A search for deviations from Newtonian gravity

A quest in a higher dimensional world

Shanna Haaker, Wouter Merbis and Loran de Vries

July 9, 2006

Abstract

Motivated by theoretical considerations involving the existence of extra compact dimensions of space, mostly developed to solve the hierarchy problem, we explore the possibility to extend our 4D space-time with more spatial dimensions. After an introduction into the mathematics of extra dimensional physics, we examine the corrections to Newton's Gravitational law, inspired by this extra dimensional physics. These deviations can be described by Yukawa-type corrections. We discuss constraints placed on these corrections from recent short-range gravity experiments and theoretical constraints that arise from astrophysics.

Contents

Preface

Over the last century the hectic world of theoretical physics has flourished like never before. The discovery of special relativity and the development of quantum mechanics paved the way for a chain of new theoretical breaktroughs and discoveries. In a relatively short period of time, theories grew to give very precise descriptions of not only the world around us, but also the universe at scales that are unimaginable small or astronomically big.

One of the most recent discoveries in this ever expanding web of theoretical physics is string theory. Without getting into too much detail about this promising young theory, we will explore one of its features in this article: extra dimensions. To be more precise, we will explore some of the possibilities extra dimensions have to offer, but most of all: the constraints that restrict their existence.

This article was created as a part of the second year project for physics students at the University of Amsterdam (UvA). Our project group originally consisted out of six second year students. Together we worked through a couple of exercise sets that made us calculate and fully grasp every step mentioned in sections 2 - 5 and parts of section 6. These first couple of sections will be more or less the same in their and our report, as we were first studying as a group of six. We would like to thank them for their help on these sections. After the introduction we were divided into two groups. One group went on to study another fascinating phenomena concerning extra dimensions; the possibility to create miniature black holes in particle accelerators. We devoted this spring to the study of short-range gravity and indications that this subject can give us for the existence of extra dimensions.

We want to express our deep gratitude to Nick Jones, our project leader, for his patience, devotion and his ever clear explanations.

1 Introduction

Before we begin our journey through the physics of higher dimensions, we should make some things clear. To begin with: what do we mean with a dimension?

You can define the number of dimensions as the number of independent directions along which you can travel. When we look around us, we see one direction in front of us, one aside of us and one above. We count three directions and the conclusion is rapidly drawn that we must live in three dimensions. You can also define the number of dimensions as the (precise) number of coordinates you need to completely pin down a point in space. Mathematicians use in this context the notation (x, y, z) and add a new symbol for every new dimension. We can visualize the three dimensions we just counted by three coordinate axes. One extra dimension simply requires adding another axis, independent of the first three axis. This created 4D coordinate system may be hard to draw, but must not be hard to imagine. When the extra dimensions come along we better stop visualizing and continue with words and equations.

Everything around us, since the day we were born, suggests that we are living in three spatial dimensions. This state of mind remains for almost every person on earth. A small group, who is introduced to special relativity, discovers the combination of our three dimensions of space with one dimension of time, forming the space-time continuum. In this construct space and time become inseparable. This group of privileged people is offered time as the 4th dimension. Mathematically speaking this means that you need four coordinates to describe an event in space-time. Now we are getting to the even smaller crowd of people who are considering even more spatial dimensions. This last gang consists of well educated people, most of them physicists. These are normally not the kind of folks who would believe in science fiction. What has gotten into them? Why seriously think about extra dimensions that you cannot even see?

First we have to note that not having seen another dimension does not mean that there is none. In addition, no physical theory states that there must be only three dimensions of space. Still, this does not explain why we would go through a whole lot of trouble working out the mathematics in a more dimensional world. The best motivation comes from the search for the holy grail of physics: unification. Our heavyweight theory of today is the standard model. This theory describes three out of four fundamental forces: the strong, weak and electromagnetic forces. However the standard model is not a complete theory. The theory does not include the fourth fundamental force: gravity.

To give an accurate description of gravity and how it behaves, we have to turn to Einstein's general relativity. The problem is that this theory breaks down at very short distances due to quantum mechanical effects. In addition, relativity is uncapable to describe all of the other forces. The unification of all the fundamental forces into one theory is the holy grail of physics: the fusion of Einstein's general relativity, with the standard model. The only serious candidate that can make this dream come true is string theory. In this model of theoretical physics, particles and forces are presented as tiny extended object: strings. It appears that string theory can only be consistent if there are up to seven additional spatial dimensions. [1]

But surely these extra dimensions cannot be of any shape or size? What are the constraints that we have to put on extra dimensions so that their existence does not contradict experimental evidence? The main focus of this article will be on the bounds that have to be put on extra dimensions, in order for them to stay consistent with astrological observations, experiments and theoretical frameworks.

In order to explain how these bounds can be obtained by experiment, we first have to explore the mathematical environment of higher dimensional physics to get an idea of what to measure. First we will discover how Newtonian gravity behaves in arbitrary number of dimensions. Then, in sections 3 and 4 we will introduce compactification, the idea that the additional spatial dimensions are wrapped into a very small volume, and we will work out the Newtonian gravitational potential in this new compactified space-time continuum. After this we turn to the hierarchy problem, which is believed to be solved if we assume extra dimensions. This will give a good impression on the consequences that extra dimensions have to our four-dimensional world. Finally, we turn to the experimental part of this article. We will consider various experiments, both laboratory work and astrophysical observations. They all have one thing in common: they restrict the size of the extra dimensions. At the end we will make a definite conclusion that will clearly show the conditions under which extra dimensions can exist.

2 Newton's law of Gravitation in more than 4D

From the preceding section we came up with the why of our extra dimensions quest. The next step we are going to take is looking at physics in a world with more spatial dimensions than the three we are used to in everyday life.

Let's consider Newton's gravitational force law. This icon of classical physics tells us how the gravitational force depends on the distance, r, between two massive objects m and M:

$$F(r) = \frac{G_N mM}{r^2} \tag{1}$$

with G_N the gravitational proportionality constant.

Every grown-up multidimensional theory which includes gravity should reproduce this formula. This will form a check point along the way. The manner in which this *inverse square law* depends on distance is strongly linked with the number of spatial dimensions. This number tells us how gravity diffuses as it spreads in space. Before we adjust our formula to more than three spatial dimensions it might be nice to have some picture of how this spreading takes place.

As a descriptive explanation we picture the problem of watering a plant in a garden. We distinguish between giving the plant the water through a nozzle or through a sprinkler. In figure 1 we can see the differences between the two methods. When we use the spout, all the water will end up

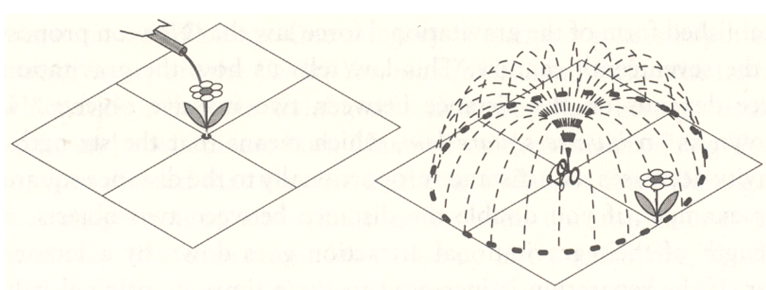

Figure 1: The amount of water delivered by the sprinkler is less than the amount delivered by the nozzle. (Figure from page 44 of [1])

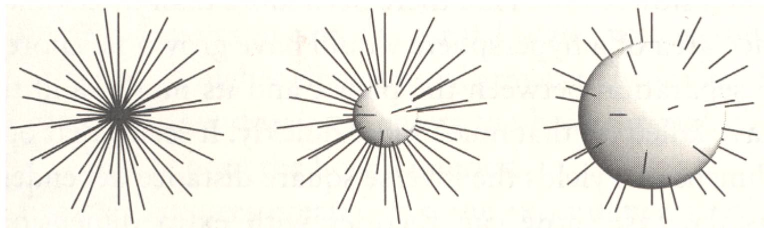

Figure 2: The same number of field lines intersect a sphere of any radius (Figure from page 45 of [1])

on the plant, whereas using the sprinkler the water is spread. Accordingly not every drop will reach the plant. Besides that, the distance between the sprinkler and the plant does matter as well. With the nozzle this is not the case. The importance of this illustration is to recognize the fundamental distinction between the two watering methods, that is, jumping to a higher dimension. The spout only gives water to a point (one dimension), other than the sprinkler, which distributes the water on to a surface (two dimensions). In general we can say that anything that is spread in more than one direction will have a lower impact on objects that are farther away. Similarly, gravity will spread more quickly with increasing distance.

We will represent the strength of gravity by field lines (in analogy with the sprinkler: the water flux). The line-density indicates the strength of the gravitational force at a given point. Because of the fact that gravity attracts all the surrounding mass isotropically, the field lines will go radially outwards. Therefore, as you can see from figure 2, the same number of field lines will intersect a sphere of any radius and the field lines become more diffuse along the way. Due to the fact that the fixed number of gravitational field lines are spread over a sphere's surface, we can conclude that the gravitational flux has to decrease with the radius squared (see page 42 - 46 of [1]).

The mathematical description of the above is given by Gauss' law, which, in a gravitational field, gives the relation between the gravitational flux flowing out of a closed surface and the mass enclosed by this surface. We will use Gauss' law to derive Newton's law of gravitation in more than four dimensions (we are considering space-time):

$$\int_{\text{surface}} \vec{g} \cdot d\vec{S} = -4\pi G_N M \tag{2}$$

where, $\vec{g}$, is the acceleration due to gravity, caused by a point mass, M. The integral is taken over any surface which completely surrounds the mass.

To use this law we have to realize a few things. First we choose the surface around the point mass to be a sphere. Secondly, $d\vec{S}$ is a unit vector pointing radially out of this sphere, whereas the direction of $\vec{g}$ is the negative radial direction and thus they are antiparallel. Now the integral becomes easy to solve, because we can pull out g (it has the same value everywhere on the surface and additionally we can work with the norm) and the integral is just the surface area of a sphere. Hence,

$$-g \int\limits_{\text{surface}} dS = -g \cdot 4\pi r^2 = -4\pi G_N M \tag{3}$$

To get equation (1) all there is left to do is substitute Newton's second law $F = m \cdot a$ with g as a and we obtain the desired result.

We are interested in expanding this relation to more dimensions. Assuming that (2) holds for more than three dimensions, we can easily rewrite (1) for d dimensions in terms of the volume of a $(d-1)$-dimensional sphere (note: $(d-1)$-sphere's live in d dimensions).
Further, if (2) holds in d dimensions, than (3) must hold as well. For 3 dimensions the integral on the left was just the surface area of a sphere, however now that we are in d dimensions it becomes the surface area of a $(d-1)$-sphere: $V_{d-1}(r)$ (We shall refer to this as their "volume"). We obtain,

$$gV_{d-1}(r) = 4\pi G_N M \Rightarrow F = \frac{4\pi G_N mM}{V_{d-1}(r)} \tag{4}$$

The last hurdle we have to take in order to produce Newtonian gravity in d dimensions is evaluating the volume of a $(d-1)$-dimensional sphere.

2.1 The "volume" of a $(d-1)$-dimensional sphere

Several times we were confronted with the surface or "volume" of a hypersphere. Let's look at this concept more precisely. A point in a d-dimensional Euclidean space is represented by $(x_1, x_2, \ldots x_d)$. The surface of a d-dimensional sphere of radius R is then defined by the equation:

$$x_1^2 + x_2^2 + \ldots + x_{d+1}^2 = R^2. \tag{5}$$

For a 0-dimensional sphere (5) reads $x_1^2 = R^2$, so $x_1 = \pm R$. The volume of a 0-dimensional sphere consists of two points at $+R$ and $-R$, living in a 1-dimensional world (we will not evaluate the "volume" just yet, but we shall see later on that it is 2).

A 1-dimensional sphere is given by the equation $x_1^2 + x_2^2 = R^2$. This is a circle in two dimensions and its volume can be obtained by integrating a infinitesimal bit of the circle's circumference $Rd\phi$ over the entire circle:

$$V_1 = \int_0^{2\pi} Rd\phi = 2\pi R \tag{6}$$

The 2-dimensional sphere is what we commonly hold for a sphere and its surface is given by $x_1^2+x_2^2+x_3^2 = R^2$. The volume of this sphere can also be obtained by integration. Only one more variable is needed, because there is one more dimension in this problem. We must integrate the infinitesimal bit $R\sin(\theta)d\theta$ over half a circle (the other integral will take it around the whole sphere). We obtain:

$$V_2 = \int_0^{2\pi} Rd\phi \int_0^{\pi} R\sin(\theta)d\theta = 4\pi R^2 \tag{7}$$

From all this we can make the generalization that the volume of a $(d-1)$-sphere must depend on $r^{(d-1)}$. Moreover from (4) we see that the gravitational force F must depend on $r^{-(d-1)}$.
We can ask ourselves how the preceding is applied to the volumes of spheres with dimensions higher than 2? We will answer this by deriving a formula for $V_{d-1}(r)$ in order to complete (4). To do this we use a trick. We will evaluate the integral I given by:

$$I = \int\limits_{\text{all space}} d^d r \; e^{-r^2}$$

in two ways; in Cartesian coordinates and in polar coordinates.
In Cartesian coordinates we know that $r^2 = x_1^2 + x_2^2 + \ldots + x_d^2$. Since the integral must be over all space, which is from $-\infty$ to $+\infty$ for every variable x_i with $1 \leq i \leq d$, the integral I becomes:

$$I = \int_{-\infty}^{+\infty} dx_1 dx_2 \ldots dx_d \; e^{-(x_1^2+x_2^2+\ldots+x_d^2)} \tag{8}$$

This is the known integral $\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi}$ in d dimensions. Hence, $I = \sqrt{\pi}^d = \pi^{\frac{d}{2}}$.

Now we will look at the problem in polar coordinates. To evaluate the integral over all space is the same as taking the volume of a $(d-1)$-sphere as a function of r and integrating that over all $r's$ (from 0 to ∞).

$$I = \int_0^{\infty} dr\; e^{-r^2} V_{d-1}(r) \tag{9}$$

We already know how $V_{d-1}(r)$ depends on r and d (quantitatively: $V_{d-1}(r) = c \cdot r^{d-1}$, where c is a constant). We can substitute this into the equation above.

$$I = \int_0^{\infty} dr\; e^{-r^2} c r^{d-1} \tag{10}$$

We can solve this in terms of the Gamma function:

$$\Gamma(x) \equiv \int_0^{\infty} t^{x-1} e^{-t} dt.$$

if we substitute $t = r^2$, we obtain

$$I = \frac{c}{2} \int_0^{\infty} dt\; e^{-t} t^{\frac{d}{2}-1} \tag{11}$$

(here we use that if $t = r^2$, then $dr = \frac{dt}{2r}$ and $r = t^{\frac{1}{2}}$). The integral is just the Gamma function for $x = \frac{d}{2}$ and we already know from the Cartesian coordinates what the answer should be.

$$I = \frac{c}{2} \Gamma(d/2) = \pi^{\frac{d}{2}} \tag{12}$$

This can be solved for c and that gives us a result for V_{d-1}:

$$V_{d-1}(r) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} r^{d-1}. \tag{13}$$

We can check this equation for $d = 1, 2, 3$ (remember: $\Gamma(n) = (n-1)!$, when n is an integer, $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = \sqrt{\pi}/2$). We find the same results we ran into earlier and now we see why the volume of a 0-dimensional sphere equals 2. In the rest of the article we will be running into this expression occasionally, but then we will refer to the volume of a unit sphere, which we will also call V_{d-1}. So from here we will drop the r-dependence.

We have almost achieved our goal. We can substitute our expression for $V_{d-1}(r)$ in equation (4) to end up with the expression for F in a d-dimensional space.

$$F = \frac{2\pi G_N m M \Gamma(\frac{d}{2})}{\pi^{\frac{d}{2}} r^{d-1}} \tag{14}$$

2.2 An application: Planetary orbits in higher dimensions

With the expression for the gravitational force in more dimensions, which we obtained in the preceding section, we can check if there can be stable planetary orbits in higher dimensions. To do so, we consider the total energy of the planet:

$$E = \frac{1}{2} m v^2 + V(r), \tag{15}$$

where the gravitational potential $V(r)$ is a function of r and alters when we change the number of spatial dimensions. By looking at the shape of the potential, we can draw conclusions about the stability of planetary orbits in higher dimensions. We can evaluate the potential by integrating the gravitational force:

$$V(r) = -\int_r^\infty F(r) dr = \begin{cases} k\frac{1}{(2-d)r^{d-2}} & d \neq 2 \\ k \ln r & d = 2 \end{cases} \tag{16}$$

where k is a constant which is different for different dimensions:

$$k = \frac{2\pi G_N m M \Gamma(\frac{d}{2})}{\pi^{\frac{d}{2}}} \geq 0$$

Because we are considering planetary orbits, we can be satisfied with a 2-dimensional description of the orbit. We can therefore write $\vec{r}$ and $\vec{v}$ in terms of 2-dimensional polar coordinates $(\hat{r}, \hat{\theta})$. $\vec{r}$ becomes $r\hat{r}$, while $\vec{v}$ (the time derivative of $\vec{r}$) becomes $\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$.
The angular momentum of the system must be constant, because we are looking at an orbit.

$$\vec{l} = \vec{r} \times \vec{p} = r\hat{r} \times m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) \tag{17}$$

$$l = 0 + mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{l}{mr^2}. \tag{18}$$

If we substitute the potential and $\vec{v} = (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$ in (15) we get the expression:

$$E = \frac{1}{2}m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})^2 + \begin{cases} k\frac{1}{(2-d)r^{d-2}} & d \neq 2 \\ k\ln r & d = 2 \end{cases} \tag{19}$$

working out the quadratic factor and substituting(18) for $\dot{\theta}$ we obtain:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + \begin{cases} k\frac{1}{(2-d)r^{d-2}} & d \neq 2 \\ k\ln r & d = 2 \end{cases} \tag{20}$$

We have reduced the problem to a one-dimensional situation (there is only r-dependance) and we have split (20) in two parts; one dependent of $\dot{r}$, associated with the kinetic energy of the system and one dependent on r, associated with the potential of the system. In order to see if a planetary orbit could be stable, we can look at the potential of the system in different dimensions.

$$V(r) = \frac{l^2}{2mr^2} + \begin{cases} k\frac{1}{(2-d)r^{d-2}} & d \neq 2 \\ k\ln r & d = 2 \end{cases} \tag{21}$$

We plotted the curve for $d = 3$ and $d = 5$, they are shown in figure 3. This potential can only have a minimum value for $d - 1 < 3$. For higher dimensions than three there is either no extreme value (for $d = 4$) or a maximum, which means that planetary orbits are not stable in higher dimensions. They

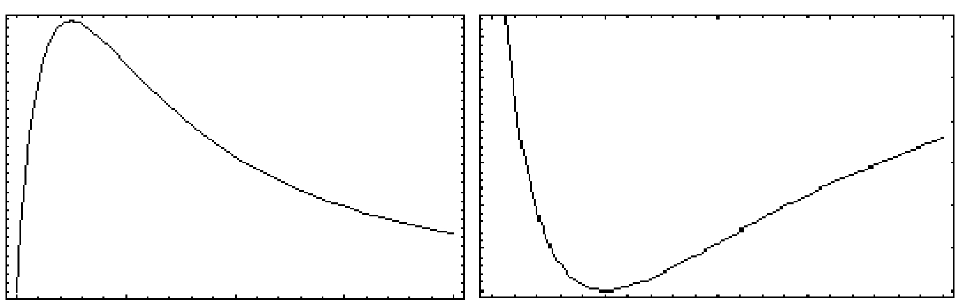

Figure 3: A plot of the gravitational potential for $d = 5$ and $d = 3$ at the left, respectively the right.

could exist, but the slightest knock would put them out of orbit. The radial position of the planets can *not* be stable.

This result should not be to surprising, since we have never seen a deviation to the inverse square law on ordinary distances. We *know* that Newton's law behaves as $\frac{1}{r^2}$ so this automatically leads to the conclusion that $d = 3$. Also the assumption that the extra dimensions are of infinite size sounds a little strange. If they were, why would we be uncapable to see them, or take a walk in them?

3 Compactification and Kaluza-Klein Reduction

3.1 Compactification

In the preceding sections we came up with some mathematical description of physics involving more spatial dimensions. Once more we can ask some questions that already arose in the introduction. How is it possible that the universe could appear to have only three dimensions of space if the fundamental underlying spacetime contains more? Why are they not visible? And if they are real, do these extra dimensions have any discernible impact on the world we see?

The idea of extra dimensions was first introduced in 1919 by the Polish mathematician Theodor Kaluza. He recognized the possibility of extra dimensions in Einstein's theory of relativity. The questions that are bothering us were the same questions asked then by Einstein. These questions remained unanswered until 1926, when the Swedish mathematician Oskar Klein shed some light on the case. He provided a reasonable explanation to these problems, by imagining these extra dimensions to be compactified, meaning that they are curled up into circles so tiny that we would not ever suspect their existence (just 10^{-35}m). This concept is now known as Kaluza-Klein reduction. Thus in the Kaluza-Klein universe space can have *not only* extended dimensions like the ones we are familiar with, but also extremely small curled-up dimensions.

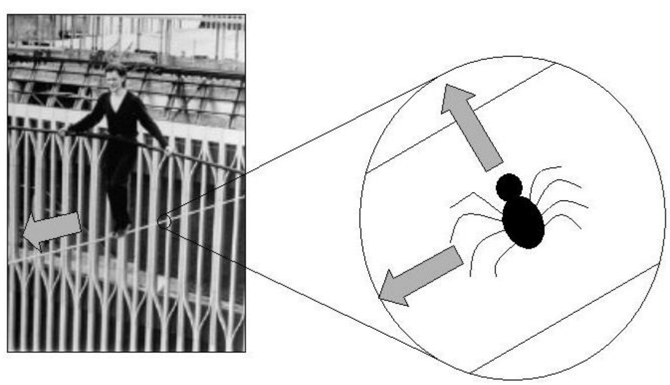

Figure 4: The tight-rope artist can move in one direction, but the spider can move in two directions along the cord. So the tight-rope appears one-dimensional for the artist and two-dimensional for the spider

In this section we will try to get a good idea on how these extra curled-up dimensions look like and how the physics works out when we assume one extra dimension to be compactified on a circle. We will see that in this new space-time, gravity and the Coulomb force can be unified in one expression.

To begin with, how can we imagine space to be curled up? The best way to make this clear might be by considering the one-dimensional world of the tight-rope artist. In figure 4 you can see that the tight-rope artist is restricted to the tight-rope. He can move in only one direction, so the rope appears 1-dimensional to him. However, if we zoom in on the tight-rope, we can see a little spider crawling along the cord. This spider is confined to the same rope, but he has two independent directions along which he can travel: he can move along the length of the cord, or go around it in circles. The same tight-rope appears two-dimensional to the spider.

This example shows that a universe that has just one extended dimension to large objects, might look multi-dimensional to small objects. We can generalize this picture to a universe with more extended dimensions, like our well know four-dimensional world, which has three extended spatial dimensions. This universe might also look more dimensional for very small objects, because the extra dimensions are curled up. Of course, these curled-up dimensions do not have to be in the shape of a circle, they can be any shape you would like to imagine them, as long a they are periodic in space. For a more detailed look into imagining compact extra dimensions see chapter 8 of [2].

3.2 KK-modes

Let's look a bit closer at the Kaluza-Klein compactification, where we assume one extra dimension to be curled-up on a circle of radius R. When we consider one relativistic particle with mass m in this 5-dimensional world, the 5-momentum for this particle is given by:

$$p^{(5)} \equiv (\frac{E}{c}, p_1, p_2, p_3, p_4). \tag{22}$$

We know from special relativity that:

$$p^{(5)} \cdot p^{(5)} = -\frac{E^2}{c^2} + p_1^2 + p_2^2 + p_3^2 + p_4^2 = -m^2c^2. \tag{23}$$

Where $(p_1, p_2, p_3) = \tilde{p}$ is the momentum in the three extended dimensions and p_4 is the momentum in the compactified dimension. We can rewrite this as an equation for the energy of the particle:

$$E^2 = m^2c^4 + \tilde{p}^2c^2 + p_4^2c^2. \tag{24}$$

When we compare this equation to the equation for the energy in 4D:

$$\frac{E^2}{c^2} = M^2c^2 + \tilde{p}^2 \tag{25}$$

where M is the mass in four dimensions, we see there is an extra term in equation (23). This means that, since E and c are the same for both 4D en 5D, the mass m in 5D has to be smaller than the mass M in 4D to compensate for the extra factor p_4^2.

We can find an explicit expression for p_4, using quantum mechanics. The direction of the particle in the curled-up dimension is periodic over time ($x_4 = x_4 + 2\pi R$). We therefore consider a particle stuck on a circle. Realizing that the particle's wavefuntion $\Psi(x,p) \propto e^{\frac{ipx}{\hbar}}$ it must be single-valued. Its wavelength has to fit a whole number of times on this circle. Hence,

$$e^{\frac{ip_4x_4}{\hbar}} = e^{\frac{ip_4(x_4+2\pi R)}{\hbar}} \tag{26}$$

leads to:

$$p_4 = \frac{n\hbar}{R} \tag{27}$$

We see that the momentum in the extra dimensions is proportional to $\frac{1}{R}$. The situation were $n = 1$ or -1 is called the first Kaluza Klein modes (or short: KK-mode). From the comparison of (24) and (25) we know that the KK-modes can be associated with mass. The momentum of the particle in the curled-up dimension translates to mass in four dimensions.

3.3 Kaluza-Klein reduction

Now we observe two particles in 5D instead of just one. If we take the 5-momenta of the two particles to be p and p', then the gravitational force between the particles, when they are at rest in the extended dimensions (so $\tilde{p}$ equals zero) and separated by $r \gg R$, can be approximated by the semi-relativistic expression:

$$F(r) \simeq \frac{G_N}{c^2} \frac{p \cdot p'}{r^2}. \tag{28}$$

We can rewrite this force by substituting $E = Mc^2$ (with M, the mass in the extended dimensions), $\tilde{p} = 0$ and equation (27) in:

$$p \cdot p' = -\frac{EE'}{c^2} + \tilde{p} \cdot \tilde{p}' + p_4 p_4' \tag{29}$$

so that (28) becomes:

$$F(r) = -\frac{G_N M' M}{r^2} + \frac{G_N n n' \hbar^2}{c^2 R^2 r^2}. \tag{30}$$

In this equation we can identify both Newton's gravitational force (the first term) and the Coulomb force (the second term). Notice that the second term can be both negative and positive (since n can be both as well), which states that this term can be both attractive and repulsive. To get all the constants right so that the second term is exactly the Coulomb force (which is given by $F_c = \frac{qq'}{4\pi\epsilon_0 r^2}$), we would have to take $n = n' = 1$ for the elementary charge q and then work out what R must be. It turns out that the size of the extra dimension must be $1.8 \cdot 10^{-34}$ m for the second term in (30) to represent the Coulomb force!

4 Gravitational potential in n compactified dimensions

The result we obtained in the last section that combined Newton's gravity and Coulombs electromagnetic force in one expression is promising for further unification. This provides an excellent motivation for considering the gravitational potential in more than one compactified dimension.

In section 2.1 en 2.2 we have derived Newton's gravitational potential in d spatial dimensions (equation 16):

$$V(r) = k\frac{1}{r^{d-2}} \tag{31}$$

where $k = \frac{2\pi G_N mM\Gamma(\frac{d}{2})}{(2-d)\pi^{\frac{d}{2}}}$. We can express the potential in $4+n$ spacetime dimensions as, using G_{4+n}: Newton's gravitational constant in $n+4$ spacetime dimensions:

$$V(r) = -\frac{G_{4+n}mM}{r^{n+1}} \tag{32}$$

We now turn to the question what the gravitational potential would look like, if the additional n dimensions were compactified on to a circle with radius R. We can imagine this as we consider a mass M on a cylinder. The length of the cylinder represents the three unfolded spatial dimensions and the radius the n compactified dimensions. We can solve the problem using the method of images (just like the electrodynamic problem of a point charge on a cylinder).
If we first imagine n to be 1, we can unfold the extra dimension, so to get an infinite extra dimension, with the mass M repeated every $2\pi R$. Equation (32) now becomes an infinite sum over all the masses. The distance to the mass becomes $\sqrt{r^2 + (b2\pi R)^2}$, where b is an integer going from $-\infty$ to ∞. Following this reasoning, the potential becomes:

$$V(r) = -\sum_{b=-\infty}^{\infty} \frac{G_{n+4}mM}{[r^2 + (b2\pi R)^2]^{\frac{1}{2}}} \tag{33}$$

If we generalize this to n compactified dimensions, we get an expression in terms of n infinite sums:

$$V(r) = -\sum_{b_1=-\infty}^{\infty} \cdots \sum_{b_n=-\infty}^{\infty} \frac{G_{n+4}mM}{[r^2 + (b_1 2\pi R)^2 + \cdots + (b_n 2\pi R)^2]^{\frac{n+1}{2}}} \tag{34}$$

In the limit of $r \gg R$ these sums can be replaced by integrals. This is because the pieces of $2\pi R$ are small in comparison to r, that it can be approximated as infinitely. Now (34) can be written as:

$$V(r) = -\int\limits_{b_1=-\infty}^{\infty} \cdots \int\limits_{b_n=-\infty}^{\infty} \frac{G_{n+4}mM}{\left[r^2 + (b_1 2\pi R)^2 + \cdots + (b_n 2\pi R)^2\right]^{\frac{n+1}{2}}} db_1 \dots db_n \tag{35}$$

To clean up this expression we divide the denominator by r^2 and substitute $x_i = \frac{b_i 2\pi R}{r}$ to get:

$$V(r) = -\frac{G_{n+4}mM}{r(2\pi R)^n} \int\limits_{x_1=-\infty}^{\infty} \cdots \int\limits_{x_n=-\infty}^{\infty} \frac{1}{(1 + x_1^2 + \cdots + x_n^2)^{\frac{n+1}{2}}} dx_1 \dots dx_n \tag{36}$$

Now we can switch to polar coordinates using the volume of a $n-1$ dimensional sphere ($V_{n-1}(\rho)$). The integral can be written (We will use ρ for the radial variable, because we already have a different r in the expression):

$$V(r) = -\frac{G_{n+4}mM}{r(2\pi R)^n} \int_0^{\infty} V_{n-1}(\rho) \frac{1}{(1+\rho^2)^{\frac{n+1}{2}}} d\rho \tag{37}$$

If we now substitute u for ρ^2, we can solve this in terms of the Beta function (from [17]);

$$B(p+1, q+1) = \int_0^{\infty} \frac{u^p du}{(1+u)^{p+q+2}} = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)} \tag{38}$$

Working out the integral and writing the Beta function in terms of Gamma-functions yields:

$$V(r) = -\frac{V_{n-1}G_{n+4}mM}{2\Sigma_n} \frac{1}{r} \tag{39}$$

where V_n is the volume of a n dimensional unit sphere given by (13) and Σ_n is the volume of the extra dimensions (in this case $\Sigma_n = (2\pi R)^n$). The equation looks just like the gravitational force we know in 4D. This is what we would expect, since we are working in the limit of $r \gg R$. We can acquire an expression for G_{n+4} in terms of G_N 39. We obtain,

$$G_{n+4} = \frac{2G_N \Sigma_n}{V_{n-1}} \tag{40}$$

If we assume the extra n dimensions to be compactified on a circle, we obtain the usual inverse square law for distances much greater than the size of the extra dimensions. However, if we look at distances close to R, the approximation of a sum by an integral no longer stands. Therefore we would expect the gravitational force to deviate from Newtonian gravity and pick up an extra correction term. These are often referred to as Yukawa-type corrections (see, for instance [6]).

$$V(r) \sim \frac{1}{r}(1 + \alpha e^{\frac{-r}{\lambda}} + \ldots) \tag{41}$$

4.1 The exact gravitational potential for $n = 1$

If we take n to be 1, we can work out (34), obtaining the exact potential. We can then check that the expression for the potential satisfies our expectations. If we calculate the limit of $r \gg R$ we want to find the $\frac{1}{r}$ potential, while taking the other limit ($R \gg r$) we would expect the potential, derived in section 2.2 to appear. In the last case we are looking at such small distances that the extra dimensions appear very large. For $n = 1$ equation (34) becomes:

$$V(r) = -\sum_{b=-\infty}^{\infty} \frac{G_{4+1} mM}{r^2 + (b2\pi R)^2} \tag{42}$$

Now we divide the denominator by $2\pi R$ and use the identity:

$$\sum_{m=-\infty}^{\infty} \frac{1}{m^2 + a^2} = \frac{\pi}{a} \coth(\pi a). \tag{43}$$

to get:

$$V(r) = -\frac{G_{4+1} mM}{2rR} \coth\left(\frac{r}{2R}\right) \tag{44}$$

We will first check the limit $R \gg r$, so we will need the limit $\lim_{x \downarrow 0} \coth x = \frac{2+x^2}{2x}$, where we can forget about the quadratic term, because in this case it would be extremely small.

$$V(r) = -\frac{G_{4+1} mM}{r^2} \tag{45}$$

which is exactly what we would expect!

Now we use the limit $\lim_{x\to\infty}\coth x = 1$ to get the other limit:

$$V(r) = -\frac{G_{4+1}mM}{2Rr} \tag{46}$$

which is exactly (39) for $n = 1$.
We have seen that for $n = 1$ the potential satisfies what we would physically expect. The exponential power in (41) can be obtained by calculating the first order of corrections to equation (46). We can calculate this corrections, by keeping the first correction term of the approximation we made to get from (44) to (46). To simplify matters, we write

$$V(r) \sim \frac{1}{r}\coth\left(\frac{r}{2R}\right) \tag{47}$$

When we work out $\coth$ and multiply by $e^{\frac{-r}{2R}}$ we obtain

$$V(r) \sim \frac{1}{r}\left(\frac{1+e^{\frac{-r}{R}}}{1-e^{\frac{-r}{R}}}\right) \tag{48}$$

In the limit where $r \gg R$, the term $e^{\frac{-r}{R}}$ goes to zero. This is where we made the approximation (see [18])

$$V(r) \sim \frac{1}{r}(1+e^{\frac{-r}{R}})(1+e^{\frac{-r}{R}}+\ldots) \tag{49}$$

Working out the brackets and omitting the last term leaves us with

$$V(r) \sim \frac{1}{r}(1+2e^{\frac{-r}{R}}+\ldots) \tag{50}$$

which is essentially the result stated in 41.

5 Deviations from Newton's inverse square law

In the preceding section we have derived an expression for the gravitational potential in n compact dimensions. We have also stated that, looking at the problem in four dimensions, this leads to a deviation from the $\frac{1}{r}$ Newtonian potential for distances close to the size of the extra dimensions. This new potential is given by:

$$V(r) \sim \frac{1}{r}(1 + \alpha e^{\frac{-r}{\lambda}}) \tag{51}$$

This is what we stated in equation (41) (see, for instance [6]).

This equation is very useful for experimental physicists, because it gives a description of how the gravitational potential would change if there were compactified extra dimensions. If someone would be able to measure this deviation from Newton's law, this would be a strong proof that extra dimensions do in fact exist.

What is the physical interpretation of this deviation? Moreover, why is it an exponential power, and not some factor of $\frac{1}{r^{n+1}}$ as in equation (32)?

To answer the first question is that we just have to look closely at the formula above. The exponential power tells us that gravity will become stronger at shorter distances. Here we can see that the parameter α accounts for the strength of the deviation. It tells you *how much* stronger it gets. While the λ parameter tells you at what distance the deviation begins to have effect. Intuitively, this parameter is related to the size of the extra dimensions, because the deviation only starts to have effects at distances close to the size of the extra dimension.

We can try to measure these two parameters in order to put experimental constraints on the size of the compactified dimensions. High precision short range measurements of the inverse square law that do not show any deviation from the gravitational force or potential give us strong upper bounds on α and λ and thereby on the size of the extra dimensions. We can also get experimental bounds on the size of extra dimensions from astrophysical constraints, as we will show in section 8.

The rest of this section is devoted to the answer of the second question mentioned above. We will derive (51) in two ways. First we consider the case of n extra dimension compactified on a circle (or torus) as we did in section 4. This will give us theoretical values for α and λ. Next we will consider the more general case, where the geometry of the extra dimensions is unknown. Research on this subject is done in [4].

5.1 Deviations from compactification on a n torus

Just like before we consider $4+n$ dimensional space-time and we let the extra n dimensions to be compactified on a circle of radius R. We assume all the extra dimensions to be of the same size, although it does not change anything in the derivation if they are not (then you just have to read R_i where we put R). As shown in section 4 the gravitational potential is given by n infinite sums (see equation (34)).

Unlike before, now we consider not only the points in the same 4D direction as the mass M, but we also consider points that have a direction in the extra dimensions. We write $\{x_i\}_{i=1}^n = \mathbf{x}$ for the direction vector in the extra dimensions. It is not that hard to see that (34) now becomes:

$$V(r) = -\sum_{\boldsymbol{b}\in\mathbf{Z}} \frac{G_{n+4}mM}{\left[r^2 + \sum_{i=1}^n (x_i - 2\pi R b_i)^2\right]^{\frac{n+1}{2}}} \tag{52}$$

(see (2) of [4])where we can consider $\boldsymbol{b} = \{b_1, b_2, \ldots, b_n\}$ to be a vector in the n-dimensional lattice, that is: every coefficient of $\boldsymbol{b}$ can take every integer number.

The trick we use to evaluate these sums is called Poisson resummation. It allows you to rewrite the sum over a periodic function $f(nR)$ with $n \in \mathbf{Z}$ to a sum over its Fourier transform $\tilde{f}$ (from [15]).

$$\sum_{n=-\infty}^{\infty} f(nR) = \sum_{n=-\infty}^{\infty} \frac{\tilde{f}(\frac{n}{R})}{2\pi R} \tag{53}$$

We will apply Poisson resummation to our potential in (52) and although our function is periodic in n directions, the procedure is straightforward. Realizing that:

$$\tilde{f}(\boldsymbol{m}) = \int d^n\mathbf{y} \frac{G_{n+4}mMe^{-i\boldsymbol{m}\cdot\mathbf{y}}}{\left[r^2 + \sum_{i=1}^n (x_i - y_i)^2\right]^{\frac{n+1}{2}}} \tag{54}$$

where $\boldsymbol{m} = \{\frac{b_1}{R}, \frac{b_2}{R}, \cdots, \frac{b_n}{R}\}$ and $\mathbf{y}$ is a vector in n space ($\mathbf{y} = \{y_i\}_{i=1}^n$). We see from (53) that our potential becomes:

$$V(r) = -\frac{G_{n+4}mM}{\Sigma_n} \sum_{\boldsymbol{b}\in\mathbf{Z}} \int d^n\mathbf{y} \frac{e^{-i\boldsymbol{m}\cdot\mathbf{y}}}{[r^2 + \sum_{i=1}^n (x_i - y_i)^2]^{\frac{n+1}{2}}} \tag{55}$$

(see (4) of [4]) This can be evaluated if we shift the $\mathbf{y}$ coordinate to $\mathbf{y}+\mathbf{x}$ (which is allowed because we integrate over all space) and then change to polar coordinates. When we change to polar coordinates, we have to realize that the exponential depends on θ, where θ is the angle between $\boldsymbol{m}$ and $\mathbf{y}$. This is because its argument hold an inner product between two vectors in n space. Because the function holds a θ dependance, the change to polar coordinates is given by:

$$\int d^n\mathbf{y} = \int_0^\infty d\rho \int_{-\pi}^{\pi} d\theta \rho^{n-1} (\sin\theta)^{n-2} V_{n-2} \tag{56}$$

with V_{n-2} the volume of a $(n-2)$-dimensional unit sphere and $\rho = |\mathbf{y}| = \sqrt{y_1^2 + \ldots + y_n^2}$. It follows that:

$$V(r) = -\frac{V_{n-2}G_{n+4}mM}{\Sigma_n} \sum_{\boldsymbol{b}\in\mathbf{Z}} e^{-i\boldsymbol{m}\cdot\mathbf{x}} \int_0^\infty d\rho \frac{\rho^{n-1}}{[r^2+\rho^2]^{\frac{n+1}{2}}} \int_{-\pi}^{\pi} d\theta e^{-i|\boldsymbol{m}|\rho\cos\theta} (\sin\theta)^{n-2} \tag{57}$$

Substituting u (and $du = \sin\theta d\theta$) for $\cos\theta$ and taking the real part gives:

$$V(r) = -\frac{V_{n-2}G_{n+4}mM}{\Sigma_n} \sum_{\boldsymbol{b}\in\mathbf{Z}} e^{-i\boldsymbol{m}\cdot\mathbf{x}} \int_0^\infty d\rho \frac{\rho^{n-1}}{[r^2+\rho^2]^{\frac{n+1}{2}}} \int_{-1}^{1} du \cos(|\boldsymbol{m}|\rho u)(1-u^2)^{\frac{n-3}{2}} \tag{58}$$

(see (7) of [4]) Note that $|\mathbf{m}| = (\frac{b_1^2}{R^2} + \frac{b_2^2}{R^2} + \cdots + \frac{b_n^2}{R^2})$ are the masses of the KK-modes derived in section 3.2. This equation might seem a handful, however, by use of the following integrals (from 8.411(8), 6.565(3) of [14]) it is quite easy to get to the final answer.

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})} \int_{-1}^{1} (1-t^2)^{\nu-\frac{1}{2}} \cos(zt) dt \tag{59}$$

$$\int_0^\infty x^{\nu+1}(x^2+a^2)^{-\nu-\frac{3}{2}} J_\nu(bx) dx = \frac{b^\nu\sqrt{\pi}}{2^{\nu+1} a e^{ab} \Gamma(\nu+\frac{3}{2})} \tag{60}$$

here J_ν is the Bessel function of order ν. After performing the integrals we find:

$$V(r) = -\frac{G_4 mM}{r} \sum_{\boldsymbol{b}\in\mathbf{Z}} e^{-r|\mathbf{m}|} e^{-i\boldsymbol{m}\cdot\mathbf{x}} \tag{61}$$

with G_4 defined in (40).

At this point we need to remember that we are searching for an expression for how gravity would behave in 4D, assuming extra toroidal compactified dimensions. Since all point particles in four-dimensional space-time can be taken to have $\mathbf{x} = \mathbf{0}$, we get to the four dimensional gravitational potential:

$$V_4(r) = -\frac{G_4 mM}{r} \sum_{\boldsymbol{b}\in\mathbf{Z}} e^{-r|\mathbf{m}|} \tag{62}$$

(see (9) of [4]) Obviously, the term with $\boldsymbol{b} = 0$ results in Newton's expression. As stated above, the second term (the one with $|\boldsymbol{b}| = 1$) can be associated with the lightest KK-mode. There are $2n$ of these modes (two in every extra dimensions, since b_i can be either 1 or -1.) and all of them have mass $\frac{1}{R}$. Thus the gravitational potential can be approximated by:

$$V_4(r) = -\frac{G_4 mM}{r}(1 + 2n\, e^{-r/R}) \tag{63}$$

Here we see that for compactification on a n-torus, we get explicit values of α and λ in equation (51). Especially nice is the fact that $\lambda = R$ in this case, thus the deviations from the $\frac{1}{r}$ potential are starting to have a noticeable effect at distances in the order of R. In the case of $n = 2$ we will show in section 6.2 that R must be in the order of millimeters. Therefore, sub-millimeter tests of the inverse square law must be able to strongly bound the existence of two extra toroidal dimensions.

5.2 Deviations from the 1/r potential in a n-dimensional compact manifold

We have no idea how the extra dimensions would look like, if they exist at all. It is easy to picture them as being rolled up into a finite size, but for all we know it might be a very complex geometrical shape. However, without saying something about the precise geometry of the extra dimensions, we can still work out an expression for the 4-dimensional potential that gives a correction to Newton's potential.

In the Newtonian limit the $4+n$ potential obeys the Poisson equation (this is of course in $3+n$ dimensions, since the Laplacian does not cover time):

$$\nabla^2_{3+n} V_{4+n} = (n+1) V_{n+2} G_{4+n} M m \delta^{(n+3)}(\mathbf{x}) \tag{64}$$

(see (14) of [4])In a flat and uncompactified space, this equation is solved by (32), but we are looking for an expression for V_{4+n} with n dimensions on a compact manifold M^n. We do not know the precise geometry of the n-dimensional compact manifold, but we can still define a set of functions $\{\Psi_{\boldsymbol{m}}\}$ as eigenfunctions of the Laplacian operator in n dimensions.

$$\nabla^2_n \Psi_{\boldsymbol{m}} = -\mu^2_{\boldsymbol{m}} \Psi_{\boldsymbol{m}} \tag{65}$$

(see (13) of [4])This set is orthogonal:

$$\int\limits_{M^n} \Psi_{\boldsymbol{n}}(\mathbf{x}) \Psi^*_{\boldsymbol{m}}(\mathbf{x}) = \delta_{\boldsymbol{n},\boldsymbol{m}}, \tag{66}$$

(see (11) of [4]) and complete. ($\boldsymbol{n}$ and $\boldsymbol{m}$ are vectors the n-dimensional lattice, just like $\boldsymbol{b}$ was in the last section.)

Now the trick is to solve equation (64) for V_{4+n} with n dimensions compactified by separation of variables. We will expand V_{4+n} in terms of the basis of eigenfunctions of the Laplace operator in the n-dimensional compact manifold, $\{\Psi_{\boldsymbol{m}}\}$.

$$V_{4+n} = \sum_{\boldsymbol{m}} \Phi_{\boldsymbol{m}}(r) \Psi_{\boldsymbol{m}}(\mathbf{x}) \tag{67}$$

(see (15) of [4]) Note that $\Phi_{\boldsymbol{m}}$ is only dependent of r. Substituting this in equation (64) yields for the left hand side:

$$\sum_{\boldsymbol{m}} [\Psi_{\boldsymbol{m}}(\mathbf{x}) \nabla^2_3 \Phi_{\boldsymbol{m}}(r) + \Phi_{\boldsymbol{m}}(r) \nabla^2_n \Psi_{\boldsymbol{m}}(\mathbf{x})] \tag{68}$$

$$\sum_{\boldsymbol{m}} \Psi_{\boldsymbol{m}}(\mathbf{x}) [\nabla^2_3 \Phi_{\boldsymbol{m}}(r) - \mu^2_{\boldsymbol{m}} \Phi_{\boldsymbol{m}}(r)] \tag{69}$$

Now we use the orthogonality of $\{\Psi_{\boldsymbol{m}}\}$ by multiplying both sides of (64) with $\int_{M^n} \Psi^*_{\boldsymbol{m}}(\mathbf{x})$. We obtain:

$$\begin{aligned}\nabla_3^2\Phi_{\boldsymbol{m}}(r) - \mu_{\boldsymbol{m}}^2\Phi_{\boldsymbol{m}}(r) &= \textstyle\int_{M^n} \Psi_{\boldsymbol{m}}^*(\mathbf{x})(n+1)V_{n+2}G_{n+4}Mm\delta^{(n+3)}(\mathbf{x}) \\ &= (n+1)V_{n+2}\Psi_{\boldsymbol{m}}^*(\mathbf{0})G_{n+4}Mm\delta^{(3)}(r)\end{aligned}$$

(for the above two lines of equations see (16) of [4])To solve this equation, you can fill in $\Phi_{\boldsymbol{m}}$ as a Fourier transform: $\Phi_{\boldsymbol{m}}(r) = \int d^3\mathbf{k}\tilde{\Phi}_{\boldsymbol{m}}(\mathbf{k})\ e^{i\mathbf{k}\cdot\mathbf{r}}$. The Fourier transform of the delta function equals one, so we get an expression of $\tilde{\Phi}_{\boldsymbol{m}}(\mathbf{k})$.

$$\tilde{\Phi}_{\boldsymbol{m}}(\mathbf{k}) = -\frac{(n+1)V_{n+2}\Psi_{\boldsymbol{m}}^*(\mathbf{0})G_{n+4}Mm}{k^2+\mu_{\boldsymbol{m}}^2} \tag{70}$$

Plugging this back into the definition of the Fourier transform would give us an expression for $\Phi_{\boldsymbol{m}}(r)$, we would just have to solve:

$$\Phi_{\boldsymbol{m}}(r) = -(n+1)V_{n+2}\Psi_{\boldsymbol{m}}^*(\mathbf{0})G_{n+4}Mm\int d^3\mathbf{k}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2+\mu_{\boldsymbol{m}}^2} \tag{71}$$

Once again we can change to polar coordinates and pay attention to the inner product in the exponential, it will give us a $\cos\theta$ term. We have looked up the final integral over k and found for $\Phi_{\boldsymbol{m}}(r)$ (from 3.723(3) of [14]):

$$\Phi_{\boldsymbol{m}}(r) = -\frac{V_n\Psi_{\boldsymbol{m}}^*(\mathbf{0})G_{n+4}Mm}{2}\frac{1}{r}e^{-\mu_{\boldsymbol{m}}r} \tag{72}$$

(see (17) of [4]) Now equation (67) looks like:

$$V_{n+4} = -\frac{V_nG_{n+4}Mm}{2r}\sum_{\boldsymbol{m}}\Psi_{\boldsymbol{m}}^*(\mathbf{0})\Psi_{\boldsymbol{m}}(\mathbf{x})e^{-\mu_{\boldsymbol{m}}r} \tag{73}$$

(see (18) of [4]) Just like before, we can set $\mathbf{x} = \mathbf{0}$. Also like before, we would expect some of the exponential powers in the sums to be the same. Remember that we had a degeneracy of $2n$ for the first KK-mode in the case of the n-torus: we could go around the torus in one direction and the other way around. Consequently, symmetry in the compact manifold M^n would also lead to some kind of degeneracy. We can change this sum above to a sum which is only over all the irreducible representations $\boldsymbol{m}_{ir}$

$$V_n = -\frac{G_nMm}{r}\sum_{\boldsymbol{m}_{ir}}d_{\boldsymbol{m}_{ir}}e^{-\mu_{\boldsymbol{m}_{ir}}r} \tag{74}$$

(see (19) of [4]) We have used the theoretical result that the sum of $|\Psi_{\boldsymbol{m}_{ir}}|^2$ over all representatives equals $d_{\boldsymbol{m}_{ir}}/\Sigma_n$. Now $\Sigma_n \neq (2\pi R)^n$, but it gives the volume of the compact manifold M^n. Additionally, $d_{\boldsymbol{m}_{ir}}$ is the degeneracy of each irreducible representation.

This is about as far as it goes without getting into the actual shape of M^n. However, if you have a nice idea in mind for the shape of your extra dimensions, all you have to do is calculate the eigenvalues of the Laplace operator $\mu_{\boldsymbol{m}}$ and its degeneracy $d_{\boldsymbol{m}}$, and you will know the theoretical values for α and λ.

6 Fundamental energy scales and the Hierarchy Problem

Before we dive into the experimental evidence for constraints on large extra dimensions, we will take a look at some important energy scales at which physical processes take place. We will see that the fundamental scale at which gravity becomes strong is much bigger than the fundamental scale at which the other forces operate. We will show in section 6.2 how extra dimensions can nullify this difference, but first we take a look at these energy scales and how they are related to length.

6.1 Terminology

To get a good idea of the relation between energy and length, picture the following. Only particles whose wavefunctions differ over very small scales will be affected by physical processes taking place at short distance. However, when we recall the de Broglie relation, particles whose wavefunction involve short wavelengths also have large momenta. Consequently we can conclude that you need high momenta, and hence high energies, to be sensitive to the physics of short distances ([1] page 143).

Therefore, particle physicists use the term energy scale for the place on the ladder of energy that indicates the amount of energy they need to probe the physical processes they want to study. We can convert an energy into a corresponding length with the formula $E = \frac{hc}{\lambda}$ and into mass using $E = mc^2$. This length we just acquired illustrates the range of the associated force. The "ladder of energy" just mentioned is pictured in figure 5 along with the corresponding length scale. We see that climbing up the ladder in energy, means climbing down in length.

Now we have had some conceptual thought of energy scales, we should write down equations. The fundamental units G_N, $\hbar$ and c can be combined into a new quantity with units of mass, and another with units of length. These quantities are called the *Planck mass* (M_p) and the *Planck length* (l_p):

$$M_p \equiv \sqrt{\frac{\hbar c}{G_N}} = 2.2 \cdot 10^{-8} kg \sim 1.2 \cdot 10^{28} eV. \tag{75}$$

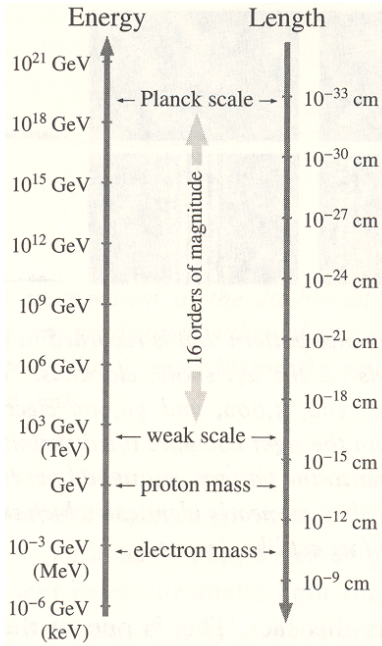

Figure 5: Some important length and energy scales in particle physics. (Figure from page 136 of [1])

and:

$$l_p \equiv \sqrt{\frac{\hbar G_N}{c^3}} = 1.6 \cdot 10^{-35} m \tag{76}$$

We see from equation (75) that Newton's gravitational constant is inversely proportional to the Planck mass and thus also related in this way to a concept called Planck energy.

In nature there are at least two fundamental energy scales. On the one hand we have the electroweak energy scale ($m_{EW} \sim 10^2$ GeV). This is the scale at which the Standard Model operates, it determines the mass of the elementary particles and above this scale the symmetry associated with the Standard Model is spontaneously broken. Current experiments in particle accelerators are operating around this energy and have found results verifying the Standard Model with great precision. When the LHC will become operational in 2007 at CERN we will be able to do experiments above the weak scale energy.

On the other hand we have the already mentioned Planck scale energy ($M_p \sim 10^{19}$ GeV). We saw that Newton's gravitational constant is inversely proportional to Planck energy. Therefore gravity is weak because the Planck scale energy is large. Moreover the Planck scale energy is the amount of energy that particles would need to have for gravity to be a strong force. As told before we can convert an energy scale into a corresponding length scale, which tells us about the range of the force in question. The enormous gap (note the 16 orders of magnitude difference in figure5) between the Plank scale and the electroweak scale bothers a lot of physicists, and they have called this the hierarchy problem.

The supporters of the Grand Unified Theory like to think of gathering all physics in one theory. However you can expect particles that experience similar forces, to be somewhat similar. The enormous desert in between the two energy scales does not help them much. For example, this gap gives a huge dissimilarity in the mass of the particles. Therefore many physicists take a lot of care in solving this mystery.

6.2 Compactified dimensions to the rescue

One possible solution of the hierarchy problem was proposed by Arkani-Hamed, Dimopoulos and Dvali in [3] (further we will refer to them as ADD). They propose a model where the weakness of gravity is explained by assuming extra compactified spatial dimension. In their reasoning the gravitational and Standard Model interactions become united at the weak scale, which will then be the only fundamental scale in physics. The Planck scale would only "look" large to us, because it can spread into the compactified dimensions.

Before we take a look at the Planck mass and length in $4 + n$ dimensions and see how the Planck mass evolves, we must say something about the Standard Model forces in $4 + n$ dimensions. The Standard Model is a theory which works in four dimensions. In order not to mess with the electroweak scale, but to bring down only the Planck scale, we must assume the Standard Model particles and forces to be confined to our 4-dimensional world. We can picture them as being stuck to a 4-dimensional membrane (or "brane") that lives in a $(4 + n)$-dimensional universe. Gravity is not confined to this brane and free to spread in the extra dimensions. This is why the Planck scale seems much bigger than it is (remember from section 3.2

that momentum in the extra dimensions translates to additional mass in the 4-dimensional world).

With the above in mind we must go back to the compactified dimensions. We can generalize the quantities from the beginning of this section to come up with a Planck length and mass in n dimensions. We already know from section 4 that Newton's constant has different units in more dimensions. In 4 dimensions (that is: three spatial and one time dimension) it has units of $m^3s^{-2}kg^{-1}$. In $n+4$ dimensions G_{4+n} has units of $m^{n+3}s^{-2}kg^{-1}$. You can easily check that the Planck length now becomes:

$$l_p = \left(\frac{\hbar G_{4+n}}{c^3}\right)^{\frac{1}{n+3}} \tag{77}$$

Now we can find an expression for the Planck mass in 4D (75), in terms of the Plank mass in $(4+n)$ dimensions (M_{4+n}), given by:

$$M_{4+n} = \left(\frac{\hbar^{n+1}}{G_{4+n}c^{n-1}}\right)^{\frac{1}{n+2}} \tag{78}$$

Substituting G_{n+4} from (40) in (78) and using (75) works out to be (we will denote M_p as M_4 for the 4 dimensional Planck mass to avoid confusion):

$$M_4^2 = \left(M_{4+n}\right)^{n+2}\frac{2\Sigma_n c^n}{V_{n-1}(\hbar^n)} \tag{79}$$

With this result we can work out how large the extra dimensions must be as a function of the number of extra dimensions, if we know the value of M_{4+n}. If we rewrite (79), pull out R^n from Σ_n, use the volume of a unit sphere and fill in $M_4 = 10^{16}$TeV, which we know from measurements, we get an explicit formula of R in terms of n and M_{4+n}^{2+n}:

$$R = \frac{10^{\frac{32}{n}}TeV^{\frac{2}{n}}}{M_{4+n}^{1+\frac{2}{n}}2\sqrt[n]{\pi^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}} \tag{80}$$

If we write $M_{4+n}^{1+\frac{2}{n}}$ in TeV, (80) has dimensions TeV^{-1}. This comes from the convention to set $\hbar = c = 1$. To express R in meters we use $E = \frac{2\pi\hbar c}{\lambda}$, this gives us:

$$R = \frac{1.98}{2\sqrt[n]{\pi^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}} \frac{10^{\frac{32}{n}-19}}{M_{4+n}^{1+\frac{2}{n}}} m \tag{81}$$

Only $n < 7$ are the relevant extra dimensions for our purpose, since this is the maximum known number of extra dimensions still consistent with a theory on physics in extra dimensions. The factor $\frac{1.98}{2\sqrt[n]{\pi^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}}$ can be left out, since we only want to get an indication of the size of the possible extra dimensions. Hence,

$$R \sim \frac{10^{\frac{32}{n}-19}}{M_{4+n}^{1+\frac{2}{n}}} m \tag{82}$$

Remember that M_{4+n} has to be given in TeV. The first way to put this relation into practice is by assuming $M_{4+n} = m_{EW} \approx 1$TeV, as proposed by ADD. This is the value M_{4+n} should have to solve the hierarchy problem. With this we can calculate the size of the extra dimensions:

n	1	2	3	4	5	6
$R(m)$	10^{13}	10^{-3}	10^{-9}	10^{-11}	10^{-13}	10^{-14}

From the table above we can draw some conclusions. For $n = 1$ we have a size for the extra dimensions of $R \sim 10^{13}$ m implying deviations from Newtonian gravity over solar system distances, so this case is ruled out. Gravity is very well studied over this kind of distances. If this deviation would be true, then we would have noticed this. However, for all $n \geq 2$ the corrections to gravity only become noticeable at distances smaller than those currently probed by experiment. Recent experiments actually allow us to study gravity at smaller distance scales than ever before. In the next section we will examine two experiments of this kind. They will help us to say something about $n = 2$.

7 Short-Range Tests of the gravitational inverse square law

7.1 Gravity measurement

Gravity was the first of the four known fundamental forces to be understood quantitatively since 1687. Despite this fact, gravity cannot compete in any way to the detailed study of the other three forces at short distances. There is an evident lack of experimental data in this range. More concrete, until recently, nothing could be said about gravity for distances below 1 mm!

In the preceding sections we have brought the weakness of gravity several times to your attention. The moment you want to carry out an experiment concerning the strength of gravity at short distances, you walk right into serious technical difficulties due to this weakness. To illustrate this once more: in any gravitational experiment the 'cancellation' of the electromagnetic interaction between test masses must be at the level of roughly one part in 10^{40} to leave any sensitivity to gravity. On short distance scales, local charge inhomogeneities and magnetic impurities in the materials of the experiment quickly become important. A careful analysis of subtle systematic effects is therefore crucial for any measurement of gravity at this scale.

Fortunately, experimental physicists like a challenge and came up with several ways to tackle this problem. This was not an easy task. The main reason for giving this subject some serious thought came from the speculations about potential deviations from Newtonian gravity at short distances as discussed in section 5.2. This adds yet another difficulty, because deviations from something weak must be even weaker. In the following sections we will describe some ingenious experiments that probe the gravitational interactions below the millimetre.

These experiments can be divided into two groups. The first group consists of "low frequency" experiments. In this category fall (among others) the classic torsion balance used by Cavendish and the experiment described below, carried out by the Eot-wash group. The name of this category is chosen only in contrast with the other one. In this second category fall the "high frequency" experiments. We will also describe one of them briefly. In this experiment, they use a kilohertz resonant-oscillator technique. Now the choice of the categories is clear. We used the articles of the various research groups [5]-[8].

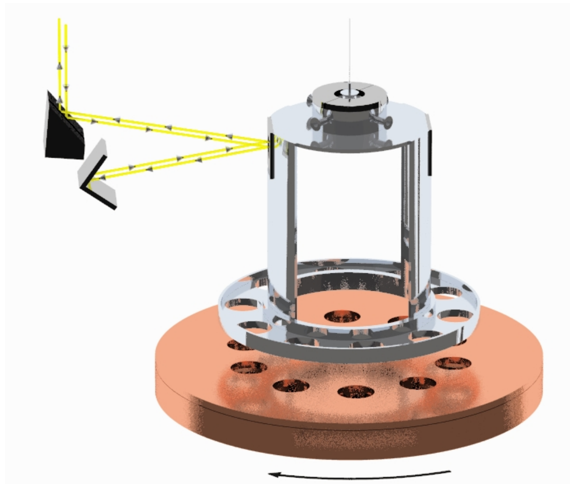

Figure 6: The apparatus of the Eöt-Wash group. The distance between the pendulum and the attractors has been exaggerated. (Figure from [9])

7.2 The Eöt-Wash experiment

The first experiment we will encounter comes from the research carried out by the Eot-Wash group. In this experiment they used a torsion pendulum. A schematic representation is shown in figure 6.
The setup consists of a ring suspended by a fiber above two disks (the attractors). During the experiment the two disks are set into slow rotational motion. As all torsion pendulums, the pendulum oscillates (i.e. twists and untwists) after being given an initial torque When these attractors rotate, they cause the ring to twist back and forth due to gravitational interaction. The ring and the two disks have ten cylindrical holes drilled into them evenly spaced about the azimuth. These holes of "missing mass" are the key-elements of this experiment. When the holes would not be there, gravity from the disks would pull directly down on the ring and therefore would not be able to twist it. The holes in the lower attractor are placed 18° rotated compared to those in the upper one. In this way they are rotated "out of phase". This means that they lie halfway between the holes in the upper disk as can be seen from figure 7. By attaching this second thicker disk in this intelligent way, the lower holes produce a torque on the ring that substantially reduces the signal due to Newtonian gravity.

However, now the holes are we may be able to measure an extra damping

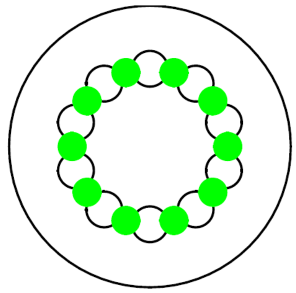

Figure 7: Top view of the attractors. The solid circles represent the in-phase upper-plate holes. The open circles show the ones out-of-phase. Figure from page 6 of [5]

force, due to other additional forces. According to the theory described in section 5, there may be deviations from Newton's gravitational force law of the form 63. Note that these correction terms make the gravitational force stronger than predicted by Newtonian theory alone. In addition, they will die away very quickly with increasing distance, because they are exponential functions.

As mentioned earlier the second disk significantly reduces the Newtonian gravity. However, when we assume that the deviations are correct, than, if gravity becomes stronger at short distances due to this deviations, the torque induced by the lower disk to cancel the torque from the upper disk will be relatively smaller than expected from Newtonian gravity alone. This will happen because the lower attractor is farther from the pendulum ring. However, torques from a short-range interaction with a length-scale less than the thickness of the upper attractor disk will not be canceled. This results in an extra damping force on the pendulum, due to the possible Yukawa interactions. Thus the geometry reduces the torque from Newtonian gravity. but will have little effect on a induced short-range torque.

In order to diminish other forces, such as the electrostatic interactions between the attractor and the pendulum, they put a stiff conducting membrane between them. Additionally the pendulum was surrounded by an almost complete Faraday cage.

The amount of torque was measured by shining a laser beam on a mirror installed on the pendulum, as can be seen from figure 6. The variation of the magnitude of the torque with changing separation between the ring and the disks provided a signature for any new gravitational or other short-range phenomena. These measured torques were then compared to calculations of the expected Newtonian and possible damping Yukawa effects. From this they could say something about α and λ from equation (41). (For this subsection ([9] was used)

7.3 The Colorado experiment

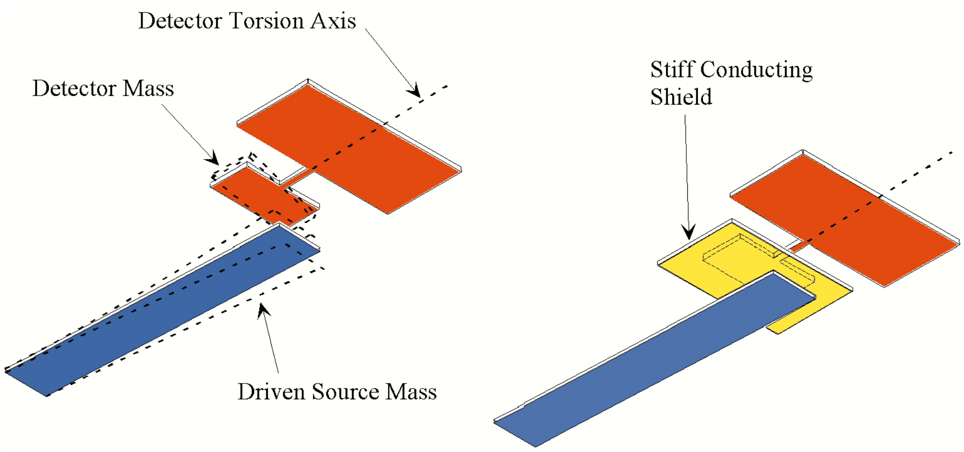

Figure 8: Active components of the experiment(From [11])

As mentioned above there is another type of experiment for testing gravity at short distances. Here so-called high frequency techniques are used. The torsion balance experiment discussed, has limits for the minimum practical test mass separation due to background effects. They become more problematic with decreasing distance. However, experiments using high frequency techniques do alow us to operate at smaller test mass separation. A research group at Colorado used this technique. We shall discuss their experiment briefly.

The active components of the apparatus used by the Colorado group is pictured in figure 8. The experiment uses a tungsten torsional oscillator (the detector in the picture). The source of gravitational field is a tungsten vibrating cantilever (the source mass in the picture). The source and detector are separated by a stiff conducting shield to eliminate background forces due to electrostatics and residual gas in the vacuum chamber surrounding the experiment. A planar test mass geometry is chosen to concentrate as much mass as possible at the scale of interest. This is done because of the short range of the Yukawa interaction. When you would use a thicker device, the Newtonian interactions would be relatively stronger than the Yukawa force using the thin plate, because this force is dominated by nearby mass due to its exponential character. The best test masses for Yukawa interactions are heavy thin plates concentrating as much mass as possible at the closest range. Therefore a thin plate of tungsten is used. The choice of this material is due to its relatively high density.

Similar to the Eöt-Wash group, the Colorado group used a null geometry experiment with respect to $\frac{1}{r^2}$ forces. That is, one that should result in a zero output for Newtonian gravity if all is well (but we hope it is not!). The Newtonian gravity exerted by the detector on the reed almost cancels the force exerted by the reed on the detector. However, this is not the case for the Yukawa interactions, which become stronger with increasing distance. When the vibrating reed comes closer to the detector there might be a small Yukawa force exerted that would influence the oscillating detector. We might be able to measure this distortion. The extensive apparatus arrangement is pictured schematically in figure 9.
The attractor formed by the tungsten reed of size 35 mm $\times$ 7 mm $\times$ 0.305 mm was brought into vibration at the natural resonance frequency of the detector mass, which was around 1 kHz. The detector consists of two coplanar rectangles joined along their central axes by a short segment. The attractor was positioned so that its front end was aligned with the back edge of the detector rectangle and a long edge of the attractor was aligned above the detector torsion axis. The attractor, detector and electrostatic shield were mounted on separate vibration-isolation stacks to minimize any mechanical couplings, as can be seen from 9. While our board vibrates in the resonant mode of interest, it induces a torque on the detector oscillator, causing the detector to counter-rotate in a torsional mode around the axis that can be seen in . They searched for distortions of the vibration modes after corrections for unintended effects such as thermal noise. When an extra signal in the oscillation is noticed this might be due to Yukawa interaction. The minimal test mass separation was 108 μm.

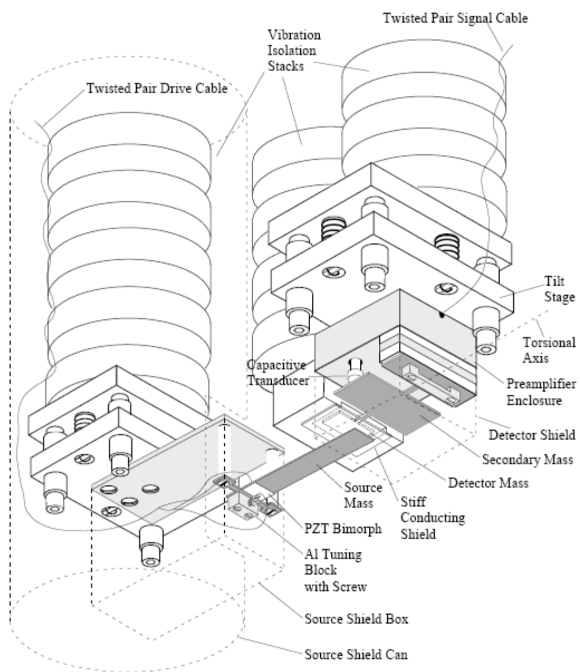

Figure 9: Apparatus (From page 21 of [10])

7.4 Results

7.4.1 Eöt-Wash group

The Eöt-Wash group found no deviation from Newtonian gravity and interpret these results as constraints on extensions of the Standard Model that predict Yukawa forces. They set a constraint on the largest single extra dimension (assuming toroidal compactification and that one extra dimension is significantly larger than all the others) of R $\leq$ 160 μm, and on two equal-sized large extra dimensions of R $\leq$ 130 μm. Yukawa interactions with $\alpha \geq$ 1 are ruled out with a confidence level of 95% for $\lambda \geq$ 197 μm. We can see this graphically in figure 10.

7.4.2 Colorado group

The null results from the experiment taken at the minimal separation were turned into constraints on α and λ using a maximum-likelihood estimation. For various assumed values of λ, the expected Yukawa force was calculated numerically 400 times, each calculation using different values for experimental parameters that were allowed to vary within their measured ranges. A likelihood function was constructed from these calculations and was used to extract limits on α from the results with a confidence level of 95%. This leads to a constraint for the largest extra dimension (again assuming toroidal compactification and that one extra dimension is significantly larger than all the others) of R $\leq$ 108 μm,The constraints from the Colorado group on α and λ are graphically represented in figure 10.

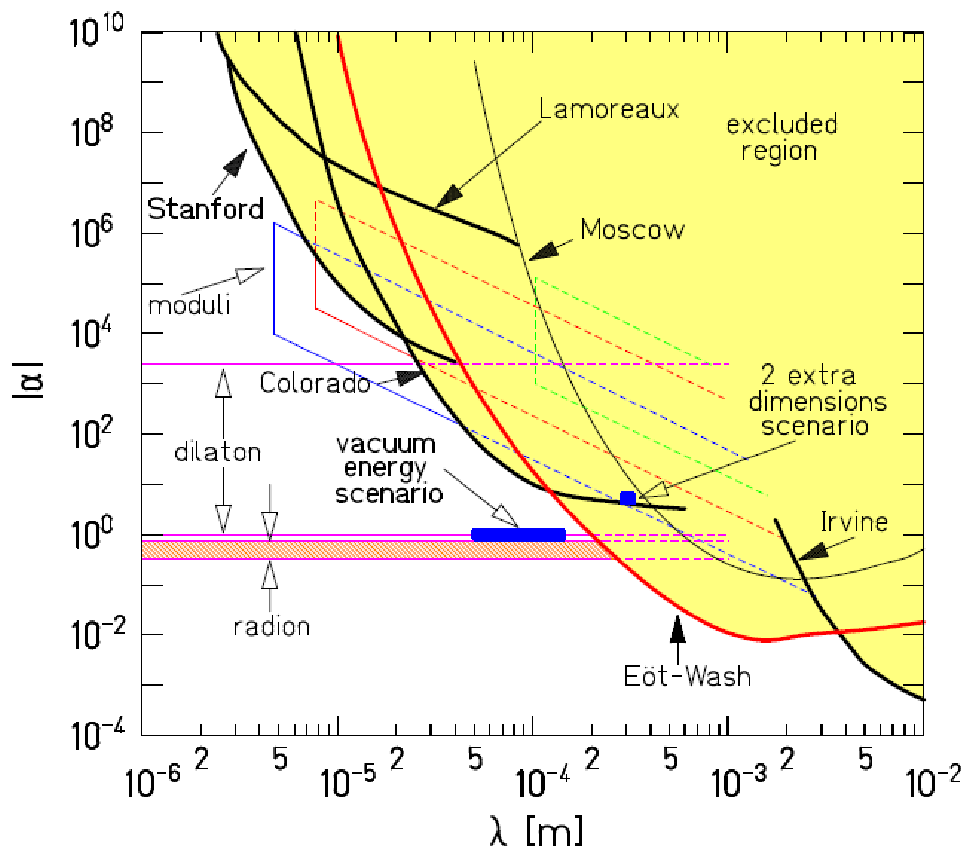

Figure 10: The results for the Yukawa constraints from the Eöt-Wash group are graphed in red. Figure from page 29 of [5]

8 Astrophysical bounds

In this section we consider bounds placed on the size of the extra dimensions by looking at stellar objects. In addition we hope to say something about solving the hierarchy problem. For instance, is $M_{4+n} \sim 1TeV$ not excluded? The chosen objects are the Sun, red giants and a supernova named SN1987A.

The reason why we look at these objects is that they have very high temperatures and we expect KK gravitons to be produced, which can escape to higher dimensions by several reactions, to be discussed later on. Those gravitons have momentum and therefore carry energy, which they take with them while escaping into the extra dimensions. After escaping, they have a very small probability of returning to the brane , since the brane only covers a small region of the higher dimensions. For more information on branes review section 6.2.

8.1 Cross section of graviton

When a reaction between SM-particles takes place, there is a probability that a graviton is emitted, which can escape into the extra dimensions (i.e. a KK mode) [12]. In particle physics we usually do not talk about probabilities, instead we discuss cross sections. Let's start with calculating the cross section of one graviton, σ_1. We want to get some sort of probability of a graviton being emitted. Each KK mode is very weakly coupled $\sim \frac{1}{M_4}$. In a diagram this looks like figure 11. Therefore the probability at which gravitons are being emitted by a certain reaction involving SM-particles, is given by

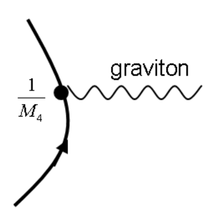

Figure 11: Coupling of a graviton to a Standard Model particle.

$$\sigma_1 \sim \left|\frac{1}{M_4}\right|^2 = \left(\frac{1}{M_4}\right)^2 \tag{83}$$

To get the total cross section of the gravitons, we consider the number of KK modes possibly emitted, when a reaction between SM-particles takes place, with energy E. The KK modes have momentum p in the extra dimensions $\sim \frac{b}{R}$ (see section 3.2), and so $\frac{b}{R} \lesssim E$. First consider an 1-dimensional sphere with radius E. Since p is quantized with spacings of $\frac{1}{R}$, the number of KK modes would be ER. This means, for an n-dimensional sphere with radius E, that the number of KK modes is given by

$$\#_{KKmodes} \sim (ER)^n \tag{84}$$

Combining (84) with σ_1 gives

$$\sigma \sim (ER)^n \left(\frac{1}{M_4}\right)^2 \tag{85}$$

If we look at possible graviton production by a stellar object, first of all we look at the total energy emitted. This is usually associated with $\dot{\epsilon}$, which is the energy emitted per unit time per unit mass. Secondly we look at $\dot{\epsilon}_{grav}$, the energy lost to gravitons per unit time per unit mass. This $\dot{\epsilon}_{grav}$ depends linearly on σ [12]. The next thing, to take into account, is the temperature T of the object. Temperature is associated with energy by setting $k_B = 1$, where k_B is Boltzmann's constant. We can even write $T \sim E$. This can be explained by looking at the Planck spectrum, which tells us about the particles a black body radiates. It gives us a relation between the frequency ν and the intensity of electromagnetic radiation. This depends on the temperature of the black body. The Planck spectrum looks like figure 12. The frequency corresponding to the maximum is given by Wien's displacement law [16]

$$\nu = \frac{2.821 k_B T}{h} \tag{86}$$

We can rewrite this to

$$E = h\nu = 2.821 k_B T \tag{87}$$

And by setting $k_B = \hbar = 1$ and neglecting the numerical factor we obtain

$$E \sim T \tag{88}$$

If $T \ll R^{-1}$, KK modes cannot be produced. If $T \gg R^{-1}$, they can be produced. The latter case gives a number of KK modes that can be produced [12]. Now by use of $T \sim E$ and (79), we can rewrite (85) and obtain

$$\sigma \sim \frac{T^n}{M_{4+n}^{n+2}} \tag{89}$$

This is exactly what we need, to say something about $\dot{\epsilon}_{grav}$, since it depends on σ. The exact derivation of the cross sections of the relevant processes taking place in stellar objects, which can produce gravitons, is beyond the scope of this paper. It can be found in [12]. We just state the relevant cross sections:

- Gravi-Compton scattering: $\gamma + e \rightarrow e + grav$

$$\sigma v \sim \delta e^2 \frac{T^n}{M_{(4+n)}^{n+2}} \beta^2 \tag{90}$$

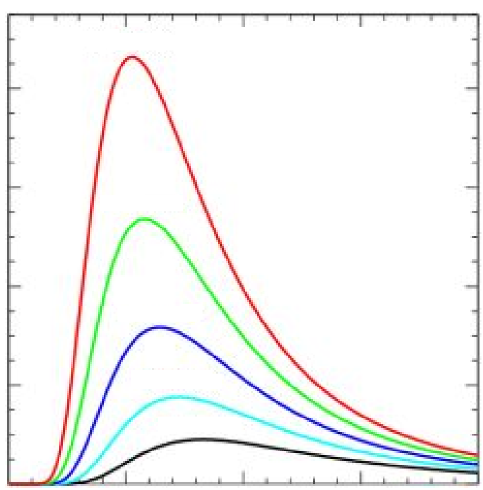

Figure 12: Planck spectrum of a black body for different values of T[16]

- Gravi-brehmstrahlung: $(e + Z \rightarrow e + Z + grav)$

$$\sigma v \sim \delta Z^2 e^2 \frac{T^n}{M_{(4+n)}^{n+2}} \tag{91}$$

- Graviton production in photon fusion: $\gamma + \gamma \rightarrow grav$

$$\sigma v \sim \delta \frac{T^n}{M_{(4+n)}^{n+2}} \tag{92}$$

- Gravi-Primakoff process: $\gamma +$EM field of nucleus $Z \rightarrow grav$

$$\sigma v \sim \delta Z^2 \frac{T^n}{M_{(4+n)}^{n+2}} \tag{93}$$

- Nucleon-Nucleon Brehmstrahlung: $N + N \rightarrow N + N + grav$ N can be a neutron or a proton.

$$\sigma v \sim (30 millibarn) \times (\frac{T}{M_{(4+n)}})^{n+2} \tag{94}$$

Where $\delta = \frac{1}{16\pi}$ and $\beta = \frac{v}{c}$. We have finally obtained enough information to take a closer look to three different stellar objects: the Sun, red giants and SN1987A.

8.2 The Sun and red giants

The first stellar object that allows us to place a bound on M_{4+n} is the Sun. The Sun has temperature $\sim 1keV$ and

$$\dot{\epsilon} \sim 10^{-45} TeV \tag{95}$$

The relevant particles are protons, electrons and photons [12]. To consider $\dot{\epsilon}_{grav}$ we let $\dot{\epsilon} \gtrsim \dot{\epsilon}_{grav}$. In words this means that the energy loss by gravitons can never exceed the normal energy loss. This looks like a nonsense demand, but in this way we can place a lower bound on M_{4+n} and it can be only greater than this bound.

The only two relevant processes, by which gravitons are being produced in the Sun, are Gravi-Compton scattering and photon fusion. There are no high-Z nuclei, so Gravi-brehmstrahlung is excluded. The Gravi-Primakoff is comparable to photon fusion, but in this case the latter is more dominant than the former. And last, Nucleon-Nucleon Brehmstrahlung is excluded, since the temperature of the Sun is too low [12]. Now, let's consider the relevant processes.

- Gravi-Compton scattering: $\gamma + e \rightarrow e + grav$ gives us

$$\dot{\epsilon}_{grav} \sim 4\pi\alpha\delta\frac{T^{n+5}}{m_p m_e M_{4+n}^{n+2}} \tag{96}$$

By use of (95) and $T \sim 1keV$, we can rewrite this to

$$M_{4+n} \gtrsim 10^{\frac{10-9n}{n+2}} TeV \tag{97}$$

n	2	3	4	5	6
$M_{4+n}(TeV)$	10^{-2}	10^{-4}	10^{-5}	10^{-5}	10^{-6}
$R(m)$	10	10^{-3}	10^{-5}	10^{-6}	10^{-7}

The highest lower bound is given by $n = 2$ and reads $M_6 \gtrsim 10^{-2}TeV$. This bound does not exclude $M_{4+n} \sim 1TeV$. It does give a possible radius to the extra two dimensions of $R \sim 10m$. This is not a problem, since this is the upper bound for R and R can only be smaller, as it has to be.

- Graviton production in photon fusion: $\gamma + \gamma \rightarrow grav$ gives us

$$\dot{\epsilon}_{grav} \sim \delta\frac{T^{n+7}}{\rho M_{4+n}^{n+2}} \tag{98}$$

Which gives a lower bound of

$$M_{4+n} \gtrsim 10^{\frac{12-9n}{n+2}} TeV \tag{99}$$

n	2	3	4	5	6
$M_{4+n}(TeV)$	$3 * 10^{-2}$	10^{-3}	10^{-4}	10^{-5}	10^{-6}
$R(m)$	1	10^{-4}	10^{-5}	10^{-6}	10^{-7}

For $n = 2$, M_{n+4} can still be low enough, but notice that this lower bound is slightly higher and so is a stronger bound.

Now we turn to red giants. The only difference in this case is the temperature $T \sim 10keV$ [12]. This is only slightly higher than the sun's temperature and so the same two processes as with the sun apply to red giants. We will not repeat all the calculations and we will just conclude that for all n $M_{4+n} \sim 1TeV$ is still a good possibility.

8.3 SN1987A

As we have seen for the Sun and red giants, $M_{4+n} \sim 1TeV$ has not yet been in danger. To push up the lower bound on M_{4+n} we need to look at stellar objects with higher temperature. Therefore, the collapse of the iron core of SN1987A is useful for our search to put a strong bound on M_{4+n}. During the collapse the energy emitted was

$$\dot{\epsilon} \sim 10^{-44} TeV \tag{100}$$

The temperature T of the object is being estimated at $30 - 70MeV$ [13]. To produce a lower bound, we will choose $T \sim 30MeV$. The two main processes which took place are, Nucleon-Nucleon Brehmstrahlung and the Gravi-Primakoff process [12]. We will only consider the former, since this will provide us the strongest bound

$$M_{4+n} \gtrsim 10^{\frac{15-4.5n}{n+2}} TeV \tag{101}$$

With this we can calculate the lower bound on M_{4+n} and upper bound on R.

n	2	3	4	5	6
$M_{4+n}(TeV)$	30	2	$3 * 10^{-1}$	$8 * 10^{-2}$	$3 * 10^{-2}$
$R(m)$	10^{-6}	10^{-9}	$6 * 10^{-11}$	$8 * 10^{-12}$	$2 * 10^{-12}$

The bound for $n = 2$ is somewhat above $1TeV$. The interpretation of this bound is a bit more tough. The authors of [12] argue that $M_6 \sim 30TeV$ is still consistent with a string scale $\sim fewTeV$. While the authors of [13], who use a different technique and find $M_6 \sim 50TeV$, argue this is too large to be measured now or in the nearby future.

From this astrophysical perspective we can draw some conclusions about the viability of solving the hierarchy problem and the actual measurement

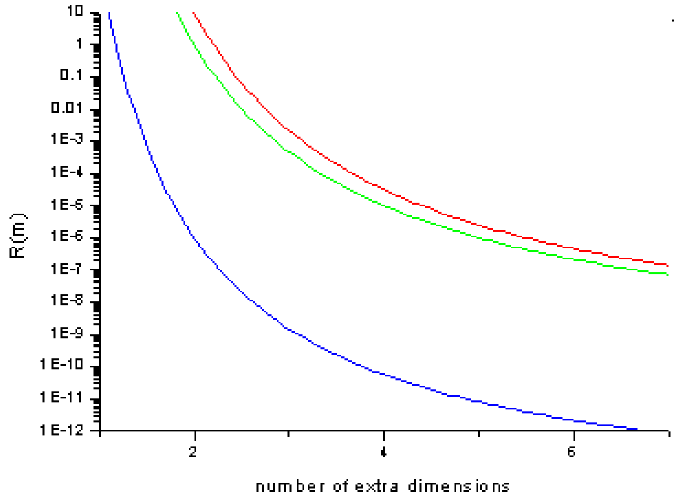

Figure 13: Upper bounds placed on the size of the possible extra dimensions by astrophysics. The red and green line represent Gravi-Compton scattering, respectively photon fusion, in the Sun. The blue line comes from Nucleon-Nucleon Brehmstrahlung in SN1987A.

of any extra dimensions in the laboratory. The results are combined and can be seen in figure 13. The strongest bound on M_{4+N} is given to us by observations on SN1987A. For $n \geq 3$, $M_{4+n} \sim 1Tev$ is not excluded. For $n = 2$ the analysis becomes a bit more tricky. It is a matter of perspective and assumptions if $n = 2$ can be excluded. So for now, we will not exclude this possibility either. This means the LHC might provide answers in solving the hierarchy problem. Nevertheless, finding deviations on the gravitational force, by sub-mm tests, does seem implausible. The upper bound on the size of the extra dimensions reads $R \lesssim 10^{-6}m$, this is far beyond the reach of tests described in section 7.

9 Conclusion

From the short-range gravity experiments we found no evidence for violations of Newtonian gravity due to Yukawa interaction. From this we can only exclude some possibilities by setting upper bounds for the size of the extra toroidal compactified dimensions. From the torsion balance experiments we found the strongest bound to be R $\leq$ 160 μm. The High frequency experiment showed an even stronger bound of R $\leq$ 108 μm. From the astrophysical considerations we found the strongest bound to be $R \lesssim 10^{-6}m$, for $n = 2$. For higher n, R will only decrease further.

For the future there is still a lot of work to do. The torsion-balance experiment group try to further improve their apparatus. However, this will be with of lot of modifications. Among other things they will use a larger number of smaller sized holes. They hope to provide good results for length scales down to 50 μm. The high frequency group hope to probe this same range of interest. An experiment is planned with a stretched membrane shield in place of a sapphire plate. If the backgrounds can be controlled, this experiment could improve limits for R to fall between 10 - 50 μm.

From astrophysics we saw that observations on stellar objects, show no inconsistency with $M_{4+n} \sim 1TeV$, the manner in which ADD try to solve the hierarchy problem. Even the strongest bound, obtained by observations on SN1987A for $n = 2$, on M_{4+n} can still be considered within the range of $\sim 1TeV$. This bound reads $M_6 \gtrsim 50TeV$. We have to take into account that the bounds from astrophysics are insecure. How insecure exactly is hard to tell, but it is comforting that $M_{4+n} \sim fewTeV$ is not excluded yet. This means future particle colliders could start measure signs of extra dimensions. Nevertheless, It seems improbable for sub-mm tests, to find deviations on the gravitational force. It does look good though that both ways (i.e. sub-mm tests and astrophysics) of considering deviations have not contradicted each other. Sub-mm tests have not shown deviations yet and from astrophysics we know, that deviations cannot be seen at the distances these tests measure.

References

[1] Lisa Randall. Warped Passages: Unraveling the mysteries of the universe's hidden dimensions. Page 11 - 21. New York 2005

[2] The Elegant Universe, Brian Greene. Vintage Books, 2000

[3] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, "The Hierarchy Problem and New Dimensions at a Millimeter." Phys.Lett. B429 (1998) 263-272

[4] A. Kehagias and K. Stefsos, "Deviations from the $1/r^2$ Newton law due to extra dimensions." Phys.Lett. B472 (2000) 39-44 hep-ph/9905417

[5] C.D. Hoyle et al. Sub-millimeter Tests of the Gravitational Inverse-square Law. Phys.Rev. D70+ (2004) 042004 hep-ph/0405262

[6] Joshua C. Long et al. New Experimental Limits on Macroscopic Forces Below 100 Microns. Nature 421, 922-925 (2003) hep-ph/0210004

[7] Joshua C. Long, John C. Price. Current Short-Range Tests of the Gravitational Inverse Square Law. Comptes Rendus Physique 4 (2003) 337-346 hep-ph/0303057

[8] E.G. Adelberger, B.R. Heckel, A.E. Nelson. Tests of the Gravitational Inverse-Square Law. Ann.Rev.Nucl.Part.Sci. 53 (2003) 77-121 hep-ph/0307284

[9] http://www.npl.washington.edu/eotwash/shortr.html

[10] Joshua C. Long, Hilton W. Chan, John C. Price. Experimental Status of Gravitational-Strength Forces in the Sub-Centimeter Regime. Nucl.Phys. B539 (1999) 23-34 hep-ph/9805217

[11] http://spot.colorado.edu/ pricej/gravity.html

[12] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, "Phenomenology, Astrophysics and Cosmology of Thearies with Sub-Millimeter Dimensions and TeV Scale Quantum Gravity." Phys.Rev. D59 (1999) 086004 hep-ph/9807344

[13] S. Cullen and M. Perelstein, "SN1987A Constraints on Large Compact Dimensions." Phys.Rev.Lett. 83 (1999) 268-271 hep-ph/9903422

[14] I.S. gradshteyh, I.M. Ryzhik. ”Table of Integrals, Series and Products” San Diego 2000

[15] Borwein, J. M. and Borwein, P. B. ”Poisson Summation.” Page 36-40 in Pi and the AGM: A Study in Analytic Number Thery and Computational Complexity. New York: Wiley, 1987.

[16] http://scienceworld.wolfram.com/physics/WiensDisplacementLaw.html

[17] http://mathworld.wolfram.com/BetaFunction.html

[18] http://mathworld.wolfram.com/BinomialSeries.html