1}\langle(\eta_n)^k\rangle,$$

where η_n is a primitive n-th root of unity. Now a multiplication rule for these special divisors is proposed, inspired by complex multiplication of roots of unity. The E_k form a ring with multiplication rule

$$E_k E_l = E_{[k,l]},$$

where $[k,l]$ denotes the least common multiple of k and l. With this notation the divisor of $\Delta(t)$ associated to the Milnor fibration of $F^{-1}(0)$ as in ((2.28) reads

$$\text{divisor}\Delta = \prod_{k=1}^{n+2}\left(r_k E_{s_k} - 1\right). \tag{2.45}$$

The Betti numbers $b_{n+1} = b_n$ of the link $L = F^{-1}(0) \cap S^{2n+3}$ are equal to the number of factors of $(t-1)$ in $\Delta(t)$ [70].

Recapitulating, the weights and degree of a weighted homogeneous polynomial F determine the Milnor number μ of the hypersurface $F^{-1}(0)$. This number counts the number of deformations of the singularity or in other words, the multiplicity of the critical point at the singularity. As such, it is related to Landau-Ginzburg models, counting the number of (c,c) primary states (see section 3.3.1). But μ also gives the dimension of the middle integral homology of the Milnor fiber $\Phi \to M \to S^1$; $\Phi \simeq S^{n+1} \vee \ldots \vee S^{n+1}$. The total space of the Milnor fibration M is obtained by gluing Φ along the base, twisting it by the characteristic map h. The boundary of M is the link $F^{-1} \cap S^{2n+3}$. Its Betti numbers $b_n(L) = b_{n+1}(L)$ are determined, employing the h, in terms of the weights and degree of F. The link itself is a circle fibration over a projective variety $S^1 \to L \to Z$.

For the A-type hypersurfaces, $Z \simeq \mathbb{P}^1$, which admits a Kähler-Einstein metric of positive curvature. This is in agreement with the observation that the F_{ADE} in table 2.2 are precisely those weighted homogeneous polynomials that satisfy,

$$\sum_{i=1}^{3} a_i \geq d + 1. \tag{2.46}$$

The F_{ADE} even saturate this inequality.

One can consider other weighted homogeneous hypersurfaces $F^{-1}(0)$, as 'cones' in $\mathbb{C}^3$ or projective surfaces in a weighted projective space $(\mathbb{C}^3\backslash\{0\})/\mathbb{C}^*[\mathbf{a}]$. Notably, one might consider projective hypersurfaces with $c_1 \leq 0$. The corresponding cones will not be suitable to serve as supersymmetric compactifications by themselves, only the ADE-cones do. Yet there are still some interesting points to note.

The simplest of ADE-hypersurfaces are those of Brieskorn-type: the A_n-series together with E_6 and E_8. These are of the form $z_1^{r_1} + z_2^{r_2} + z_3^{r_3} = 0$. Intersected with $S^5_{r=1} \subset \mathbb{C}^3$ these define the Brieskorn manifolds $M(r_1, r_2, r_3)$. The three dimensional Brieskorn manifolds were studied by Milnor [69]. He demonstrated that $M(r_1, r_2, r_3)$ are homogeneous spaces which fall into three categories, depending on the canonical class of the corresponding projective hypersurface.

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} > 1 \qquad c_1 = 1, \tag{2.47}$$
$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1 \qquad c_1 = 0, \tag{2.48}$$
$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} < 1 \qquad c_1 < 1. \tag{2.49}$$

In the cases (2.47) the homogeneous spaces $M(r_1, r_2, r_3)$ are of the form $SU(2)/\Gamma$, as familiar from the quotient description. In the case (2.49) the spaces $M(r_1, r_2, r_3)$ are $\widetilde{PSL}(2;\mathbb{R})/\Gamma$, quotients of the universal cover of the projective version of $SL(2;\mathbb{R})$ by discrete subgroups. The case (2.48) is different, there $M(r_1, r_2, r_3) \simeq G/H$ where G is the Heisenberg group, with elements the matrices

$$[a, b, c] = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b, c \in \mathbb{R}, \tag{2.50}$$

and H are subgroups where $a, b, c \in k\mathbb{Z}$ for some integer k, see [69]. .

The polynomials which define Brieskorn manifolds of type (2.48) are

$$\begin{aligned} F_{\tilde{E}_6}(z_1, z_2, z_3) &= z_1^3 + z_2^3 + z_3^3 \quad (+\alpha z_1 z_2 z_3), \\ F_{\tilde{E}_7}(z_1, z_2, z_3) &= z_1^2 + z_2^4 + z_3^4 \quad (+\alpha z_1^2 z_2^2), \\ F_{\tilde{E}_8}(z_1, z_2, z_3) &= z_1^2 + z_2^3 + z_3^6 \quad (+\alpha z_1^4 z_2). \end{aligned} \tag{2.51}$$

The conformal field theories defined by these polynomials, as Landau-Ginzburg models (see section 3.3.1) have $\hat{c} = 1$, with the Brieskorn polynomials ($\alpha = 0$) corresponding to cft's with enhanced symmetry. The polynomials in (2.51) define tori in the appropriate weighted projective spaces, as is seen from the adjunction formula[10]. The links of the singularities are circle bundles over tori, in these cases, and are homogeneous spaces.

[10] In fact, this enhanced symmetry of the cft can be interpreted as the tori being at the self-dual radius. This will not be discussed. The connection between Landau-Ginzburg models, which a priori have no geometric interpretation, and sigma models, is discussed in section 3.3.

There is an interesting correspondence between the polynomials in (2.51) that define curves with trivial anticanonical class and thus cannot be used to make supersymmetric cones directly, and the del Pezzo surfaces dP_6, dP_7 and dP_8, that not only can be used to construct supersymmetric cones (as metric cones over regular Sasaki-Einstein manifolds), but also have descriptions as projective hypersurfaces, but are not homogeneous.

2.3.3 Kähler-Einstein hypersurfaces

The 4d supersymmetric singularities are classified, have different but equivalent descriptions, and are related, via the ADE classification, to an enormous number of apparently very different objects that appear in mathematics. Each different description of one singularity highlights different aspects. For example, the metric cone shows there is a $U(1)$ isometry, degenerating at the apex. The quotient description relates singularities to homogeneous spaces It also relates metric cones and hypersurfaces to one another, at least in the case of the complex surface singularities.

In the hypersurface description possible deformations are more apparent. In addition, important for our purposes, the defining polynomials of hypersurfaces play a rôle in worldsheet conformal field theories describing strings moving on a hypersurface and also T-dual spaces. Finally, many weighted homogeneous hypersurfaces give rise to Sasaki-Einstein manifolds, mostly non-regular ones. This section deals with the relation between hypersurfaces and metric cones in dimension $d > 4$.

Affine Calabi-Yau Hypersurfaces

The condition on a metric cone to be part of a supersymmetric string vacuum, i.e. a Calabi-Yau cone, is that its link is Sasaki-Einstein. Is there an analogous condition on hypersurfaces? The answer is: “yes”. Consider an affine hypersurface $\mathcal{C} = F^{-1}(0) \subset (\mathbb{C}^{n+2} \backslash \{0\})$ defined by a weighted homogeneous polynomial,

$$F(\lambda^{a_1} z_1, \dots, \lambda^{a_{n+2}} z_{n+2}) = \lambda^d F(z_1, \dots, z_{n+2}), \tag{2.52}$$

with a singularity only at the origin. If the weights a_i and the weighted degree d of F are such that

$$\mathfrak{I} = -d + \sum_{i=1}^{n+2} a_i > 0, \tag{2.53}$$

then $\mathcal{C}$ is Calabi-Yau [27].

Note that the condition (2.53) is different from the Calabi-Yau condition for hypersurfaces in a projective space. Such hypersurfaces are Calabi-Yau iff $\mathfrak{I} = 0$, as a consequence of the adjunction formula and Yau's proof of the conjecture of Calabi. But (2.53) deals with affine hypersurfaces, not projective ones. Nevertheless, since F is a weighted homogeneous polynomial, one may consider the hypersurface $\mathcal{P} = F^{-1}(0)$ in an appropriate weighted projective space. Such hypersurfaces, which satisfy (2.53) are called Fano. In terms of the first Chern class, $c_1 > 0$ for a Fano manifold.

Such a projective hypersurface is Kähler, since it is embedded holomorphically in a weighted projective space. It can be positively curved, as $c_1 > 0$. So maybe it can be the leaf space of a Sasaki-Einstein manifold. But this is only possible if the hypersurfaces admits a positive Kähler-*Einstein* metric (possibly with orbifold singularities).

One important question is: "What are necessary and sufficient conditions that such a $\mathcal{P}$ admit a positive Kähler-Einstein metric?". And a following question is: "Can a Sasaki-Einstein manifold be constructed from a $\mathcal{P}$ that admits such a metric, and if so, how?".

The latter question can be answered affirmatively. Given a hypersurface that has a Kähler-Einstein with positive scalar curvature, and at worst cyclic orbifold singularities, a Sasaki-Einstein manifold can be constructed, using the $\mathbb{C}^*$ action on the weighted homogeneous polynomial F [33, 32, 37]. The answer to the former question is a lot more involved. It is possible to find sufficient conditions, that $\mathcal{P}$ admit a Kähler-Einstein metric with at worst cyclic quotient singularities, but part of these conditions is likely to be too strict [39, 34, 35]. Many hypersurfaces which are interesting from the perspective of string theory do not satisfy all of these sufficient conditions.

WEIGHTED PROJECTIVE BASICS

First, let us recall some basic definitions and properties of weighted projective spaces; see, for example [75]. Weighted projective spaces $\mathbb{P}[a_1, \ldots, a_{n+2}]$ are generalizations of ordinary projective spaces $\mathbb{P}^{n+1} = \mathbb{P}[1, \ldots, 1]$. Points in $\mathbb{C}^{n+2} \setminus \{0\}$ are identified by the weighted $\mathbb{C}^*$ action,

$$(z_1, \ldots, z_{n+2}) \sim (\lambda^{a_1} z_1, \ldots, \lambda^{a_{n+2}} z_{n+2}),$$

where $\lambda \in \mathbb{C}^*$. Unlike ordinary projective spaces, weighted projective spaces can have singularities. These are seen in the affine coordinate patches where $z_i \neq 0$. In such a patch, one can set $z_i = 1$ by a weighted $\mathbb{C}^*$ transformation. The coordinates is such a patch are $\zeta_j^{(i)} = z_j / z_i$ If the weight a_i of the coordinate z_i is larger than one, then a $\mathbb{Z}_{a_i}$ subgroup of the weighted $\mathbb{C}^*$ action leaves invariant $z_i = 1$, but does act on the other coordinates:

$$(z_1, \ldots, z_i = 1, \ldots, z_{n+2}) \mapsto (\eta^{a_1} z_1, \ldots, z_i = 1, \ldots, \eta^{a_{n+2}} z_{n+2}),$$

where η is a primitive a_i-th root of unity. So the affine coordinate patches where $z_i \neq 0$ can have cyclic quotient singularities. These singularities occur at the so-called vertices P_i of the weighted projective space. The vertex P_i is the point $\{z_j = 0\}$, $j \neq i$. The singularity at P_i is said to be of type $\frac{1}{a_i}(a_1, \ldots, \hat{a}_i, \ldots, a_{n+2})$. A hat over an element means that that element is omitted from the list. If some of the weights have common factors, there may also be singular lines, planes etc. The singular lines occur at edges $P_i P_j$ (i.e $z_k = 0$, $i \neq k \neq j$) and are of type $\frac{1}{\gcd(a_i, a_j)}(a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_{n+2})$, an so on.

Clearly the weight vectors $(a_1, \ldots, a_{n+2})$ and $(ka_1, \ldots, ka_{n+2})$ correspond to isomorphic weighted projective spaces, for any integer k. So one can assume that all a_i's are relatively prime. In fact, there are further isomorphisms between weighted projective spaces, and every weighed projective space is isomorphic to a well formed one, so says a theorem by Delorme[11]. A well formed projective space $\mathbb{P}[a_1, \ldots, a_{n+2}]$ has a weights such that

$$\gcd(a_1, \ldots, \hat{a}_i, \ldots, a_{n+2}) = 1 \quad 1 \leq i \leq n+2. \tag{2.54}$$

A hat over an element means that the element is omitted. In a well formed projective space, the affine coordinate charts $(z_i \neq 0)$ have $\mathbb{Z}_{a_i}$ quotient singularities. Some examples of some weighted projective spaces are

$$\begin{aligned} \mathbb{P}[p,q] &\simeq \mathbb{P}[1,1] \quad \forall\, p,q \\ \mathbb{P}[6,10,15] \simeq \mathbb{P}[6,2,3] &\simeq \mathbb{P}[3,1,3] \simeq \mathbb{P}[1,1,1] \end{aligned} \tag{2.55}$$

A hypersurface in a weighted projective space inherits singularities from the embedding space if it passes through vertices, singular lines, etc. In general a hypersurface cannot avoid all vertices. It can avoid all vertices if

$$a_i \mid d \quad \forall i. \tag{2.56}$$

A hypersurface with singularities that are all due to the singularities of $\mathbb{P}[a_1, \ldots, a_{n+2}]$ alone[12] is called quasi-smooth. Its singularities are all cyclic quotient singularities. Mathematicians know how to deal with such 'mild' sorts of singularities, and objects familiar from the algebraic geometry in ordinary projective spaces can be generalized [75]. In particular there is an adjunction formula if a hypersurface does not contain any singularities of codimension 2. Such a hypersurface is called well formed. A hypersurface $\mathcal{P} = F^{-1}(0)$, defined by a polynomial of weighted degree d in $\mathbb{P}[a_1, \ldots, a_{n+2}]$ is called 'well formed' iff the following conditions are satisfied,

$$\begin{aligned} &\mathbb{P}[a_1, \ldots, a_{n+2}] \text{ is well formed, and} \\ &\gcd(a_1, \ldots, \hat{a}_i, \ldots, a_{n+2}) \mid d \quad \forall\, i. \end{aligned} \tag{2.57}$$

The adjunction formula gives the canonical class of the $\mathcal{P}$ in terms of the weights a_i and the weighted degree d of F, which can be seen as a section of the sheaf $\mathcal{O}_{\mathbb{P}}(d)$. The adjunction formula tells us

$$K_{\mathcal{P}} \simeq \mathcal{O}(d - \sum_{i=1}^{n+2} a_i). \tag{2.58}$$

[11]Consider a weighted projective space $\mathbb{P}[a_1, \ldots, a_{n+2}]$. It can be shown that this space is isomorphic to $\mathbb{P}[a_1, a_2/g, a_3/g, \ldots, a_{n+2}/g]$, where $g = \gcd(a_2, \ldots, a_{n+2})$. Making use of this equivalence at most $n+2$ times produces a well formed projective space.

[12]It is to say, that there are no singularities due to the way the hypersurface is embedded, i.e $F = \mathrm{d}F = 0$ has no solutions in $\mathbb{P}[a_1, \ldots, a_{n+2}]$.

Surface $F^{-1}(0) \subset \mathbb{P}[a_1,a_2,a_3,a_4]$	$F=0$	$\mathbb{P}[a_1,a_2,a_3,a_4]$
$\mathbb{P}^2$	$z_1+z_2+z_3+z_4=0$	$\mathbb{P}[1,1,1,1]$
$\mathbb{P}^1\times\mathbb{P}^1$	$z_1^2+z_2^2+z_3^2+z_4^2=0$	$\mathbb{P}[1,1,1,1]$
dP_6	$z_1^3+z_2^3+z_3^3+z_4^3=0$	$\mathbb{P}[1,1,1,1]$
dP_7	$z_1^4+z_2^4+z_3^4+z_4^2=0$	$\mathbb{P}[1,1,1,2]$
dP_8	$z_1^6+z_2^3+z_3^2+z_4^2=0$	$\mathbb{P}[1,2,3,3]$

Table 2.6: Smooth del Pezzo hypersurfaces admitting a Kähler-Einstein metric.

A well formed hypersurface is Fano iff $\mathfrak{I}\equiv -d+a_1+\ldots+a_{n+2}>0$. Such hypersurfaces stand a chance of having positive Kähler-Einstein metrics, thus providing a connection with metric cones.

Hypersurfaces Admitting Kähler-Einstein Metrics

Which quasi-smooth hypersurfaces admit a Kähler-Einstein metric? A general answer is not known, but there are many examples, in various dimensions. First of all, there are the complex curves defined by the ADE polynomials, in table 2.2. As discussed earlier, all the ADE polynomials define a $\mathbb{P}^1$ hypersurface, which of course admits a Kähler-Einstein metric. Next, we know from section 2.2 which smooth complex surfaces admit positive Kähler-Einstein metrics. These are $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$ and the del Pezzo surfaces dP_n for $3\le n\le 8$. Of these, the ones that can be realized as hypersurfaces in weighted projective space are listed in table 2.6.

In addition to these smooth surfaces, there are many more quasi-smooth cases. Quasi-smoothness and well-formedness, see (2.54) and (2.57), impose conditions on the weights and degree similar to the smoothness condition (2.56). These conditions[13] are not quite strong enough to determine all surfaces. It is possible to determine all surfaces and three-folds that satisfy one more condition, which is that they be anticanonically embedded,

$$\mathfrak{I}\equiv -d+a_1+\ldots+a_{n+2}=1, \tag{2.59}$$

All the conditions impose a set of linear relations among the weights a_i, which were organized in such a way [35, 34] that all solutions were found using a computer program.

The authors of [35, 34] also discuss the existence of Kähler-Einstein orbifold metrics on these hypersurfaces. The criteria that are used are sufficient but not necessary. Many

[13]The conditions are the following, see [34] **2.** Quasi-smoothness requires that for every i there exist a j and a monomial $z_i^{m_i}z_j$ of weighted degree d. The case $i=j$ gives the smoothness condition (2.56). Well-formedness furthermore requires that if $\gcd(a_i,a_j)>0$, then there must be a monomial $z_i^{b_i}z_j^{b_j}$ of weighted degree d. Also, if every hypersurface of weighted degree d contains a coordinate axis $z_k=z_l=0$, then a general such hypersurface must be smooth along it, or have only a singularity at the vertices. This is the case if for all i,j there is either a monomial $z_i^{b_i}z_j^{b_j}$ of degree d or a pair of monomials $z_i^{c_i}z_j^{c_j}z_k$ and $z_i^{d_i}z_j^{d_j}z_l$ of degree d.

hypersurfaces which are very interesting from the point of view of string theory are not anticanonically embedded. For example, the hypersurfaces defined by

$$F(z_1, \ldots, z_{n+2}) = H(z_1, \ldots, z_n) + z_{n+1}^2 + z_{n+2}^2 \tag{2.60}$$

are not, except for those defined by the A_k polynomials of table 2.2. Yet such polynomials have a special rôle in chapter 4.

In fact, from the point of view of string theory the single essential condition on an affine hypersurface is

$$\mathfrak{I} = -d + \sum_{i=1}^{n+2} a_i > 0, \tag{2.53}$$

which ensures that it is Calabi-Yau, assuming that the only singularity is at the origin. Actually, it ensures that the cone without the apex at the origin is Calabi-Yau. For string theory one would also like that there are deformations of the singular hypersurface to a smooth one and that the smooth hypersurfaces as well as the singular limit are Calabi-Yau. This is indeed the case [77]. It would be interesting to know to what extent (2.53) is sufficient for the existence of a Kähler-Einstein metric (with singularities) on the projective hypersurface that it defines, and what additional conditions are necessary and sufficient.

A sufficient condition, based on [39] and [35, 34] and references therein, is given in [78]. They consider a Brieskorn hypersurface $F^{-1}(0)$, i.e. one defined by a polynomial of the form

$$F = \sum_{i=1}^{n+2} z_i^{r_i}, \tag{2.61}$$

with $F = \mathrm{d}F = 0$ only at the origin. F has weighted degree

$$d = R \equiv \mathrm{lcm}\{a_i\}. \tag{2.62}$$

The weighted homogeneous action on the coordinates z_i is

$$(z_1, \ldots, z_{n+2}) \simeq (\lambda^{R/r_1} z_1, \ldots, \lambda^{R/r_{n+2}} z_{n+2}). \tag{2.63}$$

Actually, they consider any deformation of such a hypersurface by a polynomial

$$f(z_1, \ldots, z_{n+2})$$

of weighted degree d,

$$\widetilde{F} = F + f,$$

provided that the intersections with any number of hyperplanes $z_l = 0$ are smooth away from the origin. The condition of [78] that a hypersurface admit a Kähler-Einstein orbifold metric of positive scalar curvature, is

$$1 < \sum_{i=1}^{n+2} \frac{1}{r_i} < 1 + \frac{n+1}{n} \min_{i,j} \left\{ \frac{1}{r_i}, \frac{1}{b_i, b_j} \right\}. \tag{2.64}$$

Here the b_i are somewhat complicated expressions, in terms of the a_j,

$$C^j \equiv \mathrm{lcm}\left\{r_1, \ldots, \hat{r}_j, \ldots, r_{n+2}\right\},$$
$$b_j \equiv \gcd\left(r_j, C^j\right).$$

The lower bound is a necessary condition. It is the requirement that the hypersurface be Fano. The upper bound is a sufficient condition. It derives from certain estimates that guarantee the existence of a Kähler-Einstein metric [39]. These will not be discussed. The estimates are related to those used to find smooth Kähler-Einstein metrics on del Pezzo surfaces [38]. Essentially, it comes down to the question if a particular nonlinear partial differential equation has a solution, similar to the reformulation of the Calabi conjecture in the proof of Yau.

NOT ANTICANONICALLY EMBEDDED: KÄHLER-EINSTEIN?

These estimates discussed above are not sharp enough to determine if a Kähler-Einstein metric exists on many interesting hypersurfaces. For example

$$z_1^{r_1} + z_2^{r_2} + z_3^2 + z_4^2 = 0$$

does not satisfy (2.64). Unfortunately no sharper criteria are known to determine if a Kähler-Einstein orbifold metric exists. It would be especially interesting to find a way to determine if such metrics exist for hypersurfaces of the form

$$F(z_1, \ldots, z_{n+2}) = H(z_1, \ldots, z_n) + z_{n+1}^2 + z_{n+2}^2,$$

which are important in chapter 4. However, if one has a hypersurface in weighted projective space that does have a Kähler-Einstein metric with at worst cyclic quotient singularities, then there is always a Sasakian-Einstein metric on the link $L = F^{-1}(0) \cap S^{2n+3}$ of the corresponding affine hypersurface [33]. Basically, the weighted projective $\mathbb{C}^*$ action restricts to a weighted S^1 action on $S^{2n+3} \subset \mathbb{C}^{2n} \backslash \{0\}$, and also on the link. This weighted S^1 action is that of a characteristic vector field of a Sasakian manifold. There is a Sasakian structure on the link with such a characteristic vector field that also has a compatible metric that is an Einstein metric.

2.4 SUMMARY AND CONTEXT

WHAT HAVE WE DONE?

Various spaces have been discussed which can feature as part of a supersymmetric string vacuum of the form

$$\mathbb{R}^{9-2m,1} \times \mathcal{C}_{2m}.$$

All $\mathcal{C}_{2m}$ must preserve some supersymmetry and have a metric with a vanishing Ricci tensor. Also, $\mathcal{C}_{2m}$ are non-compact and have an isolated singularity. There are numerous different ways to describe such spaces, among those discussed the most prominent two are metric cones and (weighted homogeneous) affine hypersurfaces.

Any particular exponent of a space $\mathcal{C}_{2m}$ may have a description in both of these ways, in just one of the two, or in neither of them. Either way of describing a $\mathcal{C}_{2m}$ emphasizes some characteristics of the space. A metric cone has a characteristic S^1 isometry which degenerates at the apex. This isometry is interesting for T-duality of such a space.

But possible deformations of the singularity are obscured in the description as a metric cone. On the other hand, a description as a hypersurface manifests some possible deformations, to be specific, deformations of the complex structure. Some such deformations can even smooth out a singularity completely, without affecting the asymptotic form of the space.

As we have seen, the number of such deformations is indicated by the Milnor number of the singularity. But this number also describes aspects of the topology of the hypersurface away from the singularity. It does so in two different ways. First, the hypersurface $\mathcal{C}_{2m}$ cuts out a link in a S^{2m-1} surrounding the singular point. This link is a fiber bundle with a circle fiber. The Milnor number roughly speaking indicates how far the fibration is from being trivial. Second, the complement of the link is a fiber bundle with a circle as a base. The fiber is a special manifold, the Milnor fiber and the Milnor number determines its complete homology. Finally, in a somewhat different context, the Milnor number counts the number of ground states in certain superconformal field theories, as discussed in section 3.3.1.

So these two descriptions, metric cones and affine hypersurfaces highlight different aspects and obscure others. Is it possible to construct one description from the other? A connection between metric cones and hypersurfaces is clearly present in some cases, most notably the $\mathcal{C}_4$ ADE singularities. In those instances, there is a direct connection via the quotient description $\mathbb{C}^2/\Gamma_{ADE}$. In higher dimensional cases, if there is a connection at all, it is more indirect.

In specific cases, a connection can be established. The most obvious similarity between the metric cones and the hypersurfaces, is that both admit a special $\mathbb{C}^*$ action. For metric cones, this comes partly from the definition, the $\mathbb{R}_+$ scaling, and partly from the requirement of supersymmetry, the S^1 of the characteristic isometry of a Sasakian base. The Sasakian base of a supersymmetric metric cone is itself a circle bundle over a Kähler manifold (possibly with quotient singularities). On the other hand, a weighted homogeneous polynomial, such as defines the affine hypersurfaces under consideration, also defines a hypersurface in a weighted projective space. Such a hypersurface is Kähler.

If it is Kähler-Einstein, then the affine hypersurface is Calabi-Yau, and it can be viewed as a metric cone. There is also a sufficient condition, due to Tian and Yau, that an affine hypersurface be Calabi-Yau. If is phrased in terms of the scaling weights a_i and the weighted degree d of the defining polynomial: $\mathfrak{I} \equiv -d + \sum a_i > 0$. Some of these Calabi-Yau hypersurfaces $\mathcal{C}$ certainly give rise to Kähler-Einstein $\mathcal{C}/\mathbb{C}^*$ and can thus be viewed as metric cones, with a S^1 isometry. It is not known what the minimal sufficient conditions are, for this to be the case. It would be interesting to know such conditions, so that metric cones and hypersurfaces can be related.

From the point of view of the T-duality of chapter 4 and further string applications, there are many affine hypersurfaces (or actually, discrete quotients of hypersurfaces, see section 4.4) which are not known to be connected to Kähler-Einstein hypersurfaces with the present status of mathematical knowledge.

WHY ARE WE DOING THIS?

Ultimately the interest of the connection of metric cones and hypersurfaces might be motivated from the T-duality of Calabi-Yau singularities, in chapter 4, which, where 'understood', relates almost all objects which have an ADE classification. A broad question would be:"If, as it seems, such a T-duality holds for a wider range of singularities, what objects does it relate, and how can these objects be interpreted in string theory, particularly from a stringy geometric point of view?"

But this met get ahead of the ideas presented to this point. Let us put hypersurfaces and metric cones in some perspective. Both metric cone and hypersurface descriptions emphasize certain objects which are important in another context, that is not discussed much in this chapter, but becomes more important in later ones. These objects have to do with worldsheet descriptions of string backgrounds. The weighted homogeneous polynomials that describe hypersurfaces, also describe Landau-Ginzburg conformal field theories. These can be used to build worldsheet conformal field theories that do not have a direct target space interpretation. However, in some cases, Landau-Ginzburg models are related to a target space.

Often Landau-Ginzburg models can be considered to describe string backgrounds that are compact Calabi-Yau hypersurfaces in weighted projective space. Or rather, a Landau-Ginzburg (-orbifold) describes a "Kähler" deformation of such a background to a non-geometric 'phase'. It may be that a similar connection exists to non-compact Calabi-Yau hypersurfaces in affine space. A very different geometric interpretation of a Landau-Ginzburg model exists in a much more limited collection of cases. Sometimes a Landau-Ginzburg model has an interpretation as a coset model, and a coset model may have a geometric target space interpretation when the levels of the Kač-Moody algebras are large, so that stringy modifications to ordinary geometric concepts are small. In particular, the coset models that preserve the same amount of supersymmetry as the $\mathcal{C}_{2m}$ of this chapter, are so-called Hermitean symmetric space coset models. Even if the levels are large so that there is a classical geometric target space interpretation, the target space of the Hermitean symmetric space coset models is very different from the geometry of the Hermitean symmetric spaces, which feature in the present chapter as particular examples of Kähler-Einstein manifolds. From these, Sasaki-Einstein manifolds can be built and from these, metric cones $\mathcal{C}_{2m}$.

3

SUPERCONFORMAL FIELD THEORIES

Where chapter 2 deals with different views of geometric objects, in the sense of different descriptions, the present chapter could be said to deal with a different kind of geometry altogether. The way that strings probe their ambient space cannot be described simply by 'ordinary' geometry. An intuitively clear reason for this, is that strings are extended objects. The basic tool in the description of string theory in a perturbative formulation, is the worldsheet conformal field theory. One important observation regarding 'string geometry' that can be made using worldsheet cft, is that some backgrounds that look different from the point of view of 'ordinary' geometry, are indistinguishable for strings.

Before dealing with such issues in chapter 4, here we present aspects and formulations of two dimensional quantum field theories, which appear as worldsheet models in string theory. Some have a quite direct 'ordinary geometric' target space interpretation: think of sigma models when $\alpha' \to 0$, for instance. Others may have a somewhat 'fuzzy' target space interpretation, like general WZW models, for example. Also, there are abstractly constructed conformal field theories which at best have an indirect target space interpretation. The kinds of these which we will be concerned with most, are Landau-Ginzburg models and more generally, conformal field theories which are defined as the low energy endpoint of the renormalization group flow of non-conformal 'ultraviolet' field theories.

The various models are important for us, not so much each in their own right, but because there exist remarkably useful interconnections, and strong evidence even for equivalences between different formulations of some conformal models. These equivalences are put to use in chapter 4.

The worldsheet conformal field theory corresponding to a generic Type II string background is $\mathcal{N} = (1, 1)$ superconformal field theory. More specifically, a background that is a vacuum preserving some spacetime supersymmetry is described by a worldsheet theory with more (super-) symmetry. In the cases of interest to us, 'compactifications' on Calabi-Yau cones, the worldsheet theories have $\mathcal{N} = (2,2)$ superconformal symmetry. A number of different constructions to obtain such theories are reviewed, such as Landau-Ginzburg effective field theories and coset models. Even though such conformal field theories have no direct geometric interpretation as sigma models, in some cases they can be related to sigma models in various ways. Often, the theory is a marginal deformation of one with a geometric interpretation.

Different objects in this chapter which play a role in the construction of 'non-geometric' conformal field theories show intriguing resemblances to the objects in the differential and algebraic geometric constructions of chapter 2.

3.1 $\mathcal{N} = (2,2)$ SUPERSYMMETRY

This section serves to set notation regarding $\mathcal{N} = (2,2)$ supersymmetry, superfields and R-charges which appear in later discussions.

$\mathcal{N} = (2,2)$ superspace has bosonic coordinates $x^\pm = x^0 \pm x^1$ and fermionic coordinates $\theta^\pm$ and $\overline{\theta}^\pm$. The spacetime signature is Minkowski. The $\overline{\theta}^\pm$ are seen as complex conjugates to $\theta^\pm$. Write $\partial_\pm$ for $\partial/\partial x^\pm$. Differential operators generating supersymmetry transformations on superspace are

$$\begin{aligned} \mathcal{Q}^\pm &= \frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm\partial_\pm, \\ \overline{\mathcal{Q}}^\pm &= -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm\partial_\pm. \end{aligned}$$

The operators

$$\begin{aligned} D_\pm &= \frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm\partial_\pm, \\ \overline{D}_\pm &= -\frac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm\partial_\pm \end{aligned}$$

commute with the $\mathcal{Q}^\pm$, $\overline{\mathcal{Q}}^\pm$ and generate supersymmetry transformations on superfields, which are functions on superspace. The superfields can be expanded as a finite sum of monomials in the fermionic coordinates, due to their anticommuting nature. The non-zero anticommutators are

$$\begin{aligned} \{\mathcal{Q}^\pm, \overline{\mathcal{Q}}^\pm\} &= -2i\partial_\pm \\ \{D_\pm, \overline{D}_\pm\} &= +2i\partial_\pm. \end{aligned}$$

So supersymmetry transformations square to Lorentz transformations.

The supersymmetry algebra has a vector and an axial $U(1)$ R-symmetry. On the fermionic coordinates and on general superfields with vector (axial) charge q_V (q_A) these symmetries act as

$$e^{i\alpha F_V} : \mathcal{F}(x^\pm, \theta^\pm, \overline{\theta}^\pm) \to e^{i\alpha q_V} \mathcal{F}(x^\pm, e^{-i\alpha}\theta^\pm, e^{+i\alpha}\overline{\theta}^\pm),$$
$$e^{i\beta F_A} : \mathcal{F}(x^\pm, \theta^\pm, \overline{\theta}^\pm) \to e^{i\beta q_A} \mathcal{F}(x^\pm, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\overline{\theta}^\pm).$$

Chiral superfields obey $\overline{D}_\pm \Phi = 0$. The conjugate field $\overline{\Phi}$ is an antichiral superfield and is annihilated by $D_\pm$. A twisted chiral superfield satisfies $\overline{D}_+ Y = 0 = D_- Y$ and its conjugate is twisted antichiral and annihilated by $\overline{D}_-$ and D_+. Due to the linearity of the differential operators, the product of two chiral superfields is a chiral superfield. Similar statements hold for twisted chirals and the conjugate fields. The notions 'untwisted' and 'twisted' are interchanged as the notions of vector and axial $U(1)$ rotations are interchanged. Such a change corresponds to an outer automorphism of the supersymmetry algebra that interchanges the generators

$$\begin{aligned} F_V &\leftrightarrow F_A \\ Q_- &\leftrightarrow \overline{Q}_-. \end{aligned} \tag{3.1}$$

In a quantum theory the conserved charges become operators. Their commutation relations follow from the classical commutation relations of the symmetry generators. However, it is possible to have central charges in the anticommutation relations

$$\{\overline{Q}_+, \overline{Q}_-\} = Z$$
$$\{Q_-, \overline{Q}_+\} = \tilde{Z}.$$

These central charges break some of the R-symmetry. If the vector symmetry is conserved, Z must vanish and $\tilde{Z} = 0$ if F_A is conserved. The automorphism (3.1) also exchanges Z and $\tilde{Z}$.

Using superfields it is straightforward to write down Lagrangians which are invariant under supersymmetric transformations. The general D-term

$$\int \mathrm{d}^2x \int \mathrm{d}^2\theta \int \mathrm{d}^2\overline{\theta} K(\mathcal{F}_i) \tag{3.2}$$

is invariant when K is a function of general superfields $\mathcal{F}_i$. More specifically, one can construct a nonlinear sigma model, taking chiral and antichiral superfields as holomorphic and antiholomorphic coordinates on a Kähler manifold M, and $K(\Phi_i, \overline{\Phi_i})$ a Kähler potential on M which defines a positive definite Kähler metric

$$g_{i,\overline{j}} = \partial_i \bar{\partial}_j K(z_m, \overline{z}_n).$$

Instead of only chiral fields, one could equivalently have taken only twisted chirals, using the automorphism of the algebra. When K is a real valued function of both chirals Φ_i

and twisted chirals Y_a with bottom component fields the bosons ϕ_i and y_a respectively, the purely bosonic term in the Lagrangian reads [48]

$$\begin{aligned} &\left[\frac{\partial^2 K}{\partial\Phi_i\partial\overline{\Phi}_j}\partial^\mu\phi_i\partial_\mu\bar{\phi}_j - \frac{\partial^2 K}{\partial Y_a\partial\overline{Y}_b}\partial^\mu y_a\partial_\mu\overline{y}_b\right] \\ +\epsilon_{\mu\nu}&\left[\frac{\partial^2 K}{\partial\Phi_i\partial\overline{Y}_a}\partial_\mu\phi_i\partial_\nu\overline{y}_a + \frac{\partial^2 K}{\partial\overline{\Phi_i}\partial Y_a}\partial_\mu\bar{\phi}_i\partial_\nu y_a\right]. \end{aligned} \tag{3.3}$$

The derivatives of K in the top line are interpreted as a sigma model metric on a target space on which the ϕ's and y's are coordinates. If there are either no chirals or no twisted chirals, the metric is a Kähler metric and the second line in (3.3) equals zero. If K depends on both chiral and twisted chiral superfields, the derivatives of K together with $\epsilon_{\mu\nu}$ behave as an antisymmetric tensor background on the target space.

In addition to a D-term it is possible to have an F-term,

$$\int \mathrm{d}^2x \int \mathrm{d}^2\theta\, W(\Phi_i) + c.c.\,,$$

where W is a holomorphic function of the chiral superfields and c.c. stands for the complex conjugate function (of the antichiral fields). Similarly it is possible to add a twisted F-term,

$$\int \mathrm{d}^2x \int \mathrm{d}\overline{\theta}^-\mathrm{d}\theta^+\, \tilde{W}(Y_i) + c.c.\,.$$

If we are considering a sigma model on a Kähler manifold coordinatized by chiral superfields Φ_i, then $W(\Phi_i)$ is a holomorphic function on the Kähler manifold.

These various terms in a possible action are invariant under transformations generated by the four supersymmetry generators, but they are not all invariant under both F_V and F_A. Furthermore, even if F_V or F_A is a symmetry of the classical theory, it may be anomalous in the quantum theory.

First consider the symmetries at the classical level. The measure $\mathrm{d}^4\theta$ of the D-term is by itself invariant under both F_V and F_A. If the Kähler potential of a nonlinear sigma model depends on the chiral superfields only through the combination $\Phi\overline{\Phi}$, then it is invariant for any values of q_V and q_A of the chiral superfields. The measure of the F-term, $\mathrm{d}\theta^+\mathrm{d}\theta^-$ is invariant under F_A, and has vector charge -2. Choosing $q_A = 0$ for all chiral superfields in the F-term makes it F_A-invariant. If the superpotential is a weighted homogeneous function of the chiral superfields, it is possible to assign a definite vector charge to it, which compensates the transformation of the chiral measure if the q_V of the chiral superfields are chosen properly:

$$W(\lambda^{q_{V,1}}\Phi_1, \ldots, \lambda^{q_{V,n}}\Phi_n) = \lambda^2 W(\Phi_1, \ldots, \Phi_n).$$

A similar argument holds for the twisted F-term, exchanging vector and axial symmetries.

Quantummechanically the symmetries may be anomalous. For models based on chiral superfields only, sigma models on a Kähler space, possibly with a superpotential that is a holomorphic function on the target space, the following statement holds. A model based only on chiral superfields cannot have an anomaly of F_V, but it turns out that the D-term can have a F_A anomaly, see e.g. [18]. This anomaly is proportional to the first Chern class of the target space. So a nonlinear sigma model with a Calabi-Yau target space has both F_V and F_A symmetry, and the addition of a superpotential that is a weighted homogeneous function on the Calabi-Yau preserves both F_V and F_A.

In addition to the models discussed above, an important rôle is played by gauged linear sigma models, see for example [18] . The sigma model on $\mathbb{C}^m$ is invariant under linear shifts of coordinates on $\mathbb{C}^m$, Φ_k, which are chiral superfields in the sigma model. Also, it is invariant under rigid phase shifts $\Phi_k \to e^{2\pi i\alpha_k}\Phi_k$. To construct a model which is invariant under superspace dependent phase transformations of the chiral superfields,

$$\Phi_k(x^\pm, \theta^\pm) \to e^{-iq_k\Lambda(x^\pm,\theta^\pm)}\Phi_k(x^\pm, \theta^\pm), \tag{3.4}$$

parametrized by the chiral superfield Λ, the kinetic term for the chiral superfields must be changed, from $L_{kin} = \int \mathrm{d}^4\theta\Phi_1\overline{\Phi}_1 + \ldots + \Phi_m\overline{\Phi}_m$ to

$$L_{kin} = \int \mathrm{d}^4\theta \sum_{k=1}^{m} \Phi_k e^{q_k V}\overline{\Phi}_k, \tag{3.5}$$

where V is a vector superfield. It is a real superfield, $V = \overline{V}$, taking values in $U(1)$, or more generally, in $U(1)^r$, with $r < m$. The gauge group index will be suppressed for now. In order to compensate for the gauge transformations (3.4), the vector superfield transforms as

$$V \to V + i(\Lambda - \overline{\Lambda}). \tag{3.6}$$

To the vector superfield corresponds a field strength superfield which can be written as

$$\Sigma = \overline{D}_+ D_- V. \tag{3.7}$$

The field Σ is a twisted chiral superfield. It can be used to construct a gauge kinetic term in the Lagrangian,

$$L_{gauge} = \int \mathrm{d}^4\theta \frac{-1}{2e^2}\overline{\Sigma}\Sigma. \tag{3.8}$$

Given the naturally appearing twisted chiral field Σ, the field strength for a $U(1)$ gauge field, one can add a twisted F-term to the Lagrangian,

$$L_{FI} = \frac{1}{2}\left(-t \int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^- \Sigma\right) + c.c.\,, \tag{3.9}$$

where the constant $t = r - i\theta$ contains the theta angle θ and the Fayet-Iliopoulos parameter r. The Fayet-Iliopoulos parameter plays a central rôle in connecting different conformal field theories via the gauged linear sigma model [18] as is discussed in section 3.3.2. This twisted F-term is F_V-invariant choosing $q_V = 0$ for Σ and, as it is linear in Σ it is classically also F_A invariant if the axial charge of Σ is taken to be $q_A = 2$. A linear twisted superpotential is the only kind compatible with axial R-symmetry. In general it is possible to have a more general twisted superpotential, say, generated as a quantum effective superpotential, but it will break (part of) the F_A symmetry. In addition it is possible to add an F-term using the Φ_k, as long as the superpotential is gauge invariant.

The total gauge invariant Lagrangian thus consists of four terms. The D-terms L_{kin} and L_{gauge}, both kinetic terms,

$$\int \mathrm{d}^4\theta \sum_{k=1}^{m} \overline{\Phi}_i e^{q_i^{(a)} V_a} \Phi_k + \sum_{a,b=1}^{r} \frac{-1}{2e^2_{(a,b)}} \overline{\Sigma}_a \Sigma_b, \tag{3.10}$$

the Fayet-Iliopoulos (and θ-angle) term L_{FI}

$$\frac{1}{2}\int \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^- \sum_{a=1}^{r} -t\Sigma_a \ + \ c.c.\,, \tag{3.11}$$

which is a twisted F-term, and possibly a gauge invariant F-term L_W

$$\int \mathrm{d}\theta^+ \mathrm{d}\theta^- W(\Phi_k) \ + \ c.c. \tag{3.12}$$

Written out in components the superfields are

$$\begin{aligned} \Phi &= \phi + \theta^+\psi_+ + \theta^-\psi_- + \theta^+\theta^- F \\ V &= \theta^-\overline{\theta}^-(v_0 - v_1) + \theta^+\overline{\theta}^+ - \theta^-\overline{\theta}^+\sigma - \theta^+\overline{\theta}^-\overline{\sigma} \\ &\quad + i\theta^-\theta^+(\overline{\theta}^-\overline{\lambda}_- + \overline{\theta}^+\overline{\lambda}_+) + \overline{\theta}^+\overline{\theta}^-(\theta^-\lambda_- + \theta^+\lambda_+) \\ &\quad + \theta^-\theta^+\overline{\theta}^+\overline{\theta}^- D \\ \Sigma &= \sigma + i\theta^+\overline{\lambda}_+ - i\overline{\theta}^-\lambda_- + \theta^+\overline{\theta}^-(D - iv_{01}). \end{aligned} \tag{3.13}$$

Where $v_{01} = \partial_0 v_1 - \partial_1 v_0$ is the gauge field strength. The component fields of a chiral (twisted chiral) superfield depend only on the combinations of superspace coordinates $x^\pm - i\theta^\pm\overline{\theta}^\pm$ $(x^\pm \mp \theta^\pm\overline{\theta}^\pm)$. Using the convariant derivative $D_\mu = \partial_\mu + iv_\mu$, the various terms in the Lagrangian can be expressed as

$$\begin{aligned} \int \mathrm{d}^4\theta \Phi e^{qV}\overline{\Phi} = &- D^\mu\bar{\phi}D_\mu\Phi \\ &+ D|\phi|^2 + |F|^2 - |\sigma|^2|\phi|^2 \\ &+ i\left[\overline{\psi}_-(D_0 + D_1)\psi_- + \overline{\psi}_+(D_0 - D_1)\psi_+\right] \\ &- \left[\overline{\psi}_-\sigma\psi_+ + \overline{\psi}_+\overline{\sigma}\psi_-\right] \\ &+ i\left[i\bar{\phi}(\lambda_+\psi_- - \lambda_-\psi_+) + \phi(\overline{\psi}_+\overline{\lambda}_- - \overline{\psi}_-\overline{\lambda}_+)\right], \end{aligned} \tag{3.14}$$

and

$$\frac{1}{2e^2}\int \mathrm{d}^4\theta\overline{\Sigma}\Sigma = \frac{1}{2e^2}\left[-\partial^\mu\overline{\sigma}\partial_\mu\sigma + v_{01}^2 + D^2\right] + \frac{i}{2e^2}\left[\overline{\lambda}_-\partial_+\lambda_- + \overline{\lambda}_+ - \partial_-\lambda_+\right], \quad (3.15)$$

and

$$\frac{1}{2}\left(\int \mathrm{d}^2\tilde{\theta} - t\Sigma\right) + c.c. = -rD + \theta v_{01}. \quad (3.16)$$

The potential energy for the scalar component fields is found to be [18]

$$U = |q_k^{(a)}\sigma_a|^2|\phi_k|^2 + \frac{(e^{(a,b)})^2}{2}\left(q_k^{(a)}|\phi_k|^2 - r_a\right)\left(q_l^{(b)}|\phi_l|^2 - r_b\right) + \sum_{i=1}^{m}|\frac{\partial W}{\partial\phi_i}|^2. \quad (3.17)$$

Here ϕ_k are scalar components of the chiral superfields and σ_a are scalar components of the twisted chiral field strengths and $(e^{(a,b)})^2$ is the inverse of the coupling matrix, for a $U(1)^r$ gauge group, appearing in (3.10).

3.2 Generalities of $\mathcal{N} = (2,2)$ Superconformal Theories

The symmetry algebra common to most of the models discussed in this chapter is the $\mathcal{N} = 2$ superconformal algebra, or more accurately, a holomorphic and an antiholomorphic copy of this algebra. The $\mathcal{N} = 2$ superconformal algebra contains the Virasoro algebra, two sets of fermionic partners to the Virasoro generators and generators of a $U(1)$ R-symmetry which rotates the two sets of fermionic generators. The nonzero (anti-) commutation relations are the following:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{\hat{c}}{4}m(m^2-1)\delta_{m+n,0}\\
[J_m, J_n] &= \hat{c}\delta_{m+n,0}\\
[L_m, J_n] &= -nJ_{m+n}\\
[L_m, G^{\pm}_{n\pm a}] &= \left(\frac{m}{2} - (n \pm a)\right)G^{\pm}_{m+n\pm a}\\
\{G^+_{m+a}, G^-_{n-a}\} &= 2L_{m+n} + (m-n+2a)J_{m+n} + \hat{c}\left((m+a)^2 - \frac{1}{4}\right)\delta_{m+n,0}\\
[J_m, G^{\pm}_{n\pm a}] &= \pm G^{\pm}_{m+n\pm a}.
\end{aligned} \quad (3.18)$$

Note that $(G^{\pm}_m)^\dagger = G^{\mp}_{-m}$ and $\hat{c} = c/3$.

The parameter $0 \leq a < 1$ determines the boundary conditions of the fermionic currents,

$$G^{\pm}(e^{2\pi i}z) = -e^{\mp 2\pi i a}G^{\pm}(z), \quad (3.19)$$

so in fact there is a family of $\mathcal{N} = 2$ superconformal algebras, including the Ramond (integral moding) and Neveu-Schwarz (half-integral moding) cases.

All the algebras in this family are isomorphic and the isomorphism is provided by spectral flow. The $U(1)$ current can be written in terms of a free scalar boson as

$$J(z) = i\sqrt{\hat{c}}\,\partial\phi(z), \tag{3.20}$$

and a spectral flow operator as

$$U_\eta = e^{-i\sqrt{\hat{c}}\eta\phi}. \tag{3.21}$$

Spectral flow shifts operators as $\mathcal{O} \to U_\eta \mathcal{O} U_\eta^{-1}$ and on states as $|\,\rangle \to U_\eta|\,\rangle$. It changes the boundary conditions of the fermionic currents, but not the structure of the algebra.

The representations of the differently moded algebras look different as spectral flow shifts the $U(1)$ charge and conformal weight. Of special interest are chiral and antichiral states [21]. These are states annihilated by $G^+_{-1/2}$ and $G^-_{-1/2}$ respectively. Among these states the primary states, annihilated by all positive modes, take a special place. The conformal weight h and the J_0 charge q of chiral (antichiral) primaries are $h = q/2$ ($h = -q/2$), which saturate the bound $2h \geq |q|$ that holds for any state, from unitarity. In a non-degenerate unitary conformal field theory (so the spectrum of L_0 is discrete) the number of chiral primary fields is finite as their weights satisfy $h \leq \hat{c}/2$. There is a unique chiral primary which saturates this bound.

The operator algebra of chiral primary fields is special, as there are no singularities to subtract, $(\chi\phi)(z) = \lim_{z'\to z} \phi(z')\chi(z)$. Because of additivity of the $U(1)$ charge, for chiral primary fields it holds that $h_{\chi\phi} = (q_\chi + q_\phi)/2 = h_\chi + h_\phi$. The general form of the operator product is $\phi(z)\chi(z') = \sum_\psi (z-z')^{h_\psi - h_\phi - h_\chi}\psi(z)$. Note that the product of two chiral (primaries) is again chiral, but not necessarily primary. Non-primary terms vanish as $z' \to z$ while primary terms occur with finite coefficients. So operator product induces the structure of a finite commutative nilpotent ring on chiral primary states. This chiral ring can also be thought of as the operator algebra of general chiral states, modulo the equivalence relation that sets descendant states to zero. Of course a similar argumentation goes through for antichiral states. Having a holomorphic and an antiholomorphic part of the algebra, there are thus four rings (c,c), (a,c) and by conjugation (a,a) and (c,a).

Spectral flow $|\,\rangle \to U_\eta|\,\rangle$ changes the $U(1)$ charge and conformal weight

$$\begin{aligned} q &\to q_\eta = q - \hat{c}\eta \\ h &\to h_\eta = h - q\eta + \frac{\hat{c}}{2}\eta^2 \end{aligned} \tag{3.22}$$

By spectral flow of half a unit a chiral primary state $|\,\rangle_{NS}$ can be mapped to a Ramond ground state $|\,\rangle_R = U_{1/2}|\,\rangle_{NS}$, and flowing yet another half a unit, to an antichiral primary. The conformal weight and $U(1)$ charge of states connected by spectral flow varies:

$$U_{1/2}\ :\ |q,h\rangle_{NS} = |Q,Q/2\rangle_{NS} \to |Q-\hat{c}/2, \hat{c}/8\rangle_R \to |-Q, Q/2\rangle_{NS}.$$

It is possible to apply spectral flow to the holomorphic and the anti-holomorphic sectors separately. Flowing by the same amount in both sectors the difference of $U(1)$ charges $q - \overline{q}$ is unchanged. The (c, c) primary states and the Ramond-Ramond ground states are related by symmetric spectral flow. The $U(1)$ character of the Ramond ground states and the (c, c) states are related

$$\mathrm{Tr}_R \left[t^{J_0}\overline{t}^{\overline{J}_0}\right]|_{G_0^\pm=\overline{G}_0^\pm=0} = (t\overline{t})^{-\hat{c}/2}\mathrm{Tr}_{NS}\left[t^{J_0}\overline{t}^{\overline{J}_0}\right]|_{(c,c)} \equiv (t\overline{t})^{-\hat{c}/2}P(t,\overline{t}). \tag{3.23}$$

The Poincaré polynomial $P(t, \overline{t}) = \sum b_{p,q}t^p\overline{t}^q$ encodes the degeneracies $b_{p,q}$ of (c, c) primaries with $U(1)$ charge (p, q). From $P(t, \overline{t})$ one can also read off the Witten index, to which only Ramond ground states contribute,

$$\begin{aligned}\mathrm{Tr}(-1)^F &= Tr_R\left[(-1)^{J_0-\overline{J}_0}q^{L_0-\hat{c}/8}\overline{q}^{\overline{L}_0-\hat{c}/8}\right]\\ &= \sum_{k\in(c,c)} e^{i\pi(q_k-\overline{q}_k)} = P(e^{i\pi}, e^{-i\pi}).\end{aligned} \tag{3.24}$$

In case all $U(1)$ charges are integral, $P(t, \overline{t})$ looks like the Poincaré polynomial of the Dolbeault cohomology ring of a complex manifold with Hodge numbers $b_{p,q}$. When all charges are integral then in particular $\hat{c} = d$ is an integer, as there is a unique state with $q = \hat{c}$ in any $\mathcal{N} = (2, 2)$ model. Then spectral flow in the holomorphic sector relates (NS, NS) and (R, NS) states and furthermore, with integral charges it is possible to define $(-1)^{F_L} = e^{i\pi J_0}$ and $(-1)^{F_R} = e^{-i\pi\overline{J}_0}$ to make a GSO projection. Such models can be used as factors in a supersymmetric string 'compactification', also if the model has no direct geometric interpretation.

The invariance of Ramond ground states under charge conjugation implies

$$P(t,\overline{t}) = (t\overline{t})^{\hat{c}}P(1/t, 1/\overline{t}). \tag{3.25}$$

In terms of the coefficients this says $b_{p,q} = b_{\hat{c}-p,\hat{c}-q}$, which looks like Poincaré duality for a d dimensional complex manifold.

The connection between supersymmetric ground states and cohomology classes is very general. Because of the relations $\{Q, Q\} = 0$ and $\{Q, Q^\dagger\} = 2H$, supersymmetric ground states are representatives of the cohomology classes of the supercharge. In supersymmetric sigma models, the ground states necessarily must have zero momentum [57]. It suffices to consider quantize only those modes which do not depend to the spatial coordinate, so the supersymmetric ground states are determined from supersymmetric quantum mechanics.

For a $\mathcal{N} = (2, 2)$ sigma model the target space is Kähler and the Lagrangian restricted to the constant modes, written out in component fields is [59]

$$\int \mathrm{d}^4\theta K(\Phi_i, \overline{\Phi}_i) = g_{i\overline{j}}\dot{\phi}_i\dot{\phi}_{\overline{j}} + ig_{i\overline{j}}\left(\overline{\psi}^{\overline{j}}D_t\psi^i + \overline{\psi}^i D_t\psi^{\overline{j}}\right) + R_{i\overline{j}k\overline{l}}\overline{\psi}^i\psi^k\psi^{\overline{j}}\overline{\psi}^{\overline{l}}, \tag{3.26}$$

So the bosonic components ϕ_i and $\phi_{\bar{j}}$ are coordinates on the target manifold (pulled back to the worldline). The supercharges are

$$\begin{aligned} Q_+ &= g_{i\bar{j}} \psi^i \dot{\bar{\phi}}^j \\ Q_- &= g_{i\bar{j}} \bar{\psi}^i \dot{\bar{\phi}}^j, \end{aligned} \tag{3.27}$$

and their conjugates.

Now these constant[1] modes can be canonically quantized. The fermions become creation and annihilation operators,

$$\begin{aligned} \{\psi^i, \overline{\psi}^{\bar{j}}\} &= g^{\bar{j}i}, \\ \{\psi^{\bar{i}}, \overline{\psi}^j\} &= g^{\bar{i}j}. \end{aligned} \tag{3.28}$$

It is natural to map these anticommuting objects with cotangent space indices to differential forms on the target manifold, $\overline{\psi}^i \sim \mathrm{d}\phi^i$ and $\overline{\psi}^{\bar{i}} \sim \mathrm{d}\bar{\phi}^{\bar{i}}$. The adjoint operators are identified with the dual vectors, $\psi^i \sim g^{i\bar{j}} \partial_{\bar{j}}$, $\psi^{\bar{j}} \sim g^{i\bar{j}} \partial_i$, where $\partial_i = \partial / \partial \phi^i$ and $\partial_{\bar{j}} = \partial / \partial^{\bar{j}}$. The supercharges are

$$\begin{aligned} Q_+ &= \psi^i \pi_i, \\ Q_- &= \overline{\psi}^i \pi_i, \\ \overline{Q}_+ &= \overline{\psi}^{\bar{j}} \pi_{\bar{j}}, \\ \overline{Q}_- &= \psi^{\bar{j}} \pi_{\bar{j}}. \end{aligned} \tag{3.29}$$

Here π_i is the momentum canonically conjugate to ϕ_i. In the field theory this can be thought of as the functional derivative $\delta / \delta \phi_i$. Restricted to the constant modes, this reduces to the ordinary partial derivatives ∂_i and $\partial_{\bar{j}}$. The supercharges can be related to the Dolbeault operators and their adjoints,

$$\begin{aligned} Q_- &\sim \mathrm{d}\phi^i \circ \partial_i, \\ \overline{Q}_+ &\sim \mathrm{d}\phi^{\bar{j}} \circ \partial_{\bar{j}}. \end{aligned} \tag{3.30}$$

So the Ramond ground states of a Calabi-Yau sigma model correspond to harmonic forms on the Calabi-Yau. By spectral flow these can be mapped to (c, c) or (a, c) primaries.

In the following sections, various different constructions of $\mathcal{N} = (2,2)$ conformal models are briefly reviewed, most notably Landau-Ginzburg models and coset cft's. There exist such models at $\hat{c} = d \in \mathbb{Z}$, which have been used in numerous ways as supersymmetric string compactifications, of the form $\mathcal{C}(\hat{c} = d) \times M^{1,9-2d}$ to leave a Minkowski factor

[1] Of course 'constant' means 'no dependence on the spatial coordinate'.

$M^{1,9-2d}$. In the next chapter we will consider $\hat{c} \in \mathbb{Z}$ models which describe either non-compact 'decompactification' limits, or 'throat' geometries. Such models are constructed out of the conformal field theories discussed in the following section, but at non-integer values of $\hat{c}$, combined with the Euclidean black hole cft, or a Liouville model, to obtain $\hat{c}_{\text{total}} \in \mathbb{Z}$.

3.3 Infrared Limits of Nonconformal Field Theories

3.3.1 Landau-Ginzburg Models

In this section $\mathcal{N} = (2,2)$ superconformal field theories will be discussed, which arise as infrared fixed points of the renormalization group flow of nonconformal $\mathcal{N} = (2,2)$ supersymmetric quantum field theories. Consider a 2d $\mathcal{N} = (2,2)$ quantum field theory of a set of chiral superfields. Its Lagrangian contains a D-term

$$\int \mathrm{d}^2x \int \mathrm{d}^4\theta K(\Phi_i, \overline{\Phi}_i),$$

where K can be interpreted as a Kähler potential for a nonlinear sigma model. In addition the Lagrangian can also contain an F-term

$$\frac{1}{2}\int \mathrm{d}^2x \left(\int \mathrm{d}^2\theta W(\Phi_i) + c.c.\right),$$

where the superpotential W is a holomorphic function of the Φ_i, or viewing the Φ_i as coordinates on the target space of a sigma model, W is a holomorphic function on the target space. In general such a theory does not have conformal invariance, but renormalization group flow to the infrared gives some, possibly trivial, scale invariant, and hence in 2d conformally invariant, fixed point.

For Landau-Ginzburg models of the above form it is believed that any weighted homogeneous superpotential, up to analytic field redefinitions, corresponds to a unique conformal fixed point. Due to $\mathcal{N} = (2,2)$ supersymmetry, the D-terms do not enter in the renormalized F-terms. A way to convince oneself of this statement goes as follows. One may promote the couplings in the D-term to twisted chiral fields that are much heavier than the mass scale of interest. Effectively these are frozen out and act as nothing but parameters. But twisted chirals can not appear in the F-term. On the other hand, the parameters in the superpotential may be thought of as frozen out chiral superfields. The supersymmetry does not preclude these fields to enter in the D-term, and hence, they may alter the form of the D-term at low energies.

For a general superpotential which is not weighted homogeneous, the axial R-symmetry is broken. Consider the couplings in front of the various terms of different scaling weight as heavy fields, with the right axial R-charges so as to render the superpotential weighted homogeneous. As a consequence of the supersymmetry, the F-terms are not renormalized, whereas the superpotential receives only wavefunction renormalization. So under a rescaling of coordinates, $z \to \lambda z$ and $\theta \to \lambda^{-1/2}\theta$ all the chiral superfields, including the couplings, are rescaled by some factor $\Phi_i \to \lambda^{\omega_i}\Phi_i$, such that $W \to \lambda W$. Consider, for concreteness, a superpotential

$$W = g_n \Phi^n + g_{n+1}\Phi^{n+1}.$$

Let $\Phi \to \lambda^{1/n}\Phi$, then g_n does not rescale, while $g_{n+1} \to \lambda^{-1/n} g_{n+1}$. In the infrared limit $\lambda \to \infty$ the effective coupling of the term $g_{n+1}\Phi^{n+1}$ vanishes and only the coefficients multiplying terms with the lowest scaling weight survive.

The idea is that conformal fixed points are uniquely labeled by weighted homogeneous superpotentials, up to analytic field redefinitions. A suitable D-term is renormalized along the RG flow in a way dictated by the superpotential, so as to get a conformally invariant theory at the endpoint of the flow. Often as a starting point the D-term corresponding to a sigma model on flat $\mathbb{C}^{n+2}$ is taken.

The $\mathcal{N} = (2,2)$ supersymmetry is enhanced to a superconformal symmetry. Under a $2d$ rescaling the chiral superfields scale as $\Phi_i \to \lambda^{w_i}\Phi_i$ and the superpotential is weighted homogeneous,

$$W(\lambda^{w_1}\Phi_1, \ldots, \lambda^{w_m}\Phi_m) = \lambda W(\Phi_1, \ldots, \Phi_m). \tag{3.31}$$

But this scaling of the fields precisely says that their anomalous dimension, their scaling weight, is w_i. As the scaling weight satisfies $\Delta = h + \overline{h}$ the conformal dimension of a field Φ_i is

$$h = \frac{w_i}{2}. \tag{3.32}$$

Using generic properties of superconformal symmetry and unitarity it is also possible to find the central charge of the conformal field theory corresponding to (3.31) [20]. The expression for the central charge is given in (3.35). This expression is obtained consideration of the (c, c) ring, and more particularly, the element with largest $U(1)$ charge in this ring, as will be done now.

The fields Φ_i have conformal weight h and axial charge q_A related as $q_A = 2h = w_i$. This is the proportionality possessed by a chiral primary field. Furthermore, the derivatives of the superpotential $\partial W/\partial \Phi_i$ are proportional to some superderivatives acting on combinations of the chiral and antichiral primaries.In cft terminology, combinations of Φ_i's and $\overline{\Phi}_j$'s that contain a factor $\partial W/\partial \Phi_i$ are descendant fields. The chiral primary ring of the conformal field theory is then obtained as quotient ring of complex polynomials in the fields Φ_i modulo the ideal generated by the partial derivatives of the superpotential, $\partial W/\partial \Phi_i$:

$$\mathcal{R} = \frac{\mathbb{C}[\Phi_i]}{\partial_j W(\Phi_i)}. \tag{3.33}$$

Any fields appearing as Φ^2 is the superpotential, do not change the (c, c) ring.

The degeneracies of $U(1)_V$ charges of the (c,c) ring are encoded in the Poincaré polynomial. This is computed in terms of the weights of W [20]. Conventionally the scaling properties of W are characterized by the (positive) integers $a_i = w_i \cdot d$, such that the greatest common denominator of all a_i's equals one, so that

$$W(\lambda_1^a x_1, \ldots, \lambda_m^a x_m) = \lambda^d W(x_1, \ldots, x_m).$$

Then the Poincaré polynomial is[2]

$$P(t,\bar{t}) = \prod_{i=1}^{m} \frac{1-(t\bar{t})^{d-a_i}}{1-(t\bar{t})^{a_i}}. \tag{3.34}$$

There is a unique (c,c) primary state of weight $h = \bar{h} = 0$ and a unique one of highest conformal weight, which has $h/2 = \bar{h}/2 = q = \hat{c}$. From inspection of the (c,c) ring it follows that the central charge is

$$\hat{c} = \sum_{i=1}^{m} (1-2w_i)\,, \tag{3.35}$$

where $w_i = a_i/d$. The dimension of the chiral ring is also easily found from the Poincaré polynomial, as

$$\mu = P(t=1,\bar{t}=1) = \prod_{i=1}^{m} \left(\frac{1}{w_i} - 1\right). \tag{3.36}$$

Having started from a field theory with (twisted) chiral superfields, at the infrared fixed point a theory lies with a non-trivial (c,c) ring and a trivial (a,c) ring (or vice versa). The holomorphic and antiholomorphic $U(1)$ charges satisfy $q - \bar{q} = 0$ (or $q + \bar{q} = 0$ for twisted chirals). The $U(1)$ charges of states are generally multiples of $1/d$ and thus not necessarily integer. In particular, the top (c,c) primary state has $q = \hat{c}$, which is generically not an integer, see (3.35).

It is possible to project out states of non-integer q by an orbifold construction [49], for which it is important that $q - \bar{q} \in \mathbb{Z}$. This last property ensures that a projection on integer left- and right-moving $U(1)$ charges can be achieved by a left-moving operator alone. More to the point, one should orbifold by the action of

$$j = e^{2\pi i J_0}. \tag{3.37}$$

Denote by J the cyclic group generated by j. The J-orbifold projects out (c,c) states with $q \notin \mathbb{Z}$ but one also needs to add all j-twisted sectors and project onto j-invariant states in each of these sectors as well.

[2] The Poincaré polynomial in 3.34 is slightly different from the definition in 3.2. Really in 3.34 is written $P(t^d, \bar{t}^d)$. In 3.34 the term $(t\bar{t})^{qd}$ corresponds to (c,c) primaries of charge $q = \bar{q}$.

Such an orbifold projection can get rid of some (c, c) states and add extra (a, c) states. If the central charge $\hat{c} \in \mathbb{Z}$ the physical states in the orbifold model all have integer $U(1)$ charges. Such a model, W/J with $\hat{c} \in \mathbb{Z}$ satisfies the requirements for a spacetime supersymmetric string compactification [50, 49].

The action of j has an interpretation in terms of the superpotential. It is the $\mathbb{Z}_d \subset \mathbb{C}^*$ part of the weighted homogeneous scalings which leave the superpotential invariant:

$$j \,:\, W(\Phi_1, \ldots, \Phi_m) \to W(e^{2\pi i q_1}\Phi_1, \ldots, e^{2\pi i q_m}\Phi_m) = W(\Phi_1, \ldots, \Phi_m). \tag{3.38}$$

It is clear that any weighted homogeneous polynomial has such a symmetry, irrespective of whether $\hat{c} \in \mathbb{Z}$. Actually, any superpotential that is the sum of k separate weighted homogeneous pieces of weighted degree d, with appropriate charge assignments to the chiral fields, possesses k such cyclic symmetries. It can be regarded as the tensor product of k separate conformal field theories, each with their own j_ℓ, $1 \leq \ell \leq k$. Each factor theory is a superconformal model by itself and hence admits a separate J-orbifolding. The projection onto integral charges is achieved by acting with the operator in the tensor product model $j_{tot} = \otimes j_\ell$.

Let us consider a simple example of a superpotential,

$$W = \Phi^{k+2}.$$

The corresponding Poincaré polynomial is

$$P(t, \bar{t}) = \sum_{\ell=1}^{k} (t\bar{t})^\ell.$$

There is one (c, c) primary at each $U(1)$ charge $q = \ell/(k+2)$ for $0 \leq \ell \leq k$. The (a, c) ring is trivial, it consists of the vacuum only. The central charge is

$$\hat{c} = 1 - \frac{2}{k}.$$

There is a unique conformal field theory with these properties and it can be constructed in different ways. One way is as the k-th $\mathcal{N} = (2, 2)$ minimal model based on the A-type modular invariant [51]. With the explicit modular invariant partition function of the model, one can construct the j-orbifold. Only the vacuum of the original model survives the j-projection. There are $k + 1$ twisted sectors, and from each but one a single state survives the orbifold projection [49]. The states surviving in the orbifold model have $U(1)$ charges $(-q, \overline{q} = q)$, whereas the states in the unorbifolded model have $(q, \overline{q} = q)$. In other words, the orbifolded model looks like the original unorbifolded Landau-Ginzburg model with a twisted chiral field instead of a chiral field (up to an overall minus sign). But the two possibilities in the overall choice of taking chiral superfields or twisted chiral superfields are related by the $\mathbb{Z}_2$ automorphism of the super(-conformal-)algebra (3.1).

The example treated above is perhaps the simplest case that illustrates an isomorphism between somewhat different looking conformal field theories (here related by the j-orbifold) which are related by the $\mathbb{Z}_2$ automorphism of the superconformal algebra (3.1), which is known as the mirror automorphism. In the context of the defining Landau-Ginzburg models the isomorphism in this case amounts to exchanging all chiral superfields for twisted chiral superfields.

3.3.2 Phases of a Gauged Linear Sigma Model

At this point, there is no apparent geometric interpretation of the isomorphism of conformal field theories as seen in Landau-Ginzburg models of the preceding section. In order to get at such an interpretation, we shall proceed to discuss a beautiful connection between Landau-Ginzburg models and nonlinear sigma models [18], which do have a quite direct geometric interpretation. Both the Landau-Ginzburg model, or more accurately, the W/J Landau-Ginzburg orbifold, and the nonlinear sigma model arise as deep infrared limits of certain supersymmetric Abelian gauge theories, in opposite regions of the value of an order parameter: the Fayet-Iliopoulos parameter. This parameter is a modulus and labels a family of conformal field theories, some of which have a geometric interpretation.

The $U(1)^r$ gauged linear sigma model of section 3.1 has the parameter $t = r - i\theta$ in front of the twisted F-term in the classical Lagrangian. The auxiliary component field D of a $U(1)$ gauge field appears in the Lagrangian as

$$\frac{1}{2e^2}D^2 + D\left(\sum_{i=1}^{m} iq_i|\phi_i|^2 - r\right),$$

where r is the 'bare' Fayet-Iliopoulos parameter. Integrating out the high frequency modes of the ϕ_i in a range $\mu < |k| < \mu_0$, one finds

$$\langle|\phi_i|^2\rangle = q_i \int_{|k|=\mu}^{\mu_0} \mathrm{d}^2k \frac{1}{k^2 + |\sigma|^2 + \ldots} \quad ,$$

where σ denotes the expectation value $\sigma = \langle\sigma\rangle$ of the scalar field that is the bottom component of the field strength (twisted chiral) superfield Σ, as in (3.13). Consequently

$$\langle\frac{-D}{e^2}\rangle \sim -r + \log\left(\frac{\mu_0}{\mu}\right)\sum_{i=1}^{m} q_i \, .$$

Unless $\sum q_i = 0$, the physical Fayet-Iliopoulos must be specified at some scale μ_0, to be $r(\mu_0)$ and the effective FI-parameter runs with the energy scale as $r(\mu) = r(\mu_0) + \sum(q_i)\log(\mu/\mu_0)$. The RG flow is specified by the dynamically generated scale Λ according to

$$r(\mu) = \left(\sum_{i=1}^{m} q_i\right) \log\left(\frac{\mu}{\Lambda}\right).$$

From the opposite viewpoint, the classical value r_0 of the Fayet-Iliopoulos parameter in a quantum theory depends on the dynamical scale and the ultraviolet cut-off as

$$r_0 = \left(\sum_{i=1}^{m} q_i\right) \log\left(\frac{\Lambda_{UV}}{\Lambda}\right). \tag{3.39}$$

If $\sum q_i \neq 0$, the fermionic components of the charged chiral superfields induce an anomaly under axial R-transformations. Under an axial rotation by $e^{i\alpha}$ the θ-angle is shifted by $\theta \mapsto \theta - 2\alpha \sum q_i \equiv \theta + 2\gamma\alpha$, which breaks the axial R-symmetry to $\mathbb{Z}_{2\gamma}$. The value of r at any scale is determined from the dynamical scale. Once Λ is specified, $r(\mu)$ is fixed.

When the sum of charges of each $U(1) \subset U(1)^r$ vanishes, the axial symmetry is preserved. Also, the Fayet-Iliopoulos term is a genuine parameter of the quantum theory. In other words, this parameter labels an entire family of theories. In the infrared it is a modulus labeling a family of conformal field theories.

Consider the infrared limit of such a model. For the sake of simplicity, take the gauge group to be just $U(1)$ and take $m = n + 1$ chiral superfields Φ_i of gauge charge one and one chiral superfield P of gauge charge $q = -n - 1$, such that the Fayet-Iliopoulos parameter does really parametrize a family of quantum theories. Also add a gauge invariant superpotential $W = \alpha P \cdot F(\Phi_1, \ldots, \Phi_n + 1)$, which can be switched off by setting $\alpha = 0$. Also let $F = dF = 0$ have a solution only at the origin, $\Phi_i = 0\ \forall i$. In this case the scalar potential reads

$$\begin{aligned} U = \overbrace{|\sigma|^2 \left((n+1)^2|p|^2 + \sum_{k=1}^{n+1} |\phi_k|^k\right) + \frac{e^2}{2}\left(-r - (n+1)|p|^2 + \sum_{k=1}^{n+1} |\phi_k|^2\right)}^{\text{From kinetic and FI-term}} \\ \underbrace{+ |\alpha|^2|F|^2 + |\alpha|^2|p|^2 \sum_{i=1}^{n+1} |\frac{\partial F}{\partial \phi_i}|^2}_{\text{From F-term (i.e. from superpotential)}}. \end{aligned} \tag{3.40}$$

Consider the infrared limit, while taking the gauge coupling $e \to \infty$, which has the effect that the gauge field is non-dynamical. The only effect of the gauge symmetry is that it identifies different values of the chiral superfields. For the moment, switch off the F-term, that is to say, set the superpotential to zero, so the second line of (3.40) disappears. If $r \gg 0$, the vacuum manifold consists of points $\sum |\phi_i|^2 = r + |p|^2$, modulo the $U(1)$ symmetry. This is the total space of $\mathcal{O}(-n-1) \to \mathbb{P}^n$. The FI-parameter r sets the size of the base space. If $r \ll 0$, the vacuum manifold consists of $|p|^2 = |r| + \sum |\phi_i|^2$, up to gauge transformations. By a gauge transformation one can align p along the positive real axis. This completely fixes p, but there is a group $\mathbb{Z}_{n+1} \subset U(1)$ of gauge transformations which do this. These act nontrivially on the ϕ_i's. The vacuum manifold is thus $\mathbb{C}^{n+1}/\mathbb{Z}_{n+1}$.

Now we switch on the F-term. For $r \gg 0$ the coordinate on the line bundle is completely fixed to $p = 0$ and the homogeneous coordinates ϕ_i of $\mathbb{P}^n$ satisfy $F(\phi_1, \dots, \phi_{n+1}) = 0$. As the degree of F is $n+1$ this is a Calabi-Yau hypersurface. For $r \ll 0$ all ϕ_i coordinates on $\mathbb{C}^{n+1}/\mathbb{Z}_{n+1}$ are forced in the origin. As $p \neq 0$ there is however a homogeneous superpotential for the Φ_i. There is no target space, but the cft is described as a Landau-Ginzburg orbifold.

Generalizing this example, linear sigma models connect projective hypersurfaces $F^{-1}(0)$ in a weighted projective space and Landau-Ginzburg orbifolds $(W = F)/J$. Using $U(1)^r$ gauge groups, this connection extends to complete intersections in toric manifolds [18]. There are some important points to note, regarding this connection.

First of all, the sum of the gauge charges of each $U(1)$ should vanish, so that it makes sense to talk about different values of r in the quantum gauge theory. Secondly, conformal models corresponding to positive and negative values of r are part of a single moduli space of conformal field theories. In order to show this, one should not pass through the singular point $r = 0$. Fortunately, one can move around this point in the ultraviolet, using the θ-angle, which is part of the same single complex parameter [18].

Finally, for now, as long as there is no F-term, the above analysis applies to situations where m chiral fields have positive gauge charges q_i and n have negative charges $\tilde{q}_j$, as long as the sum of all charges vanishes. The corresponding vacuum manifolds are

$$\bigoplus_j \mathcal{O}(-|\tilde{q}_j|) \to \mathbb{P}[q_1, \dots, q_m]$$

and

$$\bigoplus_i \mathcal{O}(-q_i) \to \mathbb{P}[|\tilde{q}_1|, \dots, |\tilde{q}_n|]$$

But with a F-term, the situation can become more complicated. Yet, one might like to add such an F-term, hoping to describe a hypersurface in affine complex space, or an orbifold thereof.

WORLDSHEET MODELS FOR SUPERSYMMETRIC CONES

The supersymmetric cones of chapter 2 can be regarded as hypersurfaces in affine $\mathbb{C}^{n+1}$ or as line bundles over projective varieties of positive first Chern class. The coordinate on such a line bundle is the scalar component of a chiral superfield Φ_0 of negative charge.

Consider a polynomial which defines a Fano subvariety in an appropriate weighted projective space,

$$F(\lambda^{a_1} x_1, \dots, \lambda^{a_m} x_m) = \lambda^d F(x_1, \dots, x_m). \tag{3.41}$$

Chiral superfield Φ_i	$U(1)$ charge a_i
$\Phi_1, \ldots, \Phi_4$	$a_i = 1$
Φ_{-1}	$a_{-1} = -3$
Φ_0	$a_0 = -1$

Table 3.1: Charge assignments for cubic cone in $\mathbb{C}^4$.

If $F^{-1}(0)$ is Fano, the weights need to satisfy

$$d < \sum_{i=1}^{m} a_i \equiv A. \tag{3.42}$$

The total space of a Calabi-Yau line bundle over the Fano variety is given by the affine hypersurface $F^{-1}(0) \subset \mathbb{C}^m$. In the remainder of this section, a variation on the linear sigma models will be discussed. This variation can, in the infrared, be viewed as a nonlinear sigma model on a hypersurface in $\mathcal{O}(d-A) \to \mathbb{P}[a_1, \ldots, a_{n+2}]$. This line bundle has an affine coordinate patch which looks like $\mathbb{C}^{n+2}$, or actually $\mathbb{C}^{n+2}/\mathbb{Z}_{d-A}$. Therefore, such model can be useful to describe supersymmetric affine hypersurfaces.

The advantage of having an ultraviolet theory, is that it may unify conformal field theories at different points in moduli space, depending on the particular value of certain expectation values. For example, in Witten's gauged linear sigma model [18] this expectation value is the Fayet-Iliopoulos parameter and theta angle, which can be viewed as an expectation value of a spurious twisted chiral superfield. The value of the Fayet-Iliopoulos parameter determines which chiral superfields acquire an expectation value in the infrared, and if the low energy conformal field theory is in a sigma model 'phase' or Landau-Ginzburg 'phase'.

For the variation discussed below, the situation is slightly different. First we present a description as close as possible to the Witten-type linear sigma model. This description will hopefully be useful to introduce the model. In this model a Fayet-Iliopoulos parameter plays a rôle similar to that in Witten's discussion. However, there are some disturbing differences. It turns out that our first description is not correct, and it is better to consider a second, more accurate formulation. This is presented at the end of this section, and it will be used in the T-dualities in chapter 4.

Consider an affine hypersurface, defined as the zero locus of a weighted homogeneous polynomial $F(x_1, \ldots, x_{n+2})$. How to construct a linear sigma model description? The coordinates x_i become chiral superfields of charge a_i. In addition there are two more chiral superfields, Φ_{-1} of charge $-d$ and Φ_0 of charge $-(A-d)$. In all, a sigma model with such charge assignment has no axial anomaly.

For definiteness and simplicity, consider a specific example. Take a linear sigma model with $4+2$ chiral superfields with $U(1)$ charges as specified in table 3.1. The scalar potential due to the D-term reads

$$U_D = -r - 3|\phi_{-1}|^2 - |\phi_0|^2 + \sum_{i=1}^{4} |\phi_i|^2. \tag{3.43}$$

If there were no F-term, the vacuum manifold $\mathcal{M}_D$ would be

$$\mathcal{M}_D = [\mathcal{O}(-3) \oplus \mathcal{O}(-1)] \to \mathbb{P}^3,$$

if $r \gg 0$ and

$$\mathcal{M}_D = \bigoplus_{i=1}^{4} \mathcal{O}(-1) \to \mathbb{P}[1,3],$$

if $r \ll 0$. But adding a term with superpotential

$$W = \Phi_{-1}\left(\mu\Phi_0^{-3} + \sum_{i=1}^{4} \Phi_i^3\right), \tag{3.44}$$

The vacuum manifold is reduced by the additional constraint that $U_F = 0$, where

$$U_F = \left|\mu\phi_0^{-3} + \sum_{i=1}^{4} \phi_i^3\right|^2 + 9|\phi_{-1}|^2\left(|\phi_0|^{-8} + \sum_{i=1}^{4} |\phi_i|^4\right). \tag{3.45}$$

In case $\underline{r \gg 0}$, (3.45) sets $\langle\phi_{-1}\rangle = 0$, so that there is no effective superpotential. That means that $\mathcal{M}_D$ is reduced to the tautological line bundle of $\mathbb{P}^3$. The remaining restriction from 3.45 reduces the vacuum manifold to a hypersurface in this space. In terms of 'inhomogeneous' coordinates $\hat{\phi}_i = \phi_0\phi_i$, the vacuum manifold of the theory with the superpotential is

$$\mathcal{M}_F = \{\sum_{i=1}^{4} \hat{\phi}_i^3 + \mu = 0\} \subset \mathbb{C}^4 \quad (r \gg 0). \tag{3.46}$$

The singular variety is approached as $|\mu| \to 0$.

When $\underline{r \ll 0}$, the situation is more complicated than in the cases of [18]. The condition $U_D = 0$ implies that ϕ_{-1} and ϕ_0 cannot both vanish simultaneously. From $U_F = 0$ it follows that either $\phi_{-1} = 0$ or $1/\phi_0 = 0$. This means that of the base manifold $\mathbb{P}[1,3]$ only a dimension zero subspace is left over. Clearly, if the latter condition is satisfied, ϕ_0 is not a good variable and one should find a more justifiable interpretation. This will be left for later. For now, consider the situation in the present variables.

Define

$$\rho \equiv -r - |\phi_0|^2. \tag{3.47}$$

Then the condition $U_D = 0$ is expressed as

$$3|\phi_{-1}|^2 = \rho + \sum_{i=1}^{4} |\phi_i|^2. \tag{3.48}$$

If $\rho < 0$, then $U_F = 0$ requires that $\phi_{-1} = 0$. So there is no effective superpotential. Using $U_D = 0$ and the gauge symmetry one can fix

$$\phi_0 = \sqrt{|r| + \sum_{i=1}^{4} |\phi_i|^2} \equiv \epsilon^{-1}. \tag{3.49}$$

This fixes the gauge completely. Finally $U_F = 0$ says

$$\sum_{i=1}^{4} \phi_i^3 + \mu\epsilon^3 = 0. \tag{3.50}$$

If $-r \gg 1$, then $\epsilon \approx 0$ and the vacuum manifold looks like a slightly deformed hypersurface singularity, as in the $r \ll 0$ case. If $r \approx 0$, then the analysis of [18] is unjustified in this case.

If $\rho = 0$, then $U_F = 0$ implies that $\phi_{-1} = \phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$. There is no superpotential, the gauge symmetry is unbroken and the 'bad' variable ϕ_0 is pushed out to infinity. This can only be consistent when $r \to -\infty$.

The case $1/\phi_0 = 0$ may seem strange. The field ϕ_0 is pushed out to infinity in the extreme infrared. This is not entirely unlike the Liouville theory. The Liouville interaction e^{-Y} prevents low energy excitation from propagating to small values of Y. In chapter 4 this will be put into perspective. Note that if e^{-Y} has a definite $U(1)$ charge, then shifts of the imaginary part of Y are gauged. The kinetic term of such a field looks something like

$$L_{\text{linear}} = \int d^4\theta - \left(Y + \overline{Y} + V\right)^2. \tag{3.51}$$

When $\rho > 0$, then $\langle \phi_{-1} \rangle \neq 0$, so there is an effective superpotential. From $U_F = 0$ it follows that $\phi_1 = \ldots = \phi_4 = 0$. Again ϕ_0 is pushed out to infinity and $r \to -\infty$. The $U(1)$ gauge invariance can be partly fixed by setting $\langle \phi_{-1} \rangle \in \mathbb{R}_{>0}$. This leaves a $\mathbb{Z}_3$ subgroup acting on Φ_0 and the other Φ_i. The resulting theory is a Landau-Ginzburg orbifold,

$$\frac{W_0 + W_F}{\mathbb{Z}_3}. \tag{3.52}$$

Where

$$W_F = \sqrt{\rho/3}\left(\Phi_1^3 + \Phi_2^3 + \Phi_3^3 + \Phi_4^3\right), \tag{3.53}$$

And W_0 is written in 'bad' variables,

$$W_0 = \sqrt{\rho/3}\Phi_0^{-3}. \tag{3.54}$$

This example generalizes to models with a gauge invariant superpotential

$$W = \Phi_{-1}\left(\mu\Phi_0^{\frac{-d}{A-d}} + F_d(\Phi_1, \ldots, \Phi_m\right),$$

where F_d is a weighted homogeneous polynomial of weighted degree d. The sum of the weights of its arguments is $A = \sum a_i > d$. In the $r \gg 0$ phase, the vacuum manifold is a hypersurface in $\mathcal{O}(-(A-d)) \to \mathbb{P}[a_1, \ldots, a_m]$. This can also be regarded as an affine hypersurface $F_d^{-1}(-\mu)$ in $\mathbb{C}^m$, quotiented by $\mathbb{Z}_r \subset U(1)$ that remains unfixed by the gauge condition $\phi_0 \in \mathbb{R}_{>0}$. The order of $\mathbb{Z}_r$ depends on the relative divisibility of the gauge charges, $r \leq (A-d)$. For $r \ll 0$ the same geometry appears when $\langle \phi_{-1} \rangle = 0$. There is a Landau-Ginzburg orbifold regime as $r \to -\infty$ and $\phi_0 \to \infty$ such that $\rho = |r| - (A-d)|\phi_0|^2$ is a positive finite number:

$$\frac{W = \Phi_0^{\frac{-d}{A-d}} + F_d(\Phi_1, \ldots, \Phi_m)}{\mathbb{Z}_d}. \tag{3.55}$$

Where $\mathbb{Z}_d$ acts as the j-orbifold of section 3.3.1. Also, not apparent in the above notation, there is a background charge for the field $\log \Phi_0$.

The discussion above should have raised some eyebrows. Essentially, we should treat the field Φ_0 differently, as this is the cause of the problems. Actually, perhaps a clearer picture of the above type of theory is presented by really treating Φ_0 as e^{Ψ}, where Ψ is a 'shift-gauged' chiral superfield (4.72), the periodicity of Ψ being

$$\Psi \sim \Psi + 2\pi i. \tag{3.56}$$

And take a kinetic term typical of such a 'shift-gauged' field. To be explicit, consider the Lagrangian

$$\begin{aligned} L = &\int \mathrm{d}^4\theta \left[\frac{d}{4|a_0|} \left(\Psi + \overline{\Psi} + V \right)^2 + |\Phi_{-1}|^2 e^{-dV} + \sum_{i=1}^{n+2} |\Phi_i|^2 e^{a_i V} \right] \\ &+ \int \mathrm{d}^2\theta \, \Phi_{-1} \left[\mu e^{-d\Psi/|a_0|} + F\left(\Phi_1, \ldots, \Phi_{n+2}\right) \right] + \text{c.c} \end{aligned} \tag{3.57}$$

Here F is a transverse weighted homogeneous polynomial of weighted degree d, the weights of the Φ_i are a_i, and

$$a_0 = d - \sum_{i=1}^{n+2} a_i < 0. \tag{3.58}$$

There is no explicit Fayet-Iliopoulos term, since it can be absorbed in Ψ and the coefficient μ. A change of the Fayet-Iliopoulos parameter $r \to r + \delta r$ is effected by a shift

$$\begin{aligned} \Psi \to& \Psi - \frac{|a_0| \delta r}{d}, \\ \mu \to& \mu e^{-\delta r}. \end{aligned} \tag{3.59}$$

The different kinetic term results in a different scalar potential $U = U_D + U_F$,

$$\begin{aligned} U_D &= \frac{d}{|a_0|}\mathrm{Re}\,\psi - d|\phi_{-1}|^2 + \sum_{i=1}^{n+2} a_i|\phi_i|^2, \\ U_F &= |\mu e^{-d\psi/|a_0|} + F\left(\phi_1,\dots,\phi_{n+2}\right)|^2 \\ &\quad +|\phi_{-1}|^2\left(\mu^2\left(\frac{d}{a_0}\right)^2 e^{2d\psi/a_0} + \sum_{i=1}^{n+2}\left|\frac{\partial F}{\partial \phi_i}\right|^2\right). \end{aligned} \tag{3.60}$$

Perhaps the most notable difference, compared to (3.43), is that the real part of ψ appears in U_D, much like a Fayet-Iliopoulos parameter. The vacuum structure of the model depends on the sign of Re ψ.

If Re $\psi < 0$ the situation resembles the $r > 0$ case. Some of the ϕ_i acquire an expectation value. Consequently $\langle\phi_{-1}\rangle = 0$, which in turn means there is no effective superpotential, but the fields ψ and ϕ_i obey a relation that ensures the top line of U_F in (3.60) vanishes. This relation is satisfied on $F^{-1}(-\mu)$ in affine $\mathbb{C}^{n+2}$. This affine space describes $\mathcal{O}(a_0) \to \mathbb{P}[a_1,\dots,a_{n+2}]$, with 'inhomogeneous coordinates' $\xi_i = \phi_i e^{a_i\psi/|a_0|}$.

On the other hand, one could have $\langle\phi_{-1}\rangle \neq 0$. In that case all other ϕ_i must have a vanishing expectation value, in order to minimize U_F. This in turn means that Re $\psi > 0$. Actually, to really set $U_F = 0$, the potential for ψ pushes Re ψ out all the way to infinity. In this case, Ψ is somewhat of an awkward variable. The $U(1)$ gauge symmetry can be used to transform $\langle\phi_{-1}\rangle$, to that it lies along the positive real axis. This gauge condition is preserved by a $\mathbb{Z}_d$ subgroup. So the effective model is a $\mathbb{Z}_d$ orbifold of a 'Landau-Ginzburg' model, with superpotential $W = W_0 + W_F$.

The latter part of the superpotential is simply the weighted homogeneous polynomial

$$F(\Phi_1,\dots,\Phi_{n+2}).$$

The former part can be written as

$$W_0 = \mu e^{-\widetilde{\Psi}}, \tag{3.61}$$

where $\widetilde{\Psi} = \frac{d}{|a_0|}\Psi$. The periodicity of $\widetilde{\Psi}$ thus is $2\pi i d/|a_0|$. The kinetic term of Ψ then becomes

$$L_{\text{kin}} = \int \mathrm{d}^4\theta \frac{|a_0|}{2d}\left|\widetilde{\Psi}\right|^2. \tag{3.62}$$

These terms are characteristic of a Liouville theory [26], with central charge

$$\hat{c} = 1 + \frac{2}{d/|a_0|}. \tag{3.63}$$

as its infrared fixed point. The Liouville theory also has a linear dilaton, which is not apparent in the way the Lagrangian is characterized above. The slope of the linear dilaton is proportional to $-d/|a_0|$.

3.4 COSET MODELS

In addition to the sigma models and Landau-Ginzburg models, there are other ways to construct $\mathcal{N} = (2,2)$ superconformal models. In some instances apparently very different constructions may describe the same conformal field theory. This equivalence of descriptions is established most rigorously for $\hat{c} < 1$ models, where the superconformal algebra constrains the models most. The conformal field theories in this range are constructed abstractly as minimal models, but also as LG-models and as supersymmetric versions of GKO coset models, which may have a target space interpretation, to some extent, as gauged WZW models.

More $\mathcal{N} = (2,2)$ models, with $\hat{c} \geq 1$, can be constructed as G/H cosets of $\mathcal{N} = 1$ conformal field theories. A clear discussion of the properties of general coset models with $\mathcal{N} = (2,2)$ superconformal symmetry is found in [45, 55]. A particular subclass of $\mathcal{N} = (2,2)$ coset models is made up of the Kazama-Suzuki models [42, 43]. In these models the coset manifold G/H is a Hermitean symmetric space (HSS). To construct a coset conformal field theory, the levels of the Kac-Moody algebras of the various factors must be specified. Starting with a bosonic model based on $g_k^{(1)}$, the numerator of the Kazama-Suzuki (KS) model is based on the reductive subalgebra $h \subset g$. The HSS condition in particular means that the rank of g equals the rank of h. Furthermore, using the Killing form on g, write $g = h \oplus t$. The HSS condition says that t must decompose as $t = t_+ \oplus t_-$ into two separately closed Lie algebras of equal dimension, and the Killing form restricted to either subspace must vanish.

Write $h = \bigoplus_i h_i \oplus u(1)^m$ and dim $g -$ dim $h = 2d$. The levels of the h_i factors are determined by the level k of g_k and the embedding of $h \subset g$ as

$$k(h_i) = I_i(k + g^\vee) - h_i^\vee, \tag{3.64}$$

where $g^\vee$ and $h_i^\vee$ are the dual Coxeter numbers of the respective algebras and I_i is the Dynkin index of the embedding[3].

In addition to the bosonic factors above, there are fermions in the superconformal coset. These form a $so(2d)_1$ theory and can be taken as d complex free fermions, ψ^a. The index a can be viewed as a cotangent index in $T^*(G/H)$, parametrizing a basis of t_+, given by the roots in t_+.

The Kazama-Suzuki models have a Lagrangian formulation as gauged WZW models [46]. If the level is large, the gauge fields can be integrated out to one-loop to get a justified target space interpretation. The resulting target space generally may have a non-trivial B-field and a varying dilaton. Thus it looks very different from the Hermitean symmetric space G/H, which is a globally symmetric Kähler manifold. Largely this difference is due to the different action of H on G, in the symmetric space it acts as $g \sim gh$ and in the gauged WZW model as $g \sim h^{-1}gh$. The dilaton is a one-loop effect.

[3]The Dynkin index I_i is the ratio of lengths squared of the highest roots of g and $h_i \subset g$. $I_i = (\theta_g, \theta_g)/(\theta_i.\theta_i)$.

All states in the Kazama-Suzuki models have R-charges which satisfy $q - \overline{q} \in \mathbb{Z}$, like Landau-Ginzburg models. Most cannot admit LG descriptions, as the form of the (c, c) ring is such that it cannot be obtained as a a quotient ring $\mathbb{C}[x_i]/\partial W(x_i)$. The models based on simply laced algebras at level $k = 1$ however, have (c, c) rings which can be reproduced by a Landau-Ginzburg construction [21], i.e. they are of the form $\mathbb{C}[x_i]/\partial W(x_i)$. Furthermore, the (c, c) rings of these models are isomorphic to the Dolbeault cohomology rings of the corresponding Hermitean symmetric spaces. This is related to the correspondence between Ramond ground states and Dolbeault cohomology classes in the case of nonlinear sigma models, as mentioned in section 3.2. In the Kazama-Suzuki case, the situation is more involved, and explained in [21]. In the simplest case, simply laced level one models, many subtleties are inconsequential. In this case the Ramond ground states are found by considering the Lie algebra cohomology of t_+, with coefficients in some representation of g and decomposing this into irreducible representations of h. Each irrep of h corresponds to a Ramond ground state in the $G_{k=1}/H$ model. But this is precisely the way to get the generators of the cohomology of a symmetric space. The number μ of such h-irreps is independent of the chosen $g-$ representation (see [21] and references therein). This number is also the dimension of the (c, c) ring, and it is given by the ratio of dimensions of the Weyl groups,

$$\mu = \frac{|W(G)|}{|W(H)|}. \tag{3.65}$$

The vector space $H^*(G/H)$ can also be obtained from a particular representation of g [21, 56]. The algebra $h \subset g$ for a Hermitean symmetric space is obtained by deleting a node[4] from the Dynkin diagram of g and replacing it with a $u(1)$. The particular g representation $\Xi_{(1)}$ which gives the (c, c) ring of the level $k = 1$ simply laced level one Kazama-Suzuki model is obtained by putting a weight $k = 1$ on this node in the Dynkin diagram of g and zeroes on all others. The Poincaré polynomial is obtained as the character of this g-representation with respect to the $U(1)$ charge corresponding to the element $\rho_G \cdot H$ of the Cartan sub-algebra, where ρ_G is one half of the sum of the positive roots of G.

As pointed out in [21], it had been known that the grading of the cohomology ring $H^*(G/H)$ precisely coincides with the grading of the representation $\Xi_{(1)}$ with respect to this $U(1)$ charge. Similar g-representations $\Xi_{(k)}$ with a weight $k > 1$ at the 'deleted' node, with the same $\rho_G \cdot H$ grading, are generally not isomorphic to the (c, c) rings of the Kazama-Suzuki models at levels $k > 1$, nor is the author aware of a geometric cohomological interpretation of these representations. The central charges of the simply laced Kazama-Suzuki models and the $\Xi_{(k)}$ characters are collected in table 3.2. Yet in some particular cases, the (c, c) rings of $k > 1$ Kazama-Suzuki models are reproduced. In these cases, the level $k > 1$ KS models are believed to be isomorphic to level one models based on a different Hermitean symmetric space.

[4]Deleting several nodes, one can construct Kähler spaces which are also homogeneous, but are not Riemannian symmetric spaces.

$(G/H)_k$	$\hat{c} = c/3$	$\mathrm{tr}_{\Xi_{(k)}} (t\bar{t})^{\rho_G \cdot H}$
$\frac{SU(m+n)}{SU(m)\times SU(n)\times U(1)}$	$\frac{kmn}{k+m+n}$	$\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1-(t\bar{t})^{d-(i+j-1)}}{1-(t\bar{t})^{i+j-1}}$
$\frac{SO(n+2)}{SO(n)\times SO(2)}$	$\frac{kn}{k+n}$	$\frac{1-(t\bar{t})^{d-n/2}}{1-(t\bar{t})^{n/2}}\prod_{i=1}^{n-1}\frac{1-(t\bar{t})^{d-i}}{1-(t\bar{t})^{i}}$
$\frac{SO(2n)}{SU(n)\times U(1)}$	$\frac{kn(n-1)}{2(k+2n-2)}$	$\prod_{i,j=1}^{n-1}\frac{1-(t\bar{t})^{d-(i+j-1)}}{1-(t\bar{t})^{2(i+j-1)}}$
$\frac{Sp(2n)}{SU(n)\times U(1)}$	$\frac{k(n+1)}{2(k+n+1)}$	$\prod_{i,j=1}^{n}\frac{1-(t\bar{t})^{d-(i+j)}}{1-(t\bar{t})^{i+j}}$
$\frac{E_6}{SO(10)\times U(1)}$	$\frac{16k}{k+12}$	$\prod_{i=1}^{11}\frac{1-(t\bar{t})^{d-i}}{1-(t\bar{t})^{i}}\prod_{j=4}^{8}\frac{1-(t\bar{t})^{d-j}}{1-(t\bar{t})^{j}}$
$\frac{E_7}{E_6\times U(1)}$	$\frac{27k}{k+18}$	$\frac{1-(t\bar{t})^{d-9}}{1-(t\bar{t})^{9}}\prod_{i=1}^{17}\frac{1-(t\bar{t})^{d-i}}{1-(t\bar{t})^{i}}\prod_{j=5}^{13}\frac{1-(t\bar{t})^{d-j}}{1-(t\bar{t})^{j}}$

Table 3.2: G/H defining Hermitean symmetric spaces used in Kazama-Suzuki construction of $(2,2)$ superconformal field theories. The integer k denotes the level of the numerator.The number d is the denominator in the corresponding expression for $\hat{c}$.

There are various conjectured isomorphisms between $\mathcal{N} = (2,2)$ coset models. Even though explicit isomorphisms of the Hilbert spaces are lacking, the conjectures hold up to tests of varying refinement, such as

- agreement of central charges
- identical Poincaré polynomials
- isomorphic (c,c) rings

A look at table 3.2 suggests a possible isomorphism of the Grassmannian KS models

$$\frac{SU(m+n)_k \times SO(2mn)_1}{SU(m)_{n+k} \times SU(n)_{m+k} \times U(1)_{mn(m+n)(m+n+k)}} \simeq (m \leftrightarrow k). \tag{3.66}$$

Already in the original construction of these models by Kazama and Suzuki, it was shown that the form of the supercurrent is compatible with this exchange. It has also been shown [21] that for $SU(m+1)_k/SU(m)$, with arbitrary k the Poincaré polynomial, as properly determined by group theoretical considerations, is given by the $\Xi_{(k)}$ character of table 3.2 and hence coincides with that of the Grassmannian coset at level one

$$\frac{SU(m+n)_1}{SU(m) \times SU(n) \times U(1)}.$$

And a one-to-one map between the primary fields has been constructed [47] for arbitrary m, n, k as long as they have no common divisor, or only a prime common divisor.

Other isomorphisms have also been proposed and tested. In particular

$$\frac{SO(m+2)_k \times SO(2m)_1}{SO(m)_{k+2} \times U(1)_{4(m+k)}} \simeq (m \leftrightarrow k), \tag{3.67}$$

for m and k odd, when the two CFTs are based on the diagonal modular invariant. And also the case m even and k odd, with the right hand theory based on the D type modular invariant, rather than the diagonal one [44]. The latter models thus is not strictly speaking a Kazama-Suzuki model. Moving away even farther from the Kazama-Suzuki case, $\mathcal{N} = (2,2)$ coset models have been constructed for which the corresponding coset manifold G/H not a globally symmetric space, though still a Kähler manifold, e.g. see [45]. For such models 'duality' relations have been derived [44],

$$\frac{Sp(n)_k \times SO(4n-2)_k}{Sp(n-1)_{k+1} \times U(1)_{2(k+n+1)}} \simeq \frac{Sp(k+1)_{n-1} \times SO(4k+2)_{n-1}}{Sp(k)_n \times U(1)_{2(k+n+1)}}, \tag{3.68}$$

which have central charge $\hat{c} = 2n - 1 - \frac{2n^2}{(n+k+1)}$, and

$$\frac{Sp(2)_{2n+1} \times SO(6)_1}{Sp(1)_{2n+2} \times U(1)_{4n+8}} \simeq \frac{SO(2n+5)_1 \times SO(8n+6)_1}{SO(2n+1)_5 \times SU(2)_{2n+2} \times U(1)_{4n+8}}. \tag{3.69}$$

The Kazama-Suzuki models which are part of a dual pair and also have a Poincaré polynomial that can be reproduced by a Landau-Ginzburg model are particularly interesting. Notably such models are

$$\frac{SU(n+1)_k}{SU(n) \times U(1)}.$$

Note that the Poincaré polynomial specifies the superpotential only up to marginal deformations. This relation of KS and LG up to marginal deformations will be used in the next chapter.

4

SINGULARITIES OR FLUXES: NONPERTURBATIVE T-DUALITY

This chapter deals with 'impurities' in supersymmetric backgrounds of string theory. Broadly speaking, it deals with two kinds of such 'impurities'. First, there are isolated singularities of the background geometry. One might call this a 'geometric impurity'. Second, in contrast to the geometric impurities, there are 'objects' in string theory, the various branes, which are sources of gauge fields and curve the geometry. The archetypal example which will play a rôle, is the NS fivebrane.

Unlike the geometric impurities, those of the second kind are sources of gauge fields; one might call these 'flux impurity'. The distinction between the two kinds of impurity is somewhat artificial from the point of view of string theory. This is so, because there are string dualities which may relate one kind to the other.

Again, the best known example is T-duality which relates asymptotically locally Euclidean spaces with an A_k singularity to a background in which there is a stack of $k + 1$ fivebranes present. In fact, it is a general feature of T-duality, that a non-trivial circle fibration, which is a purely geometric characteristic of a background, is dual to a background with NS-flux.

T-duality can be formulated in perturbative string theory, it is an isomorphism between a pair of conformal field theories that gives rise to an equivalence of a pair of string backgrounds for perturbative string theory. In order to find a pair of T-dual string backgrounds, one should thus find the pair of isomorphic worldsheet conformal field theories, and if possible, their target space interpretations.

This can be done in a perturbation expansion in α', regarding a worldsheet cft as a nonlinear sigma model. The pair of dual worldsheet cft's is obtained as different effective theories of one overarching theory, so that manifestly the pair of theories should be isomorphic.

In practice it is hard to explicitly find the pair of effective theories, when a perturbative treatment in terms of α' is not sufficient. This is the case when T-duality is considered along a circle that degenerates. A typical situation when this occurs, is at a Calabi-Yau singularity.

When the cycle degenerates, worldsheet instanton effects must crucially be taken into account to find a correct dual theory. One important effect of the worldsheet instantons, is that they typically break a symmetry which seemed to be present classically. In the present circumstances, this symmetry can be interpreted as a translation symmetry in the flux background (which seems to come from the translation symmetry in the geometric background).

One might ask for example the following two questions. Why is this symmetry broken in the full, nonperturbative T-duality transformation? And second, what is the significance of the fact that this symmetry is broken?

To begin with the first point, a physical argumentation why a translation symmetry need not be preserved by T-duality is the following. Essentially, T-duality exchanges winding modes and momentum modes of a string. The momentum modes are like the modes of a point particle, they depend on the 'ordinary' geometry of a target space. One such 'ordinary geometric' notion, is the presence of a translation symmetry. But there is another part of geometry which is probed by strings: the geometry to which winding modes are sensitive.

At a singular point, the modes winding around the orbits of the translation symmetry can become light, as the orbits degenerate near the apex. These winding modes, worldsheet instantons, have a consequence for the 'ordinary' geometry of the T-dual background, which need not have a translation symmetry.

To reflect on the second point, why is the absence of this symmetry important, let us say this. Both kinds of 'impurities', geometric and flux, are important in for string theory for a special reason. At such impurities there is 'localized physics' which takes place just at the impurity. This local physics can be decoupled by applying appropriate scaling limits. As we are dealing with 'localized physics', it is clearly relevant if the impurity is 'localized' (there is no translation symmetry, as non-perturbative (worldsheet) effects have broken it), or if it is not localized, as this difference matters for physics 'near the impurity'.

There is another important aspect, which we will not discuss much, but is a crucial motivation for the study of these dualities. The decoupling limits near impurities can be used not only to isolate 'localized physics' at the impurity, they can also be used to construct new superstring backgrounds. Essentially, these backgrounds are related 'holographically' to the localized physics. In the construction and study of such string backgrounds, often it is very useful to know of an impurity and a scaling limit which produces this background, think for example, of D-brane setups which give rise to anti-de Sitter geometries. In this respect, it promises to be useful to know T-dual descriptions of geometric and flux impurities. These can give rise either to linear dilaton backgrounds of string theory, or anti-de Sitter, by deforming the worldsheet conformal field theory, or adding various branes.

The outline of this chapter is as follows. First we will discuss impurities, of geometric and of flux type, and their scaling limits. Next, it turns out that especially the scaling limits may admit an exact worldsheet cft descriptions, while the 'full' backgrounds, before a scaling limit, do not. Then the T-duality will be discussed. The special features of T-duality in the context of a degenerating isometry are discussed. We continue with the proposed T-duality relation for Calabi-Yau singularities which have a description as certain affine hypersurfaces or discrete quotients thereof. For quite special hypersurfaces, the T-dual admits a genuine geometric interpretation, this case involves Kazama-Suzuki models which admit a Landau-Ginzburg description, for other cases, a geometric description is not known is such concrete terms. Finally, we conclude with some final observations.

4.1 Impurities and Scaling Limits

In this section two kinds of string theory impurity are considered. One is entirely geometric: a singularity in a compactification manifold $\mathcal{M}$ that features in a supersymmetric string vacuum of the form $\mathbb{R}^{9-2m,1} \times \mathcal{M}_{2m}$, without any fluxes and a trivial dilaton. Hence $\mathcal{M}_{2m}$ is a Calabi-Yau or hyper-Kähler manifold, and an isolated singularity locally is of the sort discussed in chapter 2. The other kind of impurity is a Neveu-Schwartz fivebrane, the magnetic dual of the fundamental string, or a collection of fivebranes. This object is a source of magnetic flux and also curves space around it.

Both sorts of impurity have physical consequences at certain low energy scales. In the presence of a singularity there are special massless states coming from branes wrapping the vanishing cycles. In the presence of a stack of fivebranes there are massless states which originate from D-branes that end on the fivebranes. In either case the special massless states are 'localized' at the impurity. By appropriately tuning deformations of the impurity (blowing up a singularity or separating the fivebranes in a stack), which set the energy scale of the 'localized' states, and simultaneously tuning string parameters as g_s and ℓ_s, the region near the impurity can be isolated. States not associated with the impurity decouple, and one is left with a different string vacuum than one originally started out with before the scaling limit.

There are two important features of the string theory vacua that one ends up with after the scaling process. First of all, they have isolated the physics that has to do with the impurity. Second, they are generally simpler than the original backgrounds, and it is not uncommon that the 'near impurity' backgrounds have an exact cft description, when the 'full' global backgrounds do not have a known exact description.

4.1.1 Singularities

Consider a geometric string vacuum of the form

$$\mathbb{R}^{9-2m,1} \times \mathcal{M}_{2m}$$

which preserves some supersymmetry. This means that $\mathcal{M}_{2m}$ is a Calabi-Yau manifold, or even hyper-Kähler. Usually one takes a compact $\mathcal{M}_{2m}$. From the ten-dimensional low energy effective action

$$S_{10} = \frac{1}{2\kappa^2} \int \mathrm{d}^{10}x \sqrt{-G}\mathcal{R} + \ldots, \tag{4.1}$$

one gets a low energy effective action on $\mathbb{R}^{9-2m,1}$,

$$S_{10-2m} = \frac{1}{2\kappa^2_{10-2m}} \int \mathrm{d}^{10-2m}x \sqrt{-G_{10-2m}}\mathcal{R}_{10-2m} + \ldots \tag{4.2}$$

The couplings are related as

$$\begin{aligned} 2\kappa^2 &= (2\pi)^7 \ell_s^8 g_s^2, \\ 2\kappa^2_{10-2m} &= \frac{2\kappa^2}{\mathrm{Vol}(\mathcal{M}_{2m})\ell_s^{2m}}. \end{aligned} \tag{4.3}$$

So there is an effective low energy theory on $\mathbb{R}^{9-2m,1}$ that is gravitational, having taken the volume of $\mathcal{M}_{2m}$ finite.

A Calabi-Yau manifold is usually part of a continuous family of Calabi-Yau manifolds, labeled by the moduli. The moduli govern the size of certain homology cycles of a Calabi-Yau. At some values of the moduli, some cycles may shrink to zero size, and a singularity develops. An example of this, is found for the deformations of a (non-compact) A_k singularity, in section 2.1.2.

More explicitly, an explicit metric on a smoothed A_k singularity, is provided by the multi-centered Taub-NUT space[12, 13],

$$\mathrm{d}s^2 = U^{-1}\left(\mathrm{d}\theta + \vec{\omega}\mathrm{d}\vec{r}\right)^2 + U\,\mathrm{d}\vec{r}^2, \tag{4.4}$$

where $\vec{r}$ coordinatizes flat $\mathbb{R}^3$ and θ is a periodic coordinate. Furthermore

$$\begin{aligned} U &= 1 + \sum_{i=1}^{k+1} \frac{\lambda}{|\vec{r} - \vec{r}_i|}, \\ \vec{\nabla} U &= -\vec{\nabla} \times \vec{\omega}. \end{aligned} \tag{4.5}$$

This metric is regular at $\vec{r} = \vec{r}_i$ provided that the periodicity of θ is

$$\theta \sim \theta + 4\pi\lambda. \tag{4.6}$$

In the metric (4.4) k blown-up two-spheres are seen as circle fibrations over the line segment between $\vec{r} = \vec{r}_i$ and $\vec{r} = \vec{r}_{i+1}$. At the end points of this interval $U^{-1} = 0$ and the fiber, parametrized by θ, shrinks to zero size. The volume of S^2_{ij}, the sphere between $\vec{r} = \vec{r}_i$ and $\vec{r} = \vec{r}_{i+1}$, is given by

$$V_{S_{ij}}(|\vec{r}_i - \vec{r}_j|) = \int d\theta U^{-1/2} \int \mathrm{d}r U^{1/2} = 4\pi\lambda|\vec{r}_i - \vec{r}_j|. \tag{4.7}$$

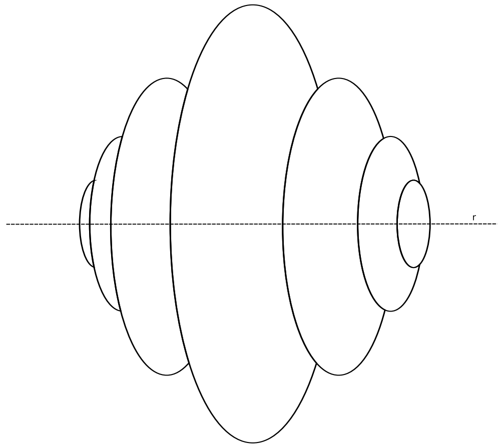


Figure 4.1: A two-sphere as a circle fibration over the line segment between $\vec{r}_i$ and $\vec{r}_j$, where the function U^{-1}, of the multi-center Taub-NUT metric vanishes, as in equation (4.4).

If $N > 1$ centers coincide, $\vec{r}_i = \vec{r}_j$ the corresponding two-spheres S_{ij} shrinks to zero size. Also, a conical singularity develops at that center, as effectively the periodicity of the θ-coordinate is reduced from $4\pi\lambda$ to $4\pi\lambda/N$.

To get a bit more feeling for this metric, consider the single center Taub-NUT metric, with

$$U(r) = 1 + \frac{\lambda}{r}. \tag{4.8}$$

Explicitly, the metric reads

$$\begin{aligned} \mathrm{d}s^2_{TN} = & \frac{1}{U(r)}\left(\mathrm{d}\theta + \lambda(1-\cos\psi)\mathrm{d}\phi\right)^2 \\ & + U(r)\left(\mathrm{d}r^2 + r^2\left[\mathrm{d}\psi^2 + \sin^2\psi\mathrm{d}\phi^2\right]\right). \end{aligned} \tag{4.9}$$

Near $r \approx 0$, redefining coordinates $r = \rho^2$ and scaling $\lambda = 1$, the metric can be written as

$$\mathrm{d}s^2 \approx \mathrm{d}\rho^2 + \rho^2\left(\mathrm{d}\psi^2 + \sin^2\psi\mathrm{d}\phi^2 + \left[\mathrm{d}\theta + (1-\cos\psi)\mathrm{d}\phi\right]^2\right).$$

The $\rho^2(\ldots)$ term is a circle bundle over S^2. It is the Hopf fibration $S^1 \to S^3 \to S^2$ precisely when the periodicity of θ is 4π, and the metric cone over S^3 is just smooth $\mathbb{R}^4$.

This reasoning sets the periodicity of θ in a smooth multi-center Taub-NUT to be $4\pi\lambda$. If N are moved on top of one another, locally the metric will look like

$$\mathrm{d}s^2 \approx \mathrm{d}\rho^2 + \rho^2\left(\mathrm{d}\psi^2 + \sin^2\psi\mathrm{d}\phi^2 + \frac{1}{N^2}\left[\mathrm{d}\theta + (1-\cos\psi)\mathrm{d}\phi\right]^2\right),$$

which is a metric on the metric cone over the lens space $S^3/\mathbb{Z}_N$.

The metric in (4.4) describes a non-compact space. The sort of singularities that can occur in a compact manifold may be restricted by global properties of the manifold, more specifically, by its homology. For example, the only family of 4d compact hyper-Kähler manifolds are the K3 surfaces, which have the following Hodge diamond,

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}. \tag{4.10}$$

So a singularity with a Milnor number $\mu > 22$ can certainly not occur in a K3 surface.

However, it is justified and interesting to not restrict the attention only to singularities that can occur in compact manifolds $\mathcal{M}_{2m}$, but also consider singularities that occur in non-compact $\mathcal{M}_{2m}$. The justification comes from the existence of a scaling limit that isolates the physics at the singularity. States localized at the singularity couple to 'far away' states through gravitational interaction. In isolating the states at the singularity, the gravitational interaction is switched off. This is a different kind of physical situation than the one which is considered in a 'compactification' as discussed above, and $\mathcal{M}_{2m}$ need not be compact for this scaling to make sense.

Let us consider a scaling limit which isolates the physics near a singularity. Around a p-cycle in some $\mathcal{M}_{2n}$ there can be wrapped Dp-branes. The tension of a Dp-brane has the following proportionality:

$$T_p \sim \frac{1}{g_s \ell_s^{p+1}}. \tag{4.11}$$

If it wraps a p-cycle of volume $V_p \ell_s^p$, the mass of the Dp-brane thus is proportional[1] to

$$M_p(V_p) \sim \frac{V_p}{g_s \ell_s}. \tag{4.12}$$

By simultaneously tuning the moduli μ in a way that the volume[2] $V_p = V_p(\mu) \to \mathbf{0}$ and 'switching off gravity', scaling the Planck length $\ell_p = \ell_s g_s^{1/4} \to 0$, while keeping the mass of a wrapped Dp-brane fixed, the states near the singularity are isolated. On the one hand, some states become very massive and can be integrated out to get the low energy dynamics, like, for example states associated to branes wrapping large cycles that are not scaled down. On the other hand the gravitational modes decouple.

Several comments are in order. First, note that it is not necessary to scale $\ell_s \to 0$ in this limit. But after the scaling one has isolated the physics near the (almost) singular point.

[1]There may also be a nontrivial B-field flux through any of the 2-cycles, corresponding to the imaginary part of the complexified Kähler class. Such a flux also contributes to the mass of a wrapping D-brane.

[2]Also, the B-field flux through the cycle should vanish.

So in a sense, it is a 'decompactification' limit that keeps only the local geometry near the singularity.

Second, the physics due to the light degrees of freedom localized near the singularity can be described in various different ways, related by dualities, which are discussed later in this chapter. For the moment, consider one particular viewpoint. Take a space of the form

$$\mathbb{R}^{5,1} \times \mathcal{M}_4,$$

where $\mathcal{M}_4$ is a multi-centered Taub-NUT space, like (4.4). Regarding this as a vacuum of IIA string theory, it can be lifted to an M-theory background

$$\mathbb{R}^{5,1} \times S^1 \times \mathcal{C}.$$

This, in turn descends [79] to another IIA vacuum, taking the M-theory circle to be the fiber coordinatized by θ; the 'original' M-theory circle can be decompactified. In this vacuum there is a D6-brane at each center of the metric. As several D6-branes move together, the fundamental strings stretching between them become light. The open strings give rise to a low energy $SU(k+1)$ gauge theory. The W-bosons from strings stretching between branes at $\vec{r_i}$ and $\vec{r_j}$ have masses that are proportional to $|\vec{r_i} - \vec{r_j}|$. This is the proportionality of masses of wrapped D2-branes in the original configuration. Indeed, the D2 branes lift to M2-branes which are extended in the θ-direction.
These descend to fundamental strings stretched between the D6 branes. In [79] an analogous analysis is also carried out for (resolved) D_{k+2} spaces. In that case the resolved geometry is somewhat more complicated, because of the additional $\mathbb{Z}_2$ of the dihedral groups. In the geometric picture this gives an extra 'center', of a different sort that the Taub-NUT centers. The 'metric link' of this center is a circle bundle over $\mathbb{RP}^2$, rather that $\mathbb{P}^1$ (see section 2.1.2). After the 9-11 flip this extra center gives on orientifold O6-plane [79, 80]. Note that in one picture the 'impurities' are purely geometric, whereas in the dual picture the impurities are manifested as branes, so there are fluxes.

4.1.2 FIVEBRANES

An interesting class of non-geometric impurities is formed by configurations of Neveu-Schwartz fivebranes[3]. The simplest configuration is formed by a stack of superimposed fivebranes that occupy a $\mathbb{R}^{5,1}$ worldvolume and have $\mathbb{R}^4$ transverse to their worldvolume. Such a stack of N coincident fivebranes curves space, is a source of 3-form H-flux and

[3]Throughout the discussion it is assumed we are dealing with fivebranes of a Type II theory. Usually we have in mind IIB theory, when we discuss fivebranes as 'flux impurities' T-dual to hyper-Kähler surface singularities. But depending on the situation, one should consider IIA theory. This is the case if the T-dual theory has a geometric singularity which is deformed, in the scaling process, by blowing up a three-cycle (like the deformed conifold) and D3 branes wrapping the three-cycle play a rôle in the scaling limit under consideration. As the discussion focuses on the bosonic sector, where the distinction between IIA and IIB is not always so important here

induces a non-trivial dilaton. More precisely, the field configuration of these fields is

$$\begin{aligned} e^{2(\Phi-\Phi_\infty)} &= h(r) = 1 + \frac{N\alpha'}{r^2}, \\ \mathrm{d}s^2 &= -\mathrm{d}t^2 + \sum_{i=1}^{5} \mathrm{d}x_i \mathrm{d}x_i + h(r)\left[\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2\right], \\ C_{(6)} &= h^{-1}(r)\left[\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_5\right], \end{aligned} \tag{4.13}$$

which can be obtained from the string equations of motion in the approximation to lowest order inα'. Here C_6 is the dual of the 2-form NS-NS gauge potential. Alternatively, in terms of the 3-form flux H, the field configuration can be written as

$$H_{mnp} = -\epsilon_{mnp}{}^{q}\partial_q \Phi, \tag{4.14}$$

where m, n, p, q are indices in the space $\mathbb{R}^4$ transverse to the fivebrane worldvolume. Several related field configurations can be obtained, by taking several parallel stacks at different points in the transverse $\mathbb{R}^4$.

As the fivebranes are BPS objects, any such configuration forms a good string vacuum, with an amount of supersymmetry that corresponds to two copies of $\mathcal{N} = 1$ in $d = 6$ (for the type II theories). If the positioning of the stacks has enough symmetry, it may be possible to sum the contributions of all stacks explicitly. The resulting field configuration then looks a lot like the 'single stack' configuration above, only with a changed harmonic function $\tilde{h}(\vec{r})$. Some such configurations will be discussed later.

Scaling Limit of a Stack of Fivebranes

There is also a way to get a simpler background. The harmonic function simplifies in the region $r \ll \sqrt{N\alpha'}$, where the constant term can be dropped, so $h(r) \sim r^{-2}$. Unlike the full background (4.13), this scaled background has a known exact worldsheet conformal field theory description [15]. This exact worldsheet cft is actually a $\mathcal{N} = (4,4)$ superconformal theory, corresponding to the $d = 6$ spacetime supersymmetry of the target space[4]. The target space of a $\mathcal{N} = 4$ superconformal model does not suffer α' corrections beyond the level at which the geometry (4.13) was derived. From this geometry it is possible to identify the exact conformal field theory.

Choosing a new radial coordinate

$$\phi = \frac{1}{\sqrt{N\alpha'}} \log \frac{r^2}{N\alpha'}, \tag{4.15}$$

[4]By dimensional reduction $d = 4$ $\mathcal{N} = 2$ supersymmetry is obtained from $d = 6$ $\mathcal{N} = 1$. The $\mathcal{N} = 2$ $d = 4$ algebra has three supercharges, which are related to three worldsheet $U(1)$ currents, similar to the argumentation in section 3.2 relating $d = 4$ $\mathcal{N} = 1$ with $\mathcal{N} = 2$ extended superconformal symmetry.

the region near the stack of fivebranes looks like a 'throat', $\mathbb{R}_\phi \times S^3$, with field configuration,

$$\begin{aligned} ds^2 &= \frac{1}{4}d\phi^2 + N\alpha' d\Omega_3^2, \\ \Phi - \Phi_0 &= -\frac{N\alpha'}{2}\phi, \\ H &= -N\alpha'\epsilon, \end{aligned} \tag{4.16}$$

where ϵ is the volume form on S^3, $\int \epsilon = 2\pi^2$.

From these expressions it can be seen that a change in the string coupling asymptotically far from the fivebranes, $g_s = e^{\Phi_0}$, accompanied by a rescaling of r, does not change the field configuration down the throat. This feature allows the physics 'localized' down the throat to be decoupled, by sending $g_s \to 0$ and simultaneously descending in the throat.

The background (4.16) is the target space field configuration of a couple of exact conformal field theories. The S^3 with N units of H-flux is the target space of a $SU(2)$ WZW model at level $k = N$. Actually, this part of the string background is described by a supersymmetric WZW model. The worldsheet fermions are free, after doing a gauge rotation. This gauge rotation is anomalous[5]. Its effect is to change the central charge of the bosonic piece of the WZW model from $c = 3N/(N+2)$ to $c = 3(N-2)/N$, i.e. the level of the $SU(2)$ current algebra of the decoupled bosonic part $SU(2)_k$, of the supersymmetric WZW model, is $k = N - 2$

The $\mathbb{R}_\phi$ part, is described by a scalar, and the linear dilaton is reflected as a background charge for this scalar, so $\mathbb{R}_\phi$ is described by (a supersymmetric analogue of) a Feigin-Fuchs cft. The background charge of the scalar is $Q = -\sqrt{1/N}$. The central charges of the Feigin-Fuchs and WZW models are

$$\hat{c}_\phi = \frac{1}{2} + \frac{N}{2}, \tag{4.17}$$

$$\hat{c}_{wzw} = \frac{1}{2} + \frac{N-2}{N}, \tag{4.18}$$

where $\hat{c} = c/3$, so the throat superconformal model has $c = 6$. A complete string vacuum is obtained by tensoring these cft's with three free chiral superfields, corresponding to the worldvolume directions of the stack.

Note that the conformal field theory description only makes sense in case the number of fivebranes is $N \geq 2$, but not for a single fivebrane. Another noteworthy point is that down the throat, $\phi \to -\infty$ the dilaton grows without bound. On the one hand, the fact that the string coupling grows in the throat, allows for a decoupling limit to exist, in which string propagation seems to be described by an exact cft. But on the other hand, where the string coupling becomes large, a worldsheet cft does not relyably reflect the string dynamics, as string loop effects may not be ignored.

[5]See, for example, [15].

OTHER FIVEBRANE CONFIGURATIONS

As mentioned earlier, any configuration of parallel fivebranes, located at different points in the transverse $\mathbb{R}^4$ has the same amount of supersymmetry as the single stack configuration. More precisely, the field configuration of (4.13) is that of extremal fivebranes, which preserve half of the supersymmetry of the string theory, and any parallel configuration of such branes is a stable one, as the branes exert no force on one another. The only change in field configuration with respect to (4.13) is manifested through a change in the harmonic function. The single center function is replaced by the superposition

$$\tilde{h}(\vec{r}) = 1 + \sum_i \frac{N_i \alpha'}{|\vec{r} - \vec{r}_i|^2}, \tag{4.19}$$

with centers at every location of a fivebrane. Of course, such configurations can be regarded as deformations of the single stack configuration. In the remainder of this section some special configurations are reviewed, which are both of physical interest, and for which the summation yields a reasonably neat result.

$SO(3) \times U(1)$ ISOMETRY

Perhaps the simplest configuration one can consider, is that of a large number of fivebranes, smeared over a transverse direction, either $\mathbb{R}$ or S^1, with a uniform density. Essentially the harmonic function that solves the four-dimensional Laplace equation $\triangle h(\vec{r}_4) = 0$ in the localized single stack configuration (4.13), is replaced here by a solution of the three-dimensional Laplace equation,

$$\tilde{h}(\vec{r}_3) = 1 + \frac{\nu \ell_s}{r_3},$$

where ν is the fivebrane density. Note that a function of the same form appears in the Taub-NUT metric. Indeed, the Taub-NUT metric and a fivebrane are related by T-duality, but in a rather more complicated way [11] than a naive application of the rules for T-duality [9] would indicate.

$SO(3) \times \mathbb{Z}$ ISOMETRY

It is also possible to get the field configuration for a stack of N coincident fivebranes with transverse space $\mathbb{R}^3 \times S^1$ by taking $\vec{r}_i = \vec{r}_0 + n\vec{e}_4$. The resulting harmonic function was obtained long ago in connection with a periodic array of instantons in $\mathbb{R}^4$ [105]. Essentially, one uses the standard expression from complex analysis

$$\sum_{n=-\infty}^{\infty} f(n) = \frac{1}{2\pi i} \oint f(z)\, \pi \cot(\pi z) \;-\; \sum \operatorname*{Res}_{z \notin \mathbb{Z}} f(z).$$

In this case $f(z) = (1+z^2)^{-1}$ has poles at $z = \pm i$, and the summation yields

$$\sum \frac{1}{1+n^2} = \pi \coth \pi .$$

Similarly,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{1}{\alpha^2 + (\beta n - \gamma)^2} &= \frac{1}{2\pi i} \oint_C \mathrm{d}z \frac{\cot\left(\frac{\pi z}{\beta}\right)}{\alpha^2 + (\beta z - \gamma)^2} \\ &\quad - \sum \operatorname*{Res}_{\frac{z}{\beta} \notin \mathbb{Z}} \frac{\frac{\pi}{\beta}\cot\left(\frac{\pi z}{\beta}\right)}{\alpha^2 + (\beta z - \gamma)^2} .\end{aligned} \tag{4.20}$$

The contour C is taken appropriately large and avoiding the poles at $z = (\gamma \pi i \alpha)/\beta$. The residues at $z = (\gamma \pm i\alpha)/\beta$ are

$$\frac{\pm 1}{2\alpha} \frac{\pi}{\beta} \cot\left(\frac{\pi\gamma}{\beta^2} \pm \frac{i\pi\alpha}{\beta^2}\right)$$

respectively. Expanding the hyperbolic tangents in exponentials, the result is

$$\frac{\sinh \frac{2\pi\alpha}{\beta^2}}{\cosh \frac{2\pi\alpha}{\beta^2} - \cos \frac{2\pi\gamma}{\beta^2}} .$$

Using this result, the harmonic function describing a stack of fivebranes on $\mathbb{R}^3 \times S^1$ is found. Let the circumference of S^1 be $2\pi R_9$, then the harmonic function for a stack of N fivebranes is

$$\tilde{h}(r_3, \xi) = 1 + \frac{N\alpha'}{2R_9 r_3} \frac{\sinh(r_3/R_9)}{\cosh(r_3/R_9) - \cos(\xi/R_9)}, \tag{4.21}$$

where ξ is a coordinate on S^1 with periodicity $2\pi R_9$.

$SO(2) \times \mathbb{Z}_n$ Isometry

Another interesting configuration is that of n stacks of q fivebranes each, the stacks being positioned at

$$\vec{r}_j = (0, 0, \rho_* \sin(2\pi j/n), \rho_* \cos(2\pi j/n)) \quad j = 0, \ldots, n-1. \tag{4.22}$$

This configuration has been considered in [25]. The $\mathbb{R}^4$ transverse to the fivebranes splits into an $\mathbb{R}^2$ with polar coordinates (r_2, θ) perpendicular to the 'ring' of fivebranes, and an $\mathbb{R}^2$ with polar coordinates (ρ, ϕ), in which the ring of fivebrane is situated, .i.e.

$$\vec{r} = (r_2 \sin\theta, r_2 \cos\theta, \rho \sin\phi, \rho \cos\phi).$$

The configuration is invariant under $SO(2)\times\mathbb{Z}_n$ shifts $(\theta,\phi)\to(\theta+\psi,\phi+2\pi ik/n)$. The harmonic function that characterizes this configuration can be obtained in a way similar to the previous case [25],

$$\tilde{h}(x,\rho,\phi)=1+\frac{nq\alpha'}{2\rho_*\rho\sinh(x)}\frac{\sinh(nx)}{\cosh(nx)-\cos(n\phi)}, \tag{4.23}$$

where

$$e^x=\frac{r_2^2+\rho_*^2}{2\rho_*\rho}+\left(\left[\frac{r_2^2+\rho_*^2}{2\rho_*\rho}\right]^2-1\right)^{1/2}.$$

This configuration in its own right may not look particularly illuminating. However, it does play an important rôle as a deformation of the single stack configuration, which exhibits a various regimes at different scales. From afar, $\rho_*^2\ll r^2=r_2^2+\rho^2\ll nq\alpha'$, a small deformation of the 'single stack' looks much like a single stack of $N=nq$ fivebranes. Near the ring, the behavior depends on the density of centers. In particular, in case n is very large, and nx is also large, so that one is not so close as to see the separate stacks, the configuration looks like a continuous ring of fivebranes. The coordinates on $\mathbb{R}^4$ transverse to all the fivebranes,

$$\begin{aligned}\vec{r}&=(r_{(1)},r_{(2)},r_{(3)},r_{(4)})\\&=(r_2\ \sin\theta,r_2\ \cos\theta,\rho\ \sin\phi,\rho\ \cos\phi),\end{aligned}$$

used above, are rewritten in more convenient coordinates,

$$\begin{aligned}r_{(1)}&=\rho_*\sinh\varrho\ \cos\chi\ \cos\tau,\\r_{(2)}&=\rho_*\sinh\varrho\ \cos\chi\ \sin\tau,\\r_{(3)}&=\rho_*\cosh\varrho\ \sin\chi\ \cos\psi,\\r_{(4)}&=\rho_*\cosh\varrho\ \sin\chi\ \sin\psi),\end{aligned}$$

so that the ring is located at $\varrho=0$ and $\chi=\pi/2$. In these coordinates, the field configuration is written as

$$\begin{aligned}ds^2&=N\alpha'\left(d\varrho^2+d\chi^2+\frac{\tan^2\chi\,d\psi^2+\tanh^2\varrho\,d\tau^2}{1+\tanh^2\varrho\ \tan^2\chi}\right),\\B&=\frac{N\alpha'}{1+\tanh^2\varrho\ \tan^2\chi}\,d\tau\wedge d\psi,\\e^{2\Phi}&=\frac{e^{2\Phi_0}}{\cosh^2\varrho\cos^2\chi+\sinh^2\varrho\sin^2\chi}.\end{aligned} \tag{4.24}$$

This configuration as it stands has various features which are familiar from relatively simple exact conformal field theories (for a good discussion, see [5]).

Some Aspects of the Deformed Throat

First of all, there is the limit $\varrho \to \infty$. There, the target space looks like a S^3 with $N = nq$ units of B-field flux, i.e. the $SU(2)$ WZW model, which is of course expected, since at large ϱ the configuration looks like a single stack. One can also consider the limit $\varrho \to 0$, i.e. the space 'at the ring of fivebranes'. One part of the metric is

$$\mathrm{d}s_I^2 = N\alpha' \left(\mathrm{d}\chi^2 + \tan^2\chi \, \mathrm{d}\psi^2\right). \tag{4.25}$$

This is a disk, with a dilaton

$$e^{\Phi - \Phi_0} = \sec\chi, \tag{4.26}$$

which diverges at the boundary of the disk, $\chi = \pi/2$, where the fivebranes are located. This is the geometry of a gauged WZW model, $SU(2)/U(1)$, at least as $N \gg 1$ (see also the examples in section 4.2), which is presently the case. The level of the $SU(2)$ current algebra is N, and when this is large, the $U(1)$ gauge field may be eliminated by its classical equations of motion to give the resulting target space geometry. At one loop the elimination of the gauge field generates the non-trivial dilaton. The coordinate τ, appropriately rescaled, corresponds to a (non-compact) $U(1)$, i.e. a free boson, but with a periodicity that has been scaled up from 2π to infinity, by the rescaling of τ. This rescaling also eliminates the B-field.

The coordinate ϱ interpolates between $SU(2)/U(1) \times U(1)$ and $SU(2)$. This is a familiar situation [81, 82]. The coordinate ϱ can be seen almost as a deformation parameter, $\alpha = \varrho^{-1}$ that deforms a $SU(2)$ WZW model by an exactly marginal deformation

$$\delta S(\alpha) \sim \alpha \int d^2z \; J\bar{J},$$

where J is a $U(1)$ current of the $SU(2)$ WZW model. However, ϱ is also a dynamical field itself, so in that sense it is not just a parameter that can be tuned 'externally'.

In speaking of a gauged WZW model $G/U(1)$, one should usually specify how the $U(1)$ acts in G. Either the vector action $g \sim h^{-1}gh$ or the axial action $g \sim hgh$ might be gauged leading to generally different anomaly free models. However, the $SU(2)/U(1)_v$ and $SU(2)/U(1)_a$ models are isomorphic. In terms of the geometry (4.25), the two are interchanged by changing $\chi \leftrightarrow \pi/2 - \chi$. This isomorphy can be seen in many different ways. For example, looking at the Landau-Ginzburg representation of the $SU(2)/U(1)$ model, $W = \Phi^N$, the mapping is effected by an orbifold by the group $J \simeq \mathbb{Z}_N$ generated by j, constructed out of the holomorphic $U(1)$ R-current, as discussed in section 3.3.1. This orbifold changes the spectrum in a way that can be undone by the mirror automorphism. This means, that if the LG model with $W = \Phi^N$, where Φ is a chiral superfield, is the $SU(2)/U(1)_v$ model, then exchanging Φ for a twisted chiral superfield Y, changes the model to the $SU(2)/U(1)_a$ one. Also, this change can be thought of in a more geometric fashion, in terms of T-duality which, for this model, is nothing but mirror symmetry. This is discussed in subsequent sections. However, do note that changing $\chi \leftrightarrow \pi/2 - \chi$ essentially inverts the radii of the circle fibers in the geometry (4.25), which is typical feature of T-duality. Interestingly, there is another way to connect $SU(2)/U(1)_v$ and $SU(2)/U(1)_a$. Starting from the undeformed $SU(2)$ WZW model at $\varrho = \infty$, or correspondingly $\alpha = 0$, the vector-gauged model arises as the limiting deformation $\alpha \to \infty$. The dual model, $SU(2)/U(1)_a$ arises in the limit $\alpha \to -\infty$. This is another way to look at the effect of T-duality [83], which has no obvious interpretation in the fat throat geometry (4.24).

Another relatively simple geometry is obtained from (4.24), not by fixing ϱ, but by taking χ fixed to a constant value. Similar to the exposition above, there are two special values of χ. At $\chi = 0$ the space looks like

$$\begin{aligned} \mathrm{d}s^2 &= N\alpha' \left(\mathrm{d}\varrho^2 + \tanh^2 \varrho^2 \, \mathrm{d}\tau^2 + \mathrm{d}\tilde{\psi}^2 \right), \\ e^{\Phi} &= \frac{e^{\Phi_0}}{\cosh \varrho}. \end{aligned} \tag{4.27}$$

Again, a change of coordinates has been done, so $\tilde{\psi}$ corresponds to a 'decompactified' $U(1)$. This limit kills the B-field. The geometry of the ϱ and τ coordinates is, at large N, that of the gauged WZW model $SL(2;\mathbb{R})/U(1)_a$ at level N. On the other hand, at $\chi = \pi/2$, the geometry looks like

$$\begin{aligned} \mathrm{d}s^2 &= N\alpha' \left(\mathrm{d}\varrho^2 + \coth^2 \varrho^2 \, \mathrm{d}\tau^2 + \mathrm{d}\tilde{\psi}^2 \right), \\ e^{\Phi} &= \frac{e^{\Phi_0}}{\sinh \varrho}, \end{aligned} \tag{4.28}$$

with a differently rescaled $\tilde{\psi}$, that again coordinatizes a 'decompactified' $U(1)$. The rest of the geometry is that of a $SL(2;\mathbb{R})/U(1)_v$ model at level N. Again, both limiting cases can be seen as exactly marginal deformations of $SL(2;\mathbb{R})$ [84]. The two geometries can again be seen as T-duals of one another, both being circle fibrations over a half line, but with reciprocally related lengths of the circle fiber.

In this section, the term T-duality has been mentioned several times. It relates various parts of the 'fat throat' geometry. It turns out that actually singular geometries and backgrounds with NS fivebranes are also related by T-duality, in quite a complicated way. Essentially, the complications are caused by the singularity. To honestly describe the T-duality completely clearly from first principles in a transparent way is quite difficult. In most cases, more can be said in a scaling limit of the geometry, where string propagation is related to an exact conformal field theory[6]. The full 'unscaled' backgrounds may not have a (known) exact cft description. Still, also the scaling limits are of interest, as there is some physics localized in the region kept by the scaling process.

[6]The string coupling may be large, in certain regimes, so it is not always justified to use the worldsheet conformal field theories as a means to compute string dynamics

4.2 GENERALITIES OF T-DUALITY

Target space configurations which look quite different from the point of view of 'classical' geometry, may be equivalent as string backgrounds. This is the case for string theory backgrounds that are related by T-duality. In that case, there is a pair of isomorphic worldsheet conformal field theories, and the spectra and scattering matrices of perturbative string theory are hence isomorphic, too. Crucially, T-duality maps a weakly coupled string theory to another weakly coupled one, so that the conformal field theories relyably reflect the string dynamics.

There are various ways to think about T-duality. Usually one has a worldsheet conformal field theory that has a target space interpretation and the target space has a $U(1)$ isometry. The archetypal case is that of bosonic string theory on a target space $\mathcal{M} = \mathcal{M}' \times S^1_R$, which is a product space, with a factor that is a circle of radius R. T duality relates this string background to one in which the circle has a new radius $\tilde{R}$ and the string coupling is changed as well, $g_s \to \tilde{g}_s$, where the relation is

$$\begin{aligned} \tilde{R} &= \frac{\alpha'}{R}, \\ \tilde{g}_s^2 &= \frac{\alpha'}{R^2}\, g_s^2. \end{aligned} \tag{4.29}$$

PATH INTEGRAL PICTURE OF T-DUALITY

The two conformal field theories are isomorphic because both arise as effective theories of one single overaching theory, see, for example [53]. This overarching theory may be obtained from the action

$$S = \frac{1}{2\pi}\int_\Sigma \mathrm{d}^2 z\, \sqrt{h}\frac{1}{2R^2} h^{\alpha\beta} B_\alpha B_\beta + \frac{i}{2\pi}\int_\Sigma B \wedge d\phi, \tag{4.30}$$

of a scalar field ϕ, with periodicity 2π, and a one-form B, defined on a worldsheet Σ. If B is eliminated by its classical equations of motion,

$$B = iR^2 \, * \mathrm{d}\phi, \tag{4.31}$$

this action reduces to

$$S_{|B} = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2 z\, \sqrt{h} R^2 h^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi, \tag{4.32}$$

which is the action of a sigma model on a circle of radius R.

On the other hand, one may eliminate ϕ to obtain an effective action for B. The classical equation of motion for ϕ sets $\mathrm{d}B = 0$. This means that B is a linear combination of an exact form and harmonic forms. It is convenient to write the harmonic piece of B in a specific fashion. On a worldsheet of genus g, the vector space of harmonic forms in $2g$-dimensional. One may choose $2g$ homology cycles $\gamma_i \in H_1(\Sigma;\mathbb{Z})$ and $2g$ dual harmonic forms $\omega^i \in H^1(\Sigma;\mathbb{Z})$, such that

$$\int_{\gamma_i} \omega^j = \delta_i^j. \tag{4.33}$$

There is a natural inner product on $H^1(\Sigma;\mathbb{Z})$,

$$\langle \omega^i, \omega^j \rangle = \int_\Sigma \omega^i \wedge \omega^j = m^{ij}. \tag{4.34}$$

Where (m^{ij}) is an matrix with only integers as its entries that has an inverse $(m_i j)$, with only integers as entries as well.

Now a generic B satisfying the equation of motion of ϕ, i.e. $\mathrm{d}B = 0$, can be written as

$$B = \mathrm{d}\theta_0 + \sum_{i=1}^{2g} a_i \omega^i. \tag{4.35}$$

Consider the term

$$\frac{1}{2\pi}\int_\Sigma B \wedge \mathrm{d}\phi$$

in the action. Recall that ϕ is a periodic field with periodicity 2π. In other words, ϕ need not be a single valued function. Or more to the point, $\mathrm{d}\phi$ integrated over any cycle need not be zero, but can be any integral multiple of 2π. That is to say, one can write

$$\mathrm{d}\phi = \mathrm{d}\phi_0 + \sum_{i=1}^{2n} 2\pi n_i \omega^i, \tag{4.36}$$

where ϕ_0 is a single valued function and the n_i are integers. Now, 'integrating out' ϕ does not only mean solving for the equation of motion coming from the variation of ϕ_0, which says $\mathrm{d}B = 0$. One also needs to sum over the lattice of n_i's. The above term in the action now reads

$$\frac{1}{2\pi}\int_\Sigma B \wedge \mathrm{d}\phi = 2\pi \sum_{i,j} a_i m^{ij} n_j. \tag{4.37}$$

The summation over the lattice of n_i's in the partition function

$$Z = \int [d\ldots] \sum_{\vec{n}\in\mathbb{Z}_{2g}} e^{iS}$$

fixes the a_i to be multiples of 2π. That is to say, one can write $B = d\theta$, where θ is not single valued, but has periodicity 2π, just as ϕ had. The effective action obtained from (4.30), after integrating out ϕ, thus becomes

$$S_{|\phi} = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2 z\, \sqrt{h}\frac{1}{R^2} h^{\alpha\beta}\partial_\alpha\theta\partial_\beta\theta, \tag{4.38}$$

which is the sigma model action with target space a circle of radius $1/R$.

Conceptually, the procedure above amounts to the following. An 'overarching' action is obtained by introducing a gauge field, so that this global symmetry is made into a local one. This gauge field is non-dynamical. In addition, an extra term is introduced in the action, which forces the gauge field strength to zero. Integrating out the gauge field, B in the example above, gives one effective theory, whereas integrating out the Lagrange multiplier, ϕ in the example above, leads to the T-dual theory. If one also takes into account one-loop effects in 'integrating out' the field from the path integral, there is also a shift in the dilaton, as in (4.29).

This can be generalized to a target space that is a circle bundle [9], provided that translations along the circle fiber are isometries of the total space. Two additional features occur in this more general setting. First, if the size of the circle fiber varies over the base, then in the T-dual model, the dilaton will vary. In particular, if the original fiber is small somewhere, then in the dual model, there will be a region where the string coupling is large. Second, if the circle bundle is not a product manifold, there will be a B-field in the dual target space. Explicitly, after T-duality along a fiber with coordinate θ the metric, B-field and dilaton are mapped $(g_ab, b_ab, \Phi) \to (\tilde{g}_{ab}, \tilde{b}_{ab}, \tilde{\Phi})$ according to the Buscher rules [9]:

$$
\begin{aligned}
\tilde{g}_{ab} &= g_{ab} - \frac{g_{\theta a}g_{\theta b} - b_{\theta a}b_{\theta b}}{g_{\theta\theta}}, & \tilde{g}_{\theta\theta} &= g_{\theta\theta}^{-1}, \\
\tilde{g}_{ab} &= g_{ab} - \frac{g_{\theta a}g_{\theta b} - b_{\theta a}b_{\theta b}}{g_{\theta\theta}}, & \tilde{g}_{\theta a} &= \frac{b_{\theta a}}{g_{\theta a}}, \\
\tilde{b}_{ab} &= b_{ab} - \frac{g_{\theta a}b_{\theta b} - b_{\theta a}g_{\theta b}}{g_{\theta\theta}}, & \tilde{b}_{\theta a} &= \frac{g_{\theta a}}{g_{\theta\theta}}, \\
\tilde{\Phi} &= \Phi - \log g_{\theta\theta}.
\end{aligned}
\tag{4.39}
$$

In the derivation of [9] effects of up to one loop are taken into account. The one-loop effect generates the shift of the dilaton, while the change of g_{ab} and b_{ab} is determined by solving classical equations of motion. This evaluation is justified, corrections of $\mathcal{O}\left((\alpha')^2\right)$ are relatively small, provided that the circle fiber does not degenerate anywhere. It is a requirement in the derivation, that translation along the circle fiber, is an isometry. Note that it is a consequence of (4.39) that the T-dual geometry also has an isometry that the dual background also has an isometry. If the target space has an isometry, then the sigma model has a corresponding conserved current. T-duality acts on this current in a way that is familiar from the $R \leftrightarrow \alpha'/R$ case.

There is a related, but somewhat different perspective to regard the $R \leftrightarrow \alpha'/R$ duality. The scalar field ϕ that solves the equation of motion of (4.32), describing string propagation on a circle, can be split into a left moving and a right moving part. T-duality is effected by changing the sign of the right-moving part. This sign flip has an effect on the zero modes of ϕ: it exchanges momentum modes and winding modes. This effect is also seen from the equation of motion (4.31), which says

$$\frac{1}{R}\,\mathrm{d}\theta = iR\ *\mathrm{d}\phi, \tag{4.40}$$

where in the sigma model on a circle, with action (4.32), $R\ \mathrm{d}\phi$ is the conserved current which measures the momentum along the circle and $iR\ *\mathrm{d}\phi$ measures the winding on the circle. Both these currents are conserved thanks to the fact that one the one hand, translations along the circle are isometries and on the other, field configurations winding along the circle are topologically stable.

In a general circle bundle $S^1 \to \mathcal{M} \to \mathcal{B}$, there may be no conserved winding, if $\pi_1(\mathcal{M}) = \{\mathrm{id}\}$. One can expect that in the T-dual space, there is non-conservation of the corresponding momentum. In other words, the isometry of (4.39) might not be present, even though there is an isometry in the original, undualized, model. From the perspective of the perturbative derivation of T-duality, which leads to the Buscher rules (4.39) this is not very clear at all. This point is discussed in section 4.3.

T-DUALITY FOR SUPERFIELDS

So far, T-duality has been discussed from the point of view of bosonic sigma models. The models relevant for supersymmetric string backgrounds have extended superconformal symmetry. There is an aesthetic way to formulate T-duality in terms on $\mathcal{N} = (2,2)$ superfields [10]. Essentially, it is a direct analogue of the bosonic path integral procedure discussed above. The supersymmetric version is illustrated in the following example.

Consider the Lagrangian

$$L = \frac{1}{2}\int \mathrm{d}^4\theta\ \left(\frac{R^2}{2}B^2 - \left(Y + \overline{Y}\right)B\right). \tag{4.41}$$

Here B is a general real superfield, $\overline{B} = B$, and Θ is a twisted chiral superfield, $\overline{D}_+ Y = 0 = D_- Y$, with a periodic imaginary part $y \sim y + 2\pi i$, so it parametrizes a cylinder. Now one can obtain a effective Lagrangian by integrating out either B or Y and $\overline{Y}$. The equations of motion coming from Y and $\overline{Y}$ read

$$\overline{D}_+ D_- B = 0 = D_+ \overline{D}_- B. \tag{4.42}$$

This says that B is the sum of a chiral superfield Φ and an antichiral superfield, and since B is real, one can write

$$B = \Phi + \overline{\Phi}. \tag{4.43}$$

The periodicity of Φ is fixed by the periodicity of Y, which in terms of component fields can be seen exactly analogously to the bosonic case treated earlier. The periodicity is $\phi \sim \phi + 2\pi i$. Substituting the above expression for B in the Lagrangian yields

$$L_{|Y,\overline{Y}} = \frac{R^2}{4}\int \mathrm{d}^4\theta\,\left(\Phi + \overline{\Phi}\right)^2 = \frac{R^2}{2}\int \mathrm{d}^4\theta\Phi\overline{\Phi}. \tag{4.44}$$

This is the supersymmetric sigma model on a cylinder with radius R. Shifts of ϕ are isometries. Alternatively, one could integrate out B. Its equation of motion reads

$$B = \frac{1}{R^2}\left(Y + \overline{Y}\right), \tag{4.45}$$

so that the effective Lagrangian becomes

$$L_{|B} = \frac{-1}{4R^2}\int \mathrm{d}^4\theta \left(Y + \overline{Y}\right)^2 = \frac{-1}{2R^2}\int \mathrm{d}^4\theta Y\overline{Y}. \tag{4.46}$$

This is the sigma model of a cylinder with radius $1/R$.

Thus the T-duality procedure eliminates a chiral superfield and introduces a twisted chiral one, or vice versa. Note that rôle of chiral and twisted chiral fields can be exchanged by the mirror automorphism of the $\mathcal{N} = (2,2)$ super algebra (3.1). Recall from section 3.1 that a sigma model with Lagrangian

$$L = \int \mathrm{d}^4\theta K(\Phi_i, \overline{\Phi}_i; Y_a, \overline{Y}_a),$$

that depends on both chiral and twisted chiral fields, is a sigma model on a target space [48] with metric

$$(G_{\mu\nu}) = \begin{pmatrix} 0 & K_{\phi_i\overline{\phi}_j} & 0 & 0 \\ K_{\phi_i\overline{\phi}_j} & 0 & 0 & 0 \\ 0 & 0 & 0 & -K_{y_a\overline{y}_b} \\ 0 & 0 & -K_{y_a\overline{y}_b} & 0 \end{pmatrix}, \tag{4.47}$$

and B-field

$$(B_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 & K_{\phi_i\overline{y}_b} \\ 0 & 0 & K_{y_b\overline{\phi}_i} & 0 \\ 0 & -K_{y_b\overline{\phi}_i} & 0 & 0 \\ -K_{\phi_i\overline{y}_b} & 0 & 0 & 0 \end{pmatrix}, \tag{4.48}$$

where a subscript denotes differentiation with respect to the corresponding variable. So a B-field is present if mixed derivatives of the Lagrangian with respect to both a chiral and a twisted chiral field do not vanish. Consequently, one cannot get rid of a B-field by acting with a mirror automorphism. On the other hand, a sigma model based on both chirals and twisted chirals need not have a B-field. In particular, if

$$\begin{aligned} L =& \int \mathrm{d}^4\theta K(\Phi_i + \overline{\Phi}_i; Y_a + \overline{Y}_a), \\ K =& K_1(\Phi_i + \overline{\Phi}_i) + K_2(Y_a + \overline{Y}_a). \end{aligned} \tag{4.49}$$

according to (4.48) there is no B-field. Moreover, such a Lagrangian describes a model which is the tensor product of a model defined by K_1 and one defined by K_2, and one can appropriately map all twisted chiral fields of K_2 to chirals, using the mirror automorphism within the model defined by K_2 alone.

Sigma models of the sort discussed above are used as part of a worldsheet conformal field theory. Therefore, they should have conformal symmetry; this means that the beta functions of the sigma model should vanish. In many cases, the conformal symmetry is kept thanks to a non-trivial dilaton, which couples to the worldsheet curvature tensor, and was not really discussed. In any case, the beta functions, determined to first order in α' read[7]

$$\begin{aligned}\beta_{G_{\mu\mu}} =& \mathcal{R}_{\mu\nu} - \frac{1}{4} H_{\mu}{}^{\rho\sigma} H_{\nu\rho\sigma} + 2\nabla_\mu \nabla_\nu \Phi + \mathcal{O}(\alpha') = 0,\\ \beta_{G_{\mu\nu}} =& \nabla_\rho H^{\rho}{}_{\mu\nu} - 2\left(\nabla_\rho \Phi\right) H^{\rho}{}_{\mu\nu} + \mathcal{O}(\alpha') = 0,\end{aligned} \tag{4.50}$$

where $H_{\lambda\mu\nu} = \partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}$. And the central charge is determined by the beta function of the dilaton,

$$c = \frac{3}{2}\left(D + \alpha'\left[4\left(\nabla\Phi\right)^2 - 4\nabla^2\Phi + \frac{1}{12}H^2 - \mathcal{R}\right]\right) + \mathcal{O}\left(\left(\alpha'\right)^2\right), \tag{4.51}$$

where $\frac{1}{2}D$ is the number of chiral and twisted chiral superfields appearing in K, or, in other words, D is the (real) dimension of the target space.

Examples of Dual Sigma Models

First consider sigma models based on a single chiral superfield, see, for example [48],

$$K = K(\Psi\overline{\Psi}).$$

These have no B-field. In addition to the case $K = \Psi\overline{\Psi}$ with a constant dilaton and central charge $\hat{c} = c/3 = 1$, the following cases occur. In the absence of a B-field, $\beta_{G_{\mu\mu}} = 0$ is solved by a dilaton

$$\Phi = \frac{1}{2}\log\det K_{\Psi\overline{\Psi}} + C\left(\Psi + \overline{\Psi}\right), \tag{4.52}$$

where C can be any real constant. First consider the case $C = 0$. The Kähler potential can then be formally expressed as

$$K(x) = \int_1^{B\,\Psi\overline{\Psi}} \frac{\mathrm{d}x}{x}\log\left(A + x\right), \tag{4.53}$$

which defines a good metric,

$$\mathrm{d}s^2 = K_{\Psi\overline{\Psi}}\mathrm{d}\psi\mathrm{d}\overline{\psi}.$$

[7]There is some variation in the literature, depending on various conventions, see for example [85] and [86]. Our expression follows [48] and references therein.

The dilaton profile is

$$\Phi = -\frac{1}{2}\log\left(A + B\Psi\overline{\Psi}\right).$$

The different possibilities are in correspondence with different choices of signs of A and B [48]; they are the following.

First, if $A > 0$ and $B < 0$, the metric is written as

$$\mathrm{d}s^2 = \frac{k}{1-|\psi|^2}\mathrm{d}\psi\mathrm{d}\overline{\psi}. \tag{4.54}$$

This is the metric of the coset $SU(2)/U(1)$. After a coordinate transformation the above metric on the disk, and the dilaton are written as

$$\begin{aligned}\mathrm{d}s^2 =& k\left(\mathrm{d}\theta^2 + \tan^2\theta\mathrm{d}\phi^2\right)\\ \Phi =& \log\cos\theta.\end{aligned} \tag{4.55}$$

The metric is invariant under phase rotations of Ψ. One can perform a corresponding T-duality, following the procedure of [10]. Phase rotations of Ψ are gauged, introducing a real superfield B and replacing $\Psi\overline{\Psi} \to \Psi\overline{\Psi}e^B$. One can gauge fix $\Psi \equiv 1 \equiv \overline{\Psi}$. An overarching Lagrangian can be written as

$$K_{\text{overarching}} = -\int^{e^B}\frac{d\alpha}{\alpha}\log(1-\alpha) + B\left(Y + \overline{Y}\right), \tag{4.56}$$

where Y is a twisted chiral superfield. On the one hand, by the equations of motion of Y and $\overline{Y}$, B is forced to be pure gauge, i.e. $B = \Theta + \overline{\Theta}$, where Θ is a chiral superfield. Writing $\Psi = e^{\Theta}$, this gives a Kähler potential

$$K_{|Y,\overline{Y}} = \int^{|\Psi|^2}\frac{\mathrm{d}\alpha}{\alpha}\log(1-\alpha).$$

On the other hand, one can use the equation of motion of B instead, which reads

$$e^B = X\overline{X} - 1,$$

where X is a twisted chiral superfield, $X = e^{-Y}$. Plugging this into $K_{\text{overarching}}$, one obtains

$$K_{|B} = \int^{1-|X|^2}\frac{\mathrm{d}\alpha}{\alpha}\log(1-\alpha) - \log\left(|X|^2\right)\log\left(1-|X|^2\right).$$

Essentially the Kähler potential for $SU(2)/U(1)$ is the dilogarithm,

$$\mathrm{Li}_2(z) = \int_z^0\frac{\mathrm{d}\alpha}{\alpha}\log(1-\alpha).$$

One of the major functional relations of the dilogarithm is

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \log(z)\log(1-z),$$

see, for example, [87]. Using this relation, the T-dual Kähler potential reads

$$K_{|B} = +\int^{|X|^2} \frac{\mathrm{d}\alpha}{\alpha}\log(1-\alpha)\,.$$

This just differs a minus sign from $K\left(|\Psi|^2\right)$, and, since X is twisted chiral, this minus sign ensures that the kinetic term for X is positive, just like that of Ψ in the original model. So this model is self-dual, the metric being written as

$$\frac{k\,\mathrm{d}\psi\mathrm{d}\overline{\psi}}{1-|\psi|^2} \leftrightarrow \frac{k\,\mathrm{d}x\mathrm{d}\overline{x}}{1-|x|^2}. \tag{4.57}$$

Here k indicates a scale of the metric. In fact, as T-duality inverts the radius of the dualized circle, it maps $\psi^2 \leftrightarrow 1-x^2$. The dilaton profile is $\Phi = -\log\left(1-|\psi|^2\right)$. Writing $\psi = \sin\chi e^{i\theta}$, this geometry is expressed as the one appearing in (4.25) in section 4.1,

$$\begin{aligned}\mathrm{d}s^2 &= k\left(\mathrm{d}\chi^2 + \tan^2\chi\mathrm{d}\theta^2\right),\\ e^{\Phi-\Phi_0} &= -\log\cos\chi.\end{aligned}$$

This is the metric the coset model $SU(2)_{k-2}/U(1)$.

As a chiral superfield is exchanged for a twisted chiral, the right-moving $U(1)_R$ charges change sign. In a coset $SU(2)/U(1)$, the change in charge assignment can be done by changing from a vector gauging to gauging of the axial action of $U(1)$. The same change of charge assignments can be accomplished by changing doing a $\mathbb{Z}_k$ orbifold, by the group generated by $j = e^{2\pi i J_0}$.

A second possible two dimensional model is obtained by taking $A < 0$ and $B > 0$. The metric can be written as

$$\mathrm{d}s^2 = \frac{k}{|\psi|^2 - 1}\mathrm{d}\psi\mathrm{d}\overline{\psi}. \tag{4.58}$$

This is the metric of $SL(2;\mathbb{R})/U(1)_v$. To be precise, if $SL(2;\mathbb{R})$ is generated by σ_1, $i\sigma_2$ and σ_3, where $\sigma_{1,2,3}$ are the ordinary Pauli matrices, then the $U(1)$ which is gauged, is generated by $i\sigma_2$. By a coordinate transformation this is written in the form of section 4.1,

$$\begin{aligned}\mathrm{d}s^2 =& k\left(\mathrm{d}r^2 + \coth^2 r\mathrm{d}\tau^2\right),\\ \Phi =& -\log\sinh\rho.\end{aligned} \tag{4.59}$$

The central charge of this model is $\hat{c} = c/3 = 1 + \frac{2}{k}$. This is the central charge of $SL(2;R)/U(1)$ at level $k+2$.

This metric also has an isometry corresponding to phase rotations of Ψ. It is not self-dual, though. Instead, in a fashion analogous to the previous case, see, e.g. [48], T-duality maps it to the metric

$$\mathrm{d}s^2 = \frac{k}{|\psi|^2+1}\mathrm{d}\psi\mathrm{d}\overline{\psi} \tag{4.60}$$

This metric corresponds to the choice $A > 0$ and $B > 0$. It is the metric of the axially gauged $SL(2;\mathbb{R})/U(1)$, also written as

$$\begin{aligned} \mathrm{d}s^2 =& k\left(\mathrm{d}r^2 + \tanh^2 r\mathrm{d}\tau^2\right), \\ \Phi =& \log\cosh\rho. \end{aligned} \tag{4.61}$$

Next, if $A = 0$ and $B > 0$, the resulting metric can be written as

$$\mathrm{d}s^2 = \frac{k}{\psi\overline{\psi}}\mathrm{d}\psi\mathrm{d}\overline{\psi}.$$

With a change of coordinates $z = \log\psi$, this is written as the standard flat metric on $\mathcal{C}$, with a linear dilaton

$$\begin{aligned} \mathrm{d}s^2 =& k\mathrm{d}z\mathrm{d}\overline{z}, \\ \Phi =& z + \bar{z}. \end{aligned} \tag{4.62}$$

Finally one can take a dilaton profile as in (4.52) with $C \neq 0$, in particular,

$$\Phi = \frac{1}{2}K_{\psi\overline{\psi}} + \frac{1}{2}\log\left(\psi\overline{\psi}\right)$$

The metric then can be written as

$$\mathrm{d}s^2 = \frac{k}{z+\overline{z}}\mathrm{d}z\mathrm{d}\overline{z}. \tag{4.63}$$

This is the T-dual of the previous case [48]. These two cases can also be obtained as an exactly marginal deformation of $SL(2;\mathbb{R})$ [84]

$$S(\alpha) = \frac{k}{4\pi}\int \mathrm{d}^2 z\, \frac{1}{2}\partial x\bar{\partial}x + \frac{e^x}{1-\alpha e^x}\partial\gamma_+\bar{\partial}\gamma_-,$$

where the duals lie at opposite extreme limits of the deformation parameter $\alpha \to \pm\infty$. The trumpet and cigar geometries,

$$\frac{k}{4\pi}\int \mathrm{d}^2 z\partial\rho\bar{\partial}\rho + f(\rho)\partial\theta\bar{\partial}\theta,$$

with $f(\rho) = \coth^2\rho$ and $\tanh^2\rho$ are also related as extreme limits of an exactly marginal deformation of $SL(2;\mathbb{R})$, however, they are deformed by a $J_2\bar{J}_2$ deformation.

All these two-dimensional examples have no fluxes. An important example of duality in a four dimensional background is [16]

$$SU(2)\times U(1)\simeq\frac{SU(2)}{U(1)}\times U(1)\times U(1).$$

The left hand side has flux. Its 'Kähler potential' has the following form:

$$K_{SU(2)\times U(1)}=-\int^{\frac{|Y|^2}{|\Psi|^2}}\frac{\mathrm{d}\alpha}{\alpha}\log\left(1+\alpha\right)+\log\Psi\ \log\overline{\Psi}\tag{4.64}$$

where Ψ is a chiral and Y a twisted chiral superfield. The metric that follows from this potential is

$$\mathrm{d}s^2=\frac{|\mathrm{d}\psi|^2+|\mathrm{d}y|^2}{|\psi|^2+|y|^2},$$

which exhibits the $SU(2)$ rotation symmetry and the scaling symmetry. The potential above can be obtained from an overarching potential

$$K_{\text{overarching}}=-\int^{e^B}\frac{\mathrm{d}\alpha}{\alpha}\log\left(1+\alpha\right)+\log\left(\Phi\overline{\Phi}\right)+\mathrm{c}\left(B+\log\left(\Phi\overline{\Phi}\right)\right)\left(\Theta+\overline{\Theta}\right),$$

dependent on chiral superfields Φ and Θ, and a real superfield B. If the equations of motion of Θ, $\overline{\Theta}$ are used, $K_{SU(2)\times U(1)}$ is recovered. On the other hand, using the equation of motion of B gives a dual model, with a potential dependent of a pair of chiral superfields, and so there is no flux. After some manipulation, see [16], the target space of this dual model turns out to be the product of $SU(2)/U(1)$ and a torus.

One particular reason why this example is important, is that the $SU(2)$ WZW model features in the description of the throat geometry of fivebranes. In fact, the complete throat geometry is described also by $SU(2)\times U(1)$, but the $U(1)$ has a background charge so that the central charge of the throat background is $\hat{c}=2$, for any number of fivebranes (recall that the number of fivebranes corresponds to the level of the $SU(2)$ current algebra and to the value of the background charge). The above duality suggests, that a throat background might be related to a purely geometric one, that is to say, one without fluxes. And actually, the dual backgrounds correspond to exact conformal field theories. More about such dualities is discussed in section 4.3.

Finally, T-duals of Kazama-Suzuki models can be constructed [28]. This is accomplished by writing a Kazama-Suzuki model as a gauged WZW model

$$\frac{G}{\left(H\times U(1)\right)_v},$$

so that there is a $U(1)$ symmetry, essentially the axial action of $U(1)$ on G. Using this symmetry an overarching Lagrangian can be constructed which reduces to either that of the original Kazama-Suzuki model or its dual, depending on which field is integrated out. Unlike in the two dimensional cases, the general procedure is dot done in an off-shell formulation, but in component fields.

Before treating the general case as it was studied in [28], let us consider the example of $SU(2)/U(1)$. This model is self-dual. T-duality essentially exchanges the chiral superfield for a twisted chiral one, as discussed earlier. This exchange basically amounts to inverting the sign of, say, the left-moving $U(1)_R$ charge. Next to exchanging the chiral field for a twisted chiral field, there is another way to flip the sign of this $U(1)$ charge of all states of the $SU(2)/U(1)$ model. It is also accomplished by taking an orbifold with respect to the $\mathbb{Z}_{k+2}$ symmetry generated by $j = e^{2\pi i J_0}$, where J_0 is the holomorphic $U(1)$ current of the $\mathcal{N} = (2,2)$ superconformal algebra. Now, T-duality relates

$$\frac{SU(2)_k}{U(1)_v} \equiv \frac{SU(2)_k}{U(1)_a \times \mathbb{Z}_{k+2}}. \tag{4.65}$$

Now, in a general gauged WZW model of Kazama-Suzuki type, one can 'gauge' the residual $U(1)_a$ symmetry, add a Lagrange multiplier term to the action which forces the gauge connection to be flat, and integrate out the gauge field, following the same philosophy as discussed in the earlier examples. The result is [28] that

$$\frac{G_k}{H \times U(1)_v} \overset{T}{\simeq} \frac{G_k}{H \times U(1)_a \times \mathbb{Z}_{k+g^\vee}}. \tag{4.66}$$

That is to say, under T-duality vector and axial gauging of the $U(1)$ are exchanged. Furthermore, an orbifold is done with respect to the $\mathbb{Z}_{k+g^\vee}$ subgroup of the $U(1)_v$ symmetry, that is a global symmetry of the axially gauged model ($g^\vee$ is the dual Coxeter number of G). While in the $SU(2)$ case, on could be somewhat sloppy with the indication of the extra orbifold, because the orbifolded and unorbifolded theories are related by a sign flip (more accurately, by action of the mirror automorphism), in a general Kazama-Suzuki model, the orbifold acts in a more complicated way. In other words, in the two dimensional models, T-duality acts as mirror symmetry: it acts quite non-trivially on the target space geometry, but almost trivially on the cft spectrum. In general Kazama-Suzuki models, it is an isomorphism which acts on the states and interactions in a more complicated fashion.

4.3 T-DUALITY AND FIVEBRANES

The Buscher rules (4.39) of the preceding section are applicable when the circle fiber along which the duality is done, is large with respect to the string length. When the fiber is not large, two problems occur. First, one must take into account corrections of higher order in the sigma model coupling; usually there are corrections which are non-perturbative in α'. Second, according to the Buscher rules, when a fiber degenerates in the dual sigma model, the string coupling becomes large, so that the utility of a worldsheet cft for the description of string dynamics is questionable.

Roughly speaking, the Buscher rules exchange B-field and 'non-product' structure of the target geometry, or in other words, the degree to which a circle bundle is non-trivial. In particular, starting with a target space that has no B-field, one expects to end up with a dual target space in which the dual circle is 'untwisted'. Furthermore, following the Buscher rules, one would expect that either space of a pair of T-dual target spaces has a circle isometry. But this is not always the case.

In the language of $R \leftrightarrow 1/R$ duality, momentum modes are mapped to winding modes and vice versa, when a T-duality transformation is done. In the case of $R \leftrightarrow 1/R$ duality, momentum along the circle is conserved because translations are isometries, and winding is conserved as well, because of the topology of a circle. However, a more general target space may well have a $U(1)$ isometry while at the same time there is no good notion of a 'winding number' along the integral curves of the isometry, due to the topology of the total space. This is a normal situation when the circle fiber degenerates somewhere, so that 'strings' winding along the fiber can be continuously contracted to a point. When this happens, 'winding' strings can become light, as they move to a region where the fiber is small. Such additional light modes, should also be taken into account in the low energy dynamics, which is not done in the derivation of the Buscher rules. So if winding is not conserved in a space that nonetheless does have a $U(1)$ isometry, one expects that the T-dual space does not have the isometry predicted by a formal application of the Buscher rules.

The Dual of Taub-NUT

A well known instance in which the Buscher rules do not yield the correct dual geometry, is in the case of a Taub-NUT space. The rules applied to the Taub-NUT metric, see also (4.4),

$$\begin{aligned} \mathrm{d}s^2 &= h(r)^{-1}\left(\mathrm{d}\theta + \vec{\omega}\cdot \mathrm{d}\vec{r}\right)^2 + h(r)\mathrm{d}\vec{r}^2, \\ h(r) &= \frac{1}{R^2} + \frac{1}{2r}, \end{aligned} \tag{4.67}$$

where the θ-circle ($\theta \sim \theta + 2\pi$) is dualized, yield a metric

$$\widetilde{\mathrm{d}s}^2 = h(r)\left(\mathrm{d}\tilde{\theta}^2 + \mathrm{d}\vec{r}^2\right) \tag{4.68}$$

and a B-field, too. The harmonic function appearing in the metric and B-field is a three-dimensional one. The geometry is just the 'transverse' geometry of a fivebrane, smeared along the dual circle, parametrized by $\tilde{\theta}$, of radius R.

However, the fundamental group of a Taub-NUT space is trivial, so there is no good notion of a winding number. Indeed, the proper T-dual geometry is not (4.68), but that of a fivebrane which is localized at a point on $\mathbb{R}^3 \times S^1$ [11]. The Buscher rules do not suffice, but get corrections which are non-perturbative in α'. These worldsheet instantons break the symmetry of translations along $\tilde{\theta}$, and the harmonic function is changed to the form,

$$\frac{1}{R^2} + \frac{1}{2r} \to 1 + \frac{1}{2r}\frac{\sinh r}{\cosh r - \cos\theta}, \tag{4.69}$$

as in (4.21).

This harmonic function can be expanded as a sum of Fourier modes with different momenta in the θ direction [14],

$$h(r_3, \theta) = \sum_{n=-\infty}^{\infty} c_n(r_3) e^{in\theta}. \tag{4.70}$$

The zero mode c_0 is the harmonic function for the smeared fivebrane. The rest of the coefficients c_n can be viewed as arising from condensates of strings with various non-zero momenta along the θ-circle. In the dual geometry, there are corresponding condensates of winding modes, which, indeed become light as the circle fiber shrinks. Alternatively, the breaking of translation symmetry can be viewed via the standard duality recipe of gauging a symmetry and integrating out the auxiliary gauge field. This is perhaps more closely related to the viewpoint of [11].

The procedure followed in [11] to determine the quantum-corrected dual of a Taub-NUT space uses a philosophy which is very powerful in two dimensional T-duality, and more generally, in mirror symmetry in higher dimensions, viewed as several T-dualities, completely dualizing the fiber of a toric variety [53]. But it also works for a single T-duality, along one circle, of Taub-NUT, as well as the asymptotically locally flat singular spaces that are obtained from putting multiple Taub-NUT centers on top of one another.

The main idea is to perform the duality transformation, à la [10], not in the (conformally invariant) non-linear sigma model, where the field configurations are complicated, but instead to find a simpler non-conformal field theory which flows to the desired conformal non-linear sigma model in the infrared limit of renormalization group flow. Essentially the simplification is obtained by introducing $U(1)$ gauge fields in the field theory, i.e. it is the same philosophy that uses gauged linear sigma models to describe non-linear sigma models at low energy, where the dynamics of the gauge field in 'frozen'.

The particular models used in [11] are $\mathcal{N} = (4,4)$ supersymmetric two dimensional theories, as they should be, describing superstring compactifications to six dimensions. It was shown that a pair of models is related by a duality transformation similar to the $\mathcal{N} = (2,2)$ one of [10]. Actually the classical vacuum manifolds of these models are Taub-NUT for one and the smeared brane geometry for the dual. But by taking into account instanton corrections, which the gauge theory for the smeared brane geometry has, Tong [11] has provided evidence [11] that the quantum corrected vacuum manifold is that of a localized fivebrane.

T-DUALITY VIA NON-CONFORMAL MODELS

It is a very powerful philosophy to do a T-duality via the following steps, which was developed in [53]. First, find a non-conformal field theory which has a $U(1)$ symmetry that is realized in a simple fashion, the $U(1)$ being the 'T-duality circle'. This theory should at low energies behave like the conformal field theory that one wishes to dualize. Typically, the ultraviolet theory is a gauged linear sigma model and at low energies the gauge dynamics is frozen out, so that the effective theory is a non-linear sigma model[8]. This low-energy non-linear sigma model has a circle symmetry, descending from a symmetry in the ultraviolet theory.

Next, one can perform a duality transformation, integrating out the appropriate auxiliary fields, in the ultraviolet theory. Integrating out the fields is slightly more subtle than in the cases discussed earlier, since the ultraviolet theory has gauge symmetries, which affect the transformations, both on the classical level, and by quantum corrections. However, these effects are under control. The quantum corrections come from vortex configurations, which are typical for $U(1)$ gauge fields in two dimensions. The modifications to the duality transformations are briefly discussed below.

The vortex corrections can break the circle symmetry that one might expect from looking only at the classical dualization procedure. Now the task is to find a description of the dual ultraviolet theory. This dual theory need not be a simple linear sigma model, as the quantum corrections typically give rise to (twisted) F-terms in the theory.

Finally, having obtained a dual ultraviolet theory, one should identify its low-energy limit. Not only may this model lack the circle isometry of the original low-energy model before the duality, but it may have no (direct) geometric interpretation whatsoever. In particular, an ultraviolet theory with a (twisted) superpotential characterizes a low-energy Landau-Ginzburg theory. That is to say, the D-terms may get renormalized in a very complicated and incomputable way, precluding a direct geometrical interpretation, as a sigma model. The twisted superpotential, which is better behaved under renormalization group flow, may still to a large extent characterize the low-energy theory as a Landau-Ginzburg model.

There are considerable classes of models for which this approach to T-duality can be carried out successfully. In the first place, T-duals of interesting two dimensional backgrounds can be constructed. In particular, there is the derivation of the duality of the 'cigar' Euclidean black hole, $SL(2;\mathbb{R})/U(1)_a$ and $\mathcal{N}=2$ Liouville theory [26]. The existence of this duality plays a significant rôle in the next section. The ultraviolet model that features in the derivation of [26] is actually not quite an ordinary gauged linear sigma model. In the model the gauge symmetry not only acts on phases of chiral fields, corresponding to D-terms of the form

$$L_{\text{phase}} = \int \mathrm{d}^4\theta \Phi e^{2qV} \overline{\Phi}, \tag{4.71}$$

but it also acts as shifts on another chiral field, corresponding to a D-term of the form

$$L_{\text{linear}} \int \mathrm{d}^4\theta \left(\Psi + \overline{\Psi} + V\right)^2. \tag{4.72}$$

The two different ways that a gauge symmetry can act, (4.71) and (4.72) result in somewhat different dualization properties, which will be briefly discussed momentarily.

[8]The low energy theory actually need not be scale invariant, it is possible to carry through the same approach for a non-linear sigma model on a positively curved target space.

Secondly, duals of geometries of more than two dimensions can be constructed. The prime example of this, is the Taub-NUT gauge theory of [11], which actually also contains a field which has a gauge symmetry acting as shifts, i.e. like in (4.72). As it stands, this model, and its cousins describing A_k-type asymptotically locally flat spaces, are quite exceptional. Much more often, mirror models have been constructed [53] of higher dimensional target spaces, but not strictly speaking T-duals. In a toric variety mirror symmetry can be seen as the composition of several T-dualities, such that every one-cycle of the toric fibers is dualized [52]. Toric varieties are naturally described in terms of gauged linear sigma models, and the ultraviolet theory corresponding to the mirror is obtained by dualizing the phase of every chiral superfield appearing in the linear sigma model. Note that a toric variety $(\mathbb{C}^{m+n}\backslash S)/(\mathbb{C}^*)^m$ is described by a linear sigma model with $m+n$ chiral superfields charged under $U(1)^m$. A T-duality along a single S^1 in the toric variety would correspond to dualizing $m+1$ combinations of phases of chiral superfields in the linear sigma model.

DUALITY WITH GAUGE SYMMETRY

Let us now explicitly recall how the duality transformations act in a model with gauge symmetries, as introduced in [53]. The quantum corrections in the ultraviolet model, due to the gauge fields, correspond to corrections to Buscher's rules, breaking isometry in the dual model, at low energies, where no gauge symmetry is visible. Consider, as an example, the following part of a D-term of a simple gauged linear sigma model with a chiral superfield of $U(1)$ charge q,

$$L_\Phi = \int \mathrm{d}^4\theta \overline{\Phi} e^{2qV} \Phi. \tag{4.73}$$

There is another part of the D-term, that gives the dynamics of the gauge field V, which reads

$$L_{\text{gauge}} = \frac{-1}{2e^2}|\Sigma_V|^2, \tag{4.74}$$

where $\Sigma = -2\overline{D}_+ D_- V$. As $e \to \infty$, the gauge field becomes non-dynamical and this term can be forgotten, which is the case in the low energy non-linear sigma model limit, discussed in section 3.3.2.

The term (4.73) can be obtained from an overarching Lagrangian

$$\int \mathrm{d}^4\theta \left(e^{2qV+B} - \frac{1}{2} B \left(Y + \overline{Y}\right)\right) \tag{4.75}$$

by integrating out the twisted chiral superfield Y and its conjugate. On the other hand, solving the classical equation of motion of the 'auxiliary gauge field' B, gives a dual Lagrangian, at least classically, that reads

$$\tilde{L}_{\text{cl}} = \int \mathrm{d}^4\theta \left(qV \left(Y + \overline{Y}\right) - \frac{1}{2}\left(Y + \overline{Y}\right) \log\left(Y + \overline{Y}\right)\right). \tag{4.76}$$

Now, the first term can be written as a twisted F-term, since Y is a twisted chiral field, $\overline{D}_+ Y = D_- Y = 0$,

$$\int \mathrm{d}^4\theta V Y = \frac{1}{2}\int \mathrm{d}^2\tilde{\theta}\Sigma_V Y.$$

Such a term looks like a Fayet-Iliopoulos/theta-angle term, with the difference that the ‘FI-parameter’ here is not a fixed number, $t = r - i\theta$, but a dynamical field Y.

In all, at the level of classical equations of motion, the linear sigma model Lagrangian

$$L = \int \mathrm{d}^4\theta \left(|\Phi|^2 e^{2qV} - \frac{1}{2e^2}|\Sigma_V|^2\right) + \frac{1}{2}\left(\mathrm{d}^2\tilde{\theta} - t\Sigma + \text{c.c.}\right) \tag{4.77}$$

is dual to

$$\begin{aligned} &\int \mathrm{d}^4\theta \left(-\frac{1}{2}\left(Y+\overline{Y}\right)\log\left(Y+\overline{Y}\right) - \frac{1}{2e^2}|\Sigma_V|^2\right) \\ &+\frac{1}{2}\int \mathrm{d}^2\tilde{\theta}\Sigma_V\left[qY - t\right] + \text{c.c.} \end{aligned} \tag{4.78}$$

Note the rôle of the twisted chiral Y, or more accurately, the real part of the expectation value of its scalar component y, as a shift of the effective Fayet-Iliopoulos parameter in the dual model,

$$r_{\text{eff}} = r_{\text{original}} - \text{Re}\,\langle y\rangle. \tag{4.79}$$

However, solving the classical equation of motion of B does not suffice to determine the effective action. There are configurations in which the phase of ϕ has winding, compensated by a vortex configuration of the gauge field B. These configurations contribute to the effective Lagrangian of the dual theory, and modify the twisted F-term [53],

$$\frac{1}{2}\int \mathrm{d}^2\tilde{\theta}\Sigma_V\left[qY - t\right] \overset{\text{vortices}}{\longrightarrow} \frac{1}{2}\int \mathrm{d}^2\tilde{\theta}\left(\Sigma_V\left[qY - t\right] + \mu e^{-Y}\right). \tag{4.80}$$

This is a generic feature that appears when a chiral superfield is dualized, of which the phase is gauged. On the other hand, if a field is dualized which has a shift gauged, like in (4.72), there are no vortex configurations, and no e^{-Y} term is generated in the twisted superpotential.

Now that a dual ultraviolet Lagrangian has been written down, consider the interpretation of the dual model it describes. Actually, the model with a single charged chiral superfield may be a bit too restricted. The target space of its non-linear sigma model limit is given by

$$0 = U = |\phi|^2 - q\log\left(\frac{\Lambda_{UV}}{\Lambda}\right),$$

modulo gauge equivalence $\phi \sim e^{i\theta}\phi$, which leaves a point for a target space. So perhaps it is better to expand the model a little, and have two charged chiral superfields and a single

$U(1)$ gauge group. Take the Lagrangian

$$L = \int \mathrm{d}^4\theta \left[\left(|\Phi_1|^2 + |\Phi_2|^2 \right) e^{2V} - \frac{1}{2e^2}|\Sigma|^2 \right] + \frac{1}{2}\int \mathrm{d}^2\tilde{\theta}\ -r\Sigma\ + \text{c.c.} \tag{4.81}$$

The scalar potential, as $e \to \infty$, reads

$$U = |\phi_1|^2 + |\phi_2|^2 - 4\log\left(\frac{\Lambda_{UV}}{\Lambda}\right), \tag{4.82}$$

so at low energy scales it behaves as a non-linear sigma model on a large $\mathbb{P}^1$.

If the phases of both chiral superfields are dualized, using two auxiliary gauge fields, the resulting Lagrangian can be written as

$$\begin{aligned} L = & \int \mathrm{d}^4\theta \left[-\frac{1}{2e^2}|\Sigma|^2 + \sum_{i=1,2} \left(Y_i + \overline{Y_i}\right) \log\left(Y_i + \overline{Y_i}\right) \right] \\ & + \frac{1}{2}\int \mathrm{d}^2\tilde{\theta} \left[\Sigma\left(4Y_1 + 4Y_2 - t\right) + e^{-Y_1} + e^{-Y_2} \right] + \text{c.c.} \end{aligned} \tag{4.83}$$

The low energy theory is then obtained by taking $e \to \infty$ and integrating out Σ, which enforces the constraint

$$Y_2 = \frac{t}{4} - Y_1.$$

The resulting model is defined in terms of a single twisted chiral superfield with a twisted superpotential, i.e. it is a Landau-Ginzburg model.

This is the viewpoint of [53]. Dualizing the phases of all chiral superfields amounts to going to the mirror description. In the present example, the sigma model on $\mathbb{P}^1$ is not scale invariant and correspondingly, the Landau-Ginzburg superpotential is not weighted homogeneous. This viewpoint of mirror symmetry is very interesting and it can be applied to toric varieties with $c_1 \geq 0$, leading to Landau-Ginzburg (-orbifold) models with superpotentials that are weighted homogeneous ($c_1 = 0$) or not ($c_1 > 0$).

For a two dimensional space, like $\mathbb{P}^1$, T-duality is mirror symmetry. As remarked earlier, the mirror transformation entails dualizing all chiral superfields. To do a genuine T-duality along a single one-cycle, one should dualize $m+1$ chirals, if the gauge group of the ultraviolet model is $U(1)^m$. In general this not only introduces $m+1$ twisted chiral fields, it also leaves some charged chiral fields. Such a field content, of both chirals and twisted chirals, in a sigma model may give rise to flux, like in [48]. However, typically a model that arises from such a duality transformation will also have (twisted) F-terms, so that finding a geometrical interpretation is more complicated than would be for a sigma model. Also, depending on which combinations of phases or shifts are dualized, it may be impossible to perform the duality transformation in a $\mathcal{N} = 2$ superfield formalism, i.e. the constraint equations coming from integrating out some auxiliary superfields may have no solution in terms of elementary functions.

Other Geometries and Fivebranes

To recapitulate, applying the classical T-duality rules to a Taub-NUT geometry, gives a geometry $\mathbb{R}^3 \times S^1$ in which a fivebrane is smeared along the circle. Taking into account quantum effects, the fivebrane turns out to be localized at a point in $\mathbb{R}^3 \times S^1$. An analogous statement is true for a stack of N coincident fivebranes. This dualizes to an asymptotically locally flat space with an A_{N-1} singularity.

It is natural to ask how this correspondence of T-dual backgrounds extends to other geometries, also higher dimensional geometries which are part of string compactifications that preserve less supersymmetry than Taub-NUT, such as Calabi-Yau three- and four-folds. In particular, it would be interesting to understand how the dual looks, presuming that a geometric interpretation exists. It is difficult to do an honest and exact quantum duality, for a variety of reasons, some of which are the following.

First, it is probably almost hopeless to consider duality of a compact geometry. However, one might consider non-compact spaces which generalize the Taub-NUT geometry. These might be smooth or have a singularity, depending on the particular situation, though most cases will be singular. In particular, there are very interesting generalizations of the Taub-NUT geometry to higher dimensions [88]. The geometry of these spaces looks like

$$\mathrm{d}s^2 = A^2(r)\mathrm{d}r^2 + C^2(r)\mathrm{d}s_Z^2 + B^2(r)\left(\mathrm{d}\theta + A\right)^2 , \tag{4.84}$$

where $\mathrm{d}s_Z^2$ is a metric on a compact homogeneous Kähler manifold, like for example the Hermitean symmetric spaces of chapter 2. Furthermore, A is a section of the cotangent bundle of Z, and is related to the Kähler form, $\Omega = dA$, locally. The coordinate θ parametrizes a circle and r is a 'radial' coordinate. The functions A, B, C depend on the radial coordinate only, and have been determined in [88]. Furthermore, these functions depend only on the dimension of Z, and on one positive parameter q, which essentially describes the size of the circle fiber at infinity. As $r \to \infty$, A and B tend to constant values, where $B \sim q$ sets the size of the fiber, and $C(r) \to r$. So the asymptotically the space looks locally like the product of a circle, times a 'cone' over the Kähler manifold Z, which is however not a metric cone. The 'center' of the space is located at $r = q$ (one can always choose $C^2(r) = r^2 - q^2$). At the center the space looks like a metric cone[9]. This space is regular only in the 'metric link' is a round sphere.

What would a background look like, obtained by T-dualizing the θ circle of such a generalized Taub-NUT? If one would take an approach similar to [11], the first step would be to find an ultraviolet gauge theory, which in the infrared flows to a nonlinear sigma model on the generalized Taub-NUT. Such a gauge theory has in general less supersymmetry than the

[9]The general expression for $A(r)$ in [88] is: $A^2(r) = \frac{(r^2-q^2)^n}{4r}\left(\int_q^r \frac{(t^2-q^2)^n}{t^2}\,\mathrm{d}t + \beta\right)^{-1}$, where n is the (complex) dimension of Z. Also $B = qA^{-1}$. If the integration constant β is chosen equal to zero, the space looks like a metric cone near the center.

$\mathcal{N} = (4,4)$ of the model of [11]. Having found a satisfying gauge theory, one should perform a $\mathcal{N} = (2,2)$ T-duality transformation, which gets rid of some chiral superfields, but not all, which would be the case for mirror symmetry, and introduces some twisted chirals. The next question would be to find a (geometric) interpretation of this model, or rather, of its infrared limit. This interpretation might, for example, be some more complicated fivebrane configuration, or something more complicated.

For example, as a four dimensional smooth space, one can take instead of a Taub-NUT, the Atiyah-Hitchin space, which should dualize to an orientifold O(5) plane, instead of a NS fivebrane, see [79] and references therein. The Atiyah-Hitchin space, combined with multi-center Taub-NUT can be used to get D-type hyper-Kähler singularities, rather than A-type. But what would be the dual geometric interpretation of an exceptional hyper-Kähler surface singularity, for example, is unclear.

There are two possible approaches to get a simpler description. First, one could consider only the classical duality. In this case, the dual background has too much isometry. An example of this situation will be considered below. Second, one could consider singular geometries, and take a scaling limit. For example, these could be metric cones, say over Hermitean symmetric spaces, as local models of the singularities of the generalized Taub-NUT geometries of [88]. But also these could be other singularities as discussed in chapter 2, which may not have a known metric description at all.

One might hope to be in a better position to find a dual description in such a scaling limit. The motivation for this hope, in part, lies in the observation that an exact conformal field theory description is known for the throat geometry of fivebranes, whereas no exact cft is known for the full 'global' background of a stack of fivebranes. If in more general scaling limits, there are exact conformal field theory descriptions as well, then one might employ known and conjectured facts about abstract conformal field theories to perform the duality, and perhaps hope that after the dust has settled, the dual cft also has a geometric interpretation. Still, one might then ask what are the 'global' backgrounds that correspond to the scaling limit conformal field theories. This second approach, a full quantum duality of a scaling limit will be considered in the next section.

Classical Duals of Generalized Taub-NUT Metrics

Consider a generalized Taub-NUT metric [88],

$$\begin{aligned} \mathrm{d}s^2 = L_q(r)dr^2 + \left(r^2 - q^2\right) G_{a\bar{b}}\mathrm{d}x^a\mathrm{d}\overline{x}^b \\ +\frac{4q^2}{L_q(r)}\left[\mathrm{d}\theta^2 + A_\mu(\mathbf{x},\overline{\mathbf{x}})\mathrm{d}x^\mu\right]^2 . \end{aligned} \tag{4.85}$$

Here $G_{a\bar{b}}$ is a Kähler metric of an $2n$ dimensional compact homogeneous Kähler manifold, Z, with coordinates x^μ, where μ runs over n holomorphic indices a and n anti-holomorphic indices $\bar{b}$. Locally, in a coordinate patch, one can obtain the Kähler metric from a Kähler potential $G_{a\bar{b}} = 2\partial_a \bar{\partial}_{\bar{b}} K(x, \overline{x})$. The coordinate θ parametrizes a circle fiber. The nontriviality of the fibration is expressed through the A_μ, which can be seen as a gauge field on Z. The gauge field A is related to the Kähler metric. In a coordinate patch, one can write $A_\mu = i\partial_\mu K$, where μ runs over holomorphic and anti-holomorphic indices.

For example, taking $Z = \mathbb{P}^1$, using spherical coordinates on $\mathbb{P}^1$, and taking $A \sim \cos\theta d\phi$, so that dA is the volume form on $S^2 \sim \mathbb{P}^1$, the resulting metric is the familiar Taub-NUT. Its classical dual is a the smeared fivebrane on $\mathbb{R}^3 \times S^1$.

For $Z \simeq \mathbb{P}^1 \times \mathbb{P}^1$, one can write a Taub-NUT like metric which is of the form

$$\begin{aligned} \mathrm{d}s^2 = L(\rho)\mathrm{d}\rho^2 + \left(\rho^2 + 2q^2\right) \left[\sum_{i=1,2} \mathrm{d}\theta_i^2 + \sin^2\theta_i \mathrm{d}\phi_i^2 \right] \\ + \frac{q^2}{L(\rho)} \left[\mathrm{d}\psi + \sum_{i=1,2} \cos\theta_i \mathrm{d}\phi_i \right]^2 . \end{aligned} \tag{4.86}$$

Where

$$L(\rho) = \frac{3}{4} \frac{(\rho + 2q)^2}{\rho\,(\rho + 4q)}. \tag{4.87}$$

The parameter q governs the size of the ψ-circle. In the (classically) T-dual geometry, it is related to the asymptotic string coupling.

Applying the Buscher rules, results in a ‘smeared’ dual background

$$\begin{aligned} \widetilde{\mathrm{d}s^2} &= L(\rho) \left(\mathrm{d}\rho^2 + q^{-2} \mathrm{d}\tilde{\psi}^2 \right) + \left(\rho^2 + 2q^2\right) \sum_{i=1,2} \left[\mathrm{d}\theta_i^2 + \sin^2\theta_i \mathrm{d}\phi_i^2 \right], \\ \widetilde{B} &= \sum_{i=1,2} \cos\theta_i \mathrm{d}\tilde{\psi} \wedge \mathrm{d}\phi_i, \\ \widetilde{\Phi} &= \Phi_0 + \log L(\rho). \end{aligned} \tag{4.88}$$

In this background, one may recognize the ‘transverse’ space of a pair of intersecting fivebranes. Both fivebranes share a common worldvolume $\mathbb{R}^{3,1}$ and both are smeared along a commonS^1. This leaves five dimensions, in which the fivebranes intersect in a point (to get a picture, say, both NS5 and NS5’ have common worldvolume directions x_{0123}, furthermore NS5 has worldvolume directions x_{45}, NS5’ has worldvolume directions x_{67} and both NS5 and NS5’ are smeared along x_9, which is a circle. The only direction in which the entire configuration is point-like is x_8). For instance, the $1/\rho$ behavior in the metric at small values of ρ, indicates there is a single direction in which the whole configuration is localized. By quantum effects, one might expect the fivebranes to localize along the $\tilde{\psi}$ direction.

CONCLUDING OBSERVATIONS ABOUT CLASSICAL DUALITY

This picture as it is presented above is obviously quite crude. There are some questions which arise immediately. For example, in a brane interpretation, two charges appear naturally, labeling the numbers of NS5 and NS5' branes. One may wonder what the corresponding interpretation is of these integers on the geometric side. A natural guess, presents itself, when zooming in on the local geometry.

In the Taub-NUT case, we have seen that the local geometry looks like a metric cone over a lens space. The degree to which the circle is fibered non-trivially over the base $\mathbb{P}^1$, i.e. the Chern class, indicates the fivebrane charge in the dual background. Similarly, there one would consider circle fibrations over $\mathbb{P}^1 \times \mathbb{P}^1$, which give rise to a pair of integers.

However, the analogy seems not to go through completely. Whereas the metric cone over any lens space $S^3/\mathbb{Z}_{N+1}$ is a supersymmetric metric cone with a smooth link, there are only two metric cones on smooth circle bundles over $\mathbb{P}^1 \times \mathbb{P}^1$. The links are T^{11} and $T^{11}/\mathbb{Z}_2$, as discussed in chapter 2. Nevertheless, one can consider many more supersymmetric singularities in six dimensions, which keep a relation to $\mathbb{P}^1 \times \mathbb{P}^1$. For instance, one can consider orbifolds of the ordinary conifold, and related spaces that are connected via blowups and blowdowns of various cycles.

There is another interesting question. It seems perfectly legitimate to consider a 'scaling limit' on the geometric side of the picture, keeping only the geometry near a singular point, similar to scaling from $(N+1)$-center Taub-NUT to an A_N singularity. The question is what such a scaling limit would correspond to in the dual background. First of all, it is clear that this question cannot be answered using the classical Buscher rules. This is so not only because the fact whether a brane configuration is smeared or not, affects what the background looks like near this configuration. But also, it is precisely the localized stack which has a throat geometry that can be decoupled from the bulk, through an appropriate scaling limit. For an intersecting configuration of fivebranes, how should one imagine taking an analogous scaling limit?

For instance, consider a configuration of two stacks of intersecting fivebranes, NS5 with worldvolume directions x_{012367} and NS5' branes with worldvolume directions x_{012389}. In such a geometry one can approach the NS5 branes while remaining far away from the NS5' branes, and it is not readily clear that there is a distinguished 'radial' direction, to perform a decoupling limit. Such a decoupling limit should yield a linear dilaton in the 'radial' direction. As it turns out, there is such a limit, which has been found, assuming a linear dilaton from the outset, in [106].

A final question for now is: 'Is it possible to find general 'flux impurity' configurations, in the same numbers as there are geometric impurities, and how should these be interpreted and a scaling limit taken?'. Or alternatively: 'Are there 'scaling limits' of flux impurities, are they described by exact conformal field theories, like the simple stack of parallel fivebranes, and how are these conformal field theories related to the geometry?. These questions form the starting point for section 4.4.

4.4 THE DUAL OF A CONE

One can take a 'local' view on the issue of quantum T-duality. In this approach, one considers only the local geometry near a singularity and only a 'throat geometry' in the dual, where there are no 'localized branes' visible, but only their effect on the nearby ambient space: fluxes and a linear dilaton. It is quite generic to consider T-duality in such a 'local' approach, since all supersymmetric singularities, discussed in chapter 2, have a (circle) isometry which degenerates at the singular point.

In order that one may consider only the 'localized' physics, it must be decoupled from the bulk through some decoupling limit. On the geometric side the decoupling limits involve deforming the singularity slightly, by a parameter μ, which is taken to zero in the decoupling limit. In order to keep the masses of localized excitations finite (think of these as branes wrapping the almost vanishing cycle), the asymptotic string coupling is scaled to zero, too.

In the case of the hyper-Kähler surface singularities, the decoupled theories are Little String Theories [3], non-gravitational theories of the worldvolume physics of fivebranes. An important way to study these theories, is via a holographic dual: linear dilaton backgrounds, like the throat geometry [4], [23, 24]. A similar view can be taken with regard to other 'impurities', which can be interpreted as Calabi-Yau singularities, or as certain 'flux impurities' which might be intersecting fivebranes, or other complicated sources of flux. An important inspiration and motivation for us to consider affine hypersurface singularities, lies in the work of Ooguri and Vafa [19], who discussed T-duality between ADE surface singularities and fivebrane throat conformal field theories in an abstract cft approach, and the work of Giveon, Kutasov and Pelc [22] who have proposed a relation between general affine hypersurface singularities to Landau-Ginzburg conformal field theories.

OUTLINE

In this section we shall begin with a discussion of a non-conformal field theory which is proposed to relate the nonlinear sigma model on an affine hypersurface (or a discrete quotient thereof, depending on details), to another conformal field theory, which we shall call a 'half-dualized' theory. The idea is that the 'half dualized' theory takes into account all non-perturbative contributions to the T-duality. Then in order to get the T-dual to the non-linear sigma model on a hypersurface, one needs only to perform a classical T-duality transformation on the 'half dualized' theory. The resulting dual theory generically contains a linear dilaton (i.e. the conformal field theory of a scalar with background charge). The rest of the theory depends much more on the hypersurface one starts out with.

We proceed to discuss some concrete examples of hypersurfaces and dual theories. First, we recover the duals to the ADE surface singularities, which consist of a dilaton and an $SU(2)$ superconformal field theory, as originally found by Ooguri and Vafa [19]. The ADE surfaces are quite special, as the are the only hypersurfaces we know that are both described exactly by our kind of ultraviolet theory, and whose duals have an interpretation with a

WZW model. As it will turn out, in general hypersurfaces of 'anticanonical type[10]' are described by our ultraviolet gauge theories. If a hypersurface is not of anticanonical type, (and $-d + \sum a_i$ divides d), then there is an ultraviolet theory which describes a cyclic quotient to the affine hypersurface.

We continue with some examples of special non-anticanonical hypersurfaces. In general their defining polynomials are of the form

$$F\left(x_1, \ldots, x_{n+2}\right) = H\left(x_1, \ldots, x_n\right) + x_{n+1}^{+} x_{n+2}^{2}.$$

When the polynomials H are of the type that defines a Landau-Ginzburg superpotential of a model that also has a Kazama-Suzuki coset model interpretation, $G/\left(H \times U(1)\right)$, then the T-dual model is of the form

$$\text{linear dilaton} \times \frac{G}{H}.$$

We conclude with some finishing remarks about hypersurfaces which have no Landau-Ginzburg/Kazama-Suzuki interpretation, and regarding Anti-de Sitter target spaces in lieu of linear dilaton backgrounds.

SIGMA MODELS FOR CONES

Consider an affine hypersurface

$$\mathcal{C} = F^{-1}(0) \subset \mathbb{C}^{n+2}, \tag{4.89}$$

defined by a weighted homogeneous defining polynomial

$$F(\lambda^{w_1} x_1, \ldots, \lambda^{w_{n+2}} x_{n+2}) = \lambda F(x_1, \ldots, x_{n+2}). \tag{4.90}$$

The 'weights' can be written as

$$w_i = \frac{a_i}{d}, \quad a_i \in \{2, 3, 4, \ldots\}, \tag{4.91}$$

where also

$$\begin{aligned} \gcd\left(\{a_i\}\right) &= 1, \\ \operatorname{lcm}\left(\{a_i\}\right) &= d \end{aligned} \tag{4.92}$$

Let $\mathcal{C}$ have only a single, isolated, singularity which is located at $\mathbf{x} = \mathbf{0}$. That is to say,

$$\left(F(\mathbf{x}) = 0 \text{ and } \mathrm{d}F(\mathbf{x}) = 0\right) \quad \Longleftrightarrow \quad \mathbf{x} = \mathbf{0}. \tag{4.93}$$

[10]Recall that by this term we refer to hypersurfaces defined by a weighted homogeneous polynomial of weighted degree d and with weights a_i such that $\sum a_i = d + 1$

According to Tian and Yau [27] this affine cone, at least without the apex $\mathbf{x} = \mathbf{0}$, is Calabi-Yau, if and only if

$$d < \sum_{i=1}^{n+2} a_i \equiv A, \tag{4.94}$$

and we assume that it is.

How is this hypersurface described via a gauged linear sigma model? The equation

$$F(x_1, \ldots, x_{n+2}) = 0$$

defining $\mathcal{C}$ is an equation in affine space $\mathbb{C}^{n+2} \setminus \{0\}$. Usually, a gauged linear sigma model is used to describe hypersurfaces in a projective space [18]. The idea is to view the affine $\mathbb{C}^{n+2}$ as a 'patch' with 'inhomogeneous' coordinates of a larger space, that does have a $U(1)$ gauge equivalence. Actually, in general, the patch described through such a gauged linear sigma model is not $\mathbb{C}^{n+2}$, but rather a cyclic quotient $\mathbb{C}^{n+2}/\mathbb{Z}_m$, in the fashion of the model of section 3.3.2 on page 61. This point of view is discussed below.

First note that $F = 0$ is also the defining equation of a hypersurface in

$$\mathbb{P}[a_1, a_2, \ldots, a_{n+2}].$$

If this hypersurface is well-formed, see (2.57),

$$\begin{aligned} &\gcd(a_1, \ldots, \hat{a}_i, \ldots, a_{n+2}) = 1 \quad 1 \le i \le n+2, \\ &\gcd(a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_{n+2}) \mid d \quad \forall\, i, j. \end{aligned} \tag{4.95}$$

then (4.94) says that it is Fano. We assume that the a_i and d are such, that it actually is Fano.

In a $U(1)$ gauged linear sigma model description of the hypersurface $F = 0$ in

$$\mathbb{P}[a_1, a_2, \ldots, a_{n+2}],$$

the variables x_i correspond to chiral superfields Φ_i of $U(1)$ (gauge) charge $q_i = a_i$. Introduce another chiral superfield, Φ_0 of charge

$$q_0 = a_0 \equiv d - A < 0. \tag{4.96}$$

In order to avoid the axial anomaly, one should introduce another chiral superfield, Φ_{-1}, with $U(1)$ charge

$$q_{-1} = -d, \tag{4.97}$$

so that the sum of all gauge charges vanishes.

Now consider the 'linear sigma model' with Lagrangian

$$L = L_D + L_F + L_{\widetilde{F}}, \tag{4.98}$$

where

$$
\begin{aligned}
L_D = \int \mathrm{d}^4\theta |\Phi_{-1}|^2 e^{-dV} + \frac{d}{2a_0^2}\left(\Psi + \overline{\Psi} + V\right)^2 \\
+ \sum_{i=1}^{2n} |\Phi_i|^2 e^{a_i V} - \frac{1}{2e^2}|\Sigma|^2,
\end{aligned}
\tag{4.99a}
$$

$$
L_F = \int \mathrm{d}^2\theta \; \Phi_{-1}\left[\mu e^{-|d/a_0 \Psi|} + F\left(\Phi_1, \ldots, \Phi_{n+2}\right)\right] + \text{c.c.}\,, \tag{4.99b}
$$

$$
L_{\widetilde{F}} = \int \mathrm{d}^2\tilde{\theta} \; - t\Sigma + \text{c.c.} \tag{4.99c}
$$

Note that this Lagrangian is not quite that of an ordinary gauged linear sigma model with a gauge invariant superpotential. The above Lagrangian has a gauge invariant superpotential, but the kinetic term for $\Phi_0 = e^{\Psi}$ is somewhat special. The field Ψ does not transform homogeneously under gauge transformations, but rather it is shifted. But on the other hand, the superpotential for Ψ, does transform homogeneously. In some respects, it is convenient to think in terms of the field Ψ, in others it is more natural to reason in terms of the condensate Φ_0. Both points of view will be used in the following.

The F-term (4.99b) is gauge invariant, and it can also be made invariant under the vectorial $U(1)$ R-transformations. This is accomplished by choosing the $U(1)_V$ charges of all Φ_i proportional to their gauge charges, $v_i = 2w_i$ except for the $U(1)_V$ charge of Φ_{-1}, which is chosen to vanish. A negative $U(1)_V$ charge for Φ_0 may seem strange, but one should keep in mind that the 'fundamental' field is Ψ.

Also, from the definition of a_0 it does not follow that $|a_0|$ should necessarily divide d. In some interesting cases, $|a_0|$ does not divide d. Some examples of such cases are discussed later. The prime case where $|a_0|$ is guaranteed to divide d, is $a_0 = -1$. Precisely in this case, the hypersurface $F = 0$ in $\mathbb{P}[a_1, a_2, \ldots, a_{n+2}]$ is anticanonically embedded. For example this is the case for the ADE hyper-Kähler surfaces, in table 2.2 and for the del-Pezzo surfaces collected in table 2.6. It is for anticanonically embedded hypersurfaces in weighted projective space, that the method of Kollár and Johnson applies to possibly determine the existence of quasi-smooth Kähler-Einstein metrics on the projective hypersurface [34, 35], which is a foundation to apply the methods of Boyer, Galicki et al.[78, 37, 32] to prove existence of Sasaki-Einstein metrics on the link, so that the affine hypersurface can be viewed as a metric cone.

Precisely when the embedding is anticanonical, one can recover the affine cone from the linear sigma model, as opposed to a cyclic quotient of the affine cone. This cone is recovered in the infrared limit, if the Fayet-Iliopoulos parameter $r \gg 0$. When the embedding is not anticanonical, $|a_0| > 1$, and $|a_0|$ divides d, then in this 'phase' of the sigma model, one recovers a $\mathbb{Z}_{|a_0|}$ quotient of the affine cone $F^{-1}(0)$. When $|a_0|$ does not divide d, a possible interpretation seems to be more subtle. Let us illustrate these cases with some examples.

EXAMPLES

ADE HYPER-KÄHLER SURFACE

The hyper-Kähler surfaces are described as hypersurfaces in $\mathbb{C}^3$, defined by the polynomials listed in table 2.2. These hypersurfaces are anticanonically embedded, so $a_0 = -1$. The gauge invariant superpotential of an ADE linear sigma model reads

$$W = \Phi_{-1}\left(\mu\Phi_0^{-d} + F_\Gamma\left(\Phi_1, \Phi_2, \Phi_3\right)\right). \tag{4.100}$$

In the non-linear sigma model phase, $r \gg 0$, the vacuum manifold (cf. equation 3.46 on page 61) is

$$\left\{\mu\Phi_0^{-d} + F_\Gamma\left(\Phi_1, \Phi_2, \Phi_3\right) = 0\right\}/U(1), \tag{4.101}$$

which is a hypersurface in $\mathcal{O}(-d) \to \mathbb{P}[a_1, a_2, a_3]$. By passing to 'inhomogeneous coordinates' $\Xi_i = \Phi_i \Phi_0^{a^i/|a_0|} = \Phi_i \Phi_0^{a^i}$, this can be viewed as the affine hypersurface

$$F_\Gamma\left(\Xi_1, \Xi_2, \Xi_3\right) + \mu = 0 \tag{4.102}$$

in $\mathbb{C}^3$. That is to say, the deformed ADE-singularity. A similar argumentation applies to any anticanonically embedded hypersurface (i.e. $a_0 = -1$).

To be more specific, consider a deformed A_{n+1} singularity. It is described as a hypersurface

$$A_{n+1} \;:\; x_1^{2+n} + x_2^2 + x_3^2 + \mu = 0. \tag{4.103}$$

The gauged linear sigma model that describes this model in its infrared regime, for large positive FI-parameter, has $U(1)$ charge assignments

$$[a_{-1}, a_0, a_1, a_2, a_3] = \left[-\alpha(n+2), -\alpha, \alpha, \frac{\alpha}{2}(n+2), \frac{\alpha}{2}(n+2)\right], \tag{4.104}$$

where α=1 (α=2) if n is even (n is odd). The superpotential reads

$$W = \Phi_{-1}\left(\mu\Phi_0^{-n-2} + \Phi_1^{n+2} + \Phi_3^2 + \Phi_4^2\right). \tag{4.105}$$

As discussed in section 3.3.2, when $r \gg 0$, the scalar potential is minimized on a hypersurface in $\mathcal{O}(-\alpha) \to \mathbb{P}\left[\alpha, \frac{\alpha}{2}(n+2), \frac{\alpha}{2}(n+2)\right]$, and the hypersurface is given by

$$\mu\Phi_0^{-n-2} + \Phi_1^{n+2} + \Phi_3^2 + \Phi_4^2 = 0. \tag{4.106}$$

This can be rewritten, defining $\Xi_i = \Phi_i\ \Phi_0^{a_i/a_0}$, so that the Ξ_i are uncharged under the gauge group, and recalling $\Phi_0 = e^\Psi$, as

$$e^\Psi\left(\mu + \Xi_1^{n+2} + \Xi_3^2 + \Xi_4^2\right) = 0. \tag{4.107}$$

The part between brackets is the defining equation of a deformed A_{n+1} singularity. Also, the Ξ_i are good coordinates on $\mathbb{C}^3$. Note that $\Psi \sim \Psi + 2\pi i$, or, in terms of Φ_0, $\Phi_0 \sim e^{2\pi i}\Phi_0$.

This periodicity could affect the interpretation of the Ξ_i's: $\Xi_i \sim e^{2\pi i}(a_i/a_0)$. Since $a_0 = -1$, the Ξ_i are 'single valued'. In other words, the Ξ_i are coordinates on $\mathbb{C}^3$, and not on some discrete quotient of $\mathbb{C}^3$, thanks to $|a_0|$ being equal to one.

There are many anticanonically embedded hypersurfaces possible. The anticanonically embedded log del Pezzo surfaces and log Fano threefold hypersurfaces in weighted projective spaces are exhaustively collected in [34] and [35] respectively. From the point of string theory, and more particularly, the T-duality discussed in this section, most of these seem not to have an apparent elegant interpretation in string theory.

Some Generalized Conifolds

Beside the anticanonically embedded hypersurfaces, there many hypersurfaces that are not anticanonically embedded, but do have an interesting interpretation, from the perspective of string theory and T-duality. Instead of discussing the widest possible kinds of classes, let us focus on the 'generalized conifolds', or actually, as subset of these.

In a generalization of the usual conifold, which can be regarded as the affine hypersurface

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \tag{4.108}$$

in $\mathbb{C}^4$, consider hypersurfaces in cyclic quotients of $\mathbb{C}^4$ or $\mathbb{C}^5$, which will be specified shortly, of the form

$$F(x_1, \ldots, x_{n+2}) = \sum_{i=1}^{n} x_i^{m_i} + x_{n+1}^2 + x_{n+2}^2, \tag{4.109}$$

taking $n = 2$ or $n = 3$. Call the corresponding surfaces $F^{-1}(0)$ generalized conifolds.

Hypersurfaces with two pure squares in the defining polynomial, like in (4.109) are never anticanonically embedded (except for the A_k surface singularities, which are defined in $\mathbb{C}^{n+2} \simeq \mathbb{C}^3$, here hypersurfaces for which $n \geq 2$ are considered). For the sake of the future interpretation of the model, restrict to the subclass of generalized conifolds with (integer) exponents $m_i \geq 2$, such that

$$\sum_{i=1}^{n} \frac{1}{m_i} = \frac{1}{m}, \tag{4.110}$$

where m is a positive integer

$$m \in \{1, 2, 3, \ldots\}. \tag{4.111}$$

Note that m may also be equal to one, unlike the m_i.

There is a 'Gepner model' sort of interpretation of this condition on the exponents. Although it may seem an observation very much disconnected from the present context, notice that a superpotential

$$W = X_0^{-m} + \sum_{i=1}^{n} X_i^{m_i} + X_{n+1}^2 + X_{n+2}^2 \tag{4.112}$$

'defines' a conformal field theory, quite analogous to the sort discussed in the seminal paper [19] of Ooguri and Vafa on T-duality of the fivebrane, and ADE surface singularities, which has a central charge

$$\hat{c} = 1 + \frac{2}{m} + \sum_{i=i} n \left(1 - \frac{2}{m_i}\right) = n, \tag{4.113}$$

using the familiar formula for the central charge of a Landau-Ginzburg model, and not worrying about the negative weight[11].

The characterization above, in terms of exponents rather than weights, a_i, is quite convenient. However, perhaps it obscures some aspects discussed earlier in terms of the a_i. Recall that from the definition (4.96) of a_0 it does not follow that this weight is necessarily a divisor of d, the weighted degree of F. Since the degree of homogeneity of x_0^m is $d = -ma_0$, integrality of m just says that $|a_0|$ is indeed a divisor of d. However, the embeddings of the generalized conifolds not being anticanonical, $|a_0|$ properly divides d, $|a_0| \geq 2$.

For concreteness, consider some particular example of a generalized conifold. First take

$$F(x_1, x_2, x_3, x_4) = x_1^{2m} + x_2^{2m} + x_3^2 + x_4^2. \tag{4.114}$$

The charge assignments in the linear sigma model, for this model are

$$[a_{-1}, a_0, a_1, a_2, a_3, a_4] = [-2m, -2, 1, 1, m, m]. \tag{4.115}$$

The F-term reads

$$L_F = \int d^2\theta \, \Phi_{-1} \left(\mu\Phi_0^{-m} + F\left(\Phi_1, \Phi_2, \Phi_3, \Phi_4\right)\right) \tag{4.116}$$

and the linear sigma model, for large positive Fayet-Iliopoulos parameter, flows to a non-linear sigma model on $F^{-1}(-\mu)$ in $\mathbb{C}^4/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on the coordinates of the covering $\mathbb{C}^4$ as

$$(\xi_1, \xi_2, \xi_3, \xi_4) \sim \left(-\xi_1, -\xi_2, (-1)^m \xi_3, (-1)^m \xi_4\right). \tag{4.117}$$

Similarly, one can consider the generalized conifold defined as $F^{-1}(-\mu)$ in $\mathbb{C}^4/\mathbb{Z}_3$, with

$$F(x_1, x_2, x_3, x_4) = x_1^{6m} + x_2^{3m} + x_3^2 + x_4^2. \tag{4.118}$$

Or, $F^{-1}(\mu)$ in $\mathbb{C}^5/\mathbb{Z}_6$ defined by a polynomial like

$$F(x_1, x_2, x_3, x_4, x_5) = x_1^{12m} + x_2^{6m} + x_3^{4m} + x_4^2 + x_5^2. \tag{4.119}$$

[11] At least, not worrying more than in [19]. Very loosely speaking, one can think of the negative weight term as a $SL(2;\mathbb{R})$ Kazama-Suzuki model, by analogy with the sound $SU(2)/U(1)$ (minimal model) interpretation of X^{m_i} terms, like Ooguri and Vafa observed (also see [91]). On the other hand, remembering the interpretation of $\Phi_0 = e^{\Psi}$, where Ψ is a 'shift-gauged' field, one can argue for the interpretation of this negative weight term, as a $\hat{c} = 1 + 2/m$ cft, through the reasoning of Hori and Kapustin [26]. In that interpretation, the negative weight term in the superpotential indicates a Liouville theory factor, while in the linear sigma model which describes the hypersurface, the field Φ_0 of negative gauge charge is part of a $SL(2;\mathbb{R})$ Kazama-Suzuki model, in the infrared.

Surfaces like these above have quite interesting duals, as will be discussed shortly. First note that, unlike the ADE hyper-Kähler surfaces, these higher dimensional varieties admit deformations by terms of the same weighted degree. That is to say, one can add monomials in the x_i's which leave the polynomial weighted homogeneous, but not all such terms can be gotten rid of by redefinition of the variables x_i. Such deformations of the a polynomial in the F-term of the linear sigma model, correspond to marginal deformations in the conformal infrared non-linear sigma model.

In the following exposition, regarding T-duality of the models, one should keep in mind such marginal deformations. Rather than performing a duality relating two precise cft's, the dual models will be related up to marginal deformations. That is to say, the 'dual' models describe string backgrounds in the same moduli space. This 'imprecision' is large a consequence of the unfortunately too poorly understood cft isomorphisms, which underlie the proposed duality relation.

DUALIZATION 1: QUANTUM EFFECTS

The (quotients of) weighted homogeneous affine cones, as discussed above, all have a characteristic $U(1)$ action, which degenerates at the apex (the singularity). Consequently, one may wonder if a corresponding T-dual description can be found, and if it has a reasonable geometric interpretation. For one, it is expected that worldsheet instantons play a crucial rôle in the dualization process, since the $U(1)$ action has a fixed point.

The characteristic $U(1)$ action of a hypersurface can be effected in the linear sigma model by a phase rotation of Φ_0,

$$\begin{aligned} W &= \Phi_{-1}\left(\mu\Phi_0^{d/a_0} + F\left(\Phi_1,\ldots,\Phi_{n+2}\right)\right) \\ &= \Phi_{-1}e^{d\Psi/a_0}\left(\mu + F\left(\Phi_1 e^{a_1\Psi/a_0},\ldots,\Phi_{n+2}e^{a_{n+2}\Psi/a_0}\right)\right), \end{aligned} \tag{4.120}$$

or, thinking of the 'shift-gauged' field Ψ, the characteristic action is achieved by simply shifting the imaginary part of Ψ. So it is natural to think to dualize shifts of Ψ in order to get a model which describes the background dualized along the characteristic $U(1)$ action.

In the 'sigma model phase' $\langle\Phi_{-1}\rangle = 0$, and it is not clear if or how the field Φ_{-1} should be involved in the duality operation. A duality operation similar to the ones discussed so far (introducing an auxiliary gauge field and an 'overarching' model and integrating out the auxiliary gauge field) would get rid of one, or perhaps more, chiral superfields and introduce twisted chirals instead. So the dual model would have a formulation involving a combination of both chiral and twisted chiral superfields, unlike the Hori-Vafa mirror symmetry dualizations [53].

In addition the (twisted) F-terms complicate matters. It is not at all obvious how the various chirals and the twisted chirals should be coupled in the dual model. In any case, this coupling would need to be consistent with $\mathcal{N} = 2$ supersymmetry. Note that in (4.99b) all the (chiral) fields are coupled to each other, although the $\Phi_1, \ldots, \Phi_{n+2}$ couple to Ψ only through Φ_{-1}.

If one were to ignore the superpotential, the dualization of Ψ is quite straightforward. As a 'shift-gauged' field (4.72), its kinetic term is replaced by one for a twisted chiral, like $\int d^4\theta - |Y|^2$, and there is a contribution to the twisted F-term, of the form $\int d^2\tilde{\theta} - Y\Sigma$, but there is no e^{-Y} term generated, as there are no vortex configurations for a 'shift-gauged' field. Taking into account the superpotential is known to be quite subtle, also in the context of 2d mirror symmetry [53] of compact or non-compact manifolds.

Clearly, it is totally incorrect to simply replace Ψ by a twisted chiral, since a direct coupling to the chiral field Φ_{-1} would be inconsistent with supersymmetry. On the other hand, one might imagine that Φ_{-1} might need to be dualized as well, yielding another twisted chiral which could be coupled to the dual of Ψ in a simple fashion. However, in that case the question presents itself how the dual of Φ_{-1} would couple to the various Φ_i.

If not along the lines of gauging Ψ and integrating out the auxiliary gauge field, how else to obtain a dual? Recall the recurring philosophy followed in the dualization procedure of backgrounds with a degenerating circle isometry. As a first step, a classical duality gives a 'smeared' dual background. This 'smeared' background has an isometry, which the exact T-dual should not have. In a second step, one gets the 'full' dual background by including the nonperturbative quantum effects, the worldsheet instantons, which break the 'unwanted' classical symmetry.

In an inversion of the order of these steps of the philosophy, could one alternatively first take into account the non-perturbative effects, in terms of some 'half-dual' model, and in a second step, get the 'full' T-dual model from a more manageable classical duality? In fact, the claim here is that this is indeed possible, and that the 'half-dual' model has a conjectured simple description in terms of the 'linear sigma model' (4.99). It is conjectured, that the non-perturbative effects of the duality give a non-zero expectation value to ϕ_{-1}.

The non-zero expectation value of ϕ_{-1} is expected to arise due to worldsheet instantons which contribute crucially in the T-duality, taken along a degenerating cycle in the cone. In a non-linear sigma model, the rôle of worldsheet instantons is conceptually clear: there are explicit field configurations in the non-linear sigma model which are interpreted as strings embedded in the target space in such a way that they are wound around the T-duality circle.

In the present model, as remarked, the situation is more subtle. Some intuition can be gained from the analogous situation which occurs with 2d mirror symmetry [53]. Note that we are at this point not discussing 'our' model, but mirror symmetry. In this case, several T-dualities are performed at once, and each T-duality corresponds to integrating out an auxiliary gauge field. The rôle of the non-linear sigma model worldsheet instantons is taken over in this case by vortex configurations of the auxiliary gauge field. The effect of these vortex contributions, is that effectively the Fayet-Iliopoulos parameter is 'shifted'. It is shifted in in the following way. First of all, the dual (twisted chiral) field couples to the $U(1)$ GLSM gauge field as a dynamical Fayet-Iliopoulos parameter. Second, in the mirror symmetry applications, a twisted superpotential is generated for this dual field. A twisted superpotential can give an expectation value to a field, in the infrared limit. If the twisted chiral field gets an expectation value due to the twisted superpotential, then effectively the Fayet-Iliopoulos parameter is shifted, because of the Fayet-Iliopoulos-like coupling of the twisted chiral to the gauge field. The shift of FI-parameter, in turn has a consequence for the expectation values of all the fields, because the scalar potential is changed.

Now consider our T-duality. In our ultraviolet gauge theory, there is a superpotential present from the outset, but no 'dynamical Fayet-Iliopoulos' coupling between the matter fields and the gauge field. When we are to dualize shifts of Ψ (if Ψ is shifted, the gauge invariant combinations $e^{a_i \Psi/|a_0|}\Phi_i$ transform in such a way that F is rotated by a phase factor, as it is expected it should), we expect that a dual field will couple as a dynamical FI-parameter to the gauge field. But from the outset, Ψ has a superpotential $e^{-d\Psi/|a_0|}$. So it is natural to expect that the dynamical FI-parameter acquires an expectation value, as a consequence of this potential. This means the effective FI-parameter is shifted or, in the language of the original ultraviolet theory, that effectively Ψ is shifted. This in turn is seen as a resulting expectation value of ϕ_{-1}, which minimizes the scalar potential part U_D as in (3.60). At this point Ψ becomes somewhat of an awkward field, as the potential pushes it out to infinity. But this is not too strange; the exponential potential defines a Liouville theory.

In terms of the formulation of the linear sigma model in terms of Φ_0, this change amounts to a drastic change of the Fayet-Iliopoulos parameter, from $r \gg 0$, to $r = -\infty$, which could qualitatively be regarded as a change of a Kähler modulus, in the non-linear sigma model, albeit a very severe change. From the point of view of Ψ, with kinetic term $\frac{d}{2a_0^2}\mathrm{d}\psi^2$ for its scalar component, it also seems like shift infinitely far away in moduli space.

Unfortunately, a clear understanding of this shift is lacking. However, loosely speaking, it is the exponential interaction of Ψ that pushes out ψ all the way to infinity, when $\langle\phi_{-1}\rangle \neq 0$, at very low energies. But in the infrared, Ψ is somewhat of an awkward variable to characterize the theory, which is actually $\mathcal{N} = 2$ Liouville theory. One can think of this theory, at large values of ψ as a sigma model on $\mathbb{R}_\phi \times S^1$, where there is a background charge for the $\mathbb{R}_\phi$ scalar. Clearly, such 'half-dual' backgrounds look nothing like the cone one started out with. In fact, as an orbifold of a product (Liouville) $\times$ (Landau-Ginzburg), it is not at all clear if a geometric description characterization of this worldsheet cft exists at all. Yet, in order to perform the usual dualization procedure, albeit only classically, one should have a sigma model interpretation.

SOME LITERATURE

Before discussing the possibility of such sigma model interpretations, and the second half of the dualization procedure, note that a considerable amount of quite related and very interesting literature exists, which connects conformal field theories such as the above, consisting of a Liouville factor and a Landau-Ginzburg factor, to geometric singularities.

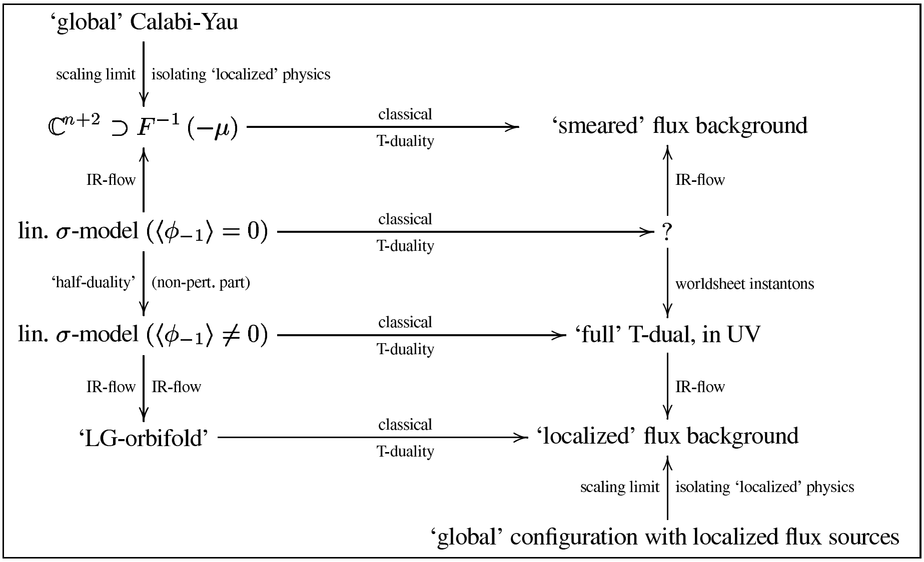


Figure 4.2: Diagram of duality relations. The pair of lines in the middle of the figure concern non-conformal (ultraviolet) models. The 'full' T-duality is in this section regarded for the 'localized' physics, like for example, the CHS fivebrane throat and A_k singularities. The 'localized' physics is isolated (by a scaling limit) from a 'global' background, such as for example a stack of fivebranes or Taub-NUT. The exactly T-dual global backgrounds appear in the top left and bottom right corners of the diagram.

A selection of salient literature is presented in the following paragraphs. There is no pretense that this selection is representative of all related and important work, but it is hoped that the following selection will provide the reader with an idea of other work in three interrelated topics. Firstly, there is work on relation between singularities and Landau-Ginzburg models. Secondly, important work exists on linear dilaton backgrounds and Little String Theory. Thirdly, such 'flux impurity' throat backgrounds are related to AdS backgrounds.

First, a major inspiration and, as far as the author is aware, the first work discussing a connection between ADE surface singularities, and deformations thereof is the paper of Ooguri and Vafa [19]. In this work Landau-Ginzburg orbifold models are taken, with superpotentials of the form

$$W_{ADE} = \mu x_0^{-d} + F_{ADE}(x_1, x_2, x_3), \tag{4.121}$$

which are proposed to describe deformed ADE surface singularities, motivated by the usual Calabi-Yau-Landau-Ginzburg correspondence [92], without worrying about the negative gauge charge of x_0. For hypersurfaces in projective spaces, the CY/LG-correspondence was put on a firm footing by Witten through the interpolating linear sigma model [18]. As discussed in section 3.3.2, the connection is found to be more involved for the affine hypersurfaces, like also the ADE surface singularities

Ooguri and Vafa take the Landau-Ginzburg orbifold as a starting point of a description of the ADE surface singularities and interpret the x_0 part as a $SL(2;\mathbb{R})/U(1)$ Kazama-Suzuki model, the 'cigar', which indeed has a $U(1)$ isometry, considering it an 'analytic continuation' of $SU(2)/U(1)$ minimal models, for their purposes. The rest of the Landau-Ginzburg superpotential defines a minimal model with a corresponding ADE $SU(2)$ modular invariant [17]. In the 'scaling limit' that isolates the local physics at a singularity, one of the parameters scaled is $\mu \to 0$. For the 'cigar' coset model, this means that the 'tip' moves ever further into the large g_s region, and a target part looks like the 'dilatonic cylinder' $\mathbb{R}_\phi \times S^1$. Taking $\mathbb{R}_\phi \times U(1)$, which has a decoupled $U(1)$, instead of $SL(2;\mathbb{R})/U(1)$, and studying carefully the partition function of the orbifold $[U(1) \times (SU(2)/U(1))_\Gamma]/\Gamma$, Ooguri and Vafa find the partition function for a $SU(2)_\Gamma$ conformal field theory. In addition to this $SU(2)_\Gamma$, the complete background also has the remaining scalar with a background charge. So the total cft, in the scaling limit, so $\mu \to 0$, has the partition function of $\mathbb{R}_\phi \times SU(2)$. That is, they are identical as conformal field theories and hence the string backgrounds can be related by T-duality. For the A-type singularities, this is nothing but the CHS throat cft of a stack of coincident fivebranes [15].

Although the work of Ooguri and Vafa is a fundamental paper, some important earlier related work, considering cft partition functions, is the earlier [91], and also [93], regarding the rôle of $SL(2)/U(1)$ in the description of the conifold singularity which, like the A_1 surface singularity, has no non-trivial cft factor coming from the polynomial F, which is simply quadric.

In a spirit like Ooguri and Vafa, considering cft partition functions, is the work of Eguchi, Sugawara and others, such as [94] and later works, which show that certain

$$[SL(2;\mathbb{R}) \times \text{Landau-Ginzburg}]\,/\Gamma$$

conformal field theories have partition functions consistent with spacetime supersymmetry (basically, they are modular invariant). A very interesting paper in particular [95], considers partition functions of models consisting of an $\mathcal{N}=1$ Liouville model and a $\mathcal{N}=1$ G/H coset models. These models are precisely cases of 'flux impurities' which admit a geometric (gauged WZW) interpretation. The G/H coset models considered are are such that the homogeneous spaces G/H lead to metric cones of special holonomy, just as in our case. Even coset conformal field theories are discussed based on current algebras G and H, such that the coset manifold G/H is a homogeneous nearly Kähler or weak G_2 manifold (leading to metric cones of G_2 and Spin(7) holonomy respectively). Our methods, using $\mathcal{N}=(2,2)$ worldsheet models, are not adept to treat such cases.

In [95], the relation with metric cones is most definitely observed, and it is a central point in that work. The spacetime supersymmetry of the 'flux impurity' cft's is found to agree with the expectation of a cone special holonomy. However, a clear connection with the 'geometric impurities', is not made. We believe that our T-duality relation, making use of the hypersurface description and an overarching ultraviolet theory provides a complementary picture to [95], as it allows to relate a geometric (hypersurface) impurity to a 'half-dualized' model. However [95] is very important, in exposing the isomorphy of the 'half dualized' models ($\mathcal{N}=2$ Liouville times a Kazama-Suzuki coset) and the true 'flux impurity' (linear dilaton times G/H coset), much in the spirit of the formal partition function considerations of Ooguri and Vafa.

The work discussed above is essentially concerned with a study of partition functions of the conformal field theories, and show equivalences between $SL(2;\mathbb{R})$ (or $\mathcal{N}=2$ Liouville) times one coset cft on one side, and a linear dilaton times another coset (essentially with a $U(1)$ factor in the 'denominator' deleted) on the other side. In some other very important work, a connection is made between supersymmetric singularities and Landau-Ginzburg models in considerable generality. Very important in this respect is the work on linear dilaton backgrounds as holographic duals to Little String Theories by Giveon, Kutasov, Seiberg and others, such as [23, 24] and [3, 4].

Perhaps the most important inspiration to consider T-duality for hypersurfaces, is the paper of Giveon, Kutasov and Pelc [22]. This proposes a general connection between hypersurface singularities and the 'half-dualized' models $\mathbb{R}_\phi \times$U(1)$\times$ Landau-Ginzburg, identifying the Landau-Ginzburg superpotential with the defining polynomial of the hypersurface. Also, hypersurfaces are discussed with a defining polynomial of a the following particular form

$$F(x_1, x_2, \ldots, x_{3+2}) = H(x_1, x_2, x_3) + x_4^2 + x_5^2.$$

The affine hypersurfaces $F^{-1}(0)$ are argued to be T-dual descriptions of a fivebrane with worldvolume $\mathbb{R}^{1,1} \times L$, where $L = H^{-1}(0) \subset \mathbb{C}^3$. The T-duality which achieves this is done fiberwise along $(e^{i\phi}u, e^{-i\phi}v)$, where $uv = x_4^2 + x_5^2$. This is not the $U(1)$ action which we consider. We consider the generic weighted homogeneous action on any weighted homogeneous polynomial. We find that our model describes not precisely affine hypersurface singularities of the form above, but discrete quotients of these, essentially because the weights and weighted degree of are such, that F does not define an anticanonically embedded hypersurface in weighted projective space. Non-compact Calabi-Yau varieties of the form $uv + H(x, y) = 0$ are also interesting from the point of view of topological string theory, see, for example [107, 108].

Finally, there is a considerable amount of very interesting work on worldsheet conformal field theories describing fivebrane backgrounds, which often have a $g_s \to \infty$ region, and deformations which keep g_s finite, such as the 'fivebrane ring' in (4.22) discussed in [25] by Sfetsos, and many other papers, mainly by Sfetsos, Kounnas, Kiritsis and others, such as [5] [96]. Some deformations involve separating the fivebranes, like the 'ring' geometry, while another possibility is to add fundamental strings, to keep the dilaton finite near the fivebranes, see, i.a. [97]. The effect of the fundamental strings is quite drastic. Not only is the dilaton made constant, rather than linearly growing down a throat, as a consequence, the decoupling limit is fundamentally changed. No longer is it required to take $g_s \to 0$, as usual for Little String Theories, but rather, there is a Maldacena type of decoupling limit, which yields $\mathrm{AdS}_3 \times \mathcal{N}$ backgrounds, rather than a throat-like (linear dilaton) background. A central paper, in this respect regarding the requirements on $\mathcal{N}$ to yield a supersymmetric background is [98] and also [101], in addition there are important papers by Elitzur, Giveon, Kutasov, Seiberg and others. Some interesting papers discussing explicit $\mathrm{AdS}_3 \times G/H$ backgrounds are [99, 100]. Deformations of the linear dilaton 'near flux impurity' backgrounds can thus lead to interesting related Anti-de Sitter backgrounds. It would be interesting to study the Anti-de Sitter backgrounds, and the dual conformal field theories in particular, obtained by deforming the 'near flux impurity' backgrounds which we obtain. These can be considered T-duals of geometric impurities with fundamental strings.

DUALIZATION 2: GEOMETRIC INTERPRETATION

Let us continue with the dualization procedure, proceeding from the 'half-dualized' models as in figure 4.2. The Liouville part of the model has a Lagrangian

$$\begin{aligned} L = &\int \mathrm{d}^4\theta \frac{|a_0|}{2d} |\widetilde{\Psi}|^2 \\ &+ \int \mathrm{d}^2\theta \mu e^{-\widetilde{\Psi}} + \text{c.c.} \end{aligned} \tag{4.122}$$

The central charge of the Liouville theory is

$$\hat{c}_{\text{Liouville}} = 1 + \frac{2}{d/|a_0|}. \tag{4.123}$$

In the region of large Re $\tilde{\psi}$, it has a target space interpretation as a 'dilatonic cylinder' $\mathbb{R}_\phi \times S^1$. The radius of the circle is quantized in units of $\sqrt{d/|a_0|}$ and determined by the periodicity of Im $\tilde{\psi}$. Actually, the periodicity of $\widetilde{\Psi}$ is $2\pi d/|a_0|$ (or an integer multiple thereof, set by the overarching 'linear sigma model' as discussed in section 3.3.2.

Now consider the Landau-Ginzburg part, with superpotential $W = F$. For a general weighted homogeneous F that describes an affine Calabi-Yau hypersurface, i.e. satisfying (4.94), the Landau-Ginzburg model with $W = F$ has no known geometric interpretation. However, for some special polynomials F it does. That is to say, some special weighted homogeneous polynomials describe (marginal deformations of) certain Kazama-Suzuki models. In particular, the Kazama-Suzuki models based on Hermitean symmetric spaces at level one have a Landau-Ginzburg formulation [21], see section 3.4. Also, some Kazama-Suzuki models at levels $k > 1$ can be related to level one Kazama-Suzuki models, utilizing the (conjectured) isomorphisms of coset models discussed in section 3.4. These Kazama-Suzuki models have a sigma model interpretation, as gauged WZW models, and they have a distinguished $U(1)$ symmetry. This symmetry is the axial action the $U(1)$ of which the vector action is gauged in

$$\frac{G}{H \times U(1)_v}.$$

The fermions in the Kazama-Suzuki models are essentially decoupled from the bosons, the fermions realize a $SO\left(\text{dim}G/(H \times U(1))\right)_1$ current algebra and the bosons realize an ordinary bosonic coset model. The dualization can be considered simply on the bosonic part

$$\left[\frac{G}{H \times U(1)} \times U(1)\right]/\Gamma \simeq \frac{G}{H}, \tag{4.124}$$

which is a generalization of the familiar duality

$$\left[SU(2)_\Gamma/U(1) \times U(1)\right]/\Gamma \sim SU(2)_\Gamma, \tag{4.125}$$

which for the A-type modular invariants has an explicit interpretation as T-duality, using a sigma model realization [16]. For general ADE modular invariants, this identity can be obtained from the consideration of partition functions [19].

EXAMPLES

In order to get a feeling for the duality, consider some specific examples, using the Landau-Ginzburg/Kazama-Suzuki equivalences of section 3.4.

A-TYPE SURFACE SINGULARITIES

An A_{k+1} surface singularity can be viewed as a metric cone on the link

$$S^1 \to S^3/\mathbb{Z}_{k+2} \to SU(2)/U(1).$$

The polynomial which defines such a singularity as a hypersurface in $\mathbb{C}^3$, see table 2.2, has such weights and weighted degree that $a_0 = -1$.

Chiral superfield Φ_i	$U(1)$ charge a_i
Φ_{-1}	$a_{-1} = -(k+2)\alpha$
Φ_0	$a_0 = -\alpha$
Φ_1	$a_1 = \alpha$
$\Phi_{2,3}$	$a_{2,3} = \frac{k+2}{2}\alpha$

Table 4.1: Charge assignments for A_{k+1} singularity, $\alpha = 1$ ($\alpha = 2$) if k is even (odd).

For $S^3/\mathbb{Z}_{k+2}$ the polynomial is

$$F_{A_{k+1}} = x_1^{k+2} + x_2^2 + x_3^2. \tag{4.126}$$

The overarching model can be characterized, roughly speaking, as a $U(1)$ 'linear sigma model' with chiral superfields with charges as in table 4.1, but strictly speaking, the chiral superfield Φ_0 should be regarded as a 'composite' field, e^{Ψ}, where Ψ appears in the Lagrangian as a 'shift-gauged' field and $\Psi \sim \Psi + 2\pi i$

The Lagrangian reads

$$\begin{aligned} L = & \int \mathrm{d}^4\theta \left[\frac{d}{4}\left(\Psi + \overline{\Psi} + V\right)^2 + |\Phi_{-1}|^2 e^{-dV} \right. \\ & \left. + |\Phi_1|^2 e^V + |\Phi_2|^2 e^{dV/2} + |\Phi_3|^2 e^{dV/2} - \frac{1}{2e^2}|\Sigma|^2\right] \\ & + \int \mathrm{d}^2\theta\, \Phi_{-1}\left(\mu e^{-d\Psi} + \Phi_1^d + \Phi_2^2 + \Phi_3^3\right) + \text{c.c.}, \end{aligned} \tag{4.127}$$

where $d = \alpha(k+2)$. The 'half-dualized' model is a $\mathbb{Z}_d$ orbifold of a product cft. One factor of the product is the Landau-Ginzburg model with the superpotential

$$W_1 = \Phi_1^d, \tag{4.128}$$

and the other factor is a Liouville model, which is the IR limit of the theory with the following Lagrangian

$$L_{\text{Liouville}} = \int \mathrm{d}^4\theta\, \frac{1}{2d}|\widetilde{\Psi}| + \int d^2\, \mu e^{-\widetilde{\Psi}} + \text{c.c.}, \tag{4.129}$$

where the periodicity of $\widetilde{\Psi}$ is $2\pi i d$. The central charge of the Landau-Ginzburg model is $\hat{c}_1 = 1 - \frac{2}{d}$ and the central charge of the Liouville model is $\hat{c}_{\text{Liouv.}} = 1 + \frac{2}{d}$.

As $\mu \to 0$, the region where the Liouville potential is weak encompasses a larger part of negative Re $\tilde{\psi}$ values. Where the Liouville potential is small, there is a target space interpretation as a dilatonic cylinder. Writing $\psi = p + idq$ the metric on the dilatonic cylinder is

$$\mathrm{d}s^2_{\text{cyl}} = \frac{1}{d}\mathrm{d}p^2 + d\mathrm{d}q^2. \tag{4.130}$$

The Landau-Ginzburg factor can be viewed as a $SU(2)/U(1)$ coset model, with metric

$$\mathrm{d}s^2_{\mathrm{KS}} = d\left(\mathrm{d}\chi^2 + \tan^2\chi \mathrm{d}\phi^2\right). \tag{4.131}$$

The coordinates q and ϕ are periodic with periodicity 2π, while $0 \leq \chi < \pi/2$ The $\mathbb{Z}_d$ orbifold acts along the integral curves of $\frac{\partial}{\partial q} + \frac{\partial}{\partial \phi}$.

The T-dual geometry is obtained by applying the Buscher rules to the above geometry, dualizing $\frac{1}{d}\left(\frac{\partial}{\partial q} + \frac{\partial}{\partial \phi}\right)$. Concretely, gauging translations along this circle using a Lagrange multiplier λ, and gauge fixing $q = 0$ this yields

$$\begin{aligned} \mathrm{d}s^2 =& d\left(\mathrm{d}p^2 + \mathrm{d}\chi^2 + \sin^2\chi \mathrm{d}\phi^2\right) + \frac{\cos^2\chi}{d}\mathrm{d}\lambda^2 \\ B =& \cos^2\chi \, \mathrm{d}\lambda \wedge \mathrm{d}\phi, \end{aligned} \tag{4.132}$$

where $\lambda \sim \lambda + 2\pi/d$ is the dual coordinate. There is also a background charge for p. Redefining $\lambda = \theta/d$, the above metric and B-field look exactly like the throat geometry, χ, ϕ and θ are coordinates on S^3, with d units of flux through it. This is in agreement with [19].

Note that the level of the $SU(2)_k$ WZW model, which is interpreted as the number of fivebranes down the throat, corresponds to the first Chern class of the bundle $S^1 \to S^3/\mathbb{Z}_{k+2} \to SU(2)/U(1)$. Curiously, for the A_1 singularity the half-dualized model has a Landau-Ginzburg part with superpotential $W_1 = \Phi_1^2$, which defines a trivial model with $\hat{c} = 0$, containing only the vacuum state. So the complete information of the A_1 singularity is contained in the Liouville factor.

Generalized Conifolds

Consider the conifold, which is a metric cone over $T^{1,1}$. The homogeneous Sasaki-Einstein manifold

$$T^{1,1} \simeq \frac{SU(2)\times SU(2)}{U(1)}$$

can be regarded as a circle bundle over

$$\frac{SU(2)}{U(1)} \times \frac{SU(2)}{U(1)},$$

as discussed in chapter 2, and it is one of the few regular Sasaki-Einstein manifolds of dimension five. It is also defined as a hypersurface in $\mathbb{C}^4$,

$$z_1 z_2 = z_3 z_4. \tag{4.133}$$

The conifold can be quotiented by the $\mathbb{Z}_n$ action

$$\begin{aligned} z_3 \sim& e^{2\pi i/n} z_3 \\ z_4 \sim& e^{-2\pi i/n} z_4. \end{aligned} \tag{4.134}$$

Define the $\mathbb{Z}_n$-invariant combinations

$$\begin{aligned} y_1 =& z_3^n, \\ y_2 =& z_4^n, \\ t =& z_3 z_4. \end{aligned} \tag{4.135}$$

The quotiented conifold is then described by the pair of equations $y_1 y_2 = t^k$ and $z_1 z_2 = t$, or by the single relation

$$y_1 y_2 = (z_1 z_2)^n. \tag{4.136}$$

So the weighted homogeneous hypersurface in $\mathbb{C}^4$ defined by $\widetilde{F} = 0$ with

$$\widetilde{F}(x_1, x_2, x_3, x_4) = (x_1 x_2)^n + x_3^2 + x_4^2 \tag{4.137}$$

is a $\mathbb{Z}_n$ quotient of the conifold. Unlike the defining polynomials of the ADE surface singularities, polynomials such as $\widetilde{F}$ admit 'marginal deformations'. That is to say, there are monomial terms δF of the same weighted degree which one can subtract from $\widetilde{F}$, such that these subtractions cannot be undone by a change of coordinates $x_i \to \tilde{x}_i(x_j)$ that respects the weights of the coordinates.

One particular 'marginal deformation' of $\widetilde{F}$ is F,

$$F(x_1, x_2, x_3, x_4) = x_1^{2n} + x_2^{2n} + x_3^2 + x_4^2. \tag{4.138}$$

This is a very interesting equation, for our purposes, though actually, not as an equation in $\mathbb{C}^4$, but as an equation in $\mathbb{C}^4/\mathbb{Z}_2$. The weights and the weighted degree of F are such, that $a_0 = -2$. Therefore, there is a $U(1)$ gauge theory, of the sort discussed earlier, that in the infrared flows to a non-linear sigma model on $F^{-1}(-\mu)$ in $\mathbb{C}^4/\mathbb{Z}_2$. The group $\mathbb{Z}_2$ acts on the coordinates x_i of $\mathbb{C}^4$ as

$$\begin{aligned} x_{1,2} \sim& (-1) x_{1,2}, \\ x_{3,4} \sim& (-1)^n x_{3,4}. \end{aligned} \tag{4.139}$$

The Lagrangian of this gauge theory reads

$$\begin{aligned} L = \int \mathrm{d}^4\theta \left[\frac{n}{4} \left(\Psi + \overline{\Psi} + V \right)^2 + |\Phi_{-1}|^2 e^{-2nV} + \sum_{i=1,2} |\Phi_i|^2 e^V + \sum_{i=3,4} |\Phi_i|^2 e^{nV} \right] \\ + \int \mathrm{d}^2\theta \Phi_{-1} \left(e^{-n\Psi} + F\left(\Phi_1, \Phi_2, \Phi_3, \Phi_4\right) \right). \end{aligned} \tag{4.140}$$

And the periodicity of the chiral superfield Ψ is $\Psi \sim \Psi + 2\pi i$.

The hypersurface admits a $U(1)$ action, as it is weighted homogeneous. The 'half-dual' model is a $\mathbb{Z}_{2n}$ orbifold of a product of three separate cft's, $\mathcal{L} \otimes \mathcal{W} \otimes \mathcal{W}$, where $\mathcal{W}$ is a

Landau-Ginzburg model with superpotential $W = \Phi^2$, i.e. a $SU(2)/U(1)$ model, and $\mathcal{L}$ is a Liouville model which is the IR fixed point of the model with a Lagrangian

$$L = \int \mathrm{d}^4\theta \, \frac{1}{2n}|\widetilde{\Psi}|^2 + \int \mathrm{d}^2\theta \, e^{-\widetilde{\Psi}} + \text{c.c} \tag{4.141}$$

and $\widetilde{\Psi} \sim \widetilde{\Psi} + 2\pi i n$. In the region where the real part of $\tilde{\psi}$ is large, the target space of the Liouville model looks like a dilatonic cylinder. In the scaling limit corresponding to the generalized conifold singularity, $\mu \to 0$, the Liouville potential is small for a larger portion of values of Re $\tilde{\psi}$.

The metric on the cylinder looks like

$$\mathrm{d}s^2_{\text{cyl}} = n\left(\mathrm{d}p^2 + \mathrm{d}q^2\right), \tag{4.142}$$

where $p \in \mathbb{R}$ is a scalar with a background charge, corresponding to the real part of $\tilde{\psi}$ and $q \sim q + 2\pi$ is a free periodic scalar. The central charge of the Liouville model (or dilatonic cylinder) is $\hat{c} = 1 + \frac{2}{n}$. The pair of $SU(2)/U(1)$ coset models have metrics

$$\mathrm{d}s^2_{1,2} = 2n\left(\mathrm{d}\chi_i^2 + \tan^2\chi_i \, \mathrm{d}\phi_i^2\right). \tag{4.143}$$

The central charge of each copy is $\hat{c} = 1 + \frac{2}{2n}$, so the total central charge is $\hat{c} = 3$. And the $\mathbb{Z}_{2n}$ orbifold identifies $(q, \phi_1, \phi_2) \sim (q - 2\pi i/n, \phi_1 + \pi i/n, \phi_2 + \pi i/n)$.

Applying the Buscher rules to this geometry, along the 'homogeneous' $U(1)$ direction (same as of the orbifolding), yields a dual geometry that looks like

$$\begin{aligned} \mathrm{d}s^2_{\text{dual}} = n &\left(\mathrm{d}p^2 + 2\sum_{i=1,2}\mathrm{d}\chi_i^2\right) \\ &+ \frac{n}{2 + \sum_{i=1,2}\tan^2\chi_i} \times \\ &\times \left(2\mathrm{d}\lambda^2 + \sum_{i=1,2}\tan^2\chi_i \mathrm{d}\phi_i^2 + \frac{\tan^2\chi_1 \, \tan^2\chi_2}{2}\left[\mathrm{d}\phi_1 - \mathrm{d}\phi_2\right]^2\right), \end{aligned} \tag{4.144}$$

with a B-field

$$B = \frac{n}{2 + \sum_{i=1,2}\tan^2\chi_i}\sum_{i=1,2}\tan^2\chi_i \mathrm{d}\lambda \wedge \mathrm{d}\phi_i. \tag{4.145}$$

The dilaton profile is also somewhat complicated. First of all, there is a linear dilaton in the p direction, already from the Liouville/dilatonic cylinder. But there are also other contributions, from the Buscher rules. In all, the dilaton profile looks like

$$\begin{aligned} \Phi = \Phi_0 &- \text{linear dilaton along } p \\ &- \sum_{i=1,2}\log\cos\chi_i - \frac{1}{2}\log\left(1 + \frac{\tan^2\chi_1}{2} + \frac{\tan^2\chi_2}{2}\right). \end{aligned} \tag{4.146}$$

Other Singularities with a Kazama-Suzuki Interpretation

Some other interesting hypersurfaces are obtained from other weighted homogeneous polynomials that (up to marginal deformations) characterize Landau-Ginzburg/Kazama-Suzuki models.

For example, the Kazama-Suzuki models

$$\frac{SU(3)_k}{SU(2)\times U(1)}$$

have a Landau-Ginzburg realization with a superpotential $W = F$ that has weights and degree such that

$$\begin{aligned} F(x_1,x_2,x_3,x_4) =& H(x_1,x_2)+x_3^2+x_4^2 \\ H(\lambda x_1,\lambda^2 x_2) =& \lambda^{k+3}H(x_1,x_2). \end{aligned} \tag{4.147}$$

The weights of the coordinates are

$$(a_1,a_2,a_3,a_4) = \begin{cases} \left(1,2,\frac{k+3}{2},\frac{k+3}{2}\right) & \text{if } k \text{ is odd} \\ (2,4,k+3,k+3) & \text{if } k \text{ is even} \end{cases} \tag{4.148}$$

The weighted degree of F is $k+3$ ($2k+6$) if k is odd (even), so $a_0 = -a_1 - a_2$

$$a_0 = \begin{cases} -3 \text{ if } k \text{ is odd} \\ -6 \text{ if } k \text{ is even} \end{cases} \tag{4.149}$$

Furthermore, to have a UV model which describes a simple cyclic quotient of a hypersurface, we need $-a_0|d$, so k should be an integer multiple of 3. Define

$$n \equiv -1 + \frac{k}{3}, \tag{4.150}$$

and

$$\alpha_n = \begin{cases} 1 \text{ if } n \text{ is odd} \\ 2 \text{ if } n \text{ is even} \end{cases} \tag{4.151}$$

With these definitions, we consider weighted homogeneous polynomials $F_n\,(x_1,x_2,x_3,x_4)$ of weighted degree $d = 3\alpha_n(n+1)$ and weights

$$(a_1,a_2,a_3,a_4) = \left(\alpha_n, 2\alpha_n, \frac{3\alpha_n(n+1)}{2}, \frac{3\alpha_n(n+1)}{2}\right).$$

then $a_{-1} = -d$ and $a_0 = -3\alpha_n$. There is a UV gauge theory which flows to a sigma model on the hypersurface $F_n^{-1}(-\mu)$ in $\mathbb{C}^4/\mathbb{Z}_{|a_0|}$, analogous to the models discussed in the earlier cases.

The 'half-dual' model is $\mathbb{Z}_{3n+3}$ orbifold of a product cft, containing a Liouville factor and a Landau-Ginzburg factor with $W = F_n$. Weighted homogeneous deformations correspond to Landau-Ginzburg cft's that differ by marginal deformations. In the moduli space of Landau-Ginzburg conformal field theories with weighted homogeneous superpotentials of the form of F_n, there is a particular point where the Landau-Ginzburg model is the $SU(3)_{3n}/\left(SU(2)\times U(1)\right)$ Kazama-Suzuki coset model. To find the precise form requires a detailed analysis, under the assumption that the 'level-rank' isomorphisms of the Kazama-Suzuki models, as discussed in section 3.4 are indeed true.

The Liouville model is the infrared fixed point of a model with Lagrangian

$$L = \int \mathrm{d}^4\theta \, \frac{1}{2(n+1)} |\widetilde{\Psi}|^2 + \int \mathrm{d}^2\theta \mu e^{-\widetilde{\Psi}}, \tag{4.152}$$

where $\widetilde{\Psi}$ is a chiral superfield with periodicity $2\pi i(n+1)$. In the scaling limit when the hypersurface develops its singularity $\mu \to 0$, the Liouville model can be interpreted as a dilatonic cylinder, as before.

When there is a Kazama-Suzuki interpretation of the Landau-Ginzburg factor, the classical T-duality that remains to be done is

$$\left[\frac{SU(3)_{3n}}{SU(2)\times U(1)} \times S^1_{n+1}\right]/\mathbb{Z}_{3n+3} \overset{T}{\simeq} \frac{SU(3)_{3n}}{SU(2)}. \tag{4.153}$$

The Kazama-Suzuki model has a canonical S^1 symmetry, which is the axial action of the $U(1)$ that appears as a vectorially gauged subgroup in the denominator (an interesting closely related duality is discussed in [28]). The T-duality transformation is taken along the combined action of the axial $U(1)$ in the coset and translations along the S^1 factor.

The above duality can be derived through a process similar to that in [28]. In [28] the extra S^1 was not considered, but rather T-duality was derived between the coset models

$$\frac{G_k}{H\times U(1)_v} \overset{T}{\simeq} \frac{G_k}{H\times U(1)_a \times \mathbb{Z}_{k+g^\vee}}. \tag{4.154}$$

Inclusion of the extra circle 'eats' the axially gauged $U(1)$. Compare this to the classic case

$$\frac{SU(2)_k}{U(1)_v} \simeq \frac{SU(2)_k}{U(1)_a \times \mathbb{Z}_k + 2} \tag{4.155}$$

(see, for example [102] for a nice exposition) versus the duality

$$\frac{SU2_k}{U(1)_v} \times U(1) \simeq SU(2)_k/\mathbb{Z}_{k+2}, \tag{4.156}$$

in, for example [19].

One can consider such a dualization for bosonic coset models, as the fermionic part of Kazama-Suzuki is essentially free (although decoupling the fermions has some effects, like shifting the level of the bosonic coset and affecting the order of the orbifold quotient by the dual Coxeter number of G, see [28]). The T-duality is performed by introducing an auxiliary gauge field which gauges the isometry of the T-duality and a Lagrange multiplier term which sets the gauge connection to be flat. In choosing a particular gauge fixing condition, one usually picks up a non-trivial dilaton profile. In the next step, of the T-duality, integrating out the gauge field, again a non-trivial dilaton profile may be generated. In [28], where the Kazama-Suzuki model alone is dualized, without the extra $U(1)$, through a cunning choice of gauge fixing condition, the generation of a dilaton is avoided. A similar tactic can be employed for the present T-duality. But also another choice of gauge fixing should not affect the end result. Indeed, the dilaton generated in the first step is cancelled in the second, thus leading indeed to the duality (4.154). Consequently, similar T-dualities can be considered for other Landau-Ginzburg-Kazama-Suzuki models, such as $SU(4)/SU(3) \times U(1)$ at arbitrary level. Also, the work of Eguchi and Sugawara [95] provides a demonstration of such dualities for many different cosets, including cases not related to Landau-Ginzburg models and cases related to G_2 and Spin(7) singularities.

4.4.1 Interpretation of the Duality

What can be said about the relation between the supersymmetric singularities and their T-dual backgrounds? First of all, the procedure that was discussed relies on the formulation of a singularity as a weighted homogeneous affine hypersurface, or as a quotient of such a hypersurface, depending on the value of a_0. It is reasonable to think of a weighted homogeneous hypersurface singularity as a metric cone, qualitatively speaking. After all, such a hypersurface admits a $\mathbb{C}^*$ action while on a (Calabi-Yau) metric cone the Euler and Reeb vector fields act in an analogous fashion.

However, explicit Sasaki-Einstein metrics on the links of supersymmetric affine hypersurfaces are scarcely known. The exceptions are mainly homogeneous spaces and and the most special cases are the hyper-Kähler surface singularities. Another reason why the ADE surface singularities are very special is, that they have no marginal deformations. This means that the polynomials which define them as hypersurfaces have no weighted homogeneous deformations other than ones which correspond to changes of variables. And in addition the ADE polynomials, see table 2.2, define Landau-Ginzburg models which have a coset interpretation. Other hypersurfaces, of dimension larger than four, either have no Landau-Ginzburg 'half-dual' model which has a coset cft interpretation, or they have $a_0 < -1$, and usually both matters occur at once. Also, they have marginal deformations.

Surface Singularities

The correspondences between the ADE surface singularities and their duals are remarkable. The surfaces can be viewed as metric cones

$$\mathbb{R}\tilde{\times}\frac{SU(2)}{\Gamma}$$

Geometric	Flux
$\mathbb{R}\tilde{\times}\frac{SU(2)}{\Gamma}$	$\mathbb{R}_\phi \times \widehat{SU(2)}_\Gamma$
$SU(2)$ isometry	$\widehat{SU(2)}$ affine symmetry
$\Gamma \subset SU(2)$ quotient by discrete subgroup of isometries	Γ modular invariant based on affine symmetry

Table 4.2: Remarkable correspondences between ADE 'geometric' and 'flux' impurities.

while the duals are conformal field theories

$$\mathbb{R}_\phi \times SU(2)_\Gamma.$$

The isometry of the homogeneous link appears as an affine symmetry in the dual. The possible links are the homogeneous $SU(2)/\Gamma$ and correspond one-to-one to the discrete subgroups Γ of $SU(2)$, which in turn correspond to the modular invariants that can be used to construct each dual $SU(2)$ conformal field theory.

The simplest geometric interpretation exists for the A-type singularities. Their links are circle bundles over $\mathbb{P}^1$, distinguished by an integer, the Chern class. The dual $SU(2)$ conformal field theories are realized as WZW models, which are labeled by one integer, the level. Because the cft is formulated as a sigma model, the target geometry can be interpreted and it is viewed as the throat of a stack of a number of fivebranes.

Put together, the remarkable correspondences are summarized in table 4.2.

KAZAMA-SUZUKI/LANDAU-GINZBURG SINGULARITIES

It is tempting to try and generalize the correspondences in table 4.2 to cones over other homogeneous spaces. Many such cones have no hypersurface description, and it is not at all clear how exactly such a correspondence would look in detail. For example, there is no one-to-one relation between discrete subgroups of $SU(3)$ and $\widehat{SU(3)}$ modular invariants. Some more comments about this will be made later. Here we wish to briefly elaborate on the 'flux impurities' which have a coset cft interpretation.

To be more specific, the proposed duality applies to flux impurities of the form

$$\text{linear dilaton} \times \frac{G}{H}$$

where $G/H \times U(1)$ is a Kazama-Suzuki model with a Landau-Ginzburg interpretation. The $U(1)$ current in G/H has a very general rôle. Any flux background $\mathbb{R}_\phi \times N$ needs to be such that N is a $\mathcal{N} = 1$ superconformal field theory with an affine $U(1)$ current, such that $N/U(1)$ is a $\mathcal{N} = 2$ superconformal theory [22], see also [98]. The Kazama-Suzuki models provide a very explicit realization of this, combined with a geometric (gauged WZW) interpretation of the 'flux impurity'.

The geometric impurities dual to the coset models which have a Landau-Ginzburg realization are cyclic quotients of affine hypersurfaces. Coset models with a LG realization are the coset models based on simply laced Hermitean symmetric spaces (see table 3.2) at level one (sometimes called SLLOHSS) , and those coset models which are related by a level/rank isomorphism to SLLOHSS models, such as notably the Grassmannian Kazama-Suzuki models at any level[12]

$$\frac{SU(m+1)_k}{SU(m)\times U(1)}.$$

The reason that they quotients, and not simply hypersurfaces $F^{-1}(0) \in \mathbb{C}^{n+2}$, is that F is not 'anticanonical', i.e.

$$F(x_1,\ldots,x_{n+2}) = H(x_1,\ldots,x_n) + x_{n+1}^2 + x_{n+2}^2, \tag{4.157}$$

and consequently $a_0 = d - \sum a_i < -1$, which causes the non-linear sigma model target space to be a $\mathbb{Z}_{-a_0}$ quotient of $F^{-1}(0) \subset \mathbb{C}^{n+2}$.

How should we think of this target space? We believe that in the same moduli space as the hypersurface quotient above, there is a particular metric cone with an interesting description, as follows. The weighted homogeneous polynomial F can be deformed by deformations, polynomial in the x_i, which preserve the weighed degree. Such deformations are marginal deformations of the Landau-Ginburg model, and correspond to deforming the geometric impurity by changing moduli. At a particular point in moduli space, where certain cycles have been blown up and others have been blown down, there is, we believe, a geometric impurity with a 'nice' metric cone / Sasaki-Einstein description.

For example, consider the generalized conifolds, defined by $F = x_1^{2m} + x_2^{2m} + x_3^2 + x_4^2$. As discussed earlier, these are marginal deformations of a $\mathbb{Z}_m$ quotient of $x_1^2+x_2^2+x_3^2+x_4^2$ which defines the $\mathbb{Z}_2$ quotient of the conifold (i..e. the metric cone on the regular Sasaki-Einstein manifold $T^{1,1}/\mathbb{Z}_2$). So the Kazama-Suzuki flux impurity

$$\mathbb{R}_\phi \times \frac{SU(2)_{2n}\times SU(2)_{2n}}{U(1)}$$

is related by T-duality to the metric cone on $T^{1,1}/\mathbb{Z}_{2n}$, if we tune the moduli appropriately. The $\mathbb{Z}_{2n}$ quotient can be regarded as a $\mathbb{Z}_2$ quotient of the fiber of $T^{1,1}$ together with a $\mathbb{Z}_n$ quotient which acts on $\mathbb{P}^1 \times \mathbb{P}^1$ as well (it is the action $u \to e^{2\pi i/n}u, v \to e^{-2pii/n}v$ on the conifold $uv = xy$).

Interestingly, if we simply take the polynomial $F = x_1^{2m} + x_2^{2m} + x_3^2 + x_4^2$ and interpret that in the appropriately weighted homogeneous space, we find, that the Hodge diamond is

[12]There are other level/rank isomorphism known, discussed in section 3.4, but these are either not based on simply laced groups, or on modular invariants other than the diagonal one, so not Kazama-Suzuki. It would be interesting to relate these models, or orbifolds of these models to Landau-Ginzburg models, but to the author's knowledge, this remains yet to be done.

[75]

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 2n & & 1. \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{4.158}$$

And $\mathbb{P} \times \mathbb{P}$, the $\mathbb{C}^*$ quotient of $\mathbb{R}\tilde{\times}T^{1,1}$ or $\mathbb{R}\tilde{\times}T^{1,1}/\mathbb{Z}_2$ has the above Hodge diamond with $n = 1$.

A final interesting feature of the Kazama-Suzuki type impurities, is that we can take the level of the coset model to be large. In that case the supergravity approximation of the gauged WZW target space is meaningful. In the example of the generalized conifolds above we see that a metric cone on the total space $S^1 \to L \to Z/\Gamma$, when the order of the discrete group Γ becomes large, is T-dual (up to marginal deformations) to the background

$$\mathbb{R}_\phi \times \tilde{Z}.$$

Here L is a circle bundle over is the homogeneous space $[SU(2)/U(1)]^2/\Gamma$ (which is only quasi-smooth, not smooth, due to fixed points of Γ), where the $U(1)$'s act from the left on the $SU(2)$'s. But on the dual side $\tilde{Z}$ is $[SU(2) \times SU(2)]/U(1)$, which is the target space of a gauged WZW model. The $U(1)$ acts vectorially (in an opposite fashion on both $SU(2)$ factors). This space has B-field flux and a non-trivial dilaton.

Other Impurities

For geometric impurities which cannot be interpreted in the above fashion, we have only some general comments to make.

The Kazama-Suzuki models are essential for us to find a geometric target space interpretation of the flux impurity. Generally speaking, the Landau-Ginzburg model which we find should be thought of as $N/U(1)$, where the flux impurity is a background

$$\mathbb{R}_\phi \times N.$$

It remains an interesting task to find such N, connected to geometric singularities, which fit in the T-duality procedure we described.

In particular it may be interesting to consider hypersurfaces which correspond to projective varieties which have a smaller group of isometries than the homogeneous spaces, like $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ etcetera. For example, it seems interesting to find duals to affine hypersurfaces that are $\mathbb{C}^*$ bundles over del Pezzo surfaces, not only smooth del-Pezzo surfaces, but also quasi-smooth log-del Pezzo surfaces found by Johnson and Kollár and Johnson [34, 35] some of these come in infinite families, which might allow for a 'supergravity approximation' on the side of the flux impurity. To do this, would require a geometric interpretation of the Landau-Ginzburg models with a superpotential that is a polynomial in the lists of Kollár and Johnson. Such an interpretation, analogous to the Kazama-Suzuki/Landau-Ginzburg relation, remains to be found.

4.5 Concluding Remarks and Directions for Future Work

We have presented a connection of geometric impurities, singular geometries realized as hypersurfaces, and flux impurities, backgrounds which contain fluxes and generically a linear dilaton. The best understood examples are the ADE surface singularities, and in particular the A-type singularities, for which the geometric interpretation is simple, both of the geometric impurity, and of the dual flux impurity.

The T-duality procedure works for more general weighted homogeneous affine hypersurfaces, with some conditions on the weights and the degree. Essentially, the condition is that the sum of weights is larger than the degree, and that the difference of this sum of weights with the degree, is a divisor of the degree. The flux impurities associated to Kazama-Suzuki coset models have a natural target space interpretation, these correspond to specific hypersurfaces. By a change of moduli, the hypersurface is related to a simple metric cone. We have seen an example of the generalized conifolds that illustrates this.

There are various interesting directions which remain to be explored. First, there is the issue to better understand ‘flux impurities’ described by coset cft’s which have no Landau-Ginzburg realization, such as the many of the cosets of Eguchi et al. [95]. Conversely there are many interesting hypersurfaces for which the LG model has no known equivalent which illuminates a target space interpretation. For example one might try to find a geometric interpretation of LG models with a superpotential that defines a del Pezzo surface. This would presumably not directly admit an interpretation in a ‘supergravity limit’ as the degree of the defining polynomial is low. Also, there are known infinite series of polynomials which define log Fano varieties [34, 35]. It seems hard to find a geometric interpretation for their ‘flux duals’, but it there is such an interpretation, there might be a ‘supergravity limit’ taking the degree of the defining polynomial large.

It would also be interesting to consider the dualization procedure for complete intersections. Another direction would be, to consider hyper-Kähler hypersurfaces with regards to non-abelian duality. The hyper-Kähler singularities can be regarded as cones on tri-Sasaki manifolds. These can be viewed as $SU(2)$ bundles on quaternionic Kähler spaces. Also, because of the three Sasaki structures, these hypersurfaces may lead to a number of connected flux backgrounds.

Finally, very generally the linear dilaton backgrounds can be deformed to AdS_3 backgrounds. On the geometric singularity side, this deformation is achieved by putting fundamental strings in the singularity. Thus, the singularities provide a way to construct a plethora of backgrounds of the form $AdS_3 \times N$ and the geometric singularity description may be a good tool to learn about AdS/CFT in such cases. The construction of AdS_3 backgrounds by means of fundamental strings in singularities can be regarded as a method which is complementary to the construction of backgrounds as a near horizon limit of F1/NS5 brane configurations, or D1/D5, by S-duality. Hopefully such constructions will provide a further understanding and intuition about $\mathrm{AdS}_3 \times N$ backgrounds and holography. In particular, using singularities may open a way to the construction of $\mathrm{AdS}_3 \times N$ vacua which cannot be obtained as near horizon limits of simple D1/D5-brane configurations.

BIBLIOGRAPHY

[1] O. Pelc, "Holography, singularities on orbifolds and 4D N = 2 SQCD," JHEP **0003**, 012 (2000) [arXiv:hep-th/0001054].

[2] P. Mayr, Fortsch. Phys. **47**, 39 (1999) [arXiv:hep-th/9807096].

[3] N. Seiberg, "New theories in six dimensions and matrix description of M-theory on T**5 and T**5/Z(2)," Phys. Lett. B **408**, 98 (1997) [arXiv:hep-th/9705221].

[4] O. Aharony, M. Berkooz, D. Kutasov and N. Seiberg, "Linear dilatons, NS5-branes and holography," JHEP **9810**, 004 (1998) [arXiv:hep-th/9808149].

[5] E. Kiritsis, C. Kounnas, P. M. Petropoulos and I. Rizos, "Five-brane configurations, conformal field theories and the strong-coupling problem," [arXiv:hep-th/0312300].

[6] D. R. Morrison and M. R. Plesser, "Non-spherical horizons. I," Adv. Theor. Math. Phys. **3**, 1 (1999) [arXiv:hep-th/9810201].

[7] C. Bär, "Real Killing Spinors and Holonomy Commun. Math. Phys. **154**, (1993)

[8] C. P. Boyer and K. Galicki, "3-Sasakian Manifolds," Surveys Diff. Geom. **7**, 123 (1999) [arXiv:hep-th/9810250].

[9] T. H. Buscher, "A Symmetry Of The String Background Field Equations," Phys. Lett. B **194**, 59 (1987).

[10] M. Roček and E. Verlinde, "Duality, quotients, and currents," Nucl. Phys. B **373**, 630 (1992) [arXiv:hep-th/9110053].

[11] D. Tong, "NS5-branes, T-duality and worldsheet instantons," JHEP **0207**, 013 (2002) [arXiv:hep-th/0204186].

[12] S. W. Hawking, "Gravitational Instantons," Phys. Lett. A **60**, 81 (1977).

[13] G. W. Gibbons and S. W. Hawking, "Classification Of Gravitational Instanton Symmetries," Commun. Math. Phys. **66**, 291 (1979).

[14] R. Gregory, J. A. Harvey and G. W. Moore, "Unwinding strings and T-duality of Kaluza-Klein and H-monopoles," Adv. Theor. Math. Phys. **1**, 283 (1997) [arXiv:hep-th/9708086].

[15] C. G. . Callan, J. A. Harvey and A. Strominger, "Supersymmetric string solitons," [arXiv:hep-th/9112030].

[16] M. Rocek, K. Schoutens and A. Sevrin, "Off-shell WZW models in extended superspace," Phys. Lett. B **265**, 303 (1991).

[17] A. Cappelli, C. Itzykson and J. B. Zuber, "Modular Invariant Partition Functions In Two-Dimensions," Nucl. Phys. B **280**, 445 (1987).

[18] E. Witten, "Phases of N = 2 theories in two dimensions," Nucl. Phys. B **403**, 159 (1993) [arXiv:hep-th/9301042].

[19] H. Ooguri and C. Vafa, "Two-Dimensional Black Hole and Singularities of CY Manifolds," Nucl. Phys. B **463**, 55 (1996) [arXiv:hep-th/9511164].

[20] C. Vafa and N. P. Warner, "Catastrophes And The Classification Of Conformal Theories," Phys. Lett. B **218**, 51 (1989).

[21] W. Lerche, C. Vafa and N. P. Warner, "Chiral Rings In N=2 Superconformal Theories," Nucl. Phys. B **324**, 427 (1989).

[22] A. Giveon, D. Kutasov and O. Pelc, "Holography for non-critical superstrings," JHEP **9910**, 035 (1999) [arXiv:hep-th/9907178].

[23] A. Giveon and D. Kutasov, "Little string theory in a double scaling limit," JHEP **9910**, 034 (1999) [arXiv:hep-th/9909110].

[24] A. Giveon and D. Kutasov, "Comments on double scaled little string theory," JHEP **0001**, 023 (2000) [arXiv:hep-th/9911039].

[25] K. Sfetsos, "Branes for Higgs phases and exact conformal field theories," JHEP **9901**, 015 (1999) [arXiv:hep-th/9811167].

[26] K. Hori and A. Kapustin, "Duality of the fermionic 2d black hole and N = 2 Liouville theory as mirror symmetry," JHEP **0108**, 045 (2001) [arXiv:hep-th/0104202].

[27] G. Tian and S.-T. Yau, "Complete Kähler Manifolds with Zero Ricci Curvature II," Inv. Math. **106**, 27 (1991) .

[28] M. Henningson, "Mirror symmetry for the Kazama-Suzuki models," Nucl. Phys. B **423**, 631 (1994) [arXiv:hep-th/9402122].

[29] B. S. Acharya, J. M. Figueroa-O'Farrill, C. M. Hull and B. Spence, Adv. Theor. Math. Phys. **2**, 1249 (1999) [arXiv:hep-th/9808014].

[30] F. Cachazo, S. Katz and C. Vafa, "Geometric transitions and N = 1 quiver theories," [arXiv:hep-th/0108120].

[31] L. Castellani, L. J. Romans and N. P. Warner, "A Classification Of Compactifying Solutions For D = 11 Supergravity," Nucl. Phys. B **241**, 429 (1984).

[32] C. P. Boyer, K. Galicki, "New Einstein metrics in dimension five," J. Differential Geom. **57** (2001), no. 3, 443-463 [arXiv:math.DG/0003174].

[33] C. P. Boyer and K. Galicki, "On Sasakian-Einstein Geometry," Int. J. Math. **11**, 873 (2000) [arXiv:math.dg/9811098].

[34] J. M. Johnson and J. Kollár, "Kähler-Einstein Metrics on log del Pezzo Surfaces in Weighted Projective 3-spaces," Ann. Inst. Fourier (Grenoble) **51** (2001), no. 1 69-79, [arXiv:math.AG/0008129].

[35] J. M. Johnson and J. Kollár, "Fano Hypersurfaces in Weighted Projective 4-spaces," Experiment. Math. **10** (2001), no. 1, 151-158, [arXiv:math.AG/0008189].

[36] T. Friedrich and I. Kath, "Einstein Manifolds of Dimension Five with Small First Eigenvalue of the Dirac Operator", J. Diff. Geom **29** (1989) 263-279.

[37] C. P. Boyer, K. Galicki and M. Nakamaye, "On the geometry of Sasakian-Einstein 5-manifolds," Math. Ann. **325** (2003), 485-524, [arXiv:math.DG/0012047].

[38] G. Tian and S. T. Yau, "Kahler-Einstein Metrics On Complex Surfaces With C(1) > 0," Commun. Math. Phys. **112**, 175 (1987).

[39] J. P. Demailly and J. Kollár, "Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds," Ann. Sci. École Norm. Sup. **34** (2001), no. 4, 525-556, [arXiv:math.AG/9910118].

[40] R. D'Auria, P. Fre and P. van Nieuwenhuizen, "N=2 Matter Coupled Supergravity From Compactification On A Coset G/H Possessing An Additional Killing Vector," Phys. Lett. B **136**, 347 (1984).

[41] L. Castellani, R. D'Auria and P. Fre, "SU(3) X SU(2) X U(1) From D = 11 Supergravity," Nucl. Phys. B **239**, 610 (1984).

[42] Y. Kazama and H. Suzuki, "New N=2 Superconformal Field Theories And Superstring Compactification," Nucl. Phys. B **321**, 232 (1989).

[43] Y. Kazama and H. Suzuki, "Characterization Of N=2 Superconformal Models Generated By Coset Space Method," Phys. Lett. B **216**, 112 (1989).

[44] J. Fuchs and C. Schweigert, "Level rank duality of WZW theories and isomorphisms of N=2 coset models," Annals Phys. **234**, 102 (1994) [arXiv:hep-th/9307107].

[45] C. Schweigert, "Poincare polynomials and level rank dualities in the N=2 coset construction," Theor. Math. Phys. **98**, 326 (1994) [Teor. Mat. Fiz. **98**, 467 (1994)] [arXiv:hep-th/9311168].

[46] T. Nakatsu, "Supersymmetric gauged Wess-Zumino-Witten models," Prog. Theor. Phys. **87**, 795 (1992).

[47] S. G. Naculich and H. J. Schnitzer, "Superconformal coset equivalence from level-rank duality," Nucl. Phys. B **505**, 727 (1997) [arXiv:hep-th/9705149].

[48] E. Kiritsis, C. Kounnas and D. Lust, "A Large Class Of New Gravitational And Axionic Backgrounds For Four-Dimensional Superstrings," Int. J. Mod. Phys. A **9**, 1361 (1994) [arXiv:hep-th/9308124].

[49] K. A. Intriligator and C. Vafa, "Landau-Ginzburg Orbifolds," Nucl. Phys. B **339**, 95 (1990).

[50] D. Gepner, "Exactly Solvable String Compactifications On Manifolds Of SU(N) Holonomy," Phys. Lett. B **199**, 380 (1987).

[51] D. Gepner and Z. a. Qiu, "Modular Invariant Partition Functions For Parafermionic Field Theories," Nucl. Phys. B **285**, 423 (1987).

[52] A. Strominger, S. T. Yau and E. Zaslow, "Mirror symmetry is T-duality," Nucl. Phys. B **479**, 243 (1996) [arXiv:hep-th/9606040].

[53] K. Hori and C. Vafa, "Mirror symmetry," [arXiv:hep-th/0002222].

[54] J. Wess and J. Bagger, "Supersymmetry And Supergravity,"

[55] J. Fuchs and C. Schweigert, "NonHermitian symmetric N=2 coset models, Poincare polynomials, and string compactification," Nucl. Phys. B **411**, 181 (1994) [arXiv:hep-th/9304133].

[56] E. J. Martinec, "Criticality, Catastrophes And Compactifications," Print-89-0373 (EFI,CHICAGO)

[57] E. Witten, "Constraints On Supersymmetry Breaking," Nucl. Phys. B **202**, 253 (1982).

[58] E. Witten, "Dynamical Breaking Of Supersymmetry," Nucl. Phys. B **188**, 513 (1981).

[59] C. Vafa and E. Zaslow, Eds. "Mirror Symmetry,"

[60] J. McKay, "Graphs, Singularities and Finite Groups," Proc. Symp. Pure Math. **37**, 183 (1980).

[61] M. R. Douglas and G. W. Moore, "D-branes, Quivers, and ALE Instantons," [arXiv:hep-th/9603167].

[62] N. J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, "Hyperkahler Metrics And Supersymmetry," Commun. Math. Phys. **108**, 535 (1987).

[63] P. B. Kronheimer, "The Construction of ALE Spaces as Hyperkähler Quotients," J. Diff. Geom. **29**, 665 (1989).

[64] P. B. Kronheimer, "A Torelli-type Theorem for Gravitational Instantons," J. Diff. Geom. **29**, 685 (1989)

[65] P. Candelas, "Lectures On Complex Manifolds"

[66] P. Nagy, "Geometrically Formal Kähler Manifolds," Université de Neuchâtel preprint, (2002), [http://www.unine.ch/math/preprints/files/Nagy20-12-02.ps]

[67] C. P. Herzog and J. McKernan, "Dibaryon spectroscopy," JHEP **0308**, 054 (2003) [arXiv:hep-th/0305048].

[68] M. J. Duff, B. E. W. Nilsson and C. N. Pope, "Kaluza-Klein Supergravity," Phys. Rept. **130**, 1 (1986).

[69] J. Milnor, "On the 3-dimensional Brieskorn Manifolds," Ann. of Math. Studies **84**, 175 (1975).

[70] J. Milnor and P. Orlik, "Isolated Singularities Defined by Weighted Homogeneous Polynomials," Topology **9**, 385 (1970).

[71] J. Milnor, "Singular Points of Complex Hypersurfaces," Ann. Math. Stud. **61**, (1968).

[72] L. Nicolaescu, "Notes on the Topology of Complex Singularities," [http://www.nd.edu/~lnicolae/vita.html]

[73] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Sasaki-Einstein metrics on S(2) x S(3)," [arXiv:hep-th/0403002].

[74] J. P. Gauntlett, D. Martelli, J. F. Sparks and D. Waldram, "A new infinite class of Sasaki-Einstein manifolds," [arXiv:hep-th/0403038].

[75] I. Dolgachev, "Weighted Projective Varieties," Group Actions and Vector Fields, Lecture Notes in Math. **956**, 35 (1982)

[76] T. Hubsch, "Calabi-Yau Manifolds: A Bestiary For Physicists,"

[77] S. Gukov, C. Vafa and E. Witten, "CFT's from Calabi-Yau four-folds," Nucl. Phys. B **584**, 69 (2000) [Erratum-ibid. B **608**, 477 (2001)] [arXiv:hep-th/9906070].

[78] C. P. Boyer, K. Galicki and J. Kollar, "Einstein Metrics on Spheres," [arXiv:math.dg/0309408].

[79] A. Sen, "A note on enhanced gauge symmetries in M- and string theory," JHEP **9709**, 001 (1997) [arXiv:hep-th/9707123].

[80] N. Seiberg and E. Witten, "Gauge dynamics and compactification to three dimensions," [arXiv:hep-th/9607163].

[81] S. F. Hassan and A. Sen, "Marginal deformations of WZNW and coset models from O(d,d) transformation," Nucl. Phys. B **405**, 143 (1993) [arXiv:hep-th/9210121].

[82] A. Giveon and E. Kiritsis, "Axial vector duality as a gauge symmetry and topology change in string Nucl. Phys. B **411**, 487 (1994) [arXiv:hep-th/9303016].

[83] A. Giveon, M. Porrati and E. Rabinovici, "Target space duality in string theory," Phys. Rept. **244**, 77 (1994) [arXiv:hep-th/9401139].

[84] S. Forste, "A Truly marginal deformation of SL(2, R) in a null direction," Phys. Lett. B **338**, 36 (1994) [arXiv:hep-th/9407198].

[85] J. Polchinski, "String Theory. Vol. 1: An Introduction To The Bosonic String," "String Theory. Vol. 2: Superstring Theory And Beyond,"

[86] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 1: Introduction," "Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology,"

[87] A. N. Kirillov, "Dilogarithm Identities," Prog. Theor. Phys. Suppl. **118**, 61 (1995) [arXiv:hep-th/9408113].

[88] F. A. Bais and P. Batenburg, "A New Class Of Higher Dimensional Kaluza-Klein Monopole And Instanton Solutions," Nucl. Phys. B **253**, 162 (1985).

[89] P. Candelas and X. C. de la Ossa, "Comments On Conifolds," Nucl. Phys. B **342**, 246 (1990).

[90] K. Hori and A. Kapustin, "Worldsheet descriptions of wrapped NS five-branes," JHEP **0211**, 038 (2002) [arXiv:hep-th/0203147].

[91] S. Mukhi and C. Vafa, "Two-dimensional black hole as a topological coset model of c = 1 string theory," Nucl. Phys. B **407**, 667 (1993) [arXiv:hep-th/9301083].

[92] B. R. Greene, C. Vafa and N. P. Warner, "Calabi-Yau Manifolds And Renormalization Group Flows," Nucl. Phys. B **324**, 371 (1989).

[93] D. Ghoshal and C. Vafa, "C = 1 string as the topological theory of the conifold," Nucl. Phys. B **453**, 121 (1995) [arXiv:hep-th/9506122].

[94] T. Eguchi and Y. Sugawara, "Modular invariance in superstring on Calabi-Yau n-fold with A-D-E singularity," Nucl. Phys. B **577**, 3 (2000) [arXiv:hep-th/0002100].

[95] T. Eguchi, Y. Sugawara and S. Yamaguchi, "Supercoset CFT's for string theories on non-compact special holonomy manifolds," Nucl. Phys. B **657**, 3 (2003) [arXiv:hep-th/0301164].

[96] D. Israel, C. Kounnas, A. Pakman and J. Troost, "The partition function of the supersymmetric two-dimensional black hole and little string theory," JHEP **0406**, 033 (2004) [arXiv:hep-th/0403237].

[97] D. Israel, C. Kounnas and M. P. Petropoulos, "Superstrings on NS5 backgrounds, deformed AdS(3) and holography," JHEP **0310**, 028 (2003) [arXiv:hep-th/0306053].

[98] A. Giveon and M. Rocek, "Supersymmetric string vacua on AdS(3) x N," JHEP **9904**, 019 (1999) [arXiv:hep-th/9904024].

[99] R. Argurio, A. Giveon and A. Shomer, "Superstring theory on AdS(3) x G/H and boundary N = 3 superconformal symmetry," JHEP **0004**, 010 (2000) [arXiv:hep-th/0002104].

[100] R. Argurio, A. Giveon and A. Shomer, "The spectrum of N = 3 string theory on AdS(3) x G/H," JHEP **0012**, 025 (2000) [arXiv:hep-th/0011046].

[101] A. Giveon and A. Pakman, "More on superstrings in AdS(3) x N," JHEP **0303**, 056 (2003) [arXiv:hep-th/0302217].

[102] B. R. Greene, "String theory on Calabi-Yau manifolds," [arXiv:hep-th/9702155].

[103] J. S. Song, "Three-dimensional Gorenstein singularities and SU(3)-hat modular invariants," Adv. Theor. Math. Phys. **4**, 791 (2000) [arXiv:hep-th/9908008].

[104] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Supersymmetric AdS(5) solutions of M-theory," [arXiv:hep-th/0402153].

[105] B. J. Harrington and H. K. Shepard, "Periodic Euclidean Solutions And The Finite Temperature Yang-Mills Gas," Phys. Rev. D **17**, 2122 (1978).

[106] K. Ohta and T. Yokono, "Linear dilaton background and fully localized intersecting five-branes," Phys. Rev. D **63**, 105011 (2001) [arXiv:hep-th/0012030].

[107] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino and C. Vafa, "Topological strings and integrable hierarchies," [arXiv:hep-th/0312085].

[108] R. Dijkgraaf, A. Sinkovics and M. Temurhan, "Universal correlators from geometry," [arXiv:hep-th/0406247].

[109] R. Boels, J. de Boer, R. Duivenvoorden and J. Wijnhout, "Nonperturbative superpotentials and compactification to three dimensions," JHEP **0403**, 009 (2004) [arXiv:hep-th/0304061].

[110] R. Boels, J. de Boer, R. Duivenvoorden and J. Wijnhout, "Factorization of Seiberg-Witten curves and compactification to three dimensions," JHEP **0403**, 010 (2004) [arXiv:hep-th/0305189].

SAMENVATTING

Dit proefschrift behandelt een onderwerp in de snaartheorie, een onderdeel van de hedendaagse theoretische fysica. Voordat wij enkele aspecten van de snaartheorie aan de orde stellen en de hoofdpunten van dit proefschrift recapituleren, zullen wij pogen een context te schetsen van ontwikkelingen in de theoretische natuurkunde waarin de snaartheorie in het algemeen, en dit proefschrift in het bijzonder, geplaatst kunnen worden.

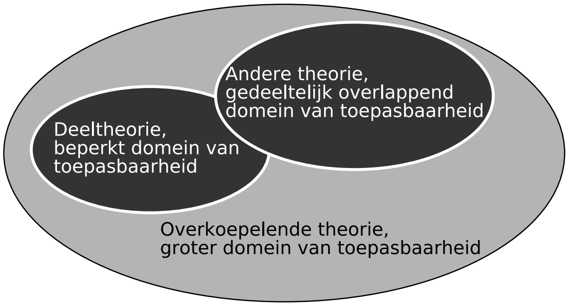
FYSICA

Wat is fysica eigenlijk? Fysici, of natuurkundigen, houden zich bezig met het het observeren van natuurverschijnselen en het begrijpen daarvan, door fundamentele wetmatigheden te zoeken en te formuleren waaraan deze natuurverschijnselen gehoorzamen. Deze wetmatigheden worden geformuleerd in een fysische theorie, die in de praktijk geformuleerd is in wiskundige termen.

Hoewel de bekende fysische theorieën geformuleerd zijn in wiskundige termen en fysici een zekere esthetiek appreciëren in de formulering van deze theorieën, waarover later meer, is natuurkunde niet een onderdeel van de wiskunde. De eerste prioriteit van een natuurkundig model, is dat het de empirische werkelijkheid accuraat beschrijft. Daarnaast wordt het zeer gewaardeerd als een model 'economisch' geformuleerd is, dat wil zeggen, dat het een klein aantal basisprincipes en regels kent, aan de hand waarvan een groot aantal diverse fenomenen beschreven kan worden. Als een model deze eigenschappen heeft, wordt het gewoonlijk aangeduid als een fysische theorie.

Er is een belangrijke manier om een fysische theorie te testen. Door de theorie zorgvuldig te bestuderen, kan men proberen zekere gevolgtrekkingen te formuleren die nog niet eerder empirisch geverifieerd werden, en dus niet deel uitmaakten van de *input* die leidde tot de formulering van de theorie. Door experimenten en empirische observaties kan men zo het domein van geldigheid van een theorie markeren.

Het is belangrijk om te beseffen, dat het domein van geldigheid van elke fysische theorie beperkt is. Per slot van rekening zijn de empirische consequenties van een theorie in de praktijk altijd slechts onder een beperkt aantal omstandigheden experimenteel gecontroleerd. Dit heeft als consequentie, dat verschillende fysische theorieën 'waar' kunnen zijn,



ook al voorspellen ze verschillende empirische consequenties. Dit is acceptabel als tenminste de twee fysische theorieën in kwestie verschillende domeinen van geldigheid hebben. Als er een domein is waar twee theorieën toepasbaar zijn, dan moeten ze daar uiteraard wel gelijke uitkomsten voorspellen.

Beschouw bijvoorbeeld de geometrische optica; hierin worden lichtstralen verondersteld in rechte lijnen te bewegen en kan weerkaatsing en breking van lichtstralen beschreven worden. Dit is heel anders dan de fysische optica, waarin licht een golfverschijnsel is, en diffractie en interferentie kan optreden: verschijnselen die niet voorkomen binnen de geometrische optica. Niettemin is zowel de fysische optica als de geometrische optica ‘waar’. De geometrische optica is bruikbaar wanneer de golflengte van het licht klein genoeg is om verwaarloosbaar te zijn. In dat geval kan reduceren de uitkomsten van de fysische optica, tot die van de eenvoudigere geometrische optica. Dit nu, is een voorbeeld van een cruciaal principe: een overkoepelende theorie, met een groter domein van geldigheid kan nieuwe fenomenen voorspellen die niet gerealiseerd zijn binnen een theorie met een ander, meer beperkt domein van geldigheid. Maar als de beperkte theorie ‘waar’ is binnen haar gelimiteerde gebied van geldigheid, dan moeten de voorspellingen van de overkoepelende theorie in dit gelimiteerde domein overeenkomen met die van de beperktere theorie.

Bij het zoeken van nieuwe, breder geldige fysische theorieën is het bovengenoemde principe van groot belang. Bekende fysische theorieën met een zeker gebied van toepasbaarheid vormen een grote hulp bij het beperken van de mogelijke eigenschappen van een overkoepelende theorie. Zo’n overkoepelende theorie moet reduceren tot de welbekende en beproefde ‘oude’ theorieën in de corresponderende deelregimes waarin de oude theorieën ook ‘waar’ zijn.

Fysische Theorieën

Wellicht is de eerste fysische theorie die een overweldigend breed gebied van toepasbaarheid heeft, de mechanica zoals ontwikkeld door Newton. De wetten van de Newtoniaanse mechanica beschrijven de beweging van objecten onder invloed van krachten. Een van die krachten is de zwaartekracht. Deze kracht werd ook door Newton theoretisch beschreven, en de Newtoniaanse mechanica in combinatie met de zwaartekrachtswet van Newton beschrijft bijvoorbeeld de dynamica van ons zonnestelsel met een zeer hoge nauwkeurigheid.

Het regime van toepasbaarheid van de Newtoniaanse mechanica en de Newtoniaanse zwaartekracht is echter beperkt. In het bijzonder vallen objecten met hoge snelheden, in de orde van de lichtsnelheid, en mechanica op zeer kleine afstands- of tijdschalen buiten het domein van de Newtoniaanse mechanica. Ook geldt de zwaartekrachtswet van Newton niet voor objecten met een zeer grote massadichtheid, en op zeer kleine en zeer grote afstandsschalen.

In sommige deelregimes zijn in de twintigste eeuw 'overkoepelende' theorieën gevonden, maar er is nog geen definitief gevestigde theorie die in elk van de deelregimes toepasbaar is. Laten we inventariseren welke theorieën er bekend zijn. Indien we de zwaartekracht buiten beschouwing laten, en ons dus enkel op de mechanica concentreren, zijn er twee richtingen mogelijk om Newton's mechanica uit te breiden.

Ten eerste, is er een speciale, hoge, snelheid in de natuur, de hoogst mogelijke snelheid. Het bestaan van een dergelijke 'universele' snelheid werd gesuggereerd door de wetten van de electrodynamica die door Maxwell werden geformuleerd in de negentiende eeuw. Daarin duikt een universele snelheid op, geïnterpreteerd als de lichtsnelheid. Het bestaan van een dergelijke universele snelheid, gelijk voor alle waarnemers ongeacht hun onderlinge eenparige beweging, is niet strikt compatibel met de wetten van Newton's mechanica. Deze wetten werden door Einstein gemodificeerd in zijn speciale relativiteitstheorie. Bij snelheden veel lager dan de lichtsnelheid, reduceren de principes van de speciale relativiteitstheorie tot die van de Newtoniaanse mechanica.

Een andere uitbreiding van het Newtoniaanse regime is in de richting van kleine afstanden of korte tijdsduren. In dit regime is de quantummechanica geldig[13].

Een belangrijk onderdeel van de hedendaagse fysica wordt gevormd door theorieën die in elk van deze regimes toepasbaar zijn, dit zijn relativistische quantumtheorieën. Specifieke exponenten van zulke theorieën beschrijven bijna alle fundamentele fysica die tot op heden is geobserveerd. Ze beschrijven echter een zeer belangrijke categorie van geobserveerde fenomenen niet, namelijk fenomenen waarin zwaartekracht een rol speelt.

Er is wel degelijk een uitbreiding van de Newtoniaanse zwaartekrachtstheorie bekend. Deze uitbreiding is de algemene relativiteitstheorie van Einstein. Bij grote massa's wijken de voorspellingen van de algemene relativiteitstheorie af van die van de theorie van Newton, evenals bij hoge snelheden. De algemene relativiteitstheorie neemt echter geen quantummechanische effecten in beschouwing.

Aangezien zowel de algemene relativiteitstheorie als quantumtheorieën een degelijke

[13]Strikt genomen wordt het regime waarin quantumeffecten van belang zijn niet aangegeven door een afstandsschaal (vergelijkbaar met een snelheidsschaal, de lichtsnelheid, die aangeeft wanneer relativistische effecten van belang zijn) maar door een schaal van *actie*, $\hbar$. In de mechanica kan aan de evolutie van een systeem van een zekere begin toestand naar een eindtoestand volgens een 'pad' van intermediaire configuraties een grootheid worden toegekend: een getal met de dimensie van $\hbar$: een 'actie'. De klassieke mechanica zegt ons, dat een systeem evolueert langs een pad waarvan de actie minimaal of maximaal is. De cruciale modificatie van de quantummechanica is, dat ook paden met een actie die niet extremaal is bijdragen aan de evolutie van het systeem.

empirische rechtvaardiging hebben, is het een legitieme vraag om een theorie te zoeken die zowel relativistische en gravitationele als ook quantumeffecten in beschouwing neemt. Niet alleen vanuit het oogpunt van volledigheid is het wenselijk om een dergelijke theorie te kennen: we weten nu eenmaal dat quantum effecten bestaan, en dat gravitationele effecten bestaan en het ligt voor de hand dat er een regime gedefinieerd kan worden waarin beide effecten een belangrijke rol spelen. Er zijn ook concrete fysische situaties denkbaar, aan de rand de domeinen van geldigheid van de empirisch beproefde fysische theorieën, waarin een theorie van quantumgravitatie nodig is om een zinnige analyse te kunnen doen. Een voorbeeld is de fysica van ons heelal in een zeer jong stadium.

Het is echter zeer moeilijk gebleken om een consistente overkoepelende theorie te formuleren die zowel gravitatie als quantumtheorie bevat. Tot op heden wordt een veelbelovend en intrigerend pad naar een dergelijke theorie gevormd door het onderzoeksgebied dat bekend is als ‘snaartheorie’.

SNAARTHEORIE

De snaartheorie is in het huidige stadium niet een fysische theorie zoals bijvoorbeeld de Newtoniaanse mechanica of de algemene relativiteitstheorie. Deze theorieën zijn ‘af’: ze hebben een duidelijke bondige logisch consistente en complete formulering en zijn fundamenteel gezien goed begrepen. Andere theorieën, zoals relativistische quantumtheorieën hebben een voor natuurkundigen acceptabele definitie, maar deze definitie heeft ons nog niet in staat gesteld om sommige belangrijke fysische fenomenen vanuit de definitie van de theorie af te leiden[14]. Bij de snaartheorie is echter zelfs een fundamentele formulering niet volledig bekend; er is ‘slechts’ een aantal formuleringen bekend die elk een zinvolle beschrijving kunnen vormen van een beperkte deelgroep van oplossingen van de, grotendeels ongekende, volledige theorie die wel wordt aangeduid met de naam ‘M-theorie’.

De naam ‘snaartheorie’ volgt uit de bekende formulering van de theorie, waarin eendimensionale objecten, ‘snaren’, de fundamentele vrijheidsgraden in de beschrijving van de theorie vormen. Dit is anders dan in de conventionele relativistische quantumtheorieën, waarin de fundamentele vrijheidsgraden worden gevormd door puntdeeltjes. Het feit dat de fundamentele vrijheidsgraden die van snaren zijn, heeft verscheidene consequenties. Ten eerste is het een bijna direct gevolg van de ruimtelijke uitgebreidheid van snaren, dat de theorie zwaartekracht kent. Het zou hier te ver voeren om dit in enig detail te bespreken, maar een essentieel punt hierin is het volgende.

Anders dan een puntdeeltje, kent een snaar oneindig veel ‘interne vrijheidsgraden’: een snaar kan op verschillende manieren trillen, zoals een vioolsnaar, bijvoorbeeld. Deze vrij-

[14]Denk hierbij aan *confinement*, of opsluiting, van kleurlading in theorieën zoals die van de sterke kernkracht. Hoewel er vele aanwijzingen zijn, uit computersimulaties en theoretische beschouwing van vergelijkbare theorieën met meer structuur, ontbreekt een goed begrip van *confinement*. Gedurende het promotieonderzoek heeft de auteur onderzoek verricht naar sommige van zulke quantumtheorieën, resulterend in de publicaties [109, 110], welke niet tot basis hebben gediend van dit proefschrift.

heidsgraden kent een puntdeeltje niet. De verschillende trillingswijzen van een snaar worden geïnterpreteerd als verschillende 'deeltjes'. In het bijzonder heeft een snaar die op een bepaalde specifieke wijze trilt, de eigenschappen van het deeltje dat verantwoordelijk is voor de zwaartekracht: het graviton.

Een quantumtheorie van gravitonen alleen, of gecombineerd met een eindig aantal andere deeltjes (zoals het geval is bij zogenoemde 'supergravitatie' theorieën) zijn tot op heden niet gebleken consistente quantumtheorieën te zijn. Dankzij het bestaan van oneindig veel verschillende trillingswijzen van een snaar (alle 'boventonen'), die gezien kunnen worden als een oneindige collectie verschillende deeltjes, is het mogelijk dat snaartheorie typische problemen van een zwaartekrachtstheorie van puntdeeltjes omzeilt.

De uitgebreidheid van snaren heeft nog andere consequenties. Een uitgebreid object als een snaar is gevoelig voor andere eigenschappen van zijn omgeving dan een puntdeeltje. Een snaar kan bijvoorbeeld ergens omheen gewonden zijn maar een puntdeeltje niet. Dit soort eigenschappen heeft grote consequenties: als een snaar de wereld 'anders ziet' dan een puntdeeltje en snaartheorie is een relevante theorie van onze wereld, dan zouden we eigenlijk ook een beschrijving van de 'wereld' willen die precies die eigenschappen onderscheidt, die door snaren 'gezien' worden. In het bijzonder kunnen twee ruimtes die er heel verschillend uitzien voor puntdeeltjes ononderscheidbaar zijn voor snaren. Het bestuderen van dit soort ruimtes, en de relaties tussen die ruimtes, staat centraal in dit proefschrift.

Het begrijpen van de onderlinge verbanden tussen schijnbaar verschillende maar feitelijk equivalente ruimtes is niet alleen interessant op zich. De verschillende beschrijvingen kunnen van pas komen bij het bestuderen van andere eigenschappen van snaartheorie: de ene beschrijving kan bepaalde aspecten duidelijk op de voorgrond brengen terwijl de andere beschrijving andere aspecten kan verhelderen. Hierover zullen wij later meer zeggen.

Snaartheorie en Dualiteiten

De titel van dit proefschrift is: 'Dual Views of String Impurities', wat in het Nederlands vertaald kan worden als: 'Duale gezichtspunten op onzuiverheden in snaartheorie.' De term 'dualiteit' kan vele verschillende betekenissen hebben in snaartheorie. Meestal kan een dualiteit gezien worden als een equivalentierelatie tussen twee theorieën. Het bestaan van een dergelijke equivalentierelatie kan erg praktisch zijn. Denk bijvoorbeeld aan de situatie dat de ene theorie, in een regime waarin berekeningen moeilijk zijn, equivalent beschreven wordt door een anders uitziende theorie in een regime waarin berekeningen veel gemakkelijker zijn. In zo'n geval kan een lastig toegankelijk regime van de ene theorie bestudeerd worden met behulp van een duale theorie.

Hoewel er vele berekeningen en analyses zijn die het bestaan van vele dualiteiten in snaartheorie ondersteunen, is het in een groot aantal gevallen tot op heden onmogelijk gebleken om het bestaan van vele dualiteiten rigoureus te bewijzen. Een uitzondering hierop vormen zogenoemde 'T-dualiteiten'. Vaak kunnen deze expliciet bewezen worden, omdat ze geformuleerd kunnen worden in de bekende en vertrouwde formulering van snaartheorie als een theorie van propagerende snaren, en wel in een regime waarin deze beschrijving de basis van zinvolle berekeningen vormt.

INTERMEZZO: SNAARTHEORIE EN MODULI

Alvorens in te gaan op enkele specifieke eigenschappen van T-dualiteit, is het nuttig om onze ideeën over snaartheorie en dualiteiten uit te breiden. Een opmerkelijke eigenschap van snaartheorie is dat snaartheorie verondersteld wordt 'uniek' te zijn. Dat wil zeggen: er zijn geen parameters die *a priori* gespecificeerd dienen te worden om te karakteriseren over welke specifieke snaartheorie we praten. Dit soort parameters is gewoonlijk wel nodig om te specificeren over welke quantum(velden-)theorie we praten: denk bijvoorbeeld aan de ladingen van de deeltjes, of meer algemeen, aan koppelingsconstanten. Er is slechts een snaartheorie!

Deze uitspraak behoeft enige nuancering. Hoewel er niet meerdere snaartheorieën zijn, geparametriseerd door de waarden van zekere a priori te specificeren parameters, zijn er wel andere belangrijke grootheden in snaartheorie: de verwachtingswaarden van verschillende toestanden van collecties, of condensaten van snaren. Deze verwachtingswaarden zijn grotendeels analoog aan de verwachtingswaarden van quantumvelden in een quantumveldentheorie. Het bijzondere van snaartheorie is dat er naast deze verwachtingswaarden geen 'externe' parameters zijn.

Er is een speciale verwachtingswaarde, de verwachtingswaarde van het zogenaamde dilatonveld, die effectief de rol speelt van een koppelings-'constante' van de snaartheorie. Wanneer deze verwachtingswaarde klein is, is de formulering van snaartheorie als een theorie van propagerende snaren zinvol en kan ermee gerekend worden.

De concrete waarde van sommige verwachtingswaarden kan worden bepaald door de vergelijkingen waaraan oplossingen van snaartheorie voldoen. Deze vergelijkingen zijn slechts ten dele bekend, en vele verwachtingswaarden worden door de nu bekende vergelijkingen niet bepaald. Er zijn dus families van oplossingen van snaartheorie, die worden geparametriseerd door deze vrije verwachtingswaarden. Deze vrije verwachtingswaarden worden 'moduli' genoemd.

Met andere woorden: hoewel er maar één fundamentele theorie is, zeg M-theorie, zonder a priori te specificeren parameters, zijn er vele verschillende oplossingen van deze theorie (voor zover het huidige begrip reikt). Bij deze oplossingen horen zekere waarden van de moduli, in het bijzonder worden de oplossingen alleen zinvol beschreven door propagerende snaren, zoals in de conventionele beschrijving van snaartheorie, wanneer een zeer specifieke modulus klein is, nl. de verwachtingswaarde van het dilaton, oftewel de effectieve snaarkoppeling.

Met dit begrip kunnen we ook dualiteiten in snaartheorie, of eigenlijk M-theorie, beter omschrijven. De dualiteiten zijn equivalentierelaties tussen een-en-dezelfde M-theorie bij verschillende waarden van de moduli. Rond specifieke waardes van de moduli is er een beschrijving van M-theorie bekend, namelijk als snaartheorie, wanneer de effectieve snaarkoppeling van de snaartheorie klein genoeg is. Sterker nog, er zijn vijf consistente snaartheorieën bekend, die elk een gedeelte van de volledige M-theorie beschrijven, en sommige dualiteiten verbinden die vijf snaartheorieën.

Snaartheorie en Dualiteiten: Vervolg

Zoals gezegd, zijn veel dualiteiten niet rigoureus bewezen. Een essentiële reden hiervoor is dat ze twee dusdanige delen van de ruimte van moduli van M-theorie verbinden, dat er niet een enkele snaartheorie is die zwak gekoppeld is in beide delen, en wanneer de snaartheorie sterk gekoppeld is, zijn er maar weinig berekeningen die vertrouwd kunnen worden.

T-dualiteit verbindt echter twee delen van de moduli ruimte van M-theorie, die zodanig zijn, dat er een afleiding van de dualiteit mogelijk is die alleen gebruik maakt van zwak gekoppelde snaartheorie. Om deze reden is het vaak mogelijk om T-dualiteit rigoureus af te leiden.

T-dualiteit heeft enkele opmerkelijke eigenschappen, waarvan we er een paar inventariseren. Ten eerste is T-dualiteit vaak rigoureus af te leiden, zoals eerder gezegd, maar in sommige gevallen is het lastiger: dit is het geval bij de ‘onzuiverheden in snaartheorie’ waar dit proefschrift over gaat. Hierover later meer. Ten tweede relateert T-dualiteit twee ruimtes, of achtergronden, waarin snaren propageren, die op een opmerkelijke manier gerelateerd zijn. Zoals eerder gezegd, ‘ziet’ een snaar niet dezelfde eigenschappen van de ruimte als een puntdeeltje. Beter gezegd: een snaar ziet alles wat een puntdeeltje ziet, en meer. Bijvoorbeeld: een snaar kan volkomen samengetrokken zijn, en er uit zien als een puntdeeltje, maar een snaar kan ook om een object in de ruimte gewonden zijn, bijvoorbeeld om een ‘onzuiverheid’. T-dualiteit relateert twee achtergronden van snaartheorie zodanig dat de eigenschappen van de ene achtergrond die alleen een gewonden snaar, maar niet een puntdeeltje ziet, gereflecteerd worden door de eigenschappen die een puntdeeltje juist wel ziet in de T-duale achtergrond.

T-dualiteit, Geometrie en Onzuiverheden

T-dualiteit relateert de ‘gewone’ geometrie van de ene ruimte aan de ‘snaar-geometrie’ van de duale ruimte en vice versa. Dit is op zichzelf een motivatie om T-dualiteiten te bestuderen, maar er zijn nog andere motivaties. Hoewel T-dualiteiten vaak rigoureus afgeleid kunnen worden, is dit niet altijd het geval. In het bijzonder is het moeilijk om een rigoureuze T-dualiteit uit te voeren in een geometrie die een singulariteit heeft. Denk bij een singulariteit bijvoorbeeld aan de punt van een kegel: overal, behalve aan de punt is een kegel glad en overal behalve aan de punt ziet een klein stukje van een kegel er bij benadering uit als een stukje van het platte vlak.

Singulariteiten in geometrieën komen veelvuldig voor in snaartheorie: een gladde geometrie die een goede achtergrond is (dat wil zeggen, die voldoet aan de vergelijkingen van snaartheorie) kan worden gedeformeerd tot een singuliere geometrie door sommige moduli op een bepaalde wijze te variëren. Zulke singuliere geometrieën worden in dit proefschrift aangeduid als 'geometrische onzuiverheden' (geometric impurities).

Niet alleen zijn geometrische onzuiverheden volkomen legitieme en 'normale' oplossingen van snaartheorie, ze hebben ook speciale eigenschappen. Typisch wordt een singuliere geometrie verkregen uit een gladde geometrie door een specifieke modulus aan te passen, nl. de modulus die het volume van een cykel bepaalt. Een cykel is een geometrisch object in de geometrie waaromheen een membraan gewikkeld kan zijn (of eigenlijk meer algemeen een p-braan, d.w.z. een p-dimensionale generalisatie van een membraan of 2-braan). Wanneer het volume van de cykel erg klein wordt, wordt de energie die nodig is om een p-braan om deze cykel te wikkelen klein. Vergelijk dit met de energie die het kost om een ballon op te blazen, deze energie hangt af van de elasticiteit van de ballon, en van het volume tot waar de ballon wordt opgeblazen; een analoog mechanisme geldt voor p-branen en p-cykels in snaartheorie. Wanneer het volume van zo'n cykel nul wordt, ontstaat er een singulariteit. Bovendien kost het geen energie om p-branen te wikkelen om een cykel met volume gelijk aan nul. De 'lichte p-branen' die om een minuscule cykel in een singulariteit gewikkeld zijn, zorgen voor nieuwe lichte fysische vrijheidsgraden die gelokaliseerd zijn op de singulariteit. Dus een bijzondere eigenschap van een geometrische onzuiverheid is, dat deze gelokaliseerde lichte vrijheidsgraden heeft.

Een andere bijzondere eigenschap van de onzuiverheden die in dit proefschrift beschouwd worden, is meer meetkundig van aard. De vergelijkingen van snaartheorie hebben speciale, zogenaamde supersymmetrische oplossingen. Deze oplossingen zijn goed onder controle, en alle geometrische onzuiverheden die supersymmetrisch zijn, zijn van een zodanige vorm, dat men een T-dualiteit zou kunnen uitvoeren. In technische termen: de supersymmetrische geometrische onzuiverheden hebben een isometrie die degenereert in de singulariteit.

Wanneer er een isometrie is, kan men pogen een T-dualiteit uit te voeren. Dit is rigoureus mogelijk wanneer de isometrie niet degenereert. Maar bij de geometrische onzuiverheden degenereert deze isometrie juist altijd, en dit compliceert het uitvoeren van de dualiteit.

In hoofdstuk 2 van dit proefschrift worden geometrische aspecten en verschillende beschrijvingen van geometrische onzuiverheden behandeld. In sommige beschrijvingen is de isometrie en andere differentiaal meetkundige eigenschappen manifest. In andere beschrijvingen zijn meer analytische of algebraïsch meetkundige eigenschappen duidelijker. In het bijzonder treden differentiaal meetkundige eigenschappen op de voorgrond in de 'metrische kegel' beschrijvingen van sectie 2.2, terwijl meer algebraïsche eigenschappen, en deformaties tot gladde geometrieën duidelijker zijn in de beschrijving als hyperoppervlakken, in sectie 2.3. In hoofdstuk 2 blijken bovendien verbanden tussen deze beschrijvingen.

T-dualiteit wordt altijd afgeleid door gebruik te maken van de beschrijving van snaartheorie als een theorie van propagerende snaren, of technisch gezegd, door gebruik te maken van de *worldsheet* conforme veldentheorie die zwak gekoppelde snaartheorie karakteriseert. Hoofdstuk 3 behandelt verscheidene karakteriseringen van dergelijke theorieën en generalisaties daarvan die van nut zijn om T-dualiteit voor geometrische onzuiverheden uit te voeren.

De verscheidene karakterisaties van de tweedimensionale conforme veldentheorieën zijn van belang, omdat het degenereren van de isometrie in de geometrische onzuiverheid tot gevolg heeft, dat de T-dualiteitstransformatie gecompliceerd is, in termen van de conforme veldentheorie. In technische termen: er zijn niet-perturbatieve bijdragen in de tweedimensionale veldentheorie. Deze kunnen, zo wordt, in hoofdstuk 4 beweerd, in aanmerking genomen worden door over te gaan van één tweedimensionale veldentheorie, met een zekere geometrische beschrijving (een sigma model, om precies te zijn) op een andere theorie met een andere, in het algemeen niet-geometrische beschrijving (een Landau-Ginzburg model). In hoofdstuk 3 worden naast sigma modellen en Landau-Ginzburg modellen, ook nog andere, niet-conforme modellen besproken, die het mogelijk maken om verschillend geformuleerde conforme veldentheorieën aan elkaar te relateren. Bovendien worden zogenoemde *coset* modellen besproken, die in sommige gevallen een geometrische interpretatie van zekere Landau-Ginzburg modellen mogelijk maken.

Hoofdstuk 4 behandelt T-dualiteit, de geometrische consequenties in het algemeen en T-dualiteit voor geometrische onzuiverheden in het bijzonder. In het algemeen relateert T-dualiteit puur metrische eigenschappen in één achtergrond aan flux in de duale achtergrond. Deze flux kan soms worden gezien als het 'magnetisch' veld dat in de achtergrond aanwezig is ten gevolge van een 'flux onzuiverheid': een object in snaartheorie, in het eenvoudigste geval een specifiek soort p-braan: de NS5-braan. Men kan zeggen dat T-dualiteit een 'geometrische onzuiverheid' relateert aan een 'flux onzuiverheid', zoals bijvoorbeeld de NS5-braan.

Ten gevolge van de niet-perturbatieve bijdragen aan de T-dualiteit heeft de flux onzuiverheid geen isometrie, terwijl de geometrische onzuiverheid wel een isometrie heeft (maar deze ontaardt in de singulariteit, en dit is precies de reden dat er niet-perturbatieve bijdragen zijn). Het ontbreken van de isometrie in de flux achtergrond maakt het mogelijk om een 'schalingslimiet' uit te voeren, die de fysica gelokaliseerd op de flux onzuiverheid isoleert. Een analoge schalingslimiet bestaat voor de corresponderende geometrische onzuiverheid, waar de lokale fysica wordt gerealiseerd door p-branen in de singulariteit). Deze schalingslimieten worden in hoofstuk 4 uiteengezet.

Een belangrijke eigenschap van de schalingslimieten is, dat deze 'nieuwe' achtergronden van snaartheorie opleveren die worden beschreven in termen van bekende exacte conforme veldentheorieën. Dit stelt ons in staat om de kennis uit hoofdstuk 3 aan te wenden, om dualiteiten in schalingslimieten van geometrische onzuiverheden uit te voeren. Er wordt een voorstel gedaan hoe de niet-perturbatieve bijdragen aan de dualiteit in aanmerking te nemen en dit wordt in verscheidene concrete situaties geïllustreerd.

In het bijzonder zijn er bepaalde 'symmetrische' onzuiverheden die aan intrigerende relaties voldoen. De geometrische onzuiverheden in deze categorie zijn metrische kegels over speciale homogene Sasaki-Einstein variëteiten G/H en hun duale flux onzuiverheden zijn snaar achtergronden die bestaan uit een lineair dilaton en een *coset* conforme veldentheorie gebaseerd op een Hermitesche symmetrische ruimte $G/(H \times U(1))$.

TOEPASSINGEN

Een belangrijke toepassing van dit werk, zou kunnen liggen in de studie van dualiteiten van een heel ander soort: equivalenties van snaartheorie in specifieke achtergronden (technisch gezegd: Anti-de Sitter ruimtes) en ‘gewone’ quantumveldentheorieën, dat wil zeggen: niet-gravitationele theorieën. Dergelijke dualiteiten (bekend als AdS/CFT dualiteiten, of meer algemeen als gesloten/open snaardualiteiten) kunnen ons niet alleen veel leren over snaartheorie, maar zeker ook over quantumveldentheorieën.

Een veelgebruikt pad om AdS/CFT dualiteiten te construeren is om de beginnen met speciale, overzichtelijke configuraties van p-branen van een speciaal soort. Er zijn dan twee manieren om tegen deze configuratie aan te kijken. Enerzijds vervormt de configuratie de ruimte waarin ze is ingebed. Er is een schalingslimiet waarin de ruimte nabij de branen ‘ontkoppelt’ van de ruimte verder weg. De ontkoppelde ruimte nabij de branen heeft een geometrie van de vorm AdS$\times N$, waarin gesloten snaren propageren en de vrijheidsgraden vormen. Hier is N een zekere ruimte, in eenvoudige gevallen een bol. Deze ruimte kan worden beschouwd als de geometrie die de braanconfiguratie omsluit. Anderzijds kunnen open snaren eindigen op de branen. In de ontkoppelingslimiet vormen deze open snaren de vrijheidsgraden van een veldentheorie.

De eigenschappen van de ruimte N hebben consequenties voor de duale veldentheorie. Men kan zich afvragen welke N gerealiseerd kunnen worden. Door de beproefde route te volgen, en te beginnen met een overzichtelijke braanconfiguratie, is maar een beperkte klasse van ruimtes N te realiseren. Men zou echter ook flux onzuiverheden zoals uit dit proefschrift kunnen gebruiken. Om precies te zijn, zijn de flux onzuiverheden van de vorm [lineair dilaton] $\times N$. Gecompliceerde lijkende flux onzuiverheden kunnen gerelateerd zijn, via T-dualiteit, aan meer overzichtelijke geometrische onzuiverheden.

De geometrische onzuiverheden op zichzelf zijn gerelateerd aan niet-graviationele theorieën die geen gewone veldentheorieën zijn, maar zogenaamde *Little String Theories*. Dit zijn niet-lokale quantumtheorieën waarover weinig bekend is. Door fundamentele snaren in de singulariteit van een geometrische onzuiverheid te plaatsen, is het mogelijk om lokale quantumveldentheorieën te krijgen, in een AdS/CFT correspondentie.

DANKWOORD

Alvorens dit proefschrift te besluiten wil ik graag enkele mensen bedanken die hebben bijgedragen aan de totstandkoming van dit proefschrift. In de eerste plaats is het mij een groot genoegen om mijn promotor te bedanken. Jan, bedankt voor je geweldige begeleiding en de vele kansen die je me geboden hebt. Het is een groot voorrecht en genoegen om met jou als begeleider te hebben gewerkt. Ik zal je altijd dankbaar zijn voor alles wat ik van je geleerd heb en ik hoop dat ik de kans zal hebben nog meer van je te leren.

Daarnaast wil ik graag de leden van het Instituut voor Theoretische Fysica bedanken, in het bijzonder alle leden van de stringtheorie groep. Ook dank ik de promovendi en postdocs van de string groep voor een stimulerende omgeving en boeiende en verhelderende discussies.

Het doet me veel plezier om mijn collega promovendi te bedanken met wie ik het grootste deel van de afgelopen vier jaar heb gewerkt, gediscussieerd en gereisd, en al lang geleden, toen de string groep nog een stuk kleiner was, een zeer nuttige Ph.D. club ben begonnen, die gelukkig de laatste tijd nieuw leven ingeblazen is. Rutger en Jeroen, bedankt voor alles, dankzij jullie was het een stuk plezieriger op het instituut en ook daarbuiten. Ik ben blij dat jullie ook op de dag van mijn promotie mij ter zijde staan. Jeroen, jij bent gedurende drieëneenhalf jaar mijn kamergenoot geweest, en wel de beste kamergenoot die ik me zou kunnen hebben gewenst. Dank je, natuurlijk in de eerste plaats voor alles wat je voor me gedaan hebt als kamergenoot, vele verhelderende discussies en ook vele die nog in de toekomst helderheid beloven te brengen (zo hoop ik), voor je niet te overschatten hulp met computers, waar ook dit proefschrift van heeft geprofiteerd en voor het prachtige programma Kile, waarmee dit proefschrift is geschreven.

Tot slot ben ik de grootste dank verschuldigd aan mijn ouders voor hun voortdurende steun en nog veel meer. Aan hen draag ik dit proefschrift op.

CURRICULUM VITAE

Robert Duivenvoorden was born in Leiderdorp, The Netherlands on March 30 1977 and spent his early childhood with his mother's family in Bulgaria. After completing his secondary education at the 'Gymnasium Haganum' in The Hague he commenced studies in physics and mathematics at Leiden University, passing his propaedeutic exams in 1996. As part of his undergraduate studies he spent one year at Keble College, at the University of Oxford, as an exchange student. He graduated in theoretical physics in 2000 under the supervision of prof. J. de Boer and prof. P. J. van Baal with a master's thesis on the subject of holography in string theory.

In the four subsequent years, the author performed his Ph. D. research under the supervision of prof. J. de Boer at the Institute for Theoretical Physics of the University of Amsterdam as an *onderzoeker in opleiding* employed by Stichting FOM. His Ph.D. research resulted in this thesis as well as the papers [109, 110], concerning three-dimensional non-perturbative gauge theories, which are not a basis for this thesis.

During this four years as a Ph.D. student, the author has tutored classes in string theory and in gravitation and cosmology. Also, he has had the opportunity to attend various schools, workshops and conferences, i.a. in Mumbai, Santa Barbara, Les Houches as well as a number of other locations.

The author is looking forward to continue to explore the fascinating realm of physics and mathematics after obtaining his Ph.D. as a post-doctoral researcher at the Center for Geometry and Theoretical Physics at Duke University in Durham, North Carolina.

Чем ночь темней - тем звезды ярче.

В.В.Розанов