

Bachelor Project: Gravitational Waves

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Abstract

The purpose of this article is to investigate the concept of gravitational radiation and to look at the current research projects designed to measure this predicted phenomenon directly. We begin by stating the concepts of general relativity which we will use to derive the quadrupole formula for the energy loss of a binary system due to gravitational radiation. We will test this with the famous binary pulsar PSR1913+16 and check the the magnitude and effects of the radiation. Finally we will give an outlook to the current and future experiments to measure the effects of the gravitational radiation

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1 Introduction

Waves are a well known phenomenon in Physics. Oscillating matter can cause waves that carry its energy away. In our daily life we can see the mechanical waves in water when you throw a stone in a pond or hear the sound waves caused by oscillating snares. Physically understanding these waves comes very natural to us. Energy stored in waves can be felt when you are blown away by a large sound system or when a tsunami wipes down a village.

Another common physical phenomenon is light. The physical understanding of light is more difficult, because of its dual character as a particle and a wave at the same time. This still is a mysterious property but the theory of electromagnetism provides us with an excellent tool to work with electromagnetic waves. The energy contained by light waves is also easy to understand because we can feel the warmth coming from a lamp or look at our electricity bill caused by our electric devices such as a microwave and a television.

When Einstein concluded in 1918 that his Theory of General Relativity also had a wave solution in the weak field regime (where there is almost no gravitation) this was not easily accepted. The concept Einstein proposed to describe gravity was counterintuitive, because in his theory gravity was no longer an attractive force caused by large amounts of mass, as the physics community had believed for about 300 years. Using the same arguments he used for special relativity and the equivalence principle (which will be explained later on) he described gravity in terms of curved spacetime. Energy and momentum curve spacetime, and the natural motion of mass in curved spacetime is what we observe as a force pulling us to the Earth or as the Earth travelling around the sun in a steady orbit.

It isn't even important whether you say that every object has its own gravitational field or that it creates its own curvature in spacetime, the effects are described by general relativity. When a massive object accelerates, the gravitational field changes and an attracted (smaller) object experiences a different gravity. One of Einstein's famous postulates states that nothing travels faster than the speed of light, including information. There has to be something that carries the information about the changed gravitational field to our attracted object. This information is stored in gravitational waves traveling with the speed of light.

The theory of general relativity is not easy to work with. The theory holds all the information we need, but it is (currently) not always possible to find an exact answer and it is not always straightforward to understand what it is trying to tell us. When physicist calculated that it was possible within the theory to have a gravitational wave solution, concluding that these waves contain information and therefore energy, a lot of physicists were sceptical. It

was doubted till the observations on the pulsar PSR1913+16, which gave good experimental evidence that gravitational waves indeed carry energy. This indirect evidence that gravitational waves exist was also rewarded with the Nobel price in 1993. Before that time the doubters argued that real vacuum could not contain energy and there were ways to rewrite the coordinate systems such that the waves seemed to disappear. Sir Arthur Eddington said that gravitational waves travel at "the speed of thought" (referring to a subset of waves which indeed are coordinate artifacts). It is ironic to say that even Einstein wrote an article in 1936 which he named "Do Gravitational Waves Exist?", with the answer "No". It is even more ironic that Physical Review refused to publish this paper, because they thought it was wrong. They were right after all, but Einstein was offended and never submitted an article to Physical Review again.

In the 1960s Joseph Weber believed firmly that he could detect passing gravitational waves with a resonance principle. He didn't succeed, however with the currently used interferometer principle physicists have good hope to find gravitational waves in a laboratory soon. To understand the concept of gravitational waves we have to study a bit of general relativity.

2 A very short introduction to the Theory of General Relativity

The theory of relativity was constructed by Einstein in the beginning of the 20th century. It was a time paradigms in physics were broken and a lot of new physical fields were explored. Einstein and his famous equation $E = MC^2$ became very popular among the general public. The general theory of relativity is not so well known, mainly because of its mathematical difficulty and somewhat counterintuitive arguments. The purpose of this thesis is not to teach you General Relativity (GR) but to show you what aspects of the theory lead to the assumption that gravitational waves exist. Some basic knowledge of GR is therefore necessary. I will not derive the Einstein equations or make it plausible that they are true, but show you the famous tensor formula (where the Greek indices run from 0 to 3) and explain what it means

$$\frac{8\pi G}{c^4} T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \quad (1)$$

2.1 On the left-hand side: The energy-momentum tensor

$T^{\mu\nu}$ is the energy-momentum tensor which has 16 components. The components describe the energy and the momentum of a macroscopic system. And just as in special relativity, where $p^\mu = (p^0, p^1, p^2, p^3) = (\frac{E}{c}, \vec{p})$, the $\mu = 0$ component relates to energy and $\mu = 1, 2, 3$, relates to the momentum. For physical understanding the Greek indices $\mu = 1, 2, 3$, are sometimes replaced by Roman indices $i = 1, 2, 3$, so $p^\mu = (\frac{E}{c}, p^i)$. The components can be interpreted as follow:

T^{00} : Energy density

T^{i0} : Momentum p^i density

T^{0j} : Energy flux in the direction of j

T^{ij} : Momentum p^i flux in the direction of j

The tensor is symmetric ($T^{\mu\nu} = T^{\nu\mu}$) and also satisfies the continuity equation in flat space

$$\frac{\partial}{\partial x^\nu} T^{\mu\nu} = 0. \quad (2)$$

Here, just as in special relativity, $x^\mu = (x^0, x^1, x^2, x^3) = (ct, \vec{x})$. The Einstein summation convention is also used (see appendix A). If you take $\mu = 0$ equation (2) reduces to the famous law of conservation of energy

$$\frac{\partial}{\partial ct} T^{00} + \frac{\partial}{\partial x^i} T^{i0} = 0, \quad (3)$$

$$\frac{\partial E/c}{\partial t} + \nabla \cdot \vec{p} = 0. \quad (4)$$

It can also be shown that $\mu = 1, 2, 3$ leads to the law of conservation of momentum [3]. The factor $\frac{8\pi G}{c^4}$ before the energy-momentum tensor is a normalization factor, where G is the universal constant of gravity known from Newtons famous inverse-square law for gravity

$$\mathbf{F} = -\frac{GMm}{r^2} \mathbf{e}_r. \quad (5)$$

This factor is found, using the fact that Einsteins theory of relativity in the weak gravitational field limit (like on earths surface) should reduce to Newtons theory of gravity. If you are interested in the derivation you can read Carroll [4] for more information.

2.2 On the right-hand side: The Einstein tensor

The Einstein equations relate the energy and momentum to the 'curvature' of spacetime. The right-hand side gives us the "shape" of the spacetime caused by the energy and momentum. This curvature influences the motion of matter and this is what we experience as gravity. This gravitational motion also influences the curvature which makes the Einstein equations nonlinear and hard to solve. A well known picture to intuitively understand curvature is the two dimensional representation of the curved space around the earth:

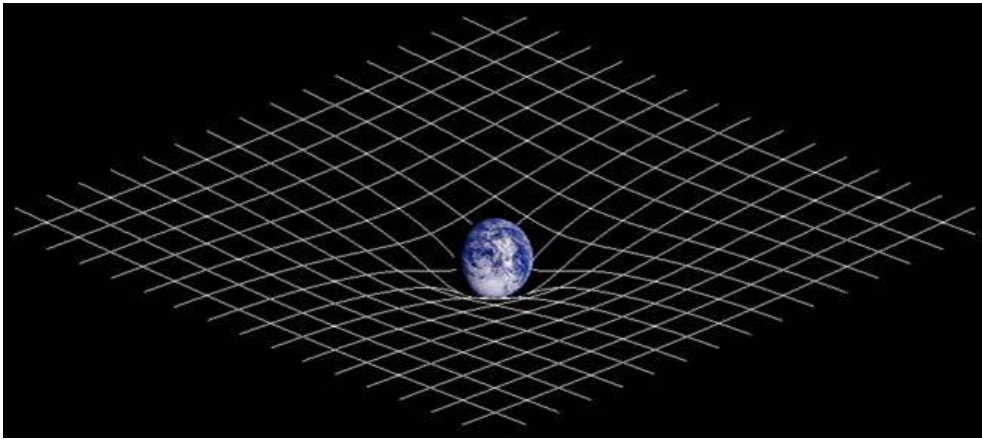


Figure 1: A two dimensional representation of the curved space caused by the earth

The right-hand side of formula (1) we can write down as one tensor, the Einstein tensor ($G^{\mu\nu}$). This tensor has to satisfy the same basic identities as the energy-momentum tensor. So in flat spacetime

$$\frac{\partial}{\partial x^\nu} T^{\mu\nu} = 0 = \frac{\partial}{\partial x^\nu} G^{\mu\nu}, \quad (6)$$

the Einstein tensor obeys the continuity equation and it is trivial it also symmetric $G^{\mu\nu} = G^{\nu\mu}$. Now we introduce the metric tensor $g_{\mu\nu}$. This quantity defines the geometry of a system and is very important (see appendix B). In flat space it is very natural to compare two vectors at different points (subtract, take the dot product and so on). When you move one vector to another, to make such a comparison, you can keep the Cartesian components constant and the value of the vector will not change because the spacetime is equal at every point (flat). The spacetime we now learn to work with is curved and therefore it is generally not possible to compare two vectors on two different places with each other. The concept of moving vectors along a path, keeping them as constant as possible all the while, is known as parallel

transport. To do this we need to construct a 'connection' between the two vectors. For more explanation on this topic you can read Carroll section 3.3. But what we have to know right now is that you need a unique combination of the metric ($g_{\mu\nu}$) which we also call the Christoffel symbol

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2}g^{\lambda\sigma}(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu}). \quad (7)$$

This symbol (it is not a tensor) is used to find the path of the shortest distance between two points and is used to construct the covariant derivative. This is the generalization of the partial derivative which we use in flat spacetime

$$\nabla_{\mu}V^{\lambda} = \partial_{\mu}V^{\lambda} + \Gamma_{\mu\nu}^{\lambda}V^{\nu}. \quad (8)$$

This generalization is needed because the partial derivative of a tensor does not generally become another tensor. Therefore we need a correction which turns out to be the Christoffel symbol [15]. Notice that in flat space ($g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$) the Christoffel symbol vanishes because of the partial derivatives. The covariant derivative then reduces to the partial derivative. Now that we know the generalization of the partial derivative we can rewrite equation (6) so it also makes sense in curved spacetime

$$\nabla_{\nu}T^{\mu\nu} = 0 = \nabla_{\nu}G^{\mu\nu}. \quad (9)$$

This equation seems simple but it yields a lot of physics. Physical phenomena do not depend on the coordinate basis we use to describe them. All the equations in a physical law should transform in the same way under general coordinate transformations. This is called generally covariant. The quantities that change in the same way under these conditions are the tensors. A tensor-equation has to have tensors with the same number of indices on both sides for it to be correct in all coordinate systems ($A_{\mu\nu} = Q_{\mu\nu}$). We want a physical law to be a tensor that is equal to another tensor with the same order or to zero, because zero can be a tensor with any number of indices ($G^{\mu\nu}$). Using the covariant derivative ensures that a tensor stays a tensor and the product remains independent of coordinates.

To construct the tensor $G^{\mu\nu}$ out of the metric and its derivatives we need to take a few steps. From a combination of the Christoffel symbol and its derivative we can construct a tensor. This is known as the Riemann tensor and it is the only tensor you can construct out of the metric tensor with at most two derivatives (either two first derivatives or one second derivative). To make the formulas more compact we use the comma notation for the partial derivatives ($\frac{\partial}{\partial x^{\mu}}F^{\nu} = \partial_{\mu}F^{\nu} = F^{\nu}_{,\mu}$) and the semicolon notation for the covariant derivative ($\nabla_{\mu}F^{\nu} = F^{\nu}_{;\mu}$). The Riemann tensor becomes

$$R_{\mu\kappa\nu}^{\lambda} = \Gamma_{\mu\nu,\kappa}^{\lambda} - \Gamma_{\mu\kappa,\nu}^{\lambda} + \Gamma_{\mu\nu}^{\eta}\Gamma_{\kappa\eta}^{\lambda} - \Gamma_{\mu\kappa}^{\eta}\Gamma_{\nu\eta}^{\lambda}. \quad (10)$$

From this (1,3)-tensor we can construct a (0,2)-tensor by contracting the first and third indices

$$R_{\mu\nu} \equiv R_{\mu\lambda\nu}^{\lambda}. \quad (11)$$

This tensor is known as the Ricci tensor. Because of antisymmetry in the Riemann tensor this is the only (0,2)-tensor you can construct from $R_{\mu\kappa\nu}^{\lambda}$ (because of symmetry $R_{\mu\nu} = R_{\nu\mu}$, see appendix C). When you contract the Riemann tensor twice you get the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R_{\mu\lambda\nu}^{\lambda} = g^{\mu\nu}g^{\kappa\lambda}R_{\kappa\mu\lambda\nu}. \quad (12)$$

It can also be proven that R is the only scalar that can be constructed from the Riemann tensor. Now we are ready to construct a symmetric (0,2)-tensor because $R_{\mu\nu}$ and $g^{\mu\nu}$ are symmetric. And when we demand $\nabla_{\nu}G^{\mu\nu} = 0$ we find

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \quad (13)$$

It is remarkable to know this tensor is uniquely determined up to normalization and is the only way to relate curvature and energy-momentum in a tensor equation

$$\frac{8\pi G}{c^4}T_{\mu\nu} = G_{\mu\nu} \quad (14)$$

3 Working with the Einstein equations of General Relativity

In the special theory of relativity Einstein used two postulates:

1. The principle of relativity
2. A constant speed of light

The first postulate states that different inertial systems, moving at a constant speed with respect to each other, should have the same laws of physics. The second states that the speed of light is the same in every inertial system and it is the maximum speed. With these two postulates Einstein realized that there is no such thing as a universal time and a three-dimensional space. Space

and time are part of a larger structure: spacetime, spanned by four coordinates (t, x, y, z) that all change according to their Lorentz-transformations. Nowadays we assume that all the laws of physics are consistent with these postulates. If they are they are called Lorentz invariant. In the beginning of the 20th century it was already known that Maxwells equations of electrodynamics were Lorentz invariant. In 1905 Einstein was able to rewrite the laws of mechanics to be Lorentz invariant. A successful combination of gravitation and Lorentz invariance was not found until Einstein came up with another principle in 1915:

3. The principle of equivalence

At every spacetime point of an arbitrary gravitational field it is possible to choose a "locally inertial coordinate system" such that, within a sufficiently small region around the point in question, the laws of nature take the same form as in unaccelerated coordinate systems in the absence of gravitation, where they can be described by the principles of special relativity. The sufficiently small region is necessary so that the gravitational field is constant throughout it. With a locally inertial coordinate system we mean a coordinate system where the point in question is in rest. Because such a coordinate system is only valid over a short period of time and in a small region of space we have to find a local coordinate system over and over again and every time the gravitational field vanishes. When you do this to describe a free falling system it is said you use free falling coordinates to describe, for example a rock falling to the earth. Now you can't determine whether the rock is in a gravitational field or not, because in your coordinate system it is at rest. This thought led Einstein to believe that gravity does not act as a force but as something that changes the curvature of space so we should use curved coordinate systems.

Gravity is now described as something that changes the curvature of spacetime and the principle of equivalence states that using local inertial systems in small enough regions gives you back the domain where special relativity applies. For an equation to be correct in curved spacetime it has to be a valid equation in special relativity if the spacetime is flat (when $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma_{\mu\nu}^{\rho} = 0$). It also has to have a tensorial form so you can change the "locally inertial coordinate system" to any other coordinate system as well as its original coordinate system. This is called general covariant. To make general covariant formulas you should write down an equation that holds in the special relativity in a tensorial form and change the $\eta_{\mu\nu}$ to $g_{\mu\nu}$ and the partial derivatives (∂) to covariant derivatives (∇).

4 A derivation of the energy loss due to gravitational radiation

The purpose of this thesis is to show that general relativity has room for gravitational waves. The theory of Electromagnetism describes the properties of electromagnetic waves. Whereas the calculations in Electromagnetism invoke the vector and scalar potentials V and \vec{A} , in general relativity it is the metric that contains the information we need. It is in general not possible to find an exact solution to the Einstein equations because they are highly nonlinear. A gravitational wave itself is a distribution of energy and momentum and influences the spacetime it travels in. To simplify the analysis we use the linearized theory of gravity. We look at gravitational waves that propagate on a flat background and do not carry enough energy and momentum to affect the spacetime they travel in. This is the weak-field approximation of the metric and it is a reasonable approximation if we are far away from any source, because gravitational radiation is generally very weak.

We look at waves that propagate in vacuum but do contain energy themselves. The amount of energy is small and does not influence the main contribution of the wave solution. The left-hand side of the Einstein equation (1) therefore reads zero. The right-hand side consists of the Ricci tensor (and scalar), which we can expand to a sum of different orders in the perturbation. The Ricci tensor to zeroth order in h holds no information. The Ricci tensor to first order in h , $R_{\mu\nu}^{(1)} = 0$, can be used to find the solution of the perturbation, which turns out to be a plane-wave solution ($\square h_{\mu\nu} = 0$) [14]. To second order of h it becomes a delicate story, but $G_{\mu\nu}^{(2)}$ can be interpreted as the stress-energy momentum tensor. Finally we want to find the formula for the energy loss due to gravitational radiation by an isolated nonrelativistic object. But we first start with the metric that can be described as the flat Minkowski metric with a small perturbation.

4.1 Finding the stress-energy tensor

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 0 \quad (15)$$

Because $|h_{\mu\nu}| \ll 0$ we will only need the first-order terms in $h_{\mu\nu}$. From the definition $g_{\nu\sigma}g^{\sigma\mu} = \delta_{\nu}^{\mu}$ we can find the inverse metric: $g^{\mu\nu} \simeq \eta^{\mu\nu} - h^{\mu\nu}$. We are free to choose any coordinate system we want. So to make life easier we choose them in such a way the metric becomes transversal and traceless. This is called the transversal-traceless gauge. So the metric is traceless: $h_{\alpha}^{\alpha} = 0$

and transversal: $\partial_\alpha h_\beta^\alpha = h_{\beta,\alpha}^\alpha = 0$. And as said before the perturbation is a plane-wave solution.

The Christoffel symbols and the Riemann tensor can be written as a sum of different orders of h

$$\Gamma = \Gamma^{(0)} + \Gamma^{(1)} + \Gamma^{(2)} + \dots \quad (16)$$

We only need the first-order Christoffel symbol for our calculations ($\Gamma^{(0)} = 0$ and the $\Gamma^{(2)}$ terms will turn out to non essential so we don't need to calculate them explicitly)

$$\Gamma^{(1)} = \frac{1}{2}\eta^{\mu\nu}(h_{\beta\mu,\nu} + h_{\beta\nu,\mu} - h_{\mu\nu,\beta}). \quad (17)$$

The Ricci tensor to 0th order in $h_{\mu\nu}$ is not interesting, because $\Gamma^{(0)} = 0$ and $R_{\mu\nu}^{(0)} = 0$. The Ricci tensor to 1st order in $h_{\mu\nu}$, $R_{\mu\nu}^{(1)} = \Gamma_{\mu\nu,\alpha}^{\alpha(1)} - \Gamma_{\mu\alpha,\nu}^{\alpha(1)} = 0$ is used to find the solution of the perturbation. To learn something about the energy contained in the gravitational field we have to look at the second-order Ricci tensor

$$R_{\mu\nu}^{(2)} = \Gamma_{\mu\nu,\alpha}^{\alpha(2)} - \Gamma_{\mu\alpha,\nu}^{\alpha(2)} + \Gamma_{\mu\nu}^{\beta(1)}\Gamma_{\alpha\beta}^{\alpha(1)} - \Gamma_{\mu\alpha}^{\beta(1)}\Gamma_{\nu\beta}^{\alpha(1)}, \quad (18)$$

$$\Gamma_{\mu\nu}^{\beta(1)}\Gamma_{\alpha\beta}^{\alpha(1)} = \Gamma_{\mu\nu}^{\beta(1)}\left(\frac{1}{2}\eta^{\alpha\rho}(h_{\rho\alpha,\beta} + h_{\rho\beta,\alpha} - h_{\alpha\beta,\rho})\right) = \Gamma_{\mu\nu}^{\beta(1)}\left(\frac{1}{2}h_{\alpha,\beta}^\alpha\right) = 0. \quad (19)$$

Here we used $\eta^{\alpha\rho}h_{\rho\beta,\alpha} = \eta^{\rho\alpha}h_{\alpha\beta,\rho}$ and the gauge condition $h_{\alpha}^\alpha = 0$. The last part of equation (18) reads

$$-\Gamma_{\mu\alpha}^{\beta(1)}\Gamma_{\nu\beta}^{\alpha(1)} = -\frac{1}{4}(h_{\mu,\sigma}^{\beta\sigma} + h_{\beta\sigma,\mu}^{\beta\sigma} - h_{\mu,\beta}^{\sigma\beta})(h_{\sigma\nu,\beta} + h_{\sigma\beta,\nu} - h_{\beta\nu,\sigma}) \quad (20)$$

$$= -\frac{1}{4}(h_{\mu,\sigma}^{\beta\sigma}h_{\sigma\nu,\beta} - h_{\mu,\beta}^{\beta\sigma}h_{\beta\nu,\sigma} + h_{\beta\sigma,\mu}^{\beta\sigma}h_{\sigma\beta,\nu} - h_{\mu,\beta}^{\sigma\beta}h_{\sigma\nu,\beta} + h_{\mu,\beta}^{\sigma\beta}h_{\beta\nu,\sigma}) \quad (21)$$

$$= -\frac{1}{4}h_{\beta\sigma,\mu}^{\beta\sigma}h_{\sigma\beta,\nu} + \frac{1}{2}h_{\mu,\sigma}^{\beta\sigma}h_{\sigma\nu,\beta} - \frac{1}{2}h_{\mu,\beta}^{\beta\sigma}h_{\beta\nu,\sigma}. \quad (22)$$

Here we used the symmetry in the metric $h^{\mu\nu} = h^{\nu\mu}$ to make some terms disappear (21) and merge some terms (22). Using partial integration on the last two terms give

$$-\frac{1}{2}h_{\mu,\sigma}^{\beta\sigma}h_{\beta\nu,\sigma} = -\frac{1}{2}(h_{\mu}^{\beta\sigma}h_{\beta\nu,\sigma})_{,\sigma} + \frac{1}{2}h_{\mu}^{\beta\sigma}h_{\beta\nu,\sigma}, \quad (23)$$

and

$$\frac{1}{2}h_{\mu}^{\beta}{}_{,\sigma}h_{\sigma\nu,\beta} = \frac{1}{2}(h_{\mu}^{\beta}h_{\sigma\nu,\beta})_{,\sigma} - \frac{1}{2}h_{\mu}^{\beta}h_{\sigma\nu,\beta}{}^{,\sigma}. \quad (24)$$

The second part of equation (24) will vanish because of the gauge condition $h_{\beta,\alpha}^{\alpha} = 0$. The tensor terms that are total derivatives in σ in equations (23, 24) and the second-order Christoffel symbol that is total derivative in α (equation 18) will we put together and will be renamed

$$S_{\mu\nu,\alpha}{}^{\alpha} = \frac{1}{2}(h_{\mu}^{\beta}h_{\alpha\nu,\beta})_{,\alpha} - \frac{1}{2}(h_{\mu}^{\beta}h_{\beta\nu,\alpha})_{,\alpha} + \Gamma_{\mu\nu,\alpha}^{\alpha(2)}. \quad (25)$$

The other second-order Christoffel symbol we rename

$$Q_{\mu,\nu} = -\Gamma_{\mu\alpha,\nu}^{\alpha(2)}. \quad (26)$$

Now we can rewrite the Riemann tensor

$$R_{\mu\nu}^{(2)} = Q_{\mu,\nu} + S_{\mu\nu,\alpha}{}^{\alpha} - \frac{1}{4}h^{\beta\alpha}{}_{,\mu}h_{\alpha\beta,\nu} + \frac{1}{2}h_{\mu}^{\beta}\square h_{\beta\nu}. \quad (27)$$

Where ${}_{,\sigma} = \partial^{\sigma}\partial_{\sigma} = \square$. The last thing we have to do before we can write down the Einstein tensor is to find

$$\left(\frac{1}{2}g_{\mu\nu}R\right)^{(2)} = \frac{1}{2}\eta_{\mu\nu}R^{(2)} + \frac{1}{2}h_{\mu\nu}R^{(1)} \quad (28)$$

$$= Q_{\mu,\mu} + S_{\mu\mu,\alpha}{}^{\alpha} - \frac{1}{4}h^{\beta\alpha}{}_{,\mu}h_{\alpha\beta,\mu} + \frac{1}{2}h_{\mu}^{\beta}\square h_{\beta\mu}. \quad (29)$$

Here $\frac{1}{2}h_{\mu\nu}R^{(1)} = 0$ because $R_{\mu\nu}^{(1)} = 0$ as said before. So we finally have

$$G_{\mu\nu}^{(2)} = R_{\mu\nu}^{(2)} - \frac{1}{2}\eta_{\mu\nu}R^{(2)}. \quad (30)$$

The Einstein tensor in zeroth and first order of $h_{\mu\nu}$ gives $G_{\mu\nu}^{(0)} = G_{\mu\nu}^{(1)} = 0$. If you take the second order to $h_{\mu\nu}$ into account as well as $G_{\mu\nu} \neq 0$ you are not able to find an exact solution to the Einstein equations. To fix this you need another correction on the metric $g_{\mu\nu}$. The Einstein tensor in first order to the second order perturbation ($G_{\mu\nu}^{(1)}[h^{(2)}]$) can be interpreted as an energy momentum tensor and we can write down

$$8\pi GT_{\mu\nu} = -G_{\mu\nu}^{(2)}. \quad (31)$$

Interpreting $T_{\mu\nu}$ as the energy momentum tensor of the gravitational waves causes problems. The equivalence principle states that at small scales we can always find a local frame of reference where spacetime is flat and all

local "gravitational fields" disappear. We can not find a local gravitational energy-momentum tensor because gravitational energy is not localizable. If we look at a macroscopic region of multiple wavelengths we can say that a gravitational wave has an effective stress-energy. We can interpret $T_{\mu\nu}$ as the such a tensor that describes the energy of gravitational waves. This tensor is symmetric and is also conserved at a flat background $\partial_\mu T^{\mu\nu} = 0$. To understand what justifies this reasoning better you can look in Weinberg 7.6 or Carroll 7.6. Averaging over a macroscopic region also gives us mathematical tools to simplify the Einstein tensor. The formula for averaging reads

$$\langle f(x) \rangle = \frac{1}{b-a} \int_a^b f(x) dx. \quad (32)$$

If we have a total derivative on the left-hand side ($f'(x)$) the right-hand side will be of the order $f(b) - f(a)$. In physics its common to work with functions that are going fast to zero when you let a and b approach infinity. In this case we also work with functions constructed out of the perturbation, which is small. The difference in value of the function on any point will also be small. If the wavelength of the gravitational wave is much bigger then the region we look at ($b - a$), $f(a)$ and $f(b)$ will also be almost equal. If you multiply this with $1/(b - a)$ this will be almost zero. When the left-hand side of (32) has no derivatives or has products of derivatives this will not be the case. This is why we made the tensors Q and S . They can be neglected after averaging

$$\langle S_{\mu\nu,\alpha} \rangle = \langle Q_{\mu,\nu} \rangle = 0. \quad (33)$$

Because the total derivative will average to zero, we can integrate by parts to get $\langle AB_{,\mu} \rangle = -\langle A_{,\mu} B \rangle$. Remember that $h_{\mu\nu}$ is a plane-wave solution so: $\square h_{\mu\nu} = 0$

$$\langle \frac{1}{4} h^{\beta\alpha}_{,\mu} h_{\alpha\beta,\mu} \rangle = -\langle \frac{1}{4} h^{\beta\alpha} \square h_{\alpha\beta} \rangle = 0. \quad (34)$$

At the end we are only left with a single term for the effective stress-energy of gravitational waves (using $c \neq 1$)

$$T_{\mu\nu} = \frac{c^4}{32\pi G} \langle h^{\beta\alpha}_{,\mu} h_{\alpha\beta,\nu} \rangle. \quad (35)$$

4.2 Rewriting the quadrupole formula to the TT gauge

In the previous section we derived the expression for the gravitational wave energy-momentum tensor. We know that the T_{00} component describes the

energy density. Therefore we can easily find an expression for the total energy contained in the gravitational radiation on the surface of Σ of constant time. To use the symmetries of the system optimally we place the source of the radiation in the origin. This is especially important when we work with the retarded time

$$E = \int_{\Sigma} T_{00} d^3x. \quad (36)$$

The energy loss due to radiation through a sphere with radius R per second can also be calculated because $T_{0\mu}$ describes the energy flux in the μ -direction

$$\frac{dE}{dt} = \int_S T_{0\mu} n^\mu r^2 d\Omega. \quad (37)$$

n^μ is the normal vector which points in the direction of r . So in spherical coordinates

$$T_{0\mu} n^\mu = T_{0r} = \frac{1}{32\pi G} \langle h_{\alpha\beta,0}^{TT} h_{TT,r}^{\alpha\beta} \rangle, \quad (38)$$

is the expression we need to find. We use (h^{TT}) to denote we used the transversal-traceless gauge to find equation (35), and the perturbation is therefore also transversal and traceless. To find the right metric we take a big step by introducing the quadrupole formula for which the derivation can be found in Carroll [8]

$$\bar{h}_{ij}(t, \mathbf{x}) = \frac{2G}{r} \frac{d^2 I_{ij}}{dt^2}(t_r). \quad (39)$$

Beginning on the left-hand side you see a different perturbation with only spatial components. This is the trace-reversed perturbation:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \quad (40)$$

$$\bar{h} = \eta^{\mu\nu} \bar{h}_{\mu\nu} = -h \quad (41)$$

For the derivation of the quadrupole formula it is not convenient to use the metric perturbation $h_{\mu\nu}$ with the traceless-transversal gauge which we used in the previous section. The reason these two different perturbations are used lies in the fact that they are both made of the same original perturbation, so they hold the same information. When we are in vacuum far away from the source, the place where the weak field approximation is correct, both perturbations will be equal when the traceless-transversal gauge ($h^\alpha_\alpha = h = 0$) is used

$$\bar{h}_{\mu\nu}^{TT} = h_{\mu\nu}^{TT}. \quad (42)$$

On the right-hand side of the quadrupole formula the second derivative of the quadrupole moment tensor

$$I_{ij}(t) = \int y^i y^j T^{00}(t, \mathbf{x}) d^3y, \quad (43)$$

is evaluated at the retarded time $t_r = t - r/c$. We can make an analogy with the origin of electromagnetic radiation. The multipole expansion is used to show that the major contribution for radiation comes from the changing dipole moment. A changing monopole moment is not possible because that violates the conservation of electric charge. In the theory of gravity the monopole moment can be related to the mass and the dipole moment to the center of mass. Both can not oscillate without violating conservation of momentum. The quadrupole moment, which measures the shape of a system, can vary in time and therefore yields the main contribution to the gravitational radiation. The quadrupole moment is much smaller compared to a dipole moment and gravity has a far more weaker coupling to matter, therefore gravitational radiation is generally much weaker than electromagnetic radiation.

Now we need to impose the transversal-traceless gauge to the trace-reversed perturbation so it changes to the weak field perturbation and we can use it in equation (35). We have to find a transversal-traceless tensor constructed from I^{ij} , I_{TT}^{ij} . We can use this to find $T_{0\mu}$, after which we can change I_{TT}^{ij} back to I^{ij} without information loss. We begin by projecting I^{ij} on its traceless component Q^{ij} ($\text{Tr}Q^{ij} = Q^i_i = 0$)

$$Q^{ij} = I^{ij} - \frac{1}{3}\delta^{ij}\delta_{kl}I^{kl}. \quad (44)$$

To make Q^{ij} transversal we want to project its components on a transversal image. Therefore we will use the projection operator

$$P_b^a(\mathbf{x}) = \delta_b^a - \frac{x^a x_b}{r^2}, \quad (45)$$

where $\mathbf{x} = (x^1, x^2, x^3)$ and $r^2 = |\mathbf{x}| = x^i x_i$. When you project Q^{ij} on its transversal image you can use the projection again but it stays the same transversal image. A basic property of a projection operator is therefore: $P^2 = P \wedge P_a^b P_c^a = P_c^b$. Using this quality we can check that

$$I_{TT}^{ij} = Q_{TT}^{ij} = P_a^i Q_{ab} P_b^j - \frac{1}{2} P_{ab} Q^{ab} P^{ij}, \quad (46)$$

is indeed traceless and transversal, see appendix D. Now we can insert equation (39) into equation (38)

$$h_{ij,0}^{TT} = \frac{2G}{r} \frac{\partial t_r}{\partial t} \frac{\partial}{\partial t_r} (\ddot{Q}_{ij}^{TT}(t_r)) = \frac{2G}{r} \ddot{\ddot{Q}}_{ij}^{TT}, \quad (47)$$

$$h_{TT,r}^{ij} = \frac{2G}{r} \frac{\partial t_r}{\partial t} \frac{\partial}{\partial t_r} (\ddot{Q}_{ij}^{TT}(t_r)) - \frac{2G}{r^2} (\ddot{Q}_{ij}^{TT}(t_r)) = \frac{2G}{r} \ddot{\ddot{Q}}_{ij}^{TT} - \frac{2G}{r^2} \ddot{Q}_{ij}^{TT}. \quad (48)$$

Because we are at a great distance from the source we can neglect the last term $\frac{2G}{r^2} \ddot{Q}_{ij}^{TT} \approx 0$ and we can write down an expression for the power radiated by a gravitational source

$$\frac{dE}{dt} = \left\langle \int_{S_\infty^2} \frac{4G^2 r^2}{32\pi G r^2} Q_{TT}^{ij} Q_{ij}^{TT} d\Omega \right\rangle = \left\langle \frac{G}{8\pi} \int_{S_\infty^2} Q_{TT}^{ij} Q_{ij}^{TT} d\Omega \right\rangle. \quad (49)$$

4.3 Solving the integral and getting rid of the TT's

We want to use the solution of the integral so it is convenient to transform the product $Q_{TT}^{ij} Q_{ij}^{TT}$ back to a form which is not transversal. To do this we use again the property of the projection operator $P^2 = P$

$$\ddot{\ddot{Q}}_{TT}^{ij} \ddot{\ddot{Q}}_{ij}^{TT} = (P_i^a P_j^b - \frac{1}{2} P^{ab} P_{ij}) (P_c^i P_d^j - \frac{1}{2} P_{cd} P^{ij}) \ddot{\ddot{Q}}_{ab} \ddot{\ddot{Q}}^{cd}. \quad (50)$$

We can divide this in four products

$$P_i^a P_j^b P_c^i P_d^j = P_c^a P_d^b, \quad (51)$$

$$-\frac{1}{2} P_i^a P_j^b P_{cd} P^{ij} = -\frac{1}{2} P_{cd} P^{ab}, \quad (52)$$

$$-\frac{1}{2} P^{ab} P_{ij} P_c^i P_d^j = -\frac{1}{2} P_{cd} P^{ab}, \quad (53)$$

$$\frac{1}{4} P^{ab} P_{ij} P_{cd} P^{ij} = \frac{1}{2} P_{cd} P^{ab}. \quad (54)$$

$$(55)$$

So there are two products of the projection operator we need to calculate

$$P_c^a P_d^b = \left(\delta_c^a - \frac{x^a x_c}{r^2}\right) \left(\delta_d^b - \frac{x^b x_d}{r^2}\right) \quad (56)$$

$$= \delta_c^a \delta_d^b - \delta_c^a \frac{x^b x_d}{r^2} - \frac{x^a x_c}{r^2} \delta_d^b + \frac{x^a x_c x^b x_d}{r^4}, \quad (57)$$

$$P_{cd} P^{ab} = \left(\delta^{ab} - \frac{x^a x^b}{r^2}\right) \left(\delta_{cd} - \frac{x_c x_d}{r^2}\right) \quad (58)$$

$$= \delta^{ab} \delta_{cd} - \delta^{ab} \frac{x_c x_d}{r^2} - \frac{x^a x^b}{r^2} \delta_{cd} + \frac{x^a x^b x_c x_d}{r^4}. \quad (59)$$

And

$$P_c^a P_d^b \ddot{Q}_{ab} \ddot{Q}^{cd} = \ddot{Q}_{ab} \ddot{Q}^{ab} - \frac{x^b x^d}{r^2} \ddot{Q}_{ab} \ddot{Q}_d^a - \frac{x^a x^c}{r^2} \ddot{Q}_{ab} \ddot{Q}_c^b \quad (60)$$

$$+ \frac{x^a x^c x^b x^d}{r^4} \ddot{Q}_{ab} \ddot{Q}_{cd}, \quad (61)$$

$$-\frac{1}{2} P_{cd} P^{ab} \ddot{Q}_{ab} \ddot{Q}^{cd} = -\frac{1}{2} \ddot{Q} \ddot{Q} + \frac{x_c x_d}{2r^2} \ddot{Q} \ddot{Q}^{cd} + \frac{x^a x^b}{2r^2} \ddot{Q}_{ab} \ddot{Q} \quad (62)$$

$$- \frac{x^a x^b x^c x^d}{2r^4} \ddot{Q}_{ab} \ddot{Q}_{cd}. \quad (63)$$

Remembering that the quadrupole moment of mass distribution is still traceless ($Q = 0$) we can rewrite equation (50)

$$\ddot{Q}_{TT}^{ij} \ddot{Q}_{ij}^{TT} = \ddot{Q}_{ab} \ddot{Q}^{ab} - \frac{x^b x^d}{r^2} \ddot{Q}_{ab} \ddot{Q}_d^a - \frac{x^a x^c}{r^2} \ddot{Q}_{ab} \ddot{Q}_c^b + \frac{1}{2} \frac{x^a x^b x^c x^d}{r^4} \ddot{Q}_{ab} \ddot{Q}_{cd}. \quad (64)$$

Now we use some standard integrals for a surface S with radius R

$$\int_S d\Omega x^a x^b = \eta^{ab} \frac{4}{3} \pi R^2, \quad (65)$$

$$\int_S d\Omega x^a x^b x^c x^d = \frac{4}{15} \pi R^4 (\eta^{ab} \eta^{cd} + \eta^{ac} \eta^{bd} + \eta^{ad} \eta^{bc}). \quad (66)$$

We are almost at the end when we use these integrals to calculate

$$\int_S d\Omega \ddot{Q}_{TT}^{ij} \ddot{Q}_{ij}^{TT} = 4\pi \ddot{Q}_{ab} \ddot{Q}^{ab} - \frac{4\pi}{3} \eta^{ac} \ddot{Q}_{ab} \ddot{Q}_c^b - \frac{4\pi}{3} \eta^{bd} \ddot{Q}_{ab} \ddot{Q}_d^a \quad (67)$$

$$+ \frac{4\pi}{30} (\eta^{ab} \eta^{cd} + \eta^{ac} \eta^{bd} + \eta^{ad} \eta^{bc}) \ddot{Q}_{ab} \ddot{Q}_{cd} \quad (68)$$

$$= 4\pi \ddot{Q}_{ab} \ddot{Q}^{ab} - \frac{8\pi}{3} \ddot{Q}_{ab} \ddot{Q}^{ab} \quad (69)$$

$$+ \frac{2\pi}{15} (\ddot{Q}\ddot{Q} + \ddot{Q}_{ab} \ddot{Q}^{ab} + \ddot{Q}_{ab} \ddot{Q}^{ba}) \quad (70)$$

$$= \pi \left(\frac{60}{15} - \frac{40}{15} + \frac{4}{15} \right) \ddot{Q}_{ab} \ddot{Q}^{ab} = \frac{8\pi}{5} \ddot{Q}_{ab} \ddot{Q}^{ab}, \quad (71)$$

using again $Q = 0$ and also $Q^{ab} = Q^{ba}$. We finally find the quadrupole radiation formula

$$\frac{dE_{grav}}{dt} = \frac{G}{5c^5} \langle \ddot{Q}_{ab} \ddot{Q}^{ab} \rangle \quad (72)$$

As said earlier this equation looks a lot like the one for the energy loss due to electromagnetic radiation.

$$\frac{dE_{EM}}{dt} = \frac{\mu_0 \mathbf{p}^2}{6\pi c}, \quad (73)$$

where \mathbf{p} is the dipole moment

$$\mathbf{p}(\mathbf{x}, t) = \int \rho(\mathbf{x}, t) \mathbf{y}(\mathbf{x}) d^3x. \quad (74)$$

5 Testing the quadrupole radiation formula with the pulsar PSR 1913 + 16

In the previous section we found the formula for the energy loss due to gravitational radiation. To find gravitational wave sources we have to look for objects with a nonzero third derivative of the quadrupole momentum. These objects are in general nonsymmetric and accelerating (and decelerating). A binary system is such an object. In 1974 Hulse and Taylor found a binary system with a special property, a Pulsar. By measuring changes in the pulse-repetition frequency they could measure the properties of the system needed to test the existence of gravitational radiation.

So to check these calculations we have to find the quadrupole moment of the pulsar and then take its third derivative. Beginning with the reduced quadrupole moment

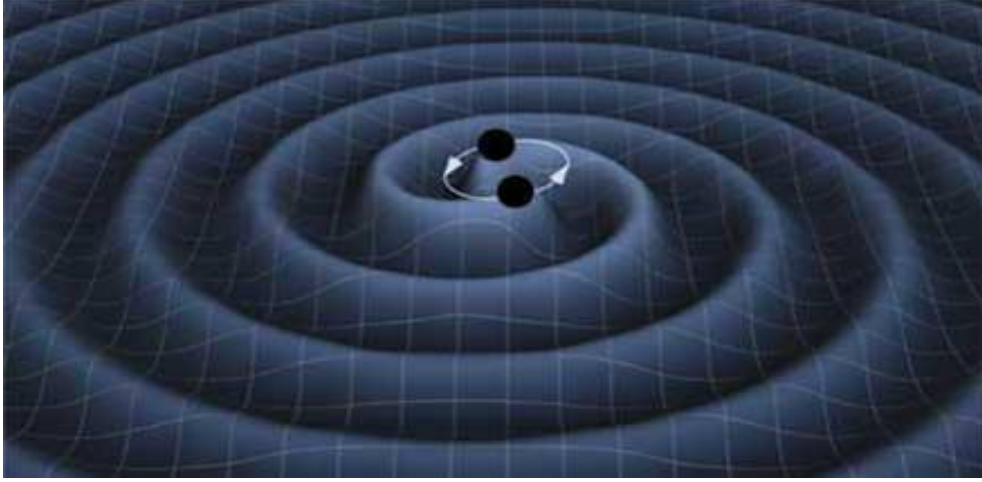


Figure 2: This is a artist impression of periodic change in the curvature of spacetime caused by a binary system. This is gravitational radiation.

$$Q^{ij} = I^{ij} - \frac{1}{3}\delta^{ij}\delta_{kl}I^{kl}, \quad (75)$$

$$I^{ij}(t) = \int y^i y^j T^{00}(t, \mathbf{y}) d^3y = \int x^i x^j \rho dV, \quad (76)$$

$$x_a^i = r(\phi) \frac{\mu}{M_a} (\cos(\phi), \sin(\phi), 0), \quad (77)$$

$$x_b^i = r(\phi) \frac{\mu}{M_b} (-\cos(\phi), -\sin(\phi), 0), \quad (78)$$

$$\rho = \delta(x^3)[M_a\delta(x^1 - x_a^1)\delta(x^2 - x_a^2) + M_b\delta(x^1 - x_b^1)\delta(x^2 - x_b^2)]. \quad (79)$$

The stars travel in an eccentric orbit and the distance between the stars depends on their position, which is related to ϕ . Because the masses are different each star needs a different correction which is related to the mass. These corrections can be found using Kepler's law: $aM_a = bM_b$, with $a + b = r$. In writing down the right parameterization we used the reduced mass: $\mu = \frac{M_a M_b}{M_a + M_b}$

$$I^{11} = \int x^1 x^1 \rho dV = x_a^1 x_a^1 M_a + x_b^1 x_b^1 M_b \quad (80)$$

$$= \left(\frac{\mu^2 r^2(\phi)}{M_a^2} M_a + \frac{\mu^2 r^2(\phi)}{M_b^2} M_b \right) \cos^2 \phi = \mu^2 r^2 \left(\frac{1}{M_a} + \frac{1}{M_b} \right) \cos^2 \phi \quad (81)$$

$$= \cos^2(\phi) r^2 \mu = \frac{1}{2} (\cos(2\phi) + 1) r^2 \mu, \quad (82)$$

$$(83)$$

where we used the delta functions from equation (79) to simplify the integral. We used a different but equal definition of the reduced mass $\mu = \frac{1}{M_a} + \frac{1}{M_b}$ and basic trigonometric functions. Similar we can calculate

$$I^{22} = \sin^2(\phi) r^2 \mu = \frac{1}{2} (1 - \cos(2\phi)) r^2 \mu, \quad (84)$$

$$I^{12} = I^{21} = \sin(\phi) \cos(\phi) r^2 \mu = \frac{1}{2} \sin(2\phi) r^2 \mu, \quad (85)$$

$$\frac{1}{3} \delta^{ij} \delta_{kl} I^{kl} = \frac{1}{3} \delta^{ij} \left(\frac{1}{2} (\cos(2\phi) + 1) + \frac{1}{2} (1 - \cos(2\phi)) \right) r^2 \mu \quad (86)$$

$$= \frac{1}{3} \delta^{ij} r^2 \mu. \quad (87)$$

Finally we can write down the reduced quadrupole moment

$$Q^{ij} = I^{ij} - \frac{1}{3} \delta^{ij} \delta_{kl} I^{kl} = \frac{1}{2} r^2 \mu J^{ij}(\phi), \quad (88)$$

with

$$J^{ij}(\phi) = \begin{pmatrix} \cos(2\phi) + \frac{1}{3} & \sin(2\phi) & 0 \\ \sin(2\phi) & -\cos(2\phi) + \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}. \quad (89)$$

Now we have to take the third derivative to the time. To make this easier we use $h = r^2 \dot{\phi}$, $u = r^{-1}$, $d/dt = hu^2 d/d\phi$ and $J''' = -4J'$. It is important to realize that h is the angular momentum per unit mass. This is a conserved quantity and can therefore be taken out of the equation to get:

$$\ddot{Q}^{ij} = (hu^2 d/d\phi)^3 \left(\frac{1}{2} \frac{\mu}{u^2} J^{ij}(\phi) \right) = \frac{\mu h^3}{2} (u^2 d/d\phi)^3 \left(\frac{1}{u^2} J^{ij}(\phi) \right) \quad (90)$$

You can calculate this by hand or use Mathematica to get

$$\ddot{Q}^{ij} = \mu h^3 (u^2 (u' u'' - u u''')) J^{ij} - 2u^3 (u'' + u) J'^{ij}. \quad (91)$$

Using $\text{Tr}(J^2) = 8/3$, $\text{Tr}(JJ') = 0$ and $\text{Tr}(J'^2) = 8$ you can calculate equation (72)

$$\frac{dE}{dt} = \frac{G}{5c^5} \langle \ddot{Q}_{ab} \ddot{Q}^{ab} \rangle. \quad (92)$$

But remember that Q is still an average so

$$\langle \text{Tr}(\ddot{Q}^2) \rangle = \frac{1}{P} \int_0^P dt \text{Tr}(\ddot{Q}^2) \quad (93)$$

$$= \frac{\mu^2 h^5}{2\pi P} \int_0^{2\pi} u^{-2} \frac{d\phi}{dt} dt [32u^4(u'u'' - uu''')^2 + \frac{8}{3}u^6(u'' + u)^2] \quad (94)$$

Using the solution for the relative coordinate

$$u = \frac{1}{a}(1 - e^2)^{-1}[1 - e \cos(\phi)], \quad (95)$$

we can simplify the integral using

$$u + u'' = \frac{1}{a}(1 - e^2)^{-1}(1 - e \cos \phi + e \cos \phi) = \frac{1}{a}(1 - e^2)^{-1} \quad (96)$$

$$u'u'' - uu''' = \frac{e^2}{a^2}(1 - e^2)^{-2}(\cos \phi \sin \phi - \cos \phi \sin \phi + \sin \phi) \quad (97)$$

$$= \frac{e^2}{a^2}(1 - e^2)^{-2} \sin \phi. \quad (98)$$

So the integral simplifies to

$$\begin{aligned} &= \frac{32}{a^6}(1 - e^2)^{-6} \int_0^{2\pi} d\phi [(1 - e \cos \phi)^4 + \frac{1}{12}(1 - e \cos \phi)^2(e \sin \phi)^2] \quad (99) \\ &= \frac{2\pi 32}{a^6} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right) (1 - e^2)^{-6}. \quad (100) \end{aligned}$$

The answer can be found using good old Eulers formula, $e^{i\phi} = \cos(\phi) + i \sin(\phi)$ to rewrite the higher order terms to sines and cosines with bigger angles. But I used our other friend Mathematica. Using $h^2 = aGM(1 - e^2)$ we finally have

$$\frac{G}{5c^5} \ddot{Q}_{ab} \ddot{Q}^{ab} = \frac{32\mu^2 M^3 G^4}{5c^5 a^5} (1 - e^2)^{-7/2} \left[1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right]. \quad (101)$$

If we would derive the energy loss of a binary system with two bodies with the same mass we would find

$$\frac{dE(e=0)}{dt} = \frac{32\mu^2 M^3 G^4}{5c^5 a^5}. \quad (102)$$

This is in agreement with (99), because a system with two bodies of equal mass would have an eccentricity $e = 0$. If you would not use the quadrupole formula but approximate the radiated power with a nonzero eccentricity you would find a wrong answer. The radiated power becomes time-dependent because of the changing distance between the two stars. Using classical approximations for the time dependence in the angular momentum and angular velocity and taking the average over one period gives us a wrong correction

$$\frac{dE_{(average)}}{dt} = \frac{dE_{(e=0)}}{dt} \times \left(1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6\right)(1 - e^2)^{-7/2}. \quad (103)$$

Using the eccentricity of our binary system $e = 0.6171313 \pm 0.0000010$ would give an answer that is 2.5 times too big, because it is not a valid way to derive the energy loss. The correction we found using the quadrupole formula is of course the right one, else we would not have gone to so much trouble. To find the answer we need some constants of nature

$$G = 6.6742 \times 10^{-8} \text{cm}^3 \text{g}^{-1} \text{s}^{-2}, \quad (104)$$

$$c = 2.997924580 \times 10^{10} \text{cm s}^{-1}, \quad (105)$$

$$M_{\odot} = 1.989 \times 10^{33} \text{g}. \quad (106)$$

Because PSR 1913+16 is a pulsar Hulse and Taylor were able to find the other values of the system which we need to find the energy loss due to gravitational radiation. The techniques they used to find these values can be found in the famous article written by Weisberg [5]

$$e = 0.6171313 \pm 0.0000010, \quad (107)$$

$$M_{pulsar} = (1.442 \pm 0.003)M_{\odot}, \quad (108)$$

$$M_{companion} = (1.386 \pm 0.003)M_{\odot}, \quad (109)$$

$$P = 27906.980895 \pm 0.000002. \quad (110)$$

Kepler's law can be used to rewrite the semi-major axis

$$a = \sqrt[3]{GM\left(\frac{P}{2\pi}\right)^2} = 1.95 \times 10^{11} \text{cm}. \quad (111)$$

Now we can write down the answer for the energy loss due to gravitational radiation as we found in equation (101)

$$\frac{dE}{dt} = (6.554 \times 10^{30})(1 - e^2)^{-7/2} \left[1 + \frac{73}{24}e^2 + \frac{37}{96}e^4 \right] \quad (112)$$

$$= 7.77 \times 10^{31} J/s. \quad (113)$$

The luminosity of the sun is $3.85 \times 10^{26} J/s$, which is clearly less than the loss due to gravitational radiation by this binary system. It is not that the radiated energy is low but the effect coupling of gravitational waves to matter that makes them hard to detect. A binary system is a heavy source with a lot of bulk motion. Our sun also produces gravitational waves but the energy loss is approximately 300 Watt.

If we want to calculate the decline of the orbital period we can use the Kepler formula for the potential energy because the stars in the binary system don't travel with relativistic speeds. Their radial velocity varies between 75 km/sec and 300 km/sec

$$E_{pot} = -\frac{GM_1M_2}{R}. \quad (114)$$

Because the distance between the two stars varies in time we can write down an expression for energy loss

$$\frac{dE}{dt} = \frac{d}{dt} \left(-\frac{GM_1M_2}{R} \right) = -GM_1M_2 \frac{d}{dt} \left(\frac{1}{R} \right). \quad (115)$$

Here we use Kepler's formula

$$\frac{1}{R} = \frac{1}{2a} = \frac{1}{2} \sqrt[3]{GM \left(\frac{2\pi}{P} \right)^2}, \quad (116)$$

to find the changing in the period

$$\frac{d}{dt} \left(\frac{1}{R} \right) = \frac{1}{2} \sqrt[3]{4\pi^2 GM} \frac{dP^{-\frac{2}{3}}}{dt} = -\frac{1}{2} \sqrt[3]{4\pi^2 GM} \frac{2}{3} P^{-\frac{5}{3}} \dot{P} = -\frac{2\dot{P}}{3RP}. \quad (117)$$

So finally we can write down an expression that relates the loss of energy due to gravitational radiation to the decrease of the orbital period

$$\frac{dE}{dt} = -\frac{2M_1M_2G}{3RP} \dot{P}. \quad (118)$$

We use this to calculate the theoretical predicted decrease

$$\dot{P} = -\frac{3RP}{2M_1M_2G} \frac{dE}{dt} \quad (119)$$

$$= -(3.09 \times 10^{-44})(7.77 \times 10^{31}) = -2.40 \times 10^{-12} s/s, \quad (120)$$

to compare with the observed data

$$\dot{P} = -(2.427 \pm 0.026) \times 10^{-12}. \quad (121)$$

We can conclude that these two results are in agreement with each other.

6 Gravitational wave detectors

The indirect observation of gravitational waves through the pulsar PSR1913+16 was a great success for the theory of general relativity. But the detection of gravitational waves in a laboratory hasn't yet occurred. This is a big challenge for experimental physicists because they are very hard to detect.

6.1 The effect of a passing gravitational wave on matter

To detect gravitational waves you have to know what you are looking for. We know we are looking for plane waves. Remember the Einstein tensor $G_{\mu\nu}$, which is symmetric so it holds 10 independent components. The Bianchi identities

$$\nabla G_{\mu\nu} = 0, \quad (122)$$

relates the components in four differential equations. Therefore we are left with $10 - 4 = 6$ degrees of freedom. The Einstein equations are made in such a way that they are independent of coordinates. We are working with a plane-wave solution and when you use a set of coordinates or a gauge condition to describe this you use four from the remaining degrees of freedom. We are left with $10 - 4 - 4 = 2$ degrees of freedom to describe the metric. In our case we name the degrees of freedom h_{\times} and h_{+} for reasons that will soon be clear. To make the solution of the metric more understandable we stick with the TT-gauge and use the spatial coordinates such that the wave travels in the z -direction. The solution of the metric becomes [8]

$$h_{\mu\nu}^{TT} = \text{Re} \left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{+} & h_{\times} & 0 \\ 0 & h_{\times} & -h_{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(t-z)} \right]. \quad (123)$$

Where ω is the frequency of the wave. Such a passing gravitational wave traveling in the z-direction distorts the relative distance in the x and y direction of matter. If a wave with the + polarization passes through a circle of independent particles it will stretch the circle into a ellipse up and down illustrated in figure (3). The h_{\times} polarization has a different effect on the particles. The distortion now happens in the shape of an \times as seen in figure (4). A linear combination of the two polarizations make it possible to describe every gravitational wave.

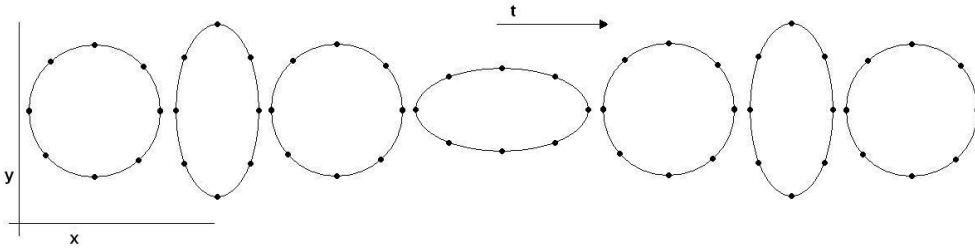


Figure 3: A gravitational waves with the + polarization will distort the circle of particle into ellipses oscillating in a + pattern

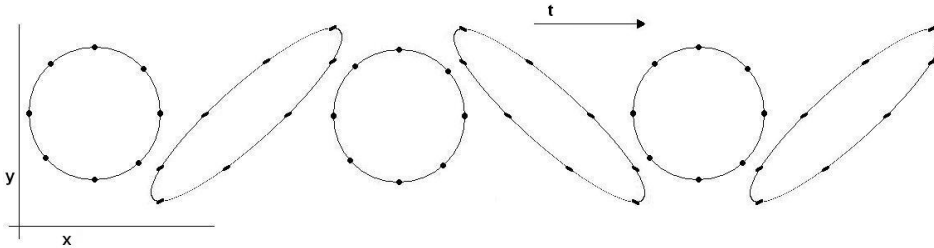


Figure 4: A gravitational waves with the \times polarization will distort the circle of particle into ellipses oscillating in a \times pattern

With the metric known it is possible to calculate the effect of a characteristic gravitational source such as a binary system. We know that it is a small effect because we don't observe it in our daily life and even the best laboratories did not detect it (yet). The distance between two test masses changes with time according to the time variation of the gravitational wave, which is described by the perturbation, $h_{ij}(t, \mathbf{x})$

$$\frac{\delta L}{L} \sim h_{ij}. \quad (124)$$

For a more extensive explanation you can consult Hartle 16.2. To find the order of magnitude we are looking for we will make rough estimates. The

quadrupole moment of binary system with $e = 0$ is $I_{ij} \sim MR^2$. The second time derivative will be

$$\ddot{I}_{ij} \sim MR^2/P^2. \quad (125)$$

Now we can use formula (39) with $c \neq 1$ and the values found by Hulse and Taylor, with the distance between the two stars averaged to 3 solar radii. The distance between the pulsar and the earth $\approx 5000pc$ [6]

$$\frac{\delta L}{L} \sim h_{ij} \sim \frac{2GMR^2}{5rc^4P^2} \sim 10^{-24}. \quad (126)$$

Suppose you have two independent test particles originally separated by a meter. If the gravitational waves from PSR 1913+16 pass this set-up, the distance between the particles will increase and decrease with a thousandths of a thousandths of a thousandths of a typical nucleus, $\delta L \sim 10^{-24}m$. The frequency of the wave is related to the frequency of the oscillation in the source. This can be approximated with Kepler's formula

$$f = \frac{\omega}{2\pi} \sim \frac{GM}{R^3} \sim 10^{-4}. \quad (127)$$

Fortunately there are also gravitational wave sources that are somewhat easier to detect. Figure (5) gives an outlook to the gravitational waves emitted by the most violent events in the universe. The collisions and collapses have typical fingerprints which can be detected. For example a black hole binary coalescence can have a theoretical maximum gravitational luminosity of $L \sim 10^{52}J/s$ but this only lasts for $10^{-2} - 10^{-3}$ seconds [7].

6.2 The resonance based wave detector

The first gravitational wave detector was built in the late 60s by Joseph Weber. He used an aluminium bar with detectors on the surface that would 'ring' when a gravitational wave passed. The little oscillation in length and height caused by a typical violent gravitational wave should cause the bar to resonate up to a measurable signal. Weber claimed he did it in 1969 but it was never scientifically accepted. The resonance principle is still used in a set of experiments to find experimental evidence for the existence of gravitational waves, but so far they haven't succeeded. Even though they are supercooled and put in superstable and supervacuum environments they are only able to measure waves with a specific frequency and really high amplitude. The International Gravitational Event Collaboration (IGEC) is a group, existing of 5 currently operating detectors all over the earth, whose aim is to produce common analysis of datasets produced by the detectors. This could filter the

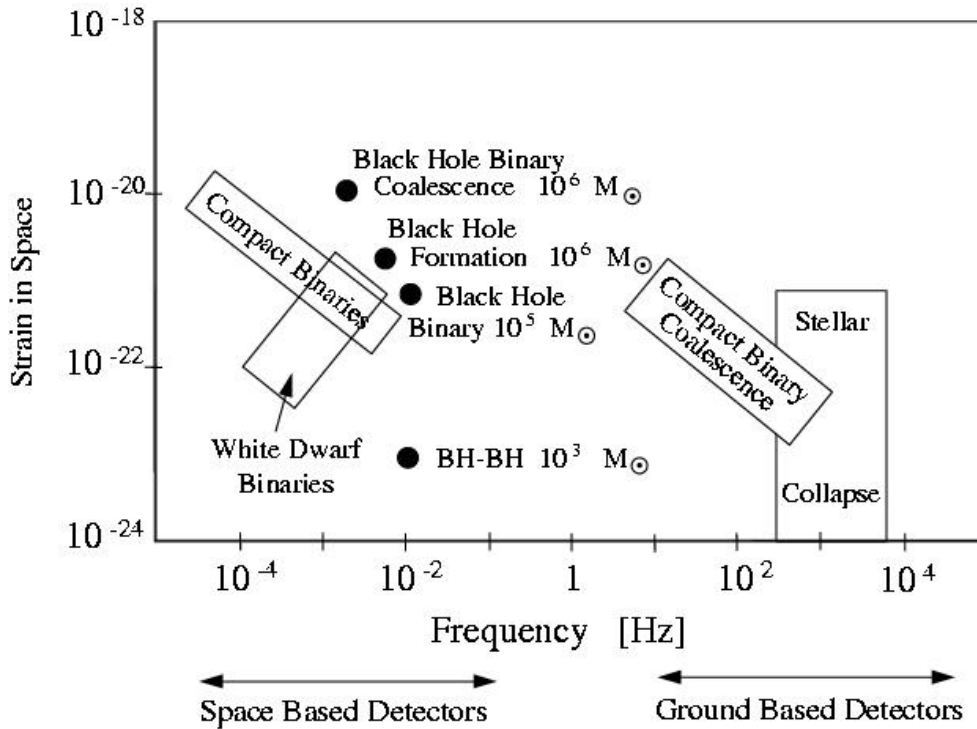


Figure 5: Different sources give different kind of gravitational waves.

noise and determine the position of the source. The detectors have been in almost continuous observation since 1997 but they haven't detected a violent event. A newer state of the art resonance based detector (MiniGRAIL) is stationed in Leiden, with the means to find waves with a frequency of 3 kHz and a $\delta L/L \sim 10^{-21}$. They hope to measure colliding small black holes and instabilities in neutron stars.

6.3 Interferometry based wave detectors

In the early 70s another detection principle was proposed. The famous Michelson and Morley interferometer could be modified to detect gravitational waves. It uses the mirrors as freely suspended test masses. When a gravitational wave passes from above the length of one arm shortens as the other stretches with a frequency and amplitude depending on the passing wave. The system is arranged such that the two laserbeams interfere destructively in absence of gravitational waves, so there is no signal observed in the photodetector. When the length of the arm is distorted by the gravitational waves, the beams won't destructively interfere anymore and send a signal to the photodiode. These signals can tell you what kind of wave

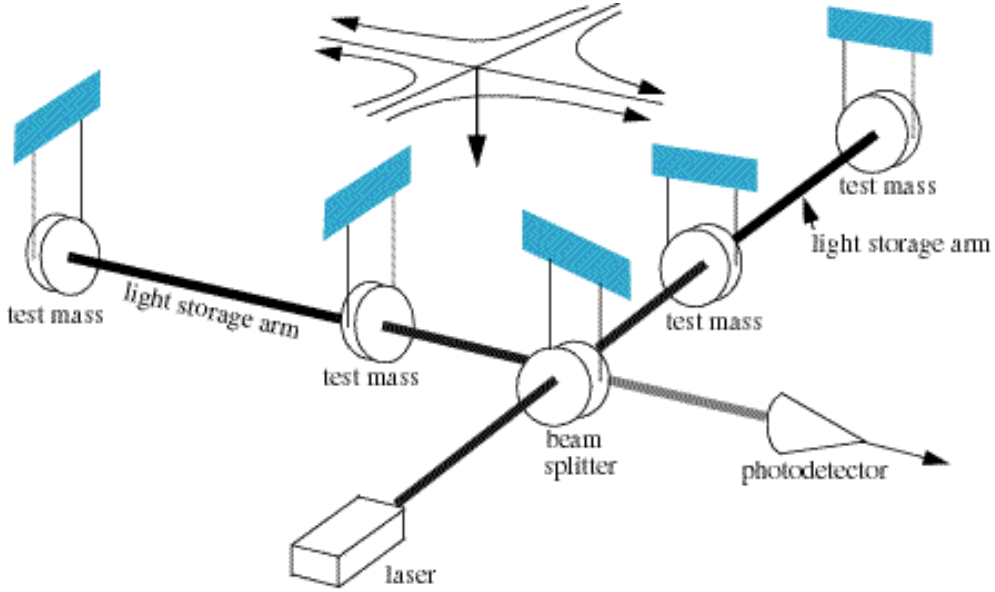


Figure 6: The interferometer detector used in a ground based laboratory

passed.

There are currently seven interferometer detectors in Europe, Japan, Australia and the US. All these projects use four mirrors, as shown in figure (6), to make the photons travel the cavity multiple times and thus effectively enlarges the cavity as well as the effect of the distortion. The most precise laboratory is the Laser Interferometer Gravitational-Wave Observatory (LIGO) located in the United States. The project has three interferometers. Two near Hanford, Washington and one in Livingston, Louisiana. The two locations are separated by 3000 km. First, it is good for statistical certainty to have more detectors. Because noise is the main difficulty, all the gravitational wave laboratories cooperate with each other, so random gravitational wavelike signals are ruled out. Second, it is necessary to determine the position of a gravitational wave source. The largest LIGO detector has two 4 km long arms where photons travel about 200 times between the mirrors. The effective path is about 10^9m . Looking at the phase difference between the two beams of a microlaser ($\lambda \approx 10^{-12}\text{cm}$) the LIGO experiment is able to see displacements of $\delta L \sim 10^{-14}\text{m}$. Without background noise they could measure amplitudes till $h \sim 10^{-23}$ [9]

$$h \sim 10^{-23} \sim \frac{\delta L}{L} \sim \frac{10^{-14}}{10^9}. \quad (128)$$

The biggest technical problem is to find the gravitational wave signal in the

background noise. An earthquake or falling tree have their own shockwave that could be misinterpreted as a gravitational wave. To keep the mechanical seismic noise as low as possible the detectors were built on reasonably quiet sites. The test masses can be isolated by their environment using pendulums and springs. This system stands also in an ultravacuum environment and is supercooled. The vacuum is necessary for the laserbeam to travel without interference and the supercooled environment lowers the thermal motion. With the state of the art techniques in isolation, cooling, vacuum, lasers, mirrors and photodiodes it is still really hard to isolate the gravitational signals from the noises from the earth. But the calculations provide good hope that a ground based detector will be able to find a gravitational signal. Especially with the planned Advanced LIGO, an upgrade that has been approved in March 2008, which enhances the sensitivity of the instruments with a factor 10, it is a matter of time till the detectors work optimally and a passing gravitational wave will be registered.

As you can see in figure (5) the ground based interferometers are limited by their range of observable frequencies. The Laser Interferometer Space Antenna (LISA) is a prestigious joint project from the ESA and the NASA that could measure different frequencies and therefore different sources. The interferometer principle still holds but the mirrors are situated in spacecrafts that will be separated by 5 million kilometers at a distance of 50 million kilometers from earth. Instead of a L shaped detector with suspended test masses it uses three spacecrafts that are freely falling. They are positioned in a triangle. With this detector it will be possible to measure the gravitational waves emitted from binary systems. Our well known binary PSR 1913+16 has, as estimated in equation (127), a frequency that will be in range of the detector $f \sim 10^{-4}$. The amplitude $f \sim 10^{-24}$ (equation 126) is too small to detect but binary systems that are closer, such as ι Boo at a distance of 11.7 pc from earth, can be detected ($h \sim 10^{-21}$).

7 Conclusion and future applications

Calculations predict that in the near future a detector will find gravitational wave signals. The detection of gravitational waves in a human laboratory will be a great success of experimental physics, with a Nobel price waiting for the discoverers, but that is only the start of a new field. Of course it is interesting if gravitational waves travel with the predicted speed of light but gravitational wave antennas used as an observation tool for astronomy is the really interesting part. There are some fundamental differences with electromagnetic waves that would allow us to 'look' at cosmic objects on a

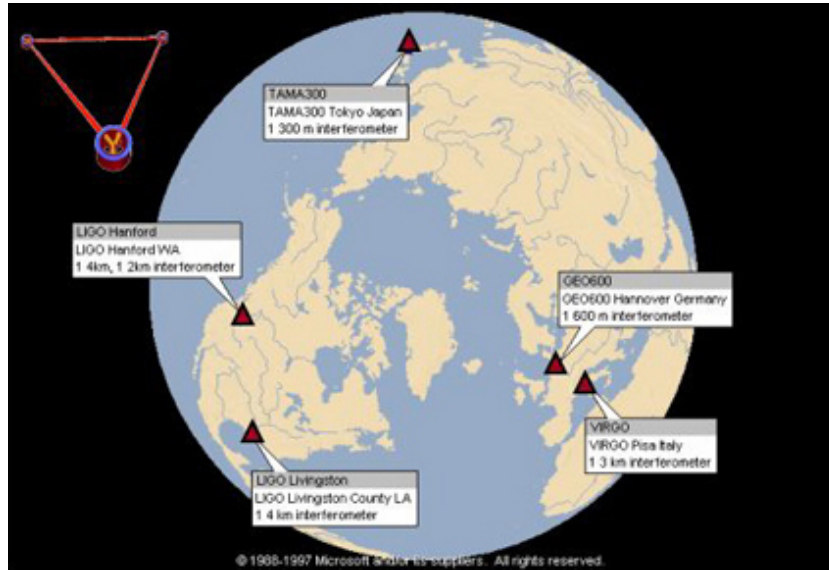


Figure 7: Global network of ground based detectors, with LISA in space

whole other level.

Electromagnetic radiation is caused by the motion of small electromagnetic particles. Therefore it can hold information about the thermodynamics of an object. They interact strongly with their environment so they are easy to detect. The downside to this is that a lot of electromagnetic waves are absorbed and information about certain cosmic systems is not available to us. Gravitational waves arise from the bulk motion of mass and contain information about the dynamics of a system. These waves interact very weak with the environment which makes them hard to detect. This also has an advantage because from their source to the earth they travel without being seriously absorbed. The gravitational waves can provide us information about cosmic objects that are hidden in the dark such as binary black holes and the collapse of a stellar core. Maybe we can find cosmic background gravitational waves analogous with the electromagnetic cosmic background microwave that can provide us with information about the big bang. About 3×10^5 years after the big bang the universe started to become transparent and electromagnetic radiation could freely propagate. Gravitational waves could give us direct information about the period before the recombination of matter into neutral atoms. Another big difference lies in the detector mechanisms. Electromagnetic astronomy uses telescopes to look very deep at a small part of the universe. A gravitational wave detector almost has a 4π steradian sensitivity to events over the sky. You could place an analogy between looking and hearing. Electromagnetic waves have small wavelengths

and can be used to make pictures and you can look very closely to a part of the universe. Workers in gravitational radiation often use sounds to describe a source.

A The Einstein summation rule

To make the equations more elegant Einstein introduced the summation convention. When the same index is used as subscript and superscript it implies that the equation is summed over all possible values of that index

$$\sum_{\alpha=0}^3 x_{\alpha} x^{\alpha} \stackrel{\text{def}}{=} x_{\alpha} x^{\alpha}, \quad (129)$$

$$\sum_{i=1}^3 x_i x^i \stackrel{\text{def}}{=} x_i x^i. \quad (130)$$

So equation (2) reads

$$\frac{\partial}{\partial x^{\nu}} T^{\mu\nu} = \frac{\partial}{\partial x^0} T^{\mu 0} + \frac{\partial}{\partial x^1} T^{\mu 1} + \frac{\partial}{\partial x^2} T^{\mu 2} + \frac{\partial}{\partial x^3} T^{\mu 3}. \quad (131)$$

B The metric

To understand some of the basic calculation rules it is good to start with the metric in flat space. We know from special relativity the four-vector which is invariant under Lorentz transformations

$$x^2 + y^2 + z^2 - c^2 t^2 = \text{constant}, \quad (132)$$

which also reads

$$\eta_{\mu\nu} x^{\nu} x^{\mu} = x_{\mu} x^{\mu} = \text{constant}, \quad (133)$$

with

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (134)$$

where will use $c = 1$ for simplicity so $x^{\mu} = (t, x, y, z)$ and $x_{\mu} = (-t, x, y, z)$.

One of the important properties of the metric is now introduced which also holds in non-flat spacetime (where the metric becomes $g_{\mu\nu}$). The metric can be used to raise or lower indices. The spacetime interval is also invariant under changes of inertial coordinates

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = \eta_{\mu\nu} \Delta x^{\mu} \Delta x^{\nu}. \quad (135)$$

For understanding it is nice to point out Δs can be interpreted as the proper time. The spacetime interval in curved spacetime is

$$(\Delta s)^2 = g_{\mu\nu} \Delta x^\mu \Delta x^\nu. \quad (136)$$

The metric has some other properties that makes some calculations easier

$$g^{\mu\nu} g_{\mu\sigma} = \delta_\sigma^\nu, \quad (137)$$

$$g = |g_{\mu\nu}| \neq 0, \quad (138)$$

$$g^{\mu\nu} = g^{\nu\mu}. \quad (139)$$

The role the metric has is too complex to explain in this short appendix, but this text from Carroll (p71) illustrates it a bit: (1) the metric supplies with a notion of "past" and "future"; (2) the metric allows the computation of path length and proper time; (3) the metric allows the computation of the "shortest distance" between two points, and therefore the motion of test particles; (4) the metric replaces the Newtonian gravitational field ϕ ; (5) the metric provides a notion of locally inertial frames and therefore a sense of "no rotation"; (6) the metric determines causality, by defining the speed of light faster than which no signal can travel; (7) the metric replaces the traditional Euclidian three-dimensional dot product of newtonian mechanics.

C The symmetries in the Ricci tensor

The symmetries in the Ricci tensor arise from the symmetries in the metric

$$g^{\mu\nu} = g^{\nu\mu}. \quad (140)$$

Therefore the Christoffel symbol is also symmetric in the two lower indices

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) = \Gamma_{\nu\mu}^\lambda. \quad (141)$$

The Riemann tensor has therefore some special properties. To make it easier to describe we write it down in the totally covariant form

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^\lambda_{\sigma\mu\nu}, \quad (142)$$

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}, \quad (143)$$

$$R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}, \quad (144)$$

$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}, \quad (145)$$

$$0 = R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu}. \quad (146)$$

The Ricci tensor is symmetric which can be seen directly from equation (145)

$$R_{\mu\nu} \equiv R^{\lambda}_{\mu\lambda\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\lambda\eta} - \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\eta}. \quad (147)$$

D Transversal and traceless Q_{TT}^{ij}

$$Q_{TT}^{ij} = P_a^i Q_{ab} P_b^j - \frac{1}{2} P_{ab} Q^{ab} P^{ij} \quad (148)$$

Let us first check that Q_{TT}^{ij} is transversal $\Rightarrow x \cdot Q_{TT}^{ij} = 0$

$$x_i Q_{TT}^{ij} = (x_i P_a^i) Q_{ab} P_b^j - \frac{1}{2} P_{ab} Q^{ab} (x_i P^{ij}), \quad (149)$$

$$x_i P_a^i = x_i \left(\delta_a^i - \frac{x^i x_a}{r^2} \right) = x_a - \frac{x^i x_i}{r^2} x_a = x_a - x_a = 0, \quad (150)$$

$$x_i P^{ij} = x_i \left(\delta^{ij} - \frac{x^i x^j}{r^2} \right) = x^j - \frac{x^i x_i}{r^2} x^j = x^j - x^j = 0, \quad (151)$$

$$x_i Q_{TT}^{ij} = 0. \quad (152)$$

Now we check of it is traceless $\Rightarrow Q_i^i = 0$

$$Q_i^i = g_{ij} Q_{TT}^{ij} = P_a^i Q_{ab} P_{ib} - \frac{1}{2} P_{ab} Q^{ab} P_i^i, \quad (153)$$

$$P_a^i P_{ib} = g_{bd} P_a^i P_i^d = g_{bd} P_a^d = P_{ab}, \quad (154)$$

$$\frac{1}{2} P_i^i = \frac{1}{2} \left(\delta_i^i - \frac{x^i x_i}{r^2} \right) = \frac{1}{2} (3 - 1) = 1, \quad (155)$$

$$Q_i^i = P_{ab} Q^{ab} - P_{ab} Q^{ab} = 0. \quad (156)$$

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