Non-relativistic Holography and Renormalization

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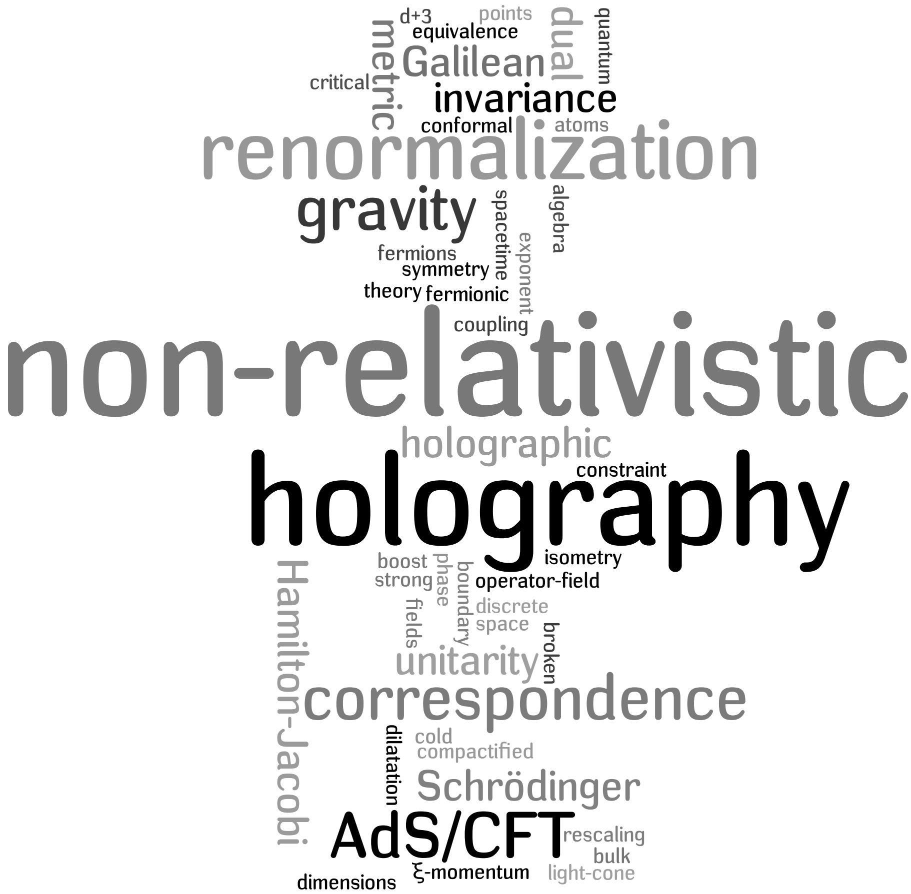

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Abstract

The present thesis reviews the recently proposed non-relativistic holographic duality and the application of the holographic renormalization procedure to this case is discussed. After a brief description of the renormalization group, we comment on the Hamilton constraints in General Relativity. We then explain the AdS/CFT correspondence and combine everything up to that point to describe the holographic renormalization procedure. Afterwards, we discuss aspects of non-relativistic holography which was put forth by Son and independently by Balasubramanian and McGreevy. We finally attempt to bring the holographic renormalization group technology in the non-relativistic framework. Some developments are described and we make some suggestions on how this program could succeed.

Contents

Historical Overview – Motivation

Mathematics and Physics have been complementary to each other for the past few centuries. Intuitions and tools developed in one field often turned out to be groundbreaking and extremely useful in the other. In the previous century, the two areas of knowledge became inextricably linked. The astonishing progress in both fields, triggered by amazing discoveries during the first part of the century, changed our view of the world in a profound and unprecedented way. Perhaps it is no accident that the two fields seem to be following similar paths.

Mathematician David Hilbert lived in that exciting era of great scientific discoveries, which initiated a scientific revolution based on radical new ideas. He was the ambassador of the notion and belief that mathematics should be a complete theory–a system of knowledge complete in itself. In his famous lecture of 1900 in Paris, he posed twenty three unsolved problems that he believed were important in determining the logical foundations of science.

However, his vision of developing a system with which one could attack any problem and answer any question, was short-lived. In 1931, a young mathematician named Kurt Gödel, published a paper that shuttered the hope for a complete structure that would be entirely self-contained and self-consistent. In his incompleteness theorems he showed that it is impossible to constructively show that an axiomatic theory is consistent, and also that in a consistent theory, there will be theorems that cannot be proved. He showed that it was as if mathematicians wanted to settle something that had the same nature as Epiminides paradox, who said that *"all Cretans are liars"*. Being from Crete himself, this puts forth an obvious inconsistency of that statement.

Interestingly, four years earlier, physicist Werner Heisenberg had made a bold statement that goes by the name of *Uncertainty Principle**. So, in a similar fashion, physicists had already started getting used to the idea that we cannot know everything; not simultaneously at least. In both fundamental mathematics and physics, what we can or cannot know became a deep and important question.

Albert Einstein on the other hand, remained until the end of his life, a strong advocate of the need for a complete, unified theory of nature. To the present day, this is still a goal for a large number of theoretical physicists. From the modern point of view, the Standard Model of fundamental particles provides a satisfactory and complete understanding of the world–despite some problems–only if we exclude gravity. It is the pinnacle of modern science and was established in the early 1970's. The remaining piece of the puzzle

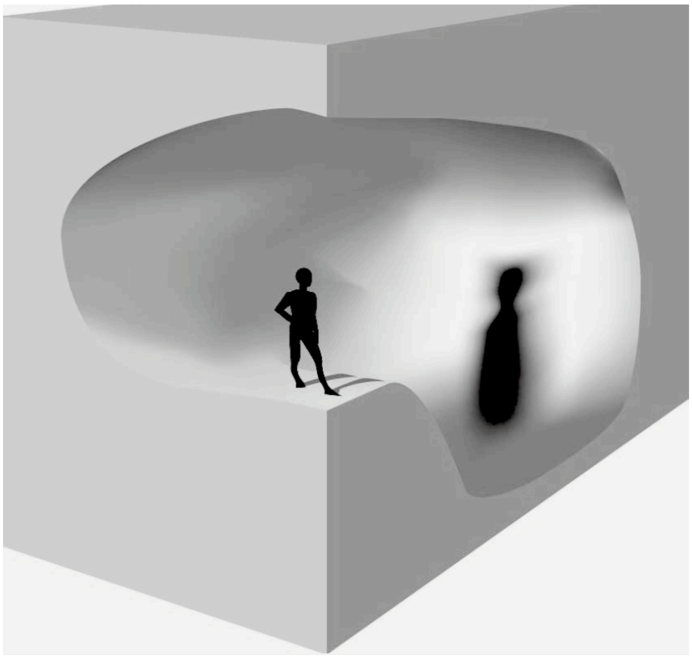
*Which is not really a principle since it is a statement that can be proved.

however, seems to be a quantum field theory of gravity that can somehow be incorporated in the Standard Model. In spite of many years of intense research, this remains unattainable. However, there have been some important developments, mainly in the context of string theory in the past decades.

Meanwhile, in the mathematics front, a new paradigm was emerging. New hope appeared when in the 1960's Robert Langlands posed a new program for a unification of seemingly different and disjoint areas of mathematics. Inspired by the Taniyama-Shimura conjecture he suggested what is now known as the "Langlands program". The Taniyama-Shimura conjecture gained its recent fame because of its role in the proof of "Fermat's last theorem" in the previous decade. Nonetheless, its implications and influence in mathematics has been much greater than that. The idea behind the Langlands program is that separate areas of mathematics can be dual (equivalent) to each other. This notion was something that physicists first came across in Maxwell's theory of electromagnetism.

Surprisingly, theoretical physicists came to a similar situation once again. Until 1995 string theory appeared to be not one, but five different unified theories of nature! How could all five of them be correct? We only observe one universe, and it should be described by only one theory. This was an embarrassing problem in string theory. Following the legacy of the Langlands program, Edward Witten pushed an existing intricate web of dualities further and showed that all five, seemingly different, string theories were different limits of the one and same theory; it was dubbed *M-theory* and this discovery is known as the "second superstring revolution".

The rekindled interest in dualities and the accumulation of knowledge and experience led, two years later, to an important discovery. In the seminal paper by Maldacena, it was conjectured that string theory, which includes gravity, is equivalent to a quantum field theory without gravity on the boundary of this space. Because the boundary has one or less dimensions, it also became linked to the holographic principle proposed a few years earlier by Gerard 't Hooft and independently by Leonard Susskind. The idea of holography dates back to Plato's cave, but the realization of a rigorous, scientific theory with predictive aspirations remains one of the most important and inspiring developments in recent years. Perhaps, this does not embody the hopes for a novel, exotic idea for a quantum field theory of gravity that some had. If the conjecture is proved to be correct, it essentially gives an interpretation of a quantum field theory of gravity with terms and notions that are already known and studied–what we could call "traditional" field theory. In this sense, the conjecture seems to render the aim for a completely new and extravagant idea somewhat unnecessary. However, a holographic description of nature is arguably very intriguing and unconventional!



In Plato's "Allegory of the Cave", people are chained and can only see their fuzzy shadows on the cave walls cast by a fire. To them, the shadows represent the totality of their existence.

After several years of research, physicists studying the conjecture have come to the conclusion that the validity and implications of the proposal go beyond the context of the originally anticipated regime. It is now generally referred to as gauge/gravity duality or holography. Recently, the rules of the original conjecture were somewhat bent and a model of gauge/gravity duality that is expected to be able to describe non-relativistic theories was proposed.

In the present work, the newly proposed extension is studied and an attempt is made to describe its behavior in different energy scales. To this end, holographic renormalization techniques are employed. Chapters 1 to 3 present the necessary tools and theoretical background. Holographic renormalization is discussed in Chapter 4. In Chapter 5, we deal with basic principles and certain aspects of non-relativistic holography. In the last chapter, the application of the holographic renormalization procedure in the non-relativistic case, as well as related problems are presented.

1 | Renormalization

The aim of this chapter is to introduce the so-called renormalization procedure. Although a few ideas are described thoroughly, familiarity of the reader with quantum field theory is assumed.

A very important problem that seemed to be inherent in quantum field theories was the ultraviolet divergences that occurred. Therefore, in the earlier days of quantum field theory (QFT) several theorists advocated the abandonment of that approach altogether. This was a reasonable conclusion at the time, since the cancellation of ultraviolet divergences is essential if a theory is to yield quantitative physical predictions.

We address some related issues by focusing on the method developed by Kenneth Wilson [1]. It is not a bottom up approach that would introduce the reader to all the basic techniques or provide sufficient background to deal with related topics. However, it does provide a physical picture based on the scale dependence of the theory's parameters and it is essential to what is discussed in the following chapters. Afterwards, we mention a few things about different renormalization schemes.

1.1 Perturbative Renormalization - A Toy Model

We start by considering an unspecified theory which yields ultraviolet divergences. For simplicity we assume that it has only one free parameter. We represent a physical quantity by $F(x)$ which is calculated perturbatively in terms of the free parameter λ_0, which is the coupling constant. If for example, we were talking about quantum electrodynamics (QED) F may represent the cross section of a scattering process of an electron on a heavy nucleus. In that case x would be the energy-momentum fourvector of the electron. So, we assume that $F(x)$ has the general form:

$$F(x) = \lambda_0 + \lambda_0^2 F_1(x) + \lambda_0^3 F_2(x) + \dots \,, \tag{1.1}$$

which up to a redefinition of $F(x)$ corresponds to something we encouter in a realistic field theory.

We further assume that the perturbative expansion is ill-defined in the sense that $F_i(x)$ are functions involving divergent quantities. For example, we might encouter the following form

$$F_1(x) = \alpha \int_0^\infty \frac{dt}{t+x} \tag{1.2}$$

which shares common features with integrals in a QFT, since it is also logarithmically divergent in the upper limit. This integral in QFT would represent the summation over virtual states and $\alpha\,(t+x)^{-1}$ would represent the probability amplitude associated with each state.

The assumption made earlier–that we are talking about a one-parameter theory–has as a consequence that only one measurement is enough fix the value of λ_0, for example at the point $x=\mu$.

It might seem redundant to parametrize the theory in terms of (the *bare* parameter) λ_0, since it only seems helpful in intermediate calculations and will be finally replaced with the *physical*, measured quantity $F(\mu)$. This freedom however is generic in physics. Moreover, in this particular case, there is a subtlety since the singular behavior of the expansion dictates a singular relationship between λ_0 and $F(\mu)$. What we would do in general is follow the following steps:

1. Decide on the Lagrangian that respects the required symmetries, locality etc.

2. Perform calculations of physical processes using the bare quantities at any required order in perturbation theory

3. Finally fix the parameter (or parameters for that matter) to reproduce and/or predict experimental results.

A theory that is ill-defined however, suggests that F needs to be reparametrized in terms of $F(\mu)$. This leads to what we might call the renormalizability hypothesis by which a reparametrization of the theory in terms of a physical quantity instead of the bare parameters, is enough to render the perturbative expansion into a well-defined one. In other words, the problem of the expansion is *not* the nature of the functions $F_i(x)$ we used, but the choice of the parameter we used to get the perturbative expansion. So, the physical quantity $F(x)$, should have a well-defined perturbative expansion once calculated in terms of the physical parameter $F(\mu)$. We are therefore led to define the renormalized coupling constant (also referred to as the physical coupling constant)

$$\lambda_R = F(\mu)\,.$$

Unfortunately, we cannot use (1.1) because it is, by assumption, ill-defined. We must somehow *regularize* the expansion to give it a well-defined meaning. We accomplish that by introducing a new set of functions F_Λ and $F_{i,\Lambda}$ that involve an new parameter Λ called the regulator.

Thus, we will work with the regularized functions F_Λ and $F_{i,\Lambda}$ which are now finite for a finite value of Λ. We now have

$$F_\Lambda(x,\lambda_0,\Lambda) = \lambda_0 + \lambda_0^2 F_{1,\Lambda}(x) + \lambda_0^3 F_{2,\Lambda}(x) + \dots \tag{1.3}$$

In order to regularize any F_i we introduce the cut-off for example in the integral

$$F_{1,\Lambda}(x) = \alpha \int_0^\Lambda \frac{dt}{t+x}\,. \tag{1.4}$$

We can now obtain a well-defined perturbation series of F_Λ in terms of the physical coupling λ_R. If indeed, this expansion makes any physical sense, it should do so even after we take the limit $\Lambda \to \infty$ because it expresses a finite physical quantity $F(x)$ in terms of λ_R. So, we change $F(x, \lambda_0)$ to $F_\Lambda(x, \lambda_0, \Lambda)$ and then rewrite F_Λ in terms of λ_R and μ and then take the limit $\Lambda \to \infty$ keeping λ_R and μ fixed. What we expect is

$$F(x) = F(x, \lambda_R, \Lambda) \overset{\Lambda\to\infty}{=} F_\Lambda(x, \lambda_R, \mu)\,.$$

We implement this order by order to see how it works exactly and what effect it has on the perturbative expansion. At order λ_0 we have

$$F_\Lambda(x) = \lambda_0 + \mathcal{O}(\lambda_0^2)\,.$$

So, we get

$$\lambda_0 = \lambda_R + \mathcal{O}(\lambda_R^2)\,.$$

To second order λ_0^2 we need to redefine λ_0 in order to eliminate the divergence of $F_\Lambda(x)$. We first expand λ_0 as a power series in λ_R and obtain

$$\lambda_0 = \lambda_R + \delta_2\lambda + \delta_3\lambda + \dots\,, \tag{1.5}$$

where $\delta_n\lambda \sim \mathcal{O}(\lambda_R^n)$. At λ_R^2 order we get

$$F_\Lambda = \lambda_R + \delta_2\lambda + \lambda_R^2 F_{1,\Lambda}(x) + \mathcal{O}(\lambda_R^3) \tag{1.6}$$

by using $\lambda_0^2 = \lambda_R^2 + \mathcal{O}(\lambda_R^3)$. By using $\lambda_R = F(\mu)$ we obtain

$$\delta_2\lambda = -\lambda_R^2 F_{1,\Lambda}(\mu) \tag{1.7}$$

which diverges for $\Lambda \to \infty$. In our case, using (1.4) we find

$$\delta_2\lambda = -\alpha\lambda_R^2 \int_0^\Lambda \frac{dt}{t+\mu} = -\alpha\lambda_R^2 \log\frac{\Lambda+\mu}{\mu}\,.$$

We substitute what we have so far back in the expansion to get

$$F_\Lambda = \lambda_R + \lambda_R^2 \left(F_{1,\Lambda}(x) - F_{1,\Lambda}(\mu)\right) + \mathcal{O}(\lambda_R^3)\,.$$

This expression will be finite for all x at this order if the divergent part $(F_{1,\Lambda}(x))$ is exactly cancelled by that of $F_{1\Lambda}(\mu)$, i.e.

$$F_{1,\Lambda}(x) - F_{1,\Lambda}(\mu) \text{ is regular in } x \text{ and } \mu \text{ for } \Lambda \to \infty\,.$$

This translates into the fact that the divergent part of $F_{1,\Lambda}(x)$ must be independent of x and essentially a constant. We can now define the *renormalized* $F(x)$ as the limit of $F_\Lambda(x)$ for $\Lambda \to \infty$. The above condition is satisfied for the integral (1.2) and so

$$F(x) = \lambda_R + \alpha(\mu - x)\lambda_R^2 \int_0^\infty \frac{dt}{(t+x)(t+\mu)} + \mathcal{O}(\lambda_R^3)\,.$$

We can now say that the theory is renormalized to this order. What we "silently" did in (1.6) is an apparent "addition of a divergent" term $\delta_2\lambda$ to cancel the divergence. These terms (which also occur in higher orders) are called counterterms. They "appear" to absorb the infinite but unobservable shifts between the bare parameters and the physical parameters. This mechanism is a generic: a divergence coming from the n^{th} term of the perturbative expansion is cancelled by the expansion in powers of λ_R of the $n-1$ preceding terms. It is noteworthy that this cancellation is possible only if the divergence of $F_{1,\Lambda}(x)$ is independent of x, i.e. just a number. If that is not the case, then $F_{1,\Lambda}(x) - F_{1,\Lambda}(\mu)$ will diverge for every $x \neq \mu$. This would entail going through the renormalization procedure (at least) once again introducing (at least) one more independent coupling constant, which does not agree with the fact that our theory is defined to have only one free parameter.

The analysis follows through for higher orders in perturbation theory. So we would have

$$F_\Lambda(x) = \lambda_R + \delta_2\lambda + \delta_3\lambda + (\lambda_R^2 + 2\lambda_R\delta_2\lambda)F_{1,\Lambda}(x) + \lambda_R^3 F_{2,\Lambda}(x) + \mathcal{O}(\lambda_R^4)\,,$$

where we have used $\lambda_0^3 = \lambda_R^3 + \mathcal{O}(\lambda_R^4)$ and $\lambda_0^2 = \lambda_R^2 + 2\lambda_R\delta_2\lambda + \mathcal{O}(\lambda_R^4)$. Imposing $\lambda_R = F(\mu)$ once again, we obtain

$$\delta_3\lambda = 2\lambda_R^3\left(F_{1,\Lambda}(\mu)\right)^2 - \lambda_R^3 F_{2,\Lambda}(\mu)\,.$$

Substituting back, we obtain

$$\begin{aligned} F_\Lambda(x) = \lambda_R + \lambda_R^2\left(F_{1,\Lambda}(x) - F_{1,\Lambda}(\mu)\right) + \lambda_R^3\,[F_{2,\Lambda}(x) - F_{2,\Lambda}(\mu) - \\ 2F_{1,\Lambda}(\mu)\left(F_{1,\Lambda}(x) - F_{1,\Lambda}(\mu)\right)] + \mathcal{O}(\lambda_R^4)\,. \end{aligned}$$

To eliminate the divergence, we require

$$F_{2,\Lambda}(x) - F_{2,\Lambda}(\mu) - 2F_{1,\Lambda}(\mu)\left(F_{1,\Lambda}(x) - F_{1,\Lambda}(\mu)\right) \tag{1.8}$$

to be regular in x and μ when $\Lambda \to \infty$. The new feature now is that we also have $F_{1,\Lambda}$ in our constraint. For convenience we split up the $F_{i,\Lambda}$'s in a regular and a singular part for this limit

$$F_{i,\Lambda}(x) = F_{i,\Lambda}^s(x) + F_{i,\Lambda}^r(x)\,.$$

This decomposition is not unique since adding anything to infinity yields infinity again. So, the $F^s_{i,\Lambda}(x)$'s are defined up to a regular part. It is convenient to make the choice

$$F^s_{1,\Lambda}(x) - F^s_{1,\Lambda}(\mu) \stackrel{\Lambda\to\infty}{\longrightarrow} 0 \tag{1.9}$$

which is also implied by the constraint in the analysis for the expansion to the second order in λ. This choice is always possible if the previous constraint is fulfilled[2]. This means that the divergent part of $F_{1,\Lambda}$ is independent of x. We can go a step further and impose a tighter constraint to $F^s_{1,\Lambda}$ and choose it to be completely independent of x for every Λ, because we can tune the regular part of $F_{1,\Lambda}$. So, we define

$$F^s_{1,\Lambda}(x) = f_1(\Lambda)\,. \tag{1.10}$$

For our integral (1.2) we can choose

$$f_1(\Lambda) = \alpha \log \Lambda \quad \text{and} \quad F^r_{1,\Lambda}(x) = \alpha \log\left(\frac{\Lambda + x}{\Lambda x}\right)\,.$$

We now substitute back in our constraint (1.8) and obtain

$$F^s_{2,\Lambda}(x) - F^s_{2,\Lambda}(\mu) - 2f_1(\Lambda)\left(F^r_{1,\Lambda}(x) - F^r_{1,\Lambda}(\mu)\right) \stackrel{\Lambda\to\infty}{\longrightarrow} 0$$

which can be rewritten as

$$\left(F^s_{2,\Lambda}(x) - 2f_1(\Lambda)F^r_{1,\Lambda}(x)\right) - \left(F^s_{2,\Lambda}(\mu) - 2f_1(\Lambda)F^r_{1,\Lambda}(\mu)\right) \stackrel{\Lambda\to\infty}{\longrightarrow} 0\,.$$

The structure is the same as (1.9) up to the replacement $F^s_{1,\Lambda} \to F^s_{2,\Lambda} - 2f_1(\Lambda)F^r_{1,\Lambda}$ and can therefore have the same solution as (1.10). We have:

$$F^s_{2,\Lambda}(x) = 2f_1(\Lambda)F^r_{1,\Lambda}(x) + f_2(\Lambda)$$

where $f_2(\Lambda)$ is any function of Λ independent of x. Apparently, the divergent part of $F_{2,\Lambda}$ does depends on x, unlike $F^s_{1,\Lambda}$. This dependence however is determined by the first order of the perturbative expansion. The $\delta_2\lambda$ term used to remove the $\mathcal{O}(\lambda_0^2)$ divergence, has produced an x-dependent divergent term at order λ_R^3, namely $2\lambda_R\delta_2\lambda F_{1,\Lambda}(x)$. This kind of dependence is generic in renormalization. The counterterms that remove divergences at a given order produce divergences in higher orders. If the theory is renormalizable, these divergences contribute to the cancellation of divergences that appear at higher order. In this sense, this procedure suggests a precise structure of the divergent parts of the successive terms of the perturbative expansion. At n^{th} order, the singular part of $F_{n,\Lambda}$ involves x-dependent terms entirely determined by the preceding terms plus one new that is x-dependent. In our case, we find

$$F^s_{2,\Lambda}(x) = 2\alpha^2 \log\Lambda \log\left(\frac{\Lambda + x}{\Lambda x}\right) + f_2(\Lambda)\,.$$

Going to higher orders makes it increasingly technical without adding to the circle of ideas already presented. As a rule of thumb, renormalized perturbation theory is technically easier, especially for multiloop diagrams (higher order expansion).

What we essentially did so far is the following. We just reparametrized our theory in terms of the physical quantity λ_R. By renormalizing F we pay the price of letting λ_0 go to infinity for the limit $\Lambda \to \infty$ (if we plug (1.7) in (1.5)). But this is not a problem because λ_0 is not a measurable physical parameter. It is merely something helpful for intermediate calculations. The interpretation of the procedure we followed however is rather obscure. It seems like we first introduced unphysical (bare) quantities that make everything infinite, and then rewrite everything in a way that it looks like we "added" other divergent quantities (counterterms) to compensate for the original divergences.

Nonetheless, the program works! In a realistic, renormalizable theory, we can follow the procedure outlined in our toy model and get sensible results that agree with experiments. Until the physical interpretation of the renormalization procedure was developed by Ken Wilson many people felt uneasy about renormalization. At first, it seems like it is nothing more that a well-organized mathematical trick to hide what is not well understood. The physical picture that we will eventually discuss in the following is that of renormalization group flows in the space of theories.

1.2 Wilsonian Renormalization Theory

To discuss Wilson's analysis it suffices to work in ϕ^4 theory. This will provide the basic qualitative results of the renormalization group (RG) program. We will further make this discussion more intelligible by using a sharp momentum cutoff instead of the method of dimensional regularization which is briefly described in Appendix A. This is also more closely related to the RG procedure in the context of the gauge/gravity duality which is discussed in Chapter 4.

The starting point is the construction of Green's functions of the ϕ^4 theory in terms of a functional integral representation of the generating functional $Z[J]$

$$Z[J] = \int_\Lambda \mathcal{D}\phi \, e^{i\int [\mathcal{L}+J\phi]}.$$

We then impose a sharp ultraviolet cutoff Λ by integrating only over the field configurations $\phi(x) = \int \frac{d^dk}{(2\pi)^d} e^{ikx}\phi(k)$ such that $\phi(k) = 0$ for $|k| > \Lambda$. This amounts to taking into account the influence of quantum fluctuations at very short distances ($L = 1/\Lambda$) or equivalently very large momenta. But to ensure that large momenta are controlled we have to ensure that we are working in Euclidean space. This is crucial. If we remain in Minkowski space, the lightlike components of k can be large while k^2 remains very small. In order to obtain Euclidean momenta we perform a Wick rotation. This also makes this treatment relevant to statistical mechanics*. Incidentally, when the need to rescale time and space differently as done in condensed matter physics (e.g. the surface growth problem), the dynamical exponent z is employed. This measures the difference in the rescaling and is discussed in section 5.1.

The next step is to to carry out the integration over the high-momentum degrees of freedom ϕ. We set $J = 0$ and we let $\Lambda \to \Lambda - \delta\Lambda$ (with $\delta\Lambda > 0$). We then write $\phi = \phi_s + \phi_f$ (let us use s for "slow" and f for "fast"). The fields ϕ_s and ϕ_f are defined in such a way that the Fourier components are non zero for $|k| \leq (\Lambda - \delta\Lambda)$ and $(\Lambda - \delta\Lambda) \leq |k| \leq \Lambda$ respectively. For convenience we introduce a real number $b < 1$ and rewrite $\Lambda - \delta\Lambda = b\Lambda$.

The generating functional for the ϕ^4 theory then becomes

$$\begin{aligned} Z = & \int \mathcal{D}\phi_s \int \mathcal{D}\phi_f \, \exp\left\{ - \int d^d x \left[\frac{1}{2} \left(\partial_\mu \phi_s + \partial_\mu \phi_f \right)^2 + \frac{1}{2} \left(\phi_s + \phi_f \right)^2 \right.\right. \\ & \left.\left. + \, \frac{\lambda}{4!} \left(\phi_s + \phi_f \right)^4 \right] \right\} \\ = & \int \mathcal{D}\phi_s \, e^{-\int \mathcal{L}(\phi_s)} \int \mathcal{D}\phi_f \, \exp\left\{ - \int d^d x \left[\frac{1}{2} \left(\partial_\mu \phi_f \right)^2 + \frac{1}{2} \phi_f^2 \right.\right. \\ & \left.\left. + \, \lambda \left(\frac{1}{6} \phi_s^3 \phi_f + \frac{1}{4} \phi_s^2 \phi_f^2 + \frac{1}{6} \phi_s^3 \phi_f + \frac{1}{4} \phi_f^4 \right) \right] \right\} \end{aligned} \tag{1.11}$$

It should be noted that terms quadratic in $\phi_s \phi_f$ vanish, since Fourier components of different wavelengths are orthogonal. Obviously, in the final expression all terms independent of ϕ_f are gathered in $\mathcal{L}(\phi_s)$.

After an integration over ϕ_f the result should be of the form

$$Z = \int [\mathcal{D}\phi]_{b\Lambda} \exp\left(- \int d^d x \, \mathcal{L}_{\text{eff}} \right) .$$

Here $\mathcal{L}_{\text{eff}}$ has only the Fourier components of $\phi(k)$ with $|k| < b\Lambda$. As it turns out, the effective Lagrangian density is equivalent to the original with some added corrections proportional to powers of λ. These corrections emerge so that they will compensate for the removal of the large momentum Fourier components ϕ_f. This is achieved by the interactions among the remaining $\phi(k)$ that were previously mediated by fluctuations of the ϕ_f.

We now wish to compare the original functional integral $Z[J = 0] = \int_\Lambda \mathcal{D}\phi \, e^{i \int \mathcal{L}}$ and $Z = \int [\mathcal{D}\phi]_{b\Lambda} \exp\left(- \int d^d x \, \mathcal{L}_{\text{eff}} \right)$. We will treat the terms after the first one (the kinetic term) as small perturbations. This is a valid approximation as long as the coupling constants are small. In order to do this we introduce rescaled distances and momenta in the latter

$$k' = k/b \qquad \text{and} \qquad x' = xb \,, \tag{1.12}$$

*The Euclidean functional integral for ϕ^4 theory has precisely the same form as the continuum description of the statistical mechanics of a magnet, where the field $\phi(x)$ is interpreted as the fluctuating spin field $s(x)$.

so that the integral is carried out over $|k'| < \Lambda$. As we already mentioned the $\mathcal{L}_{\text{eff}}$ is the original Lagrangian density with added corrections which are essentially a sum of connected diagrams. This means that we can write (dropping the subscript of the field)

$$\int d^dx\,\mathcal{L}_{\text{eff}} = \int d^dx\left[\frac{1}{2}(1+\Delta Z)(\partial_\mu\phi)^2 + \frac{1}{2}\left(m^2+\Delta m^2\right)\phi^2 \right.$$
$$\left. +\frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \Delta C(\partial_\mu\phi)^4 + \Delta D\phi^6 + \ldots\right].$$

Using the rescaled variable x' we obtain

$$\int d^dx\,\mathcal{L}_{\text{eff}} = \int d^dx\,b^{-d}\left[\frac{1}{2}(1+\Delta Z)\,b^2\left(\partial'_\mu\phi\right)^2 + \frac{1}{2}\left(m^2+\Delta m^2\right)\phi^2 \right.$$
$$\left. +\frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \Delta C b^4(\partial_\mu\phi)^4 + \Delta D\phi^6 + \ldots\right],$$

where Z is the so-called *field-strength renormalization.*

The original functional integral (1.11) gives rise to the propagator

$$\begin{aligned}
\overbrace{\phi_f(k)\phi_f}(p) &= \frac{\int \mathcal{D}\phi_f\,\exp\left(-\int\mathcal{L}_0\right)\phi_f(k)\phi_f(p)}{\int\mathcal{D}\phi_f\,\exp\left(-\int\mathcal{L}_0\right)} \\
&= \frac{1}{k^2}(2\pi)^d\delta^{(d)}(k+p)\Theta(k)\,,
\end{aligned}$$

where

$$\Theta(k) = \begin{cases} 1 & \text{if } b\Lambda < |k| < \Lambda \\ 0 & \text{otherwise.}\end{cases}$$

It is reasonable to demand that the new functional should lead to the same propagator. This is done by rescaling the field according to

$$\phi' = \left[b^{2-d}(1+\Delta Z)\right]^{1/2}\phi\,.$$

The rescaling leads to a transformation of the perturbations:

$$\int d^dx\,\mathcal{L}_{\text{eff}} = \int d^dx\left[\frac{1}{2}\left(\partial'_\mu\phi'\right)^2 + \frac{1}{2}m'^2\phi'^2 + \frac{1}{4!}\lambda'\phi^4 + \Delta C'\left(\partial'_\mu\phi'\right)^4 \right.$$
$$\left. +\Delta D'\phi'^6 + \ldots\right].$$

Now, the new parameters of the Lagrangian are

$$\begin{aligned}
m'^2 &= \left(m^2+\Delta m^2\right)(1+\Delta Z)^{-1}\,b^{-2} \qquad (1.13)\\
\lambda' &= (\lambda+\Delta\lambda)(1+\Delta Z)^{-2}\,b^{d-4}\\
C' &= (C+\Delta C)(1+\Delta Z)^{-2}\,b^d\\
D' &= (D+\Delta D)(1+\Delta Z)^{-3}\,b^{2d-6}\,.
\end{aligned}$$

The combination of integrating out of the high momentum degrees of freedom and the rescaling (1.12) we can viewed as an operation acting on the Lagrangian as a transformation. This procedure can be repeated and integrate over another shell of momentum space. By doing so, another set of transformations is inevitably introduced, analogous to (1.13). In the limit where the parameter b is close to 1 the transformation becomes a continuous one. In this sense, the result of this procedure we view this integration of a field theory as a trajectory or a *flow* in the space of all possible Lagrangians and here lies the core of the renormalization in the Wilsonian sense. The term *renormalization group* is attributed due to the continuous generated transformations of Lagrangians. This of course does not constitute a group in any formal way, since the integration of degrees of freedom is not invertible.

1.3 Renormalization Group Flows

Let us consider a simple case where the Lagrangian is in the vicinity of the point where all the perturbations vanish. This means that we are close to the point where $m^2, \lambda, C, D\ldots$ are equal to zero. The transformations will leave this point unchanged. So, we say that the free-field Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$$

is a *fixed point* of the renormalization group transformation.

Keeping only terms that are linear in perturbations gives a simple transformation law:

$$m'^2 = m^2 b^{-2}, \quad \lambda' = \lambda b^{d-4}, \quad C' = Cb^d, \quad D' = Db^{2d-6}. \tag{1.14}$$

Since $b < 1$, parameters multiplied by a negative power of b will grow, while the others will decay. The growing coefficients of the Lagrangian will carry it away from $\mathcal{L}_0$.

The various terms in $\mathcal{L}_{\text{eff}}$ can be thought of as a set of local operators that can be added as perturbations to $\mathcal{L}_0$. As we are interested in longer distance scales (or lower energies) couplings are affected in different ways. The operators with coefficients that grow during the iterative procedure of transformation are called *relevant*. Those that decay are called *irrelevant* and those that are multiplied by b^0 are called *marginal*. For example, the mass term ϕ^2 is always relevant but the ϕ^4 term is relevant only if $d < 4$. In the case of $d = 4$ the ϕ^4 term is marginal. The general formula that determines the transformation of an operator with N powers of ϕ and M derivatives is:

$$C'_{N,M} = b^{N(d/2-1)+M-d} C_{N,M}\,.$$

From a Wilsonian point of view, any quantum field theory is defined fundamentally with a cutoff Λ that has some physical significance. For statistical mechanics this is translated as the inverse atomic spacing. In Quantum Electrodynamics and other relevant to high energy physics theories, the cutoff is associated with some fundamental graininess of spacetime. What all this means is that no matter what the Lagrangian looks like at its fundamental scale, as long as the couplings are sufficiently weak, it must be described at the energies of our experiments by a renormalizable effective Lagrangian, $\mathcal{L}_{\text{eff}}$.

As an illustration we turn again to the renormalization group flows near $\mathcal{L}_0$ for the case of ϕ^4 theory. We consider three distinct cases $d > 4$, $d = 4$ and $d < 4$ as shown in Fig. 1.1. For $d > 4$ the only relevant operators is the mass term since it increases near the point of $\mathcal{L}_0$. Meanwhile, the ϕ^4 term and all other higher order interactions decay.

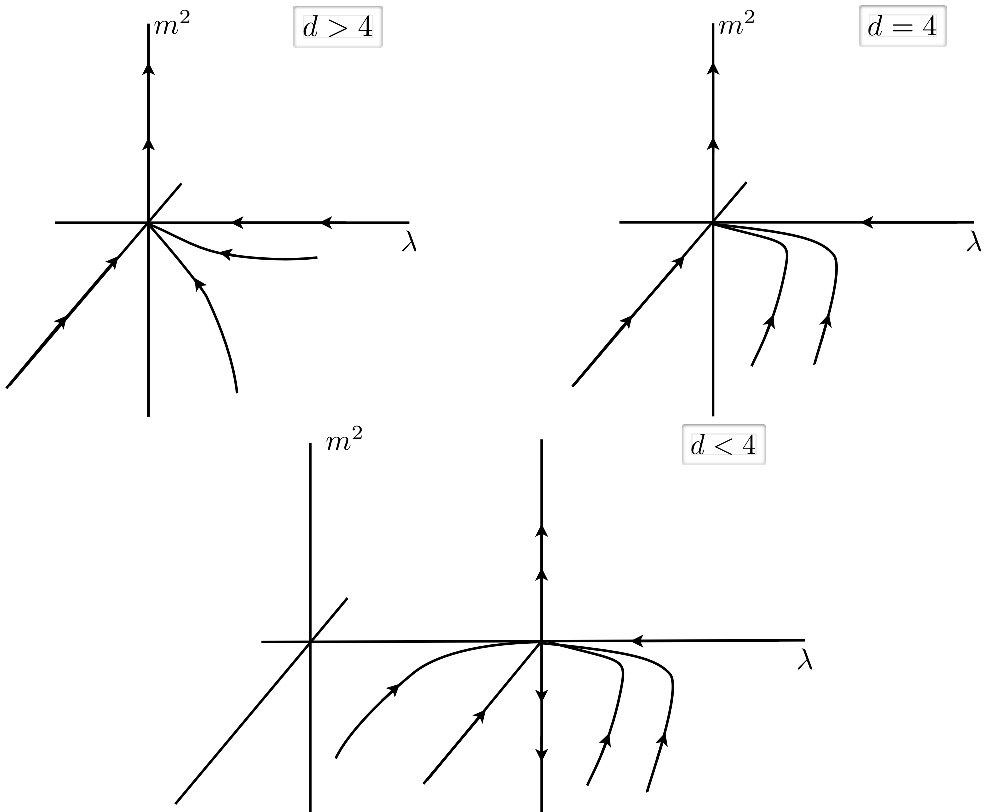


Fig. 1.1: RG flows near the free-field fixed point [3].

If we consider the $d = 4$ case the transformation law (1.14) does not suffice to decide whether the ϕ^4 term is relevant or irrelevant. We have to use the complete set of transformations (1.13). We find the transformation

$$\lambda' = \lambda - \frac{3\lambda^2}{16\pi^2} \log\left(b^{-1}\right)$$

which suggests that λ slowly decreases as we integrate out high momentum degrees of freedom.

For the final case of $d < 4$, λ becomes a relevant parameter. This means that the theory flows away from the free theory $\mathcal{L}_0$ as we integrate out degrees of freedom. In other words, in large distances the ϕ^4 interaction becomes increasingly important. But due to the nonlinear corrections that occur when λ becomes large at one-loop, we find for $d < 4$ that

$$\lambda' = \left[\lambda - \frac{3\lambda^2}{(4\pi)^{d/2}\,\Gamma\left(d/2\right)} \frac{b^{d-4}-1}{4-d}\Lambda^{d-4}\right] b^{d-4}\,,$$

What this implies is that there is a value of λ at which the decrease due to the nonlinear effect compensates the increase due to rescaling. At this point, λ remains unchanged as we integrate out degrees of freedom. This leads to a second fixed point of the renormalization group flow. By taking the limit of $d \to 4$ the new fixed point merges with the free field theory fixed point and shares the property that the mass parameter m^2 is increased by iteration. So, the mass operator will be a relevant operator near the new fixed point and the form of the flow "degenerates" to the form of the $d = 4$ case. In this region ($d < 4$), the case of $d = 2$ is especially interesting: all the operators (of any power) become relevant.

In condensed matter physics, for a given scalar field theory, the dimensions d at which most relevant interactions become marginal is known as the *critical dimension*. For example, in the case of the ϕ^6 theory, the critical dimension is 3.

In a quantum field theory in a arbitrary scale μ the coupling constant evolves according to the renormalization group flow equation

$$\mu\frac{dg}{d\mu} = \beta(g)\,.$$

For a theory with several (N) coupling constants we can write

$$\frac{dg_i}{dt} = \beta_i(g_1, \ldots, g_N)$$

upon defining $t \equiv \log(\mu/\mu_0)$. It is also known as the *Gell-Mann–Low* equation. In this picture, we think of $(g_1, \ldots, g_N)$ as the coordinate of a particle in N-dimensional space, t as time, and $\beta_i(g_1, \ldots, g_N)$ as a position dependent velocity field. The idea is to study how the particle moves or *flows* as we tamper with μ and t. Obviously, the point where β vanishes is of particular interest. This point $(g_1^*, \ldots, g_N^*)$ is called the aforementioned *fixed point.* We can distinguish three classes:

1. *Stable fixed points,* whose scaling fields are all irrelevant, or at worst marginal. These points define in condensed matter physics what we

might call "stable phases of matter". When a system is released somewhere in the parameter space surrounding any of these attractors, it will scale toowards the fixed point and eventually sit there. In other words, when the system is studied in larger and larger distance scales it will resemble the infinitely correlated self-similar fixed point configuration.

2. *Unstable fixed points*, whose scaling fields are all relevant. These fixed point represent the ancient old idea of never being able to get there! The conditions near the fixed point will force the system to *flow* away from it. Despite the lack of realizable forms of matter that correspond to this situation, they are still important as they "orient" the global RG flow of the system.

3. *Generic class of fixed points* with both relevant and irrelevant scaling fields. These points present the interesting association to phase transitions. In condensed matter at fixed points the intrinsic length ξ is either zero or infinite. The first case is not interesting. The latter however, a diverging correlation length $\xi \to \infty$, is an indicator of a second-order phase transition.

As a general remark, we mention again that once the coupling constants are fine-tuned to a fixed point, the system no longer changes under subsequent RG transformations. In particular, it remains invariant under the change of space or time scale associated with the transformation. So, they look the same no matter how closely we observe them. Systems that also share this property are fractals (Fig. 1.2).

Field theories that are insensitive to this scaling transformation, that have zero β functions, are often referred to as *conformal field theories.* There are also some other considerations involved that have to do with symmetries, but these will be addressed later[†].

If we want to study behaviour of a theory at high energies we need to find all its stable points under the renormalization group flow. Some couplings may flow toward larger values while other are flowing toward zero.

In practice however, this is difficult to implement since we have no way of calculating the functions $\beta_i(g)$. What is more, the point g^* can be quite far away from zero and this is known as a strong coupling fixed point. In this case perturbation theory and Feynman diagrams break down and are of no use in determining the properties of the theory there. In fact the fixed point structure of very few theories is known.

[†]Strictly speaking the conformal group is a much larger group that includes scaling transformations. Be that as it may, cases of quantum field theories that are scale invariant and not conformally invariant are extremely rare (see [4] for a counterexample). Therefore, the terms scale-invariant and conformally-invariant are customarily used interchangeably.

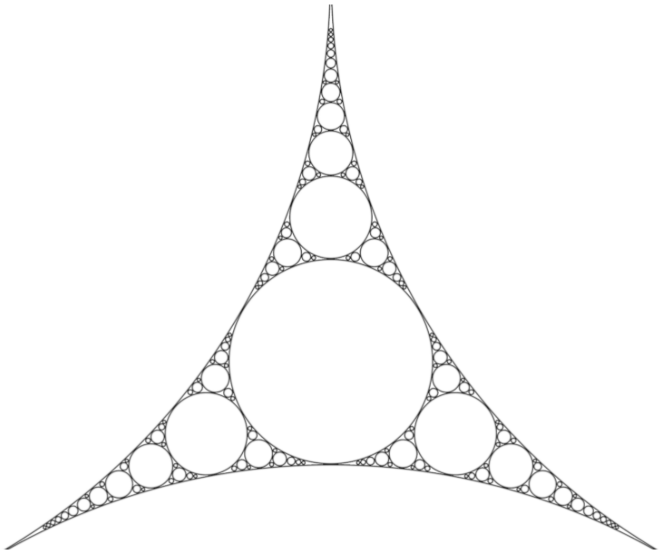


Fig. 1.2: This is an *Apollonian packing*, in which the triangular gap between three touching circles is filled with another circle. That leaves three more gaps, which are filled by three smaller circles, etc. Similar insensitivity to zooming in or out characterizes physical systems at fixed point.

1.4 The Callan-Symanzik Equation

Going back to the ϕ^4 theory let us now suppose that we want to define the same theory at a different scale μ'. This means that we are looking at a theory whose bare Green's functions

$$\langle\Omega|\, T\phi_0(x_1)\phi_0(x_2)\dots\phi_0(x_n)\,|\Omega\rangle$$

are given by the same functions of the bare coupling constant λ_0 and the cutoff Λ. Here $|\Omega\rangle$ represents the ground state. The scale dependence enters when the cutoff dependence is removed by rescaling the fields and eliminating λ_0 in favor of the renormalized coupling λ. Green's functions remain the same up to a rescaling by powers of the field strength renormalization: $Z^{-n/2}$.

Let us suppose that we shift the scale by $\delta\mu$. This leads to

$$\begin{aligned}\mu \to& \mu + \delta\mu\\ \lambda \to& \lambda + \delta\lambda\\ \phi \to& (1+\delta\eta)\phi\,.\end{aligned}$$

The induced renormalized Green's function is

$$G^{(n)} \to (1+n\delta\eta)G^{(n)}\,.$$

Treating $G^{(n)}$ as a functional of μ and λ we write

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial \mu}\delta\mu + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda = n\partial\eta G^{(n)} .$$

Conventionally though, what is used is

$$\beta \equiv \frac{\mu}{\delta\mu}\delta\lambda \qquad \gamma \equiv -\frac{\mu}{\delta\mu}\delta\eta .$$

This is the same β function that we mentioned in a more abstract framework in the discussion of the RG flows. Substituting back and multiplying by $\mu/\delta\mu$ we obtain

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right] G^{(n)}\left(x_1, \ldots, x_n; \mu, \lambda\right) = 0 ,$$

which is the so-called *Callan-Symanzik* equation. The parameters β and γ are the same for every n and must be independent of x_i. Since $G^{(n)}$ is renormalized, β and γ cannot depend Λ and therefore, by dimensional analysis, these functions cannot depend on μ. They are functions of the dimensionless variable λ only! So, Green's functions of massless ϕ^4 theory must satisfy the Callan-Symanzik equation. What it tells us is that there exist two universal functions $\beta(\lambda)$ and $\gamma(\lambda)$, related to the shifts in the coupling constant and field strength, that compensate for the shift in the renormalization scale μ. The generalization to other massless theories with dimensionless couplings is straightforward. We mention for example QED at zero electron mass gives

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(e)\frac{\partial}{\partial e} + n\gamma_2(e) + m\gamma_3(e)\right] G^{(n,m)}\left(x_1, \ldots, x_n; \mu, e\right) = 0 ,$$

where n and m are the number of electron and photon fields respectively in the Green's function and γ_2 and γ_3 are the rescaling functions of the electron and photon fields.

1.5 Renormalization Schemes

Several ways of renormalizing a theory have been developed by field theorists throughout the years. We will not discuss them thoroughly. We will only describe one of them and make some comments on scheme dependence.

The most popular renormalization schemes are the *Minimal Subtraction* (MS) and the related modified minimal subtraction ($\overline{\text{MS}}$), the *Dimensional Regularization* (DR) and the modified version of that ($\overline{\text{DR}}$). We directly present the rules for the MS scheme:

1. Dimensional regularization to control ultraviolet divergences

2. Identify the μ parameter of dimensional regularization

$$\frac{d^4 p}{(2\pi)^4} \to \mu^{4-D} \frac{d^D p}{(2\pi)^D}$$

with the energy scale of the renormalization group. This identification set the logarithm $\log\left(\mu^2/\mu_0^2\right)$ to zero.

3. The ultraviolet divergence of a L-loop amplitude is a L^{th}-degree polynomial in $1/\epsilon$:

$$\left(\frac{A_L}{\epsilon^L} + \frac{A_{L-1}}{\epsilon^{L-1}} + \ldots + \frac{A_1}{\epsilon^1} + f(p^2)\right) \tag{1.15}$$

for some constants $A_L, \ldots, A_1$ and with $f(p^2)$ being a finite function of p^2. To cancel the divergence we need a L-loop order counterterm

$$\delta_L^Z = g^{2L}\left(\frac{A_L}{\epsilon^L} + \frac{A_{L-1}}{\epsilon^{L-1}} + \ldots + \frac{A_1}{\epsilon^1} + A_0\right),$$

where the coefficients are now completely determined by the ultraviolet divergence of the L-loop diagrams, but the finite term A_0 is not. Its value is determined from the renormalization scheme we use for the amplitude 1.15 and not from the divergence.

In the MS scheme there are no conditions imposed on the amplitudes. Instead, we set $A_0 = 0$. Similarly, the finite parts of all the other counterterms are set to zero. The name comes from the fact that the counterterms simply subtract the pole at $\epsilon = 0$. The finite part of the amplitude is whatever the loop diagrams produce and the counterterms do not mess with it. In the similar modified minimal subtraction scheme $\overline{\text{MS}}$, one absorbs the divergent part plus a universal constant (which always arises along with the divergence in Feynman diagram calculations) into the counterterms.

The DR and $\overline{\text{DR}}$ schemes are more often used in supersymmetric theories. The schemes are similar to MS and $\overline{\text{MS}}$ schemes with a different approach to dimensional regularization (Appendix A) called dimensional reduction. In this case, all momenta live in $D = 4 - 2\epsilon$ dimensions but the vector fields maintain all four components. Such a reduced four-dimensional vector field comprises one species of a D-dimensional vector plus 2ϵ species of scalar with the same mass and charge. Unlike the original dimensional regularization, the dimensional reduction respects supersymmetry and that is the main difference between the two.

Scheme (in)dependence

The renormalization scheme is expected to modify the coupling constant, and in fact it does. So, for the same energy we would end up with a slightly different running coupling $\lambda'(\mu_0) \neq \lambda(\mu_0)$. The differences usually appear at one loop and are due to quantum corrections

$$\lambda'(\mu_0) = \lambda(\mu_0) + \mathcal{O}(\lambda^2(\mu_0))\,.$$

The beta function however turns out to be independent of the renormalization scheme up to three-loop level. We can prove this as follows.

We write the coupling constant as a power series

$$\lambda'(\mu_0) = \lambda(\mu_0) + c_1\lambda^2(\mu_0) + c_2\lambda^3 + \dots \tag{1.16}$$

with some constants $c_1, c_2, \dots$ The inverse coupling would be

$$\frac{1}{\lambda'(\mu_0)} = \frac{1}{\lambda(\mu_0)} - c_1 + \left(c_1^2 - c_2\right)\lambda(\mu_0) + \dots$$

The inverse coupling depends on energy according to

$$\frac{d}{d\log E}\frac{1}{\lambda(\mu_0)} = \frac{-1}{\lambda^2(\mu_0)}\frac{d\lambda(\mu_0)}{d\log\mu_0} = \frac{-1}{\lambda^2(\mu_0)}\beta(\lambda(\mu_0))\,.$$

We can write the beta function in power series

$$\beta(\lambda) = b_1\lambda^2 + b_2\lambda^3 + b_3\lambda^4 + \dots$$

and use it to get

$$\frac{d}{d\log(\mu_0)}\frac{1}{\lambda(\mu_0)} = -b_1 - b_2\lambda(\mu_0) - b_3\lambda^2(\mu_0) - \dots$$

Let us assume that from a different renormalization scheme we obtained the β-function

$$\beta'(\lambda') = b_1'\lambda'^2 + b_2'\lambda'^3 + b_3'\lambda'^4 + \dots$$

and consequently

$$\frac{d}{d\log(\mu_0)}\frac{1}{\lambda'(\mu_0)} = -b_1' - b_2'\lambda'(\mu_0) - b_3'\lambda'^2(\mu_0) - \dots \tag{1.17}$$

We now differentiate both sides of the coupling constant expansion (1.16) and obtain

$$\begin{aligned}\frac{d}{d\log(\mu_0)}\frac{1}{\lambda'(\mu_0)} &= \left[\frac{-1}{\lambda^2(E)} + \left(c_1^2 - c_2\right) + \dots\right]\Big(-b_1\lambda^2(\mu_0) - b_2\lambda^3(\mu_0)\\ &\qquad\qquad\qquad\qquad\qquad -b_3\lambda^4(\mu_0) - \dots\Big)\\ &= -b_1 - b_2\lambda(\mu_0) - \left[b_3 - b_1\left(c_1^2 - c_2\right)\right]\lambda^2(\mu_0) - \dots\\ &= -b_1 - b_2\lambda'(\mu_0) - \left[b_3 - -b_2c_1 - b_1\left(c_1^2 - c_2\right)\right]\lambda'^2(\mu_0) - \dots\end{aligned}$$

By comparison with (1.17) we see that $b_1' = b_1$ and $b_2' = b_2$ but $b_3' \neq b_3$ and we can guess that higher order coefficients are also different. A similar theorem applies to theories with more coupling constants for the three-loop and higher-order terms. Of course, physical observables are unaffected by the renormalization scheme used.

2 Action Principle and Geometry

As a universal strategy in physics, the relation of the Hamiltonian and the generator of time translations is used to determine the time evolution of a physical system. The Hamiltonian is also directly related to the total energy of a physical system. It is therefore useful to have a Hamiltonian formulation of a physical theory. However, in General Relativity, this turns out to be a non-trivial task. However, it was done in a satisfactory manner in the pivotal paper by Arnowitt, Deser, Misner [5] (ADM). They provided a consistent way of discussing energy in curved spacetime, yielding positive values and obeying fundamental conservation principles. This also opened the way for positive energy theorems [6] (an interesting alternative proof can be found in [7]) This solidified the fact that Minkowski space is stable. As we will see (Chapter 2.2), the coordinate invariance underlying the theory creates an analogous problem to gauge invariance in electromagnetic theory. This makes the required breakup of spacetime into space and time more subtle than in, for example, classical field theory.

Eventually, we will want to use the Hamilton-Jacobi theory. Since this can be viewed in a more elemental level and is not always treated in a course on (classical) mechanics nor (classical) field theory, we will discuss this first. We will also have a chance to present some notions that are useful to what is discussed in following chapters. Afterwards, we will present the Hamiltonian formulation of General Relativity and therein, discuss Hamilton constraints.

2.1 Hamilton-Jacobi Theory

We start by describing the idea of Hamilton and Jacobi in the context of classical mechanics.

We define the action for a trajectory as a functional of the independent varables q_i and p_i and the Hamiltonian

$$S = \int_{t_0}^{t_1} dt \left(p_i(t) dq_i(t) - H\left(q_i(t), p_i(t), t\right)\right). \tag{2.1}$$

According to the *modified Hamilton's principle* the true path extremizes the action for whatever independent variation of $q_i(t)$ and $p_i(t)$, for which the initial and final positions $q_i(t_1)$, $q_i(t_2)$ are fixed. The difference between the modified and the original Hamilton's principle is that we treat both position and momentum as independent variables. In order to extremize the action we vary both $q_i(t)$ and $p_i(t)$, not just $q_i(t)$.

Next, we make the observation that, even in the Lagrangian formulation, adding a total time derivative of a function $F(q,p,t)$ will not affect the true path

$$S = \int_{t_0}^{t_1} dt \sum_i \left(p_i(t) dq_i(t) - H\left(q_i(t), p_i(t), t\right) + \frac{dF(q,p,t)}{dt} \right). \tag{2.2}$$

Indeed, the true path is the same for both (2.1) and (2.2). This observation is important because it allows us to introduce a class of coordinate transformations

$$Q_i = Q_i(q,p,t)\,, \quad P_i \quad = P_i(q,p,t)$$

with the all-important property that the true path satisfies Hamilton's equations in the new coordinates:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}\,, \quad \dot{P}_i \quad = -\frac{\partial K}{\partial Q_i}\,, \tag{2.3}$$

where K is the Hamiltonian in the new coordinates that must be determined. The class of transformations that satisfy (2.3) are called *canonical transformations*. It is important to note that if the transformation satisfies:

$$p_i \dot{q}_i - H\left(q_i, p_i, t\right) = P_i \dot{Q}_i - K\left(Q_i, P_i, t\right) + \frac{dF}{dt} \tag{2.4}$$

then the true path that extremizes the action

$$S_1 = \int_{t_0}^{t_1} dt \sum_i \left(P_i dQ_i - K(Q,P,t)\right)$$

satisfies (2.3) and also extremizes

$$S = \int_{t_0}^{t_1} dt \sum_i \left(p_i(t) dq_i(t) - H\left(q_i(t), p_i(t), t\right)\right)$$

by virtue of (2.4). Therefore, equations (2.3) describe the true path that satisfies Hamilton's equations $\dot{q}_i = \partial H/\partial p_i$, $\dot{p}_i = -\partial H/\partial q_i$ but now in the new coordinates Q, P. In other words, canonical transformations must satisfy the form of (2.4).

Furthermore, (2.4) can be rewritten as

$$dF = \sum_i \left(p_i dq_i - P_i dQ_i\right) + (K - H)\, dt\,, \tag{2.5}$$

in which case we can choose a function $F_1(q, Q, t)$. Thus, for the example where F depends only on q, Q and possibly time t, we have

$$dF_1 = \sum_i \left(\frac{\partial F_1}{\partial q_i} dq_i + \frac{\partial F_1}{\partial Q_i} dQ_i \right) + \frac{\partial F_1}{\partial t} dt \,. \tag{2.6}$$

Then, the canonical transformation $(q, p) \to (Q, P)$ is indirectly defined by the relations

$$p_i = \frac{\partial F_1}{\partial q_i} \,, \qquad P_i = -\frac{\partial F_1}{\partial Q_i} \,. \tag{2.7}$$

This is so because we can determine the new coordinates as functions of the old ones $Q_i = Q_i(q, p, t)$, $P_i = P_i(q, p, t)$ by solving the algebraic equations (2.7). Comparing (2.5) and (2.6) we can see that the Hamiltonian in the new coordinates is

$$K = H + \frac{\partial F_1}{\partial t} \,.$$

The radical idea of Hamilton and Jacobi was to make a canonical transformation in which the Hamiltonian K is zero. If a transformation like that can be determined then the initial dynamical problem is immediately solved in the new coordinates since the equations of motion become

$$\dot{Q}_i = 0 \,, \qquad \dot{P}_i = 0 \,.$$

We immediately see that

$$H(q, p, t) + \frac{\partial F}{\partial t} = 0$$

through which one can define $F(q, p, t)$. But we also have $p_i = \partial F / \partial q_i$. Therefore, the differential equation that F needs to satisfy can be written as

$$\frac{\partial F}{\partial t} + H \left(q_i, \frac{\partial F}{\partial q_i}, t \right) = 0 \,. \tag{2.8}$$

Equation (2.8) that determines the required transformation is called the *Hamilton-Jacobi equation*. Solving the Hamilton-Jacobi equation is equivalent to solving Hamilton's equations.

As a partial differential equation it can have more than one solution. As an example we can work out one solution of the Hamilton-Jacobi for the simple case of the free particle Hamiltonian

$$H = \frac{p^2}{2m} \,.$$

If we take

$$F(q, p, t) = \frac{(q - Q)^2}{2m(t - t_\star)} \tag{2.9}$$

as a solution, we can see that this function satisfies the Hamilton-Jacobi equation:

$$\frac{\partial F}{\partial t} + \frac{1}{2m}\left(\frac{\partial F}{\partial q}\right)^2 = 0\,.$$

Therefore, we know that it will provide the required transformation. We can now discuss the physical interpretation of F. We can recognize in (2.9) that it describes the movement of a particle that was at the time $t_\star$ at the position Q and at time t at the position q. This is more than just a coincidence.

Let us define the *function* $S(q_1, q_2, t_1, t_2)$ as the action that emerges from the true path that goes through $q_1(t_1)$ and $q_2(t_2)$. The function

$$S(q_1, q_2, t_1, t_2) = \int_{t_2}^{t_1} dt\ (p\dot{q} - H) = \int_{q_1}^{q_2} dq\, p - \int_{t_1}^{t_2} dt\, H$$

is no longer a functional, but merely a function in position space that depends only on the initial and final positions and times, the connection of which is achieved automatically via the true path. A small spacetime variation of the initial and final points will result in the variation of the function action

$$dS = p_2 dq_2 - p_1 dq_1 - H_2 dt_2 + H_1 dt_1 \tag{2.10}$$

where $p_{1,2}$ and $H_{1,2}$ are the values of momenta and the Hamiltonian at the endpoints. From (2.10) we can deduce

$$p_2 = \frac{\partial S}{\partial q_2}, \qquad p_1 = -\frac{\partial S}{\partial q_1}$$

and

$$H_2 = -\frac{\partial S}{\partial t_2}, \qquad H_1 = \frac{\partial S}{\partial t_1}\,.$$

Therefore, the function $S(q, Q, t, t_\star)$ that corresponds to the function of the true path from Q at time $t_\star$ to q at time t is a solution of the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0\,.$$

This makes the function $S(q, Q, t, t_\star)$ the desired function $F(q, Q, t)$ that generates the canonical transformation from (q, p) to (Q, P) and makes the Hamiltonian expressed in the coordinates Q, P to equal zero.

However, determining $S(q, Q, t, t_\star)$ requires the knowledge of the true path, and therefore the idea of Hamilton-Jacobi is not a panacea for the solution of the dynamical equations. Nonetheless, when the Hamiltonian as it appears in (2.8) can be split in terms that depend on separate variables the Hamilton-Jacobi equation can provide a fast mathematical solution to the physical problem.

2.2 Electromagnetism and the Hamiltonian

Before embarking in formulating General Relativity in terms of the Hamiltonian, it is useful to see similar subtleties and problems that arise in electromagnetism in Minkowski spacetime, for similar reasons.

Although it might seem straightforward to apply the standard procedure to obtain the Hamiltonian formulation in the classical electromagnetic field in Minkowski spacetime, it is actually a bit more involved than that. Let us see how this comes about.

We may start by writing the Lagrangian density in ordinary 3-dimensional vector notation:

$$\mathcal{L}_{\mathrm{EM}} = \frac{1}{2}\left(\dot{\vec{A}} + \vec{\nabla}V\right)\cdot\left(\dot{\vec{A}} + \vec{\nabla}V\right) - \frac{1}{2}\left(\vec{\nabla}\times\vec{A}\right)\cdot\left(\vec{\nabla}\times\vec{A}\right).$$

The momentum conjugate to $\vec{A}$ is

$$\vec{\pi} = \dot{\vec{A}} + \vec{\nabla}V \equiv -\vec{E}\,.$$

However, it is immediately obvious that there is a problem: $\dot{V}$ does not appear in $\mathcal{L}_{\mathrm{EM}}$. This means that the momentum π_V conjugate to V vanishes identically:

$$\pi_V = 0\,.$$

On the other hand, the Hamiltonian density $\mathcal{H}$ is defined as

$$\mathcal{H}(q,\pi) = \pi\dot{q} - \mathcal{L}\,.$$

This means that we cannot obtain an invertible relation between π and $\dot{q}$. In other words, we cannot eliminate $\dot{q}$ in favor of π and q. This leads to a breakdown of our general prescription for obtaining a Hamiltonian. This is directly related to the arbitrariness of gauge in $\vec{A}$ which prohibits getting deterministic dynamics for the gauge field of the form

$$\dot{q} \equiv \frac{\delta H}{\delta\pi} \qquad (2.11a)$$

$$\dot{\pi} \equiv -\frac{\delta H}{\delta q}\,. \qquad (2.11b)$$

Fortunately, there is a way to resolve this. We can think of V as not a dynamical variable since π_V vanishes identically. So, we define

$$\begin{aligned}\mathcal{H}_{\mathrm{EM}} &= \vec{\pi}\cdot\dot{\vec{A}} - \mathcal{L}_{\mathrm{EM}}\\ &= \frac{1}{2}\vec{\pi}\cdot\vec{\pi} + \frac{1}{2}\vec{B}\cdot\vec{B} - \vec{\pi}\cdot\vec{\nabla}V\\ &= \frac{1}{2}\vec{\pi}\cdot\vec{\pi} + \frac{1}{2}\vec{B}\cdot\vec{B} + V\vec{\nabla}\cdot\vec{\pi} - \vec{\nabla}\cdot(V\vec{\pi})\,.\end{aligned}$$

The total divergence term at the end contributes only as a boundary term in the action and will vanish in the limit as the boundary goes to infinity for the usual asymptotic conditions imposed on V and $\vec{\pi} = -\vec{E}$. The main point is that we view H_{EM} as a functional of $\vec{A}$ and $\vec{\pi}$, with V playing the effective role of a Lagrange multiplier. This means adding the equation

$$\frac{\delta H_{\text{EM}}}{\delta V} = 0$$

to (2.11a) and (2.11b) for $\dot{\vec{A}}$ and $\dot{\vec{\pi}}$*. The newly added equation essentially suggests

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{2.12}$$

and (2.11a), (2.11b) yield

$$\begin{aligned} \dot{\vec{A}} &= -\vec{E} - \vec{\nabla} V && \text{(2.13a)} \\ \dot{\vec{\pi}} &= -\dot{\vec{E}} = -\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) && \text{(2.13b)} \end{aligned}$$

and all of them are equivalent to Maxwell's equations.

This result can be viewed as obtaining the constraint (2.12) and the evolution equations (2.11). As expected the value of H_{EM} for a solution of Maxwell's equations will be proportional to the total energy of the electromagnetic field. This type of Hamiltonian formulation where a non-dynamical variable appears in $\mathcal{H}$ and is effectively a Lagrange multiplier is called *constrained Hamiltonian formulation*. It is a general feature, expected to arise, in a theory where the field variables have a gauge arbitrariness.

2.3 The ADM Formalism

Discussing the ADM [5] formulation of the dynamics of geometry boils down to carefully formulating General Relativity as a field theory, using the Hamiltonian instead of the Lagrangian. A Hamiltonian formulation of a field theory requires a breakup of spacetime into space and time. This is subtle in the case of General Relativity and should be done with some caution. The reason for this, as we already mentioned, is that coordinate invariance underlying the theory creates analogous problems to gauge invariance in electromagnetic theory. Comprehensive discussions on this topic can be found in [8] and [9]. Here, we will primarily focus on aspects related to Hamilton constraints in the presence of gravity.

The idea of slicing spacetime into a one-parameter family of spacelike hypersurfaces is necessary not only for the analysis of the dynamics along the way, but also by the boundary conditions in an action principle. It is summed

*Here, $\vec{A}$ plays the role of q.

up in [8]: "Give the 3-geometries on the two faces of a sandwich of spacetime and adjust the 4-geometry in between to extremize the action."

Given the 3-geometry[†] of the "lower" hypersurface, $g_{ij}(t,x,y,z)dx^i dx^j$ gives the distance squared between two points on that hypersurface. Similarly for the "upper" hypersurface we have $g_{ij}(t+dt,x,y,z)$. Then a formula for the proper length is $N(t,x,y,z)dt$ of the connector at the point (x,y,z), where $N(t,x,y,z)$ is the *lapse function.* This is properly defined as

$$N = -t^\alpha n_\alpha = (n^\alpha \nabla_\alpha t)^{-1}\,,$$

where t^α is a vector field on the manifold satisfying $t^\alpha \nabla_\alpha t = 1$, and n^α is the normal unit vector. The vector field t^α can be interpreted as the *flow in time* in spacetime.

Another thing we define is the *shift vector* N^α by

$$N^\alpha = h^\alpha{}_\beta t^\beta\,,$$

where $h_{\alpha\beta}$ is the induced metric $h_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta$. In this framework, N measures the rate of flow of proper time τ with respect to coordinate time t (i.e. $d\tau = Ndt$) and N^α measures the amount of shift tangential to the 3-geometry contained in the time flow vector field t^α. The lapse function and the shift vector are not considered to be dynamical, since they merely prescribe a way to move "forward in time".

We also need a formula for the place on the "upper" hypersurface $x^i_{upper}(x^m) = x^i - N^i(t,x,y,z)dt$ where this connector is to be welded and $N^i(t,x,y,z)$ is the so-called shift function. Now, by making use of the Pythagorean theorem in its 4-dimensional form, we get

$$\begin{aligned} ds^2 &= \begin{pmatrix} \text{proper distance} \\ \text{in base 3-geometry} \end{pmatrix}^2 - \begin{pmatrix} \text{proper time from} \\ \text{lower to upper 3-geometry} \end{pmatrix}^2 \\ &= g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) - (Ndt)^2. \end{aligned}$$

This reasoning can be extended to more dimensions in a completely straightforward manner. Putting it all together the 4-metric should have the following form:

$$\begin{pmatrix} g_{00} & g_{0k} \\ g_{i0} & g_{ik} \end{pmatrix} = \begin{pmatrix} (N_s N^s - N^2) & N_k \\ N_i & g_{ik} \end{pmatrix}.$$

In order to lower the indices of the shift functions we used $N_i = g_{im}N^m$. The inverse metric is

$$\begin{pmatrix} g^{00} & g^{0m} \\ g^{k0} & g^{km} \end{pmatrix} = \begin{pmatrix} -1/N^2 & N^m/N^2 \\ N^k/N^2 & \left(g^{km} - N^k N^m/N^2\right) \end{pmatrix}.$$

[†]Here, Latin letters indicate $d-1$ geometry of a d-dimensional manifold. In this case $i,j = 1,2,3$. Greek indices are used for the d-dimensional description; in our discussion, for the 4-geometry.

We can make the following connection. A timelike unit normal vector in covariant 1-form will look like

$$n_\beta = (-N, 0, 0, 0).$$

By use of the inverse metric we can obtain the tangent vector

$$n^\alpha = \left({}^1\!/_N, -{}^{N^m}\!/_N\right) ,$$

so that the typical "perpendicular" connector will have the components

$$(dt, -N^m dt)$$

with proper length $d\tau = N dt$.

So, we can write

$$n^\alpha = \frac{1}{N}\left(t^\alpha - N^\alpha\right) \tag{2.14}$$

and express the inverse metric as

$$g^{\alpha\beta} = h^{\alpha\beta} - n^\alpha n^\beta = h^{\alpha\beta} - N^{-2}(t^\alpha - N^\alpha)(t^\beta - N^\beta)\,.$$

For convenient we will choose the spatial metric $h_{\alpha\beta}$, the lapse function N and the covariant form of the shift vector $N_\alpha = h_{\alpha\beta}N^\beta$ instead of the inverse metric $g^{\alpha\beta}$ as field variables. We see that these contain exactly the same information. We can also write

$$\sqrt{-g} = N\sqrt{h}\,.$$

We now want to express the gravitational action in terms of $(h_{\alpha\beta}, N, N_\alpha)$ with their time and space derivatives as a first step in obtaining the Hamiltonian functional for general relativity. We will ignore boundary terms for the sake of simplicity.

We start with the Hilbert action

$$S = \int_M \sqrt{-g}\mathcal{L}_\mathrm{G}$$

an express the Ricci scalar R as

$$R = 2(G_{\alpha\beta}n^\alpha n^\beta - R_{\alpha\beta}n^\alpha n^\beta)\,,$$

where for $G_{\alpha\beta}$ we have

$$G_{\alpha\beta}n^\alpha n^\beta = \frac{1}{2}\left({}^{(3)}R - K_{\alpha\beta}K^{\alpha\beta} + K^2\right),$$

where $K_{\alpha\beta}$ is the extrinsic curvature of the hypersurface and $K = K^{\alpha}{}_{\alpha}$. From the definition of the Riemann tensor, we have

$$\begin{aligned}
R_{\alpha\beta}n^{\alpha}n^{\beta} &= R_{\alpha\gamma\beta}{}^{\gamma}n^{\alpha}n^{\beta} \\
&= -n^{\alpha}(\nabla_{\alpha}\nabla_{\gamma} - \nabla_{\gamma}\nabla_{\alpha})n^{\gamma} \\
&= (\nabla_{\alpha}n^{\alpha})(\nabla_{\gamma}n^{\gamma}) - (\nabla_{\gamma}n^{\alpha})(\nabla_{\alpha}n^{\gamma}) \\
&\quad -\nabla_{\alpha}(n^{\alpha}\nabla_{\gamma}n^{\gamma}) + \nabla_{\gamma}(n^{\alpha}\nabla_{\alpha}n^{\gamma}) \\
&= K^2 - K_{\alpha\gamma}K^{\alpha\gamma} - \nabla_{\alpha}(n^{\alpha}\nabla_{\gamma}n^{\gamma}) + \nabla_{\gamma}(n^{\alpha}\nabla_{\alpha}n^{\gamma})\,.
\end{aligned}$$

The last two terms are divergences and will be discarded. Thus, we obtain

$$\mathcal{L}_{\mathrm{G}} = N\sqrt{h}({}^{(3)}R + K_{\alpha\beta}K^{\alpha\beta} - K^2)\,. \tag{2.15}$$

The extrinsic curvature is related to the "time derivative", $\dot{h}_{\alpha\beta} \equiv h_{\alpha}{}^{\gamma}h_{\beta}{}^{\delta}\pounds_t h_{\gamma\delta}$ by

$$\begin{aligned}
K_{\alpha\beta} &= \frac{1}{2}\pounds_n h_{\alpha\beta} = \frac{1}{2}\left(n^{\gamma}\nabla_{\gamma}h_{\alpha\beta} + h_{\alpha\gamma}\nabla_{\beta}n^{\gamma} + h_{\gamma\delta}\nabla_{\alpha}n^{\gamma}\right) \\
&= \frac{1}{2}N^{-1}\left[Nn^{\gamma}\nabla_{\gamma}h_{\alpha\beta} + h_{\alpha\gamma}\nabla_{\beta}(Nn^{\gamma}) + h_{\gamma\beta}\nabla(Nn^{\gamma})\right] \\
&= \frac{1}{2}N^{-1}h_{\alpha}{}^{\gamma}h_{\beta}{}^{\delta}\left(\pounds_t h_{\gamma\delta} - \pounds_N h_{\gamma\delta}\right) \\
&= \frac{1}{2}N^{-1}\left(\dot{h}_{\alpha\beta} - D_{\alpha}N_{\beta} - D_{\beta}N_{\alpha}\right),
\end{aligned}$$

where D_{α} is the derivative associated with $h_{\alpha\beta}$ and we used (2.14). If we substitute back to (2.15) we obtain the gravitational action in the desired form given in [5].

The result however gives a $\mathcal{L}_{\mathrm{G}}$ that does not contain any time derivatives of N or N_{α}. Thus, their conjugate momenta vanish identically. In analogy to electromagnetism, we interpret this as the fact that N and N_{α} are not to be viewed as dynamical variables. We define our Hamiltonian density by

$$\begin{aligned}
\mathcal{H}_{\mathrm{G}} &= \pi^{\alpha\beta}\dot{h}_{\alpha\beta}\mathcal{L}_{\mathrm{G}} \\
&= -h^{1/2}N{}^{(3)}R + Nh^{-1/2}\left(\pi^{\alpha\beta}\pi_{\alpha\beta} - \frac{1}{2}\pi^2\right) + 2\pi^{\alpha\beta}D_{\alpha}N_{\beta} \\
&= h^{1/2}\Bigg[N\left(-{}^{(3)}R + h^{-1}\pi^{\alpha\beta}\pi_{\alpha\beta} - \frac{1}{2}h^{-1}\pi^2\right) - 2N_{\beta}\left[D_{\alpha}\left(h^{-1/2}\pi^{\alpha\beta}\right)\right] \\
&\qquad + 2D_{\alpha}(h^{-1/2}N_{\beta}\pi^{\alpha\beta})\Bigg],
\end{aligned}$$

where $\pi = \pi^{\alpha}{}_{\alpha}$. The last term contributes as a boundary term and will be dropped. Variation of H_{G} with respect to N and N_{α} yields

$$-{}^{(3)}R + h^{-1}\pi^{\alpha\beta}\pi_{\alpha\beta} - \frac{1}{2}h^{-1}\pi^2 = 0 \tag{2.16a}$$

$$D_{\alpha}(h^{-1/2}\pi^{\alpha\beta}) = 0\,. \tag{2.16b}$$

The dynamical equations (2.16a), (2.16b) obtained from H_{G} are

$$\dot{h}_{\alpha\beta} = \frac{\delta H_{\mathrm{G}}}{\delta \pi^{\alpha\beta}} = 2h^{-1/2}N\left(\pi_{\alpha\beta} - \frac{1}{2}h_{\alpha\beta}\pi\right) + 2D_{(\alpha}N_{\beta)} \tag{2.17a}$$

$$\begin{aligned}\dot{\pi}^{\alpha\beta} = -\frac{\delta H_{\mathrm{G}}}{\delta h_{\alpha\beta}} = {} & -Nh^{1/2}\left({}^{(3)}R^{\alpha\beta} - \frac{1}{2}Rh^{\alpha\beta}\right) \\ & +\frac{1}{2}Nh^{-1/2}h^{\alpha\beta}\left(\pi_{\gamma\delta}\pi^{\gamma\delta} - \frac{1}{2}\pi^2\right) \\ & -2Nh^{-1/2}\left(\pi^{\alpha\gamma}\pi_{\gamma}{}^{\beta} - \frac{1}{2}\pi\pi^{\alpha\beta}\right) \\ & +h^{1/2}\left(D^{\alpha}D^{\beta}N - h^{\alpha\beta}D^{\gamma}D_{\gamma}N\right) \\ & +h^{1/2}D_{\gamma}\left(h^{-1/2}N^{\gamma}\pi^{\alpha\beta}\right) - 2\pi^{\gamma(\alpha}D_{\gamma}N^{\beta)},\end{aligned} \tag{2.17b}$$

where again boundary terms have been dropped and (2.16b) was used. Equations (2.16) and (2.17) are equivalent to the vacuum Einstein equation $R_{\mu\nu} = 0$. This provides a constrained Hamiltonian formulation of Einstein's equation.

However, there is still a gauge arbitrariness in our choice of configuration field $h_{\alpha\beta}$. This is due to diffeomorphisms. If ϕ is a diffeomorphism on the hypersurface, then $h_{\alpha\beta}$ and $\phi^* h_{\alpha\beta}$ represent the same physical configuration. What we should do is to take the configuration space of general relativity to be the set of equivalence classes $\tilde{h}_{\alpha\beta}$ of Riemannian metrics on the hypersurface, where two metrics can be considered equivalent up to a diffeomorphism transformation. So, for a vector field w^{α} on the hypersurface the conjugate momenta $\pi^{\alpha\beta}$ must satisfy

$$\int \pi^{\alpha\beta}\left(\delta h_{\alpha\beta} + D_{(\alpha}w_{\beta)}\right) = \int \pi^{\alpha\beta}\delta h_{\alpha\beta}\,.$$

This implies that $\pi^{\alpha\beta}$ automatically satisfies

$$D_{\alpha}(h^{-1/2}\pi^{\alpha\beta}) = 0$$

and therefore the constraint (2.16b) is eliminated by this choice of configuration space. The constraint (2.16a) though remains. It may be viewed as the result of the gauge arbitrariness involved in the choice of how to "slice" spacetime into space and time. It is analogous to the constraint which arises when one "parametrizes" an original unconstrained theory in a fixed background spacetime. In other words, when one introduces a time function in the Lagrangian, which defines the choice of hypersurfaces with respect to a reference surface and treats this time function as a dynamical variable. In such parametrized theories, the constraint (2.16a) is linear in the momentum conjugate to the time function [9]. If this is the case, one can "deparametrize" the theory by solving the constraint for this momentum. Unfortunately, in

Einstein's equation, the constraint is quadratic in momentum and a similar "deparametrization" appears impossible. It seems impossible to find a suitable choice of configuration space in general relativity, such that only the "true" dynamical degrees of freedom are present in its phase space. The constraint (2.16a) is an intrinsic and unavoidable feature of the Hamiltonian formulation of general relativity. This provides a serious obstacle when one tries to formulate a quantum field theory of gravitation following the canonical quantization approach (for details see [9]).

3 AdS/CFT Correspondence

The gauge-gravity refers to a fascinating equivalence between certain theories with gravity and certain theories without gravity. For a bit over a decade there has been intense research on a realization of such a duality, known as the AdS/CFT correspondence, based on a seminal paper [10] by Maldacena, which provides a duality between a theory with quantum gravity in $d+1$ dimensions and a field theory in d dimensions. In particular, the claim is that string theory in a Anti-de Sitter (AdS) background and a conformal filed theory (CFT) are equivalent. In this chapter we will review the idea discussing what motivated Maldacena to formulate the conjecture.

3.1 A first hint

The first hint of a relation between d dimensional conformal field theory in Minkowski space is the fact that it has the same symmetry group as gravity in $d+1$ dimensional gravity in AdS space. Let us see how this comes about.

Conformal Invariance

In flat d dimensions (i.e. on $\mathbb{R}^{1,\text{d-1}}$), conformal transformations are defined by

$$\begin{aligned} x_\mu &\rightarrow x'_\mu(x): \\ dx'_\mu dx'_\mu &= [\Omega(x)]^{-2} dx_\mu dx_\mu \end{aligned}$$

It is noteworthy that what we mean by conformal invariance is *not* the same as general coordinate invariance* since the metric is modified, from flat $ds^2 = dx'_\mu dx'_\mu$ to "conformally flat" $ds^2 = [\Omega(x)]^{-2} dx_\mu dx_\mu$, yet we are studying *flat space* field theories. In other words, conformal transformations are generalizations of the scale transformations that change the distance between the points with the all-important property that all angles are preserved. Hence the name conformal.

The conformal group be viewed as the set following transformations:

- Scale transformations: $x_\mu \rightarrow x_\mu + \lambda x_\mu$

*Although conformal transformations are a *subclass* of general coordinate transformations.

- Translations: $x_\mu \to x_\mu + a_\mu$
- Lorentz (rotations): $x_\mu \to x_\mu + \omega_\mu{}^\nu x_\nu$ with $\omega_{\mu\nu} = -\omega_{\nu\mu}$

and the less familiar

- special conformal transformation: $x_\mu \to x_\mu + b_\mu x^2 - 2x_\mu b \cdot x$.

So, the symmetry group has the following generators: P_μ for a_μ and $M_{\mu\nu}$ for $\omega_{\mu\nu}$ which together form the Poincaré group. The new generators are K_μ for the special conformal transformations parametrized by b_μ and the dilatation generator D parametrized by λ. We can assemble the generators in an antisymmetric $(d+2) \times (d+2)$ matrix

$$\bar{M}_{MN} = \begin{pmatrix} M_{\mu\nu} & \bar{M}_{\mu,d+1} & \bar{M}_{\mu,d+2} \\ -\bar{M}_{\nu,d+1} & 0 & D \\ -\bar{M}_{\nu,d+2} & -D & 0 \end{pmatrix}$$

with

$$\bar{M}_{\mu,d+1} = \tfrac{K_\mu - P_\mu}{2}, \quad \bar{M}_{\mu,d+2} = \tfrac{K_\mu + P_\mu}{2}, \quad \bar{M}_{d+1,d+2} = D \ .$$

The Lie algebra of $\bar{M}_{MN}$ shows that the metric in the $d+2$ direction is negative, thus the symmetry group is $\mathsf{SO}(2,d)$. So conformal invariance in $(1, d-1)$ dimensions $(d > 2)$ corresponds to the symmetry group $\mathsf{SO}(2,d)^\dagger$.

For completeness we write down the Lie algebra of the conformal group:

$$\begin{aligned}
[M_{\mu\nu}, P_\mu] &= i\,(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu) \\
[M_{\mu\nu}, M_{\rho\tau}] &= i\,(g_{\mu\tau}M_{\nu\rho} + g_{\nu\rho}M_{\mu\tau} - g_{\mu\rho}M_{\nu\tau} - g_{\nu\tau}M_{\mu\rho}) \\
[M_{\mu\nu}, K_\rho] &= i\,(g_{\nu\rho}K_\mu - g_{\mu\rho}K_\nu) \\
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[P_\mu, K_\nu] &= 2i\,(g_{\mu\nu}D + M_{\mu\nu})
\end{aligned}$$

with all others equal to zero.

Anti-de Sitter Space

AdS space is a space of Lorentzian signature but of constant *negative* curvature. Thus, it is the Lorentzian analogue of the so-called Lobachevski space‡. In d dimensions, de Sitter space is defined as a submanifold of Minkowski

†Strictly speaking, $\mathsf{SO}(1, d{+}1)$ is the connected component of the conformal group which includes the identity. The conformal group is an extension that also contains the *inversion* $I : x'_\mu = \frac{x_\mu}{x^2} \Rightarrow \Omega(x) = x^2$.

‡Euclidean space with a constant negative curvature.

space in one higher dimension. Let us take Minkowski space $\mathbb{R}^{1,d}$ with the standard metric:

$$ds^2 = -dx_0^2 + \sum_{i=1}^{d-1} dx_i^2 + dx_{d+1}^2 \,.$$

Then de Sitter space is the submanifold described by

$$R^2 = -x_0^2 + \sum_{i=1}^{d-1} x_i^2 + x_{d+1}^2 \tag{3.1}$$

with some positive constant R. This is the Lorentzian version of the sphere, and it is invariant under the group $\mathsf{SO}(1,d)$.

Similarly, in d dimensions, anti-de Sitter space is defined by a Lobachevski-like embedding in $d+1$ dimensions

$$ds^2 = -dx_0^2 + \sum_{i=1}^{d-1} dx_i^2 - dx_{d+1}^2 \,.$$

and

$$-R^2 = -x_0^2 + \sum_{i=1}^{d-1} x_i^2 - x_{d+1}^2 \,. \tag{3.2}$$

It is therefore the Lorentzian version of a Lobachevski space. So, AdS_5 can be thought of as product of four-dimensional Minkowski space times an extra radial coordinate. The metric on Minkowski space is however multiplied by an exponential function of the radial coordinate. AdS space is therefore an example of a warped space since in a suitable local coordinate system the metric is

$$ds^2 = r^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{dr^2}{r^2}. \tag{3.3}$$

One can see that it is invariant under the group $\mathsf{SO}(2,d-1)$ that rotates the coordinates $x_\mu = (x_0, x_{d+1}, x_1, \ldots, x_{d-1})$ by $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

So AdS space in $d+1$ dimensions has the same symmetry group as a conformally invariant field theory in d dimensional Minkowski space!

Another useful choice of coordinates for the AdS_{d+1} are the so-called Poincaré coordinates where metric is given by

$$ds^2 = \tfrac{R^2}{z^2} \left((\eta_{\mu\nu} dx^\mu dx^\nu)_{d+1} + dz^2 \right), \qquad z \geq 0 \tag{3.4}$$

The boundary at spatial infinity ($r \to \infty$) corresponds to $z = 0$ since $z \sim R^2/r$ and the horizon at $r = 0$ corresponds to $z = \infty$.

3.2 String Theory

In the previous sections we showed that conformally invariant field theories enjoy invariance under the same symmetries as AdS space in one extra dimension. From this point of view, we could suspect that the same symmetries suggest similar physics. We will now turn to string theory to argue that this is indeed true.

In string theory, the fundamental objects are no longer point-particles. Fundamental particles are understood as excitations of one-dimensional objects (either open or closed strings). It is a theory that consistently describes a quantum theory of gravity. Unfortunately, despite its mathematical rigour and robustness it seems impossible to check the theory experimentally with present day technology.

The theoretical assumption of one-dimensional fundamental objects, allows five different consistent descriptions in 10-dimensional spacetime, which lead to five flavors of string theory. These are all considered equivalent and are related through a web of dualities. In the reasoning that follows, we will start with one flavor of string theory called type IIB string theory.

Dp-branes

When open strings are considered it is necessary to impose boundary conditions. There are two kinds of boundary conditions. Namely:

- *Neumann boundary conditions.*

  In this case the component of the momentum normal to the boundary of the world sheet vanishes. i.e.

$$\left.\frac{\partial X_\mu}{\partial \sigma}\right|_{\sigma=0,\pi} = 0\,.$$

  If this holds $\forall\mu$, these boundary conditions respect d-dimensional invariance under Poincaré transformations. Physically, Neumann boundary conditions mean that no momentum is flowing through the ends of the string.

- *Dirichlet boundary conditions.*

  In this case the positions of the two string ends are fixed so that $\delta X^\mu = 0$ and

$$X^\mu|_{\sigma=0} = X_0^\mu \quad \text{and} \quad X^\mu|_{\sigma=\pi} = X_\pi^\mu$$

  with X_0^μ and X_π^μ constants and $\mu = 1, ..., d-p-1$. Neumann boundary conditions are imposed for the other $p+1$ coordinates. Dirichlet boundary conditions break Poincaré invariance. The constants X_0^μ and X_π^μ represent the positions of Dp-branes (D stands for Dirichlet). By Dp-brane we mean a hypersurface (a p-brane) on which a fundamental string can end. The presence of a Dp-brane breaks Poincaré invariance unless it is spacetime filling ($p = d - 1$).

In 1995 Polchinski [11] proved that what we would strictly call D-branes are exactly extremal p-branes (hence the *hybrid* name Dp-branes). By doing so, he showed that the dynamical endpoints of open string correspond to extremal solutions of supergravity, the low energy limit of string theory.

D-branes turn out to be dynamical objects that carry energy and therefore curve space. The so-called black D3-brane ($p = 3$) supergravity solution for N coincident D3-branes is:

$$ds^2 = H^{-1/2} d\vec{x}^2 + H^{1/2} \left(dr^2 + r^2 d\Omega_5^2\right) \tag{3.5}$$

$$H(r) = 1 + \frac{R^4}{r^4}, R = 4\pi g_s N \alpha'^2$$

Where R is the radius, α' is the Regge slope equal to the string length squared l_s^2. The first part of the metric describes the coordinates *on* the D3-branes ($d\vec{x}^2$ is the four-dimensional Minkowski metric on the brane). The second part contains the coordinates perpendicular to the branes. The metric S^5 is written $d\Omega_5^2$, as usual. The horizon is at $r = 0$ and the near horizon geometry is $\mathrm{AdS}_5 \times S^5$. This is the supergravity metric for the spacetime where open strings propagate.

In the low energy limit the excitations near $r = 0$ and at spatial infinity decouple from each other and we obtain one free supergravity theory and the near horizon region the geometry of $\mathrm{AdS}_5 \times S^5$.

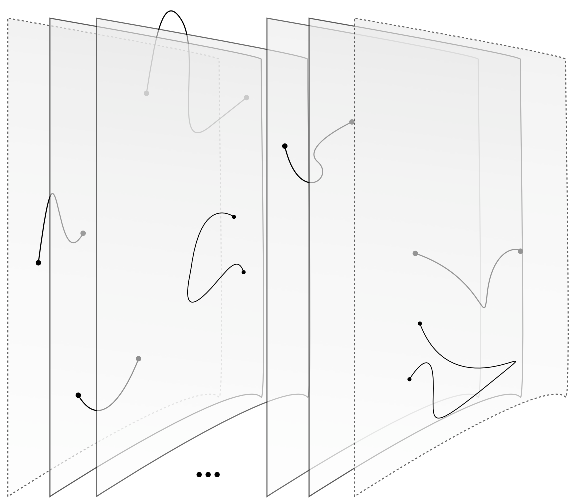


Fig. 3.1: Stack of D3-branes and open strings with endpoints attached to the hypersurfaces.

Field theory of open strings

Now let us consider the case were open strings can have endpoints on these D-branes. There will be $i = 1, \ldots, N$ possible states for each endpoint. Since an open string has two endpoints, it has two so-called Chan-Paton indexes ij. An open string state can be written as:

$$|p; a\rangle = \sum_{i,j=1}^{N} \lambda^a_{ij} \, |p; ij\rangle$$

The λ^a_{ij} matrices are called the Chan-Paton factors. Amplitudes obtained when including Chan-Paton factors are invariant under local $\mathsf{U}(N)$ transformations in spacetime. This is exactly what is required for Yang-Mills theories, so it provides a basis for including the standard model in string theory. Considering coincident D-branes gives rise to massless gauge fields in the following way:

- For N coincident Dp-branes, there are N^2 massless gauge fields[§].
- The open string massless states give an $\mathcal{N} = 4$ vector supermultiplet in $(3 + 1)$ dimensions and the low effective Lagrangian is that of a $\mathcal{N} = 4$, $d = 1 + 3$, $\mathsf{U}(N)$ Super Yang-Mills (SYM) theory on the world-volume of N coincident D-branes [12] .

For the solution (3.5) we implicitly started off with type IIB string theory and we considered a low energy limit where the theory on the D3-brane decouples from the bulk. By the low energy limit we mean the following: we keep the type IIB string coupling g_s and the energies fixed by taking the equivalent limit:

$$\alpha' \to 0, \qquad U \equiv \tfrac{r}{\alpha'} = \text{fixed} \ .$$

The $\mathcal{N} = 4$ SYM on the D3-branes however, is not enough to give a full description of the resulting physics. Two more things need to be considered.

1. There are closed strings living in the bulk. This gives supergravity coupled to the massive modes of the string. In the low energy limit only supergravity survives.

2. Two open strings living on a D-brane may collide and form a closed string. The closed string is no longer confined on the D-brane and may *move away* as Hawking radiation.

[§] A string *gains* mass from its tension. In the limit where the branes coincide the strings become massless.

This means that there should be a relation between the theory of open strings living on the D-branes ($\mathcal{N} = 4$ SYM) and the gravity theory of the bulk. Qualitatively, the action of these strings would look something like:

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{interactions}}$$

In the low energy limit $\alpha' \to 0$, the massive string modes drop out and we have $S_{\text{bulk}} \to S_{\text{SUGRA}}$ and $S_{\text{brane}} \to S_{\mathcal{N}=4\,\text{SYM}}$. Meanwhile $S_{\text{int}} \propto \sqrt{G_N} \sim g_s\alpha'^2 \to 0$. So we effectively get two decoupled systems: free gravity in the bulk and $\mathcal{N} = 4$, $d = 1 + 3$ SYM on the D3-branes.

3.3 The AdS/CFT Duality

Since we started from the same theory, namely type IIB string theory, and using two descriptions we obtained–in the low energy limit–two decoupled theories. In both cases one of the decoupled systems is supergravity in flat space. It is therefore natural to assume that we can identify the second system that appears in both descriptions.

The conjecture essentially states that since the starting point was a quantum theory that includes gravity the correspondence is valid beyond the supergravity approximation. In [10] the statement is*: "Type IIB string theory on* $\text{AdS}_5 \times S^5$ *plus appropriate boundary conditions is dual to* $\mathcal{N} = 4$*,* $d = 1+3$ $\mathsf{U}(N)$ *Super Yang-Mills".* More specifically, the $\mathsf{U}(1)$ subgroup of $\mathsf{U}(N)$ decouples as a free theory and does not participate in the duality. In particular, the $\mathsf{U}(1)$ vector supermultiplet includes six scalars which are related the center of mass of all the D-branes. From the AdS space point of view these zero modes live at the boundary, and it looks like we might or might not decide to include them in the AdS theory. Depending on this choice we could have a correspondence to an $\mathsf{SU}(N)$ or a $\mathsf{U}(N)$ theory. We therefore have the choice to interpreted $\mathsf{U}(1)$ as living on the boundary and the $\mathsf{SU}(N)$ living in the bulk, which is why the $\mathsf{U}(1)$ is not necessarily relevant. So the gauge group in the duality can effectively be $\mathsf{SU}(N)$ as a result of the large N limit[¶].

The next step in Maldacena's approach was to identify the parameters of the two theories and relate them. On the one side we have the Yang-Mills coupling constant g_{YM} and the number of colors N of the non-Abelian group $\mathsf{SU}(N)$. On the gravity side we have the type IIB solution in $\text{AdS}_5 \times S^5$ which has the string coupling constant g_s and the radius R. These parameters were related by Maldacena in the following way

$$g_s \equiv g_{YM}^2 \qquad \frac{R^4}{a'^2} \equiv 4\pi g_{YM}^2 N \equiv 4\pi\lambda$$

[¶]Essentially we can write $\mathsf{U}(N) \sim \mathsf{SU}(N) \times \mathsf{U}(1)$. This effectively means that $\mathcal{L} \to \mathcal{L}_{\mathsf{U}(1)} + \mathcal{L}_{\mathsf{SU}(N)}$. But $\mathsf{U}(1)$ is Abelian and therefore the gauge fields commute; consequently, the $\mathsf{U}(1)$ factor does not *feel* the strong gauge interactions .

where λ is the 't Hooft coupling.

We can distinguish two limits where the duality becomes useful:

1. **The large N limit**
   also known as the 't Hooft limit, for which we have $N \to \infty$ and $g_{YM}^2 N = \lambda \ll 1$ and fixed. In this limit the string coupling ($g_s = \lambda/N$) tends to zero. So it is possible to perform string calculations at tree level, the *classical* limit of string theory.

2. **The strong coupling limit**
   in this case we take $\lambda \to \infty$ which makes the string tension very large. This accounts for making all the massive modes extremely heavy. If we keep the AdS radii fixed, we are dealing with the case were $l_s^2 = \alpha' \to 0$. Strictly speaking one cannot send the dimensionful string length to zero. What one means by this is to consider energy scales for which string excitations may be neglected. If we consider the string length fixed and still take λ large, the radius of S^5 tends to infinity. Then all curvatures are "small" and quantum gravity corrections may be neglected: classical supergravity is adequate.

Another property related to the strong coupling limit is supersymmetry. Quantum field theories at strong coupling are susceptible to severe instabilities. An example of that is when particle and antiparticle pairs can appear spontaneously with their negative potential energy exceeding their positive rest and kinetic energies. Supersymmetric theories however have a stabilizing property because of the nature of the Hamiltonian[‖].

Because all this, it seems impossible to prove the conjecture. This would require non-perturbative solutions of either the $\mathcal{N} = 4$ SYM or the string theory in the $\mathrm{AdS}_5 \times S^5$. But we do not have a good definition of non-perturbative type IIB string theory. Even at tree level we do not know how to solve the theory completely. However, if the duality is true, this feature is exactly what makes the conjecture so powerful. Stated differently, it is always possible to choose the description where perturbative methods (and therefore calculations) are possible. Moreover, it is possible and advantageous to view the $\mathcal{N} = 4$ SYM as the definition of non-perturbative type IIB string theory on the $\mathrm{AdS}_5 \times S^5$ background.

More intuitively, it is useful to bear in mind that the metric (3.3) at the limit of the radial coordinate goes to infinity the exponential part blows up. That is what we call the boundary of AdS space and there is where the dual field theory lives. String theory excitations extend all the way to the boundary and in this way one obtains a map from string theory states to states in the field theory living on the boundary.

[‖]The Hamiltonian is the square of a Hermitian supercharge and therefore bounded from below.

Finally, it is worth mentioning, that the duality has not been proven wrong either. It has been argued that the conjecture is so audacious that it should be easy to disprove. Instead, qualitatively, we obtain exactly what we would physically expect, for a wide variety of situations**. So, despite a lot of attention, the conjecture has resisted invalidation. On the contrary, compelling evidence have accumulated.

3.4 The AdS/CFT Dictionary

In the original paper [10] Maldacena did not provide the exact correspondence between elements of the bulk and the boundary theory. This was done by Gubser, Klebanov and Polyakov [14] and independently by Witten [15]. We start by making some observations. A conformal field theory does not have particle (massive) states or an S-matrix. If a mass was introduced, that would automatically infuse a scale and break conformal invariance. Moreover, the lack of massive states, renders the S-matrix singular. The only physical observables, that is, well-defined and meaningful quantities, in a CFT are correlation functions of gauge-invariant operators. What is necessary is an explicit prescription for relating such correlation functions to computable quantities in the bulk theory. The operators are defined at a point. This corresponds to perturbing the gauge theory in the ultraviolet. As explained in Chapter 4 this amounts to considering the gauge theory at the boundary of the AdS space.

More concretely, if we consider the corresponding operator $\mathcal{O}$ we can add the term $\int d^4x\, \phi_0\left(\vec{x}\right)\mathcal{O}\left(\vec{x}\right)$ to the Lagrangian. It is natural to assume that this will change the boundary condition of the dilaton at the boundary of AdS to $\phi\left(\vec{x}, z\right)|_{z=0} = \phi_0\left(\vec{x}\right)$. As argued in [14, 15] we can write for the string theory full partition function:

$$\left\langle e^{\int d^4x\, \phi_0(\vec{x})\mathcal{O}(\vec{x})}\right\rangle_{\mathrm{CFT}} = \mathcal{Z}_{\mathrm{string}}\left[\, \phi\left(\vec{x}, z\right)|_{z=0} = \phi_0\left(\vec{x}\right)\right]. \tag{3.6}$$

The left hand side is the generating function of correlation functions in the field theory side. So, we can calculate correlation functions of $\mathcal{O}$ by taking functional derivatives with respect to ϕ_0 and then setting $\phi_0 = 0$. Each differentiation brings down an insertion $\mathcal{O}$, which "sends" a closed string state into the bulk. Feynman diagrams can then be used to compute the interactions of particles in the bulk. In the classical supergravity limit, the only diagrams that contribute are the tree-level diagrams of the gravity theory (Fig. 3.2).

This is a rather general method for defining the correlation functions of a field theory dual to a gravity theory in the bulk. In principle it applies to any theory of gravity [15]. Each field propagating in AdS is in a one to one correspondence with a single-trace operator in the field theory. Let us consider

**For example, the quark gluon plasma in RHIC physics is treated as an application of the AdS/CFT correspondence without knowing the gravity dual of QCD[13].

a scalar field ϕ of mass m in Euclidean AdS_5. Close to the boundary the wave equation has two independent solutions which behave like $z^{d-\Delta}$ and z^{Δ} with

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + R^2 m^2}$$

determined by the equations of motion. In order to get consistent behavior for the massive field the boundary condition for the partition function of (3.6) should be changed to

$$\phi\left(\vec{x}, \epsilon\right) = \lim_{\epsilon \to 0} \epsilon^{d-\Delta} \phi_0\left(\vec{x}\right) \, .$$

This leads to assigning a scaling dimension to ϕ_0 of the form $[\text{length}]^{\Delta - d}$ so that ϕ is indeed dimensionless. This implies that the associated operator $\mathcal{O}$ has dimension Δ.

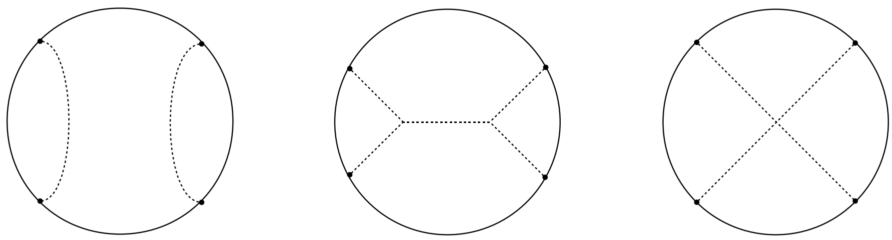


Fig. 3.2: Correlation functions can be calculated in terms of classical supergravity Feynman diagrams. Here we see the leading contribution coming from a disconnected diagram plus connected pieces involving interactions of the supergravity fields in the bulk of AdS . At tree level, these diagrams and those related to them by crossing are the only ones that contribute to the four-point function [16].

It is noteworthy that since the CFT always has a stress-energy tensor $T_{\mu\nu}$ as gauge invariant operator. This corresponds to the metric $g_{\mu\nu}$ in the bulk. Therefore the AdS/CFT correspondence always involves a gravitational theory for the bulk. Another bulk field is the dilaton which corresponds to the Lagrangian of the CFT. This is because a small change in the gauge coupling which is dual to the string coupling determined by the dilaton, adds an operator proportional to the Lagrangian. Meanwhile, massless gauge fields in the bulk correspond to global currents in the field theory.

A more general dictionary of various elements and their corresponding analogs is given in Table 3.1. This is by no means complete but not because of negligence. In fact, the known dictionary relating spacetime concepts in the bulk and field theory concepts on the boundary is still being developed. So, this is merely an example of some “entities” that have a proper corresponding description on both sides of the duality.

Corresponding elements

Gravity (bulk)	Gauge theory (boundary)
field ϕ	operator $\mathcal{O}$
graviton $g_{\mu\nu}$	Energy-momentum tensor $T_{\mu\nu}$
mass of the field	dimension of operator
gauge symmetry	global symmetry
gauge field	conserved current
Chern-Simons term	anomaly
Isometry	conformal symmetry

Table 3.1: Corresponding elements in AdS/CFT

4 Holographic Renormalization

The holographic RG is based on the idea that the radial coordinate of a spacetime with asymptotically AdS geometry can be identified with the RG flow parameter (i.e. energy scale) of the boundary field theory.

Let us consider a scale transformation of the gauge theory $x^\mu \to ax^\mu$. Scale invariance implies that, if this is accompanied by a rescaling of the energy scale $E \to E/a$, this is a symmetry. Since x^μ is identified with x^μ in the bulk, this scaling can be performed in the AdS metric (3.4). But when $x^\mu \to ax^\mu$, then $z \to z/a$ for the scaling to be a symmetry of the metric. This leads to the identification:

$$E \sim 1/z \sim r.$$

This shows that the radial coordinate in the bulk corresponds to the energy scale of the dual gauge theory (this was realized in [10, 14, 15, 17]). By dimensional analysis we get $E = krl_s^{-2}$, where k is a dimensionless constant. One way to define it is by identifying the energy of a string stretched from the horizon at $r = 0$ to a point with radial coordinate r with scale E. But regardless of how one determines the constant, the important fact is that $E_1/E_2 = z_2/z_1 = r_1/r_2$.

A reasonable question, given the identification, is where is the dual gauge theory located. A theory on

without any degrees of freedom integrated out corresponds to $E \to \infty$. In this case the dual gauge theory is located at the boundary ($r \to \infty$). When high-momentum degrees of freedom are integrated out, it is translated inwards toward the horizon.

This notion makes the correspondence more powerful. It can be generalized to bulk theories that are only AdS asymptotically as $r \to \infty$. On the other hand, the dual gauge theory need only be conformal in the sense that it should approach a conformal fixed point in ultraviolet.

Given this collection of ideas it should be apparent that talking about renormalization in the context of the AdS/CFT correspondence is in fact relevant. We are no longer looking at gauge theories were the β functions vanish at any scale. As long as $\beta \to 0$ in the ultraviolet regime, then a dual description can become relevant.

The first systematic development of holographic renormalization for bulk gravity coupled to scalar fields was given in [18]. This method involves the cancellation of all cut-off related divergences from the bulk on-shell action by the addition of counterterms on a cut-off boundary hypersurface and followed by the removal of the cut-off. Later this method was summarized by Bianchi, Freedmann and Skenderis [19].

Another approach was developed by de Boer, Verlinde and Verlinde (dBVV) [20] (for reviews see [21, 22]). In their work the Hamilton-Jacobi theory is used to separate terms in the bulk on-shell action, which can be written as local functionals of the boundary data. The remaining, in principle non-local, part was identified with the generating functional of a boundary field theory. This approach is substantially different, since the boundary field theory *lives* ***on*** the cut-off boundary and the generating functional contains logarithmic divergences. Notwithstanding, the dBVV method provides an amazingly simple bulk description of the renormalization group flow in deformed CFTs and also yields the correct gravitational anomalies. Additionally, the Hamilton-Jacobi equation directly characterizes the classical action of bulk gravity without solving the equations of motion.

Moreover, two apparent disadvantages in the original formulation of the dBVV method were dealt with in [23]. More specifically, they attempted to explain exactly how the ambiguities from the solutions of the local terms can be removed and how logarithmic counterterms can be obtained. Martelli and Mück also include $\mathsf{U}(1)$ gauge fields in their treatment which provides a useful extension to the original formulation for scalar fields coupled to gravity.

4.1 Hamilton-Jacobi applied to gravity

We now discuss the Hamilton-Jacobi formalism as applied to gravity in its canonical form. We essentially reformulate what was discussed in section 2.2 in a way relevant and useful to what follows. In the ADM formalism (section 2.2) the bulk metric can always be written as

$$ds^2 = N^2 dr^2 + g_{\mu\nu}(x,r)(dx^\mu + N^\mu dr)(dx^\nu + N^\nu dr)\,.$$

In this context the role of r will be similar to that of a time coordinate. Therefore, the Hamilton-Jacobi formalism will describe flows in the radial direction instead of the typical flow in time. Locally, diffeomorphism invariance allows us to choose $N = 1$ and $N^\mu = 0$, which is a common gauge to work in. The Hamiltonian treatment involves the time-slicing formalism, which assumes that the bulk spacetime manifold can be globally foliated into hypersurfaces specified by a time coordinate (in our case r).

The Hamilton constraint emerges by imposing the equations of motion for the lapse function N. Variation of the action with respect to N^μ will give the diffeomorphism constraint along a fixed time slice. The Hamilton constraint will provide the Hamilton-Jacobi equation.

As toy model we will consider gravity minimally coupled to scalar fields ϕ^I, a potential $V(\phi)$ and a kinetic term $\frac{1}{2}g^{\mu\nu}\partial_\mu\phi^I G_{IJ}(\phi)\partial_\nu\phi^J$. The Lagrangian density will be of the form $\mathcal{L} = V(\phi) + R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi^I G_{IJ}(\phi)\partial_\nu\phi^J$ and the variables of the theory will be $g_{\mu\nu}$ and the scalars ϕ^I at the cut-off (for the classical trajectory). The corresponding canonical momenta will be $\pi_{\mu\nu}$ and π_I given by

$$\pi^{\mu\nu} = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}, \quad \pi_I = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta \phi^I}\,. \tag{4.1}$$

The constraints in phase space are

$$\frac{1}{\sqrt{g}}\frac{\delta S}{\delta N} \equiv \mathcal{H} = \pi^{\mu\nu}\pi_{\mu\nu} - \frac{1}{d-1}\pi^\mu_\mu\pi^\nu_\nu + \frac{1}{2}\pi_I G^{IJ}\pi_J + \mathcal{L} = 0 \tag{4.2}$$

$$\frac{1}{\sqrt{g}}\frac{\delta S}{\delta N^\nu} \equiv \mathcal{H}_\nu = \nabla^\mu\pi_{\mu\nu} + \pi_I\nabla_\nu\phi^I = 0\,. \tag{4.3}$$

Given the expressions (4.1) we can recast (4.2) into the form:

$$\sqrt{g}\mathcal{L}(\phi,g) = \frac{1}{\sqrt{g}}\left(\frac{1}{d-1}\left(g^{\mu\nu}\frac{\delta S}{\delta g_{\mu\nu}}\right)^2 - g^{\mu\lambda}g^{\nu\rho}\frac{\delta S}{\delta g^{\mu\nu}}\frac{\delta S}{\delta g^{\lambda\rho}} - G^{IJ}\frac{\delta S}{\delta\phi^I}\frac{\delta S}{\delta\phi^J}\right)$$

which gives (4.2) the desired form in terms of $S[\phi, g]$.

From a solution of the Hamilton-Jacobi equation, we can now compute the radial derivatives of ϕ^I and $g_{\mu\nu}$ from the Hamilton equation of motion $\dot{q} = \partial H/\partial p$. Using (4.2) we obtain the coveted flow equations:

$$\begin{aligned}\frac{\partial\phi^I}{\partial r} &= G^{IJ}\pi_J \\ \frac{\partial g_{\mu\nu}}{\partial r} &= 2\pi_{\mu\nu} - \frac{2}{d-1}g_{\mu\nu}\pi^\lambda_\lambda\,.\end{aligned} \tag{4.4}$$

4.2 The Holographic Renormalization Group

The divergences that arise in the supergravity action in the bulk of AdS need to be dealt with. The divergent terms are local in nature which is exactly the case in renormalizable field theories. According to [20] the action is non-local at the scale of the cut-off , but at a much lower energy scale a part of it can be represented as a local action. Following this reasoning the action is decomposed in a local and a non-local part

$$S\,[\phi, g] = S_{\text{loc}}\,[\phi, g] + \Gamma\,[\phi, g]\,. \tag{4.5}$$

Here, Γ represents the effective action of the gauge theory and contains all higher derivative and non-local terms and S_{loc} includes all these local, divergent terms.

To make the discussion more concrete let us take

$$S_{\text{loc}}[\phi, g] = \int \sqrt{g}\left(U(\phi) + \Phi(\phi)R + \frac{1}{2}\partial_\mu \phi^I M_{IJ}(\phi)\partial^\mu \phi^J\right)$$

where U, Φ and M_{IJ} are local functions of the couplings. One needs to be careful when defining S, since, in general it will be non-local. To this end, a scaling procedure is required. What should not be done, is to simply see how things rescale under rescalings of the metric. The reason is that the non-locality of S will not make this work properly. The right thing to do is to assign a degree $+2$ to $g_{\mu\nu}$ and a degree $\Delta_I - 4$ to ϕ^I. In order to motivate this we should have the following in mind. In the Poincaré patch, the metric of $\text{Ad}S_5 \times S^5$ is described by

$$ds^2 = dr^2 + e^{2r/L}\eta_{\mu\nu}dx^\mu dx^\nu$$

which is invariant under

$$r \to r + a\,, \qquad \eta_{\mu\nu} \to \eta_{\mu\nu}e^{-2a/L}\,,$$

where L is the AdS_5 radius $L = \left(g_{YM}^2 N\right)^{1/4}$. A generic solution for the ϕ field is

$$\phi(r) \sim \alpha e^{(\Delta-4)r/L} + \beta e^{-\Delta r/L} + \ldots$$

and Δ satisfies $\Delta(\Delta - 4) = m^2 L^2$. For large value of r_0, where the cut-off is, we can perform the following substitutions

$$\hat{\phi} \sim e^{r(\Delta-4)}\hat{\phi}\,, \quad \hat{g} \sim e^{-2a}\hat{g}\,, \quad r_0 \to r_0 + a$$

which will leave the supergravity solution unchanged. Because of this transformation prescription we decide to assign the aforementioned degrees to $g_{\mu\nu}$ and ϕ^I.

Now, S_{loc} is defined to be the local term that contains at least the complete part of S that has a degree larger than zero. However, there is an ambiguity in (4.5) since terms with zero and negative degree, in other words, finite local terms may be shifted between S_{loc} and Γ. This is a manifestation of the usual ambiguity of choosing a renormalization scheme, which was discussed in section 1.5.

Fortunately though, the Hamilton-Jacobi theory takes care of that. As we will see, it provides a way to choose an appropriate set of local terms in S_{loc} and terms with positive degree are fixed unambiguously in this way. As expected, this procedure makes sense only for marginal and relevant perturbations. Irrelevant perturbations may appear in terms of arbitrary positive degree.

Now, the most critical step of this method is to split the on-shell action based on power divergences:

$$S = \left(S^{(0)} + S^{(2)} + \ldots + S^{(2n)}\right)_{\text{loc}} + \Gamma\,, \tag{4.6}$$

where the power divergent terms are denoted by $S^{(2k)}$, $k = 0, \ldots, n$. Given (4.1) the momentum π naturally splits into

$$\pi = \pi^{(0)} + \pi^{(2)} + \ldots + \pi^{(2n)} + \pi^{\Gamma}$$

and similarly for $\pi^{\mu\nu}$.

The key point of the dBVV method is to insert the expansion (4.6) into the Hamilton constraint (4.2) and combine the contributions of the left hand side that have the same scaling degree as the terms on the right hand side and require them to cancel. This amounts to splitting the Hamilton constraint into a derivative expansion

$$\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(2)} + \ldots + \mathcal{H}^{(2n)} + \mathcal{H}^{\Gamma}\,,$$

where $\mathcal{H}^{(2k)}$ denotes those terms in $\mathcal{H}$ that stem only from the counterterms and contain a total of k inverse metrics. This is essentially a different way of writing the Hamiltonian constraint in a form which makes the contributions from the various counterterms explicit.

For this purpose we can define

$$S^{(0)}_{\text{loc}}[\phi, g] = \int \sqrt{g}\,U(\phi) \tag{4.7}$$

$$S^{(2)}_{\text{loc}}[\phi, g] = \int \sqrt{g}\left(\Phi(\phi)R + \frac{1}{2}\partial_\mu\phi^I M_{IJ}(\phi)\partial^\mu\phi^J\right) \tag{4.8}$$

which is reasonable since

$$\mathcal{L}^{(0)}(\phi, g) = \sqrt{g}\,V(\phi)$$

$$\mathcal{L}^{(2)}(\phi, g) = \sqrt{g}\left(R + \frac{1}{2}\partial_\mu\phi^I G_{IJ}(\phi)\partial^\mu\phi^J\right)$$

according to the power-counting prescription that we have.

The Hamilton-Jacobi equations then can be cast in the form:

$$\{S, S\} + \mathcal{L}^{(0)} + \mathcal{L}^{(2)} = 0\,,$$

where the brackets here denote:

$$\{S, S\} = \frac{1}{\sqrt{g}}\left[\frac{1}{3}\left(g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}}\right)^2 - g^{\mu\lambda}g^{\nu\rho}\frac{\delta S}{\delta g^{\mu\nu}}\frac{\delta S}{\delta g^{\lambda\rho}} - \frac{1}{2}G^{IJ}\frac{\delta S}{\delta\phi^I}\frac{\delta S}{\delta\phi^J}\right].$$

By the scaling procedure explained, we have

$$\left\{S^{(0)}_{\text{loc}}, S^{(0)}_{\text{loc}}\right\} = \mathcal{L}^{(0)} \tag{4.9a}$$

$$2\left\{S^{(0)}_{\text{loc}}, S^{(2)}_{\text{loc}}\right\} = \mathcal{L}^{(2)}$$

$$2\left\{S^{(4)}_{\text{loc}}, \Gamma\right\} + \left\{S^{(2)}_{\text{loc}}, S^{(2)}_{\text{loc}}\right\} = 0\,. \tag{4.9b}$$

The ambiguity of (4.5) is also reflected here in the sense that $S^{(0)}$ can also contain terms of scaling weight$\leq$ 0. Focusing on (4.9a) and collecting the various terms we obtain

$$V = \frac{1}{3}U^2 - \frac{1}{2}G^{IJ}\partial_I U \partial_J U\,. \tag{4.10}$$

This is the usual relation between scalar potential and superpotential of supergravity (for example [24]) which was recovered here by bosonic analysis. We also obtain

$$\dot{\phi}^I = G^{IJ}\partial_J U, \qquad \dot{g}_{\mu\nu} = -\frac{1}{3}U g_{\mu\nu} \tag{4.11}$$

from (4.4) where Poincaré invariant solutions of the supergravity system alone are considered. In that case, $S \equiv S^{(0)}_{\text{loc}}$ and $\left\{S^{(0)}_{\text{loc}}, S^{(0)}_{\text{loc}}\right\} = \mathcal{L}^{(0)}$ is the only nontrivial equation. Thus, we can recover all the information about the simple flow directly from the Hamilton-Jacobi equation.

The following ansatz solves (4.11):

$$g_{\mu\nu} = a^2 \hat{g}_{\mu\nu}\,,$$

with $\hat{g}_{\mu\nu}$ independent of r and a satisfies

$$\dot{a} = -\frac{1}{6}U(\phi)a\,. \tag{4.12}$$

The parameter a determines the physical scale. We replace the r derivatives in the flow equations by derivatives with respect to a using (4.12). We obtain

$$a\frac{d}{d\phi}\phi^I = \beta^I(\phi)$$

with the following form for the β-functions

$$\beta^I(\phi) = -\frac{6}{U(\phi)}G^{IJ}(\phi)\partial_J U(\phi)$$

which are ϕ^I-dependent. Near the AdS boundary we have

$$\beta^I(\phi) \sim (\Delta_I - 4)\,\phi^I + \ldots \qquad .$$

These β-functions do indeed play the role of β-functions in the renormalization group equations.

IT should be noted that he solution of U to (4.10) is not unique. We can identify U with the superpotential for supersymmetric flows, but this is not a relation that generally holds. We can approach this issue perturbatively around a critical point of V which will also be a critical point of U. We use a basis where G_{IJ} is δ_{IJ} and (4.10) can be rewritten as

$$\frac{1}{3}U^2 - \frac{1}{2}\left(\partial_I U\right)^2 = V\,. \tag{4.13}$$

By implicitly assuming that there exists a classical solution for V that can be extended from the boundary to the interior, we get some restrictions on the potential V. Apparently, it forces it to be of a supersymmetric form. We can expand the 5-dimensional potential in powers of ϕ^I

$$V = 12 - \frac{1}{2}m_I^2\phi^I\phi^I + g_{IJK}\phi^I\phi^J\phi^K + \ldots$$

where a possible linear terms in ϕ^I can be removed by the freedom to shift the fields. We can attempt a similar expansion for U to obtain a solution to (4.13) for the 4-dimensional potential,

$$U = 6 + \frac{1}{2}\lambda_I\phi^I\phi^I + \lambda_{IJK}\phi^I\phi^J\phi^K\,,$$

where the constant term has been chosen in order to match with that of V. The β-functions are

$$\beta^I = -\left(4 - \Delta_I\right)\phi^I - c^I_{JK}\phi^J\phi^K$$

with

$$\Delta_I = 4 - \lambda_I\,, \qquad c_{IJK} = 3\lambda_{IJK} \tag{4.14}$$

the scaling dimensions and operator product coefficients of the operators O_I corresponding to the couplings ϕ^I. Inserting both expansions back to (4.13) we obtain

$$\lambda_I^2 - 4\lambda_I = m_I^2\,.$$

If we insert it back to (4.14) we obtain the standard relation ($\Delta(\Delta - 4) = m^2L^2$) between the scaling dimensions Δ_I of the 4-dimensional couplings and the corresponding masses m_I of the 5-dimensional fields. This relation implies that the 5-dimensional potential must satisfy a unitarity bound $m_I^2 \geq -4$, otherwise there will be no bounded solutions that extend to the asymptotic boundary. This is of course the Breitenlohner-Freedman bound for stability in AdS space. The existence of a perturbative solution implies that the masses of the fields need to be consistent with an AdS solution.

Going to the next order we obtain

$$(\lambda_I + \lambda_J + \lambda_K - 4\lambda)\,\lambda_{IJK} = -g_{IJK}$$

which expresses the operator product coefficients c_{IJK} in terms of the cubic term in V.

An infinitesimal variation δU respects (4.13) provided that

$$4U\delta U - 6\partial_I U \partial_I (\delta U) = 0 \quad \text{or} \quad \left(4 + \beta^I \partial_I\right) \delta U = 0\,.$$

This means that the terms that have a total dimension 4 are not determined by the Hamilton-Jacobi constraint. Interestingly, these are exactly those terms that remain finite in the continuum limit. In other words, the Hamilton-Jacobi relation apparently constraints only the divergent terms of the potential U but not the finite part. These are the terms that survive as finite local terms in the boundary effective action.

Moreover, there are discrete ambiguities in U since for example λ_I can be either Δ_I or $4 - \Delta_I$. If the space is asymptotic AdS, the boundary conditions in the infra-red limit will determine that the right solution is $4-\Delta_I$. In general however, more general U can appear.

If we move to the next level of the action expansion and use $2\left\{S^{(0)}_{\text{loc}}, S^{(2)}_{\text{loc}}\right\} = \mathcal{L}^{(2)}$ we obtain the following equations

$$\begin{aligned}
\beta^K \partial_K \Phi &= -2\Phi + \frac{6}{U} \\
-2M_{IJ} + \frac{6}{U} G_{IJ} &= -12\partial_I\partial_J\Phi + \beta^K\partial_K M_{IJ} - \beta^K \partial_I M_{KJ} - \beta^K \partial_J M_{IK} \\
\beta^I &= -6M^{IJ}\partial_J \Phi\,.
\end{aligned}$$

In [23] a perturbative approach is also employed, but for only one field ϕ but for d dimensions. The discussion then becomes a bit more involved in settling the discrete ambiguities. In their notation and conventions:

$$V = -\frac{d(d-1)}{4L^2} + \frac{1}{2}m^2\phi^2 + \frac{1}{3!}\upsilon_3\phi^3 + \frac{1}{4!}\upsilon_4\phi^4 + \mathcal{O}(\phi^5)\,,$$

where the constant part represents a negative cosmological constant. A similar expansion for U is used in order for each coefficient to be determined recursively:

$$U = u_0 + u_1\phi + \frac{1}{2}u_2\phi^2 + \frac{1}{3!}\phi^3 + \frac{1}{4!}\phi^4 + \mathcal{O}(\phi^5)\,.$$

They find a breakdown for the procedure for $\lambda = d/6$, since

$$u_3 = \frac{2L}{6\lambda - d}\upsilon_3\,,$$

which leaves the coefficient u_3 undetermined. A more general breakdown will occur for

$$\lambda = \frac{(k-2)\,d}{2k}$$

for $k > 2$. This will leave the u_k coefficient undetermined and the boundary integral of ϕ^k is finite in the limit where the cut-off goes to infinity. A similar relation also occurs at higher levels.

The main idea so far is to find the power divergent terms by writing down the most general set of covariant local counterterms up to the necessary level. Then we solve for them by asking that they satisfy the Hamiltonian constraint for arbitrary and independent values of boundary conditions of the fields. This does not prove that the method will yield all power divergences. However, there is no counter-example that suggests that the method will not.

Callan-Symanzik equation

The functional Γ contains all the information about the correlation functions of the theory via the identity

$$\langle \mathcal{O}_{I_1}(x_1)\ldots\mathcal{O}_{I_n}(x_n)\rangle = \frac{1}{\sqrt{g}}\frac{\delta}{\delta\phi^{I_1}(x_1)}\cdots\frac{1}{\sqrt{g}}\frac{\delta}{\delta\phi^{I_n}(x_n)}\Gamma\left[\phi,g\right]\,, \tag{4.15}$$

for the case of many scalars. From this, the Callan-Symanzik equation can be derived using (4.9b) which implies

$$\frac{1}{\sqrt{g}}\left(2g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}} - \beta^I(\phi)\frac{\delta}{\delta\phi^I}\right)\Gamma\left[\phi,g\right] = 4-\text{derivative terms}. \tag{4.16}$$

What we need to do, in order to derive the Callan-Symanzik equations for expectation values of local operators, we vary the last relation with respect to the fields ϕ^I as we do in (4.15). After the variations, the fields are put to their constant average value given by the couplings of the gauge theory.

If we take one extra step and assume the metric to be of the form $g_{\mu\nu} = a^2\eta_{\mu\nu}$, with a being x^μ-dependent, the 4-derivatives will drop out. Therefore, they will play no role, as long as operators at different points in space are considered. Then the resulting expression is integrated over all space and the functional derivatives are to be replaced by ordinary derivatives using

$$-2\int g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}} = a\frac{\partial}{\partial a}\,, \qquad \int\frac{\delta}{\delta\phi^I} = \frac{\partial}{\partial\phi^I}\,.$$

This way, we can obtain the standard form of the Callan-Symanzik equations from (4.16):

$$\begin{aligned}\left(a\frac{\partial}{\partial a} + \beta^I\partial_I\right)&\langle \mathcal{O}_{I_1}(x_1)\ldots\mathcal{O}_{I_n}(x_n)\rangle \\ &+ \sum_{i=1}^{n}\gamma_{I_i}{}^{J_i}\langle \mathcal{O}_{I_1}(x_1)\ldots\mathcal{O}_{J_i}(x_i)\ldots\mathcal{O}_{I_n}(x_n)\rangle = 0\,,\end{aligned} \tag{4.17}$$

where ${\gamma_I}^J = \nabla_I \beta^J$ represent the anomalous scaling dimensions of the operators $\mathcal{O}_I$.

In this procedure the Callan-Symanzik (4.17) still "carries" a finite cut-off. We need to take the limit $r \to \infty$. Thus, we write the metric and the couplings as

$$g_{\mu\nu} = \epsilon^{-2} g^R_{\mu\nu}$$

and

$$\phi^I = \phi^I(\phi_R, \epsilon)$$

where R denotes the renormalized metric and coupling which are kept fixed in the limit $\epsilon \to 0$. By integrating the RG-flow we obtain the relation between the bare and the renormalized couplings

$$\epsilon \frac{\partial \phi^I}{\partial \epsilon} = -\beta^I(\phi), \quad \phi^I = \phi^I_R, \quad \text{at } \epsilon = 1\,.$$

The renormalized effective action Γ_R is defined by

$$\Gamma_R\left[\phi_R, g_R\right] = \lim_{\epsilon \to 0} \Gamma_{\text{finite}} \left[\phi(\phi_R, \epsilon), \epsilon^{-2} g_R\right]$$

where Γ_{finite} is obtained from Γ by subtracting the divergent part. Now the action should again satisfy the Callan-Symanzik equation, but expressed in terms of the renormalized couplings and metric

$$\frac{1}{\sqrt{g}} \left(2 g^{\mu\nu}_R \frac{\delta}{\delta g^{\mu\nu}_R} - \beta^I_R(\phi_R) \frac{\delta}{\delta \phi^I_R} \right) \Gamma_R\left[\phi_R, g_R\right] = \text{local terms.}$$

Beta-functions can be viewed as vector fields in the space of couplings as was discussed in section 1.3, which means that

$$\beta^I \frac{\delta}{\delta \phi^I} = \beta^I_R \frac{\delta}{\delta \phi^I_R}\,.$$

If we also use the following definition of the renormalized operators

$$\mathcal{O}^R_I = \mathcal{O}_J \frac{\partial \phi^J}{\partial \phi^I_R}$$

it is possible to obtain the Callan-Symanzik equations for all renormalized n-point functions

$$\left(a \frac{\partial}{\partial a} + \beta^I_R \partial_I \right) \left\langle \mathcal{O}^R_{I_1}(x_1) \ldots \mathcal{O}^R_{I_n}(x_n) \right\rangle + \sum_{i=1}^{n} \gamma^{(R)J_i}_{I_i} \left\langle \mathcal{O}^R_{I_1}(x_1) \ldots \mathcal{O}^R_{J_i}(x_i) \ldots \mathcal{O}^R_{I_n}(x_n) \right\rangle = 0\,.$$

Gauge Fields and Hamilton-Jacobi

In the application of the Hamilton-Jacobi theory to the case of non-relativistic gauge/gravity duality, it is necessary to be able to treat gauge (vector) fields as well as scalar fields as we shall see in the following chapter. To this end, we present here the variation that includes vector fields in the original action. We start by considering the following action:

$$S = \int d^{d+1}x \sqrt{\tilde{g}} \left(-\frac{1}{4}\tilde{R} + \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi) + \frac{1}{4}K(\phi)F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mathcal{M}^2(\phi)A_\mu A^\mu \right) ,$$

where we used tildes and Greek letters do denote $d+1$ dimensions. The scalar potential, $V(\phi)$, has a stable fixed point at $\phi = 0$. The vector part of the action also includes a massive gauge field. We employ the ADM formalism once again, and we write the metric as

$$g_{\mu\nu} = \begin{pmatrix} n_i n^i + n^2 & n_j \\ n_i & g_{ij} \end{pmatrix} ,$$

where we treat r as the zeroth component. The action now can be written as

$$S = \int d^{d+1}x \sqrt{\tilde{g}}\, n \left(-\frac{1}{4}R + \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi) + \frac{1}{4}K(\phi)F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mathcal{M}^2(\phi)A_\mu A^\mu \right) .$$

We can gauge fix the following quantities

$$n = 1, \quad n^i = 0, \quad A_r = 0 .$$

This way the corresponding equations of motion, enter as constraints into the Hamilton formalism

$$\begin{aligned} -\frac{\delta S}{\delta n} &= \mathcal{H} = 0 \\ \frac{\delta S}{\delta n^i} &= \mathcal{H}_i = 0 \\ \frac{\delta S}{\delta A_r} &= \mathcal{G} \end{aligned}$$

which translates into

$$\begin{aligned} \mathcal{H} &= \frac{1}{4}R + \frac{1}{2}(\partial_r\phi)^2 + +\frac{1}{2}Kg^{ij}F_{ri}F_{rj} - \frac{1}{2}g^{ij}\partial_i\phi\partial_j\phi - \frac{1}{2}\mathcal{M}^2 A_i A^i \\ &\quad -\frac{1}{4}KF_{ij}F^{ij} - V(\phi) \\ \mathcal{H}_i &= -\partial_r\phi\partial_i\phi - Kg^{jk}F_{ij}F_{rk} \\ \mathcal{G} &= \nabla^i(KF_{ri}) . \end{aligned}$$

The gauge fixed action then reads

$$S = \int d^{d+1}x\,\sqrt{g}\left(-\frac{1}{4}R + \frac{1}{2}(\partial_r\phi)^2 + \frac{1}{2}g^{ij}\partial_i\phi\partial_j\phi + V(\phi) + \frac{1}{2}Kg^{ij}F_{ri}F_{rj}\right.$$
$$\left. + \frac{1}{2}\mathcal{M}^2 A_i A^i\right) .$$

The congugate momenta are

$$\begin{aligned}
\pi^{ij} &= \frac{1}{\sqrt{g}}\frac{\delta S}{\delta(\partial_r g_{ij})} \\
\pi &= \frac{1}{\sqrt{g}}\frac{\delta S}{\delta(\partial_r \phi)} \\
E^i &= \frac{1}{\sqrt{g}}\frac{\delta S}{\delta(\partial_r A_i)}
\end{aligned}$$

through which we can rewrite the constraints

$$\begin{aligned}
\mathcal{H} &= 4\pi^i_j\pi^j_i - \frac{4}{d-1}\pi^i_i\pi^j_j + \frac{1}{2}\pi^2 + \frac{1}{2K}E_iE^i + \frac{1}{4}R - \frac{1}{2}g^{ij}\partial_i\phi\partial_j\phi - V \\
&\quad -\frac{1}{2}\mathcal{M}^2 A_iA^i - \frac{1}{4}KF_{ij}F^{ij} = 0 \\
\mathcal{H}_i &= 2\nabla_j\pi^j_i - \pi\partial_i\phi - F_{ij}E^j = 0 \\
\mathcal{G} &= \nabla_i E^i = 0\,,
\end{aligned}$$

where $\mathcal{H}$ coincides with the canonical Hamiltonian density, and $\mathcal{G} = 0$ is the Maxwell equation for the electric field in the vacuum.

The bulk theory is defined on a bulk spacetime with a boundary at $r = \rho$, with ρ being the cut-off parameter. In the Hamilton-Jacobi formalism the momenta of the theory are obtained from the on-shell action S as a functional of the prescribed boundary data, $g_{ij}(x,\rho)$, $\phi(x,\rho)$ and $A_i(x,\rho)$ from the appropriate variations

$$\begin{aligned}
\pi^{ij} &= \frac{1}{\sqrt{g}}\frac{\delta S}{\delta g_{ij}} \\
\pi &= \frac{1}{\sqrt{g}}\frac{\delta S}{\delta \phi} \\
E^i &= \frac{1}{\sqrt{g}}\frac{\delta S}{\delta A_i}
\end{aligned}$$

with respect to the boundary data.

In order to discuss the equations obtained from the constraint $\mathcal{H} = 0$, it is convenient to group the terms of the local part of the on-shell action into different levels. The lowest possible counterterms for the vector sector are of level two and read

$$S_v^{(2)} = \int d^d x \sqrt{g} \left(\frac{1}{2} N(\phi) g^{ij} A_i A_j + \frac{1}{2} P(\phi) g^{ij} \partial_i \phi A_j \right) . \tag{4.18}$$

After computing the momenta that arise from the sum of (4.7), (4.8) and (4.18) and inserting them into the Hamilton constraint it is evident that the level zero equation is unchanged, the level two contributions can always be solved by setting $P(\phi) = 0$. The justification for this, is that we do not expect counterterms linear in A_i. This way the gravity-scalar part of the analysis remains the same.

The new equation to be solved is

$$\mathcal{H}_v^{(2)} = \left(\frac{U'N'}{2} - \frac{d-2}{d-1} UN + \frac{N^2}{2K} - \frac{\mathcal{M}^2}{2} \right) A^i A_i = 0 \, .$$

We similarly expand U, N and $\mathcal{M}^2$ in powers of ϕ. For the flows that are of interest to us in the last chapter, $K(\phi) = \mathcal{O}(\phi)$ and $\mathcal{M}^2(\phi) = \mathcal{O}(\phi^2)$. If we use the solution we obtained for U we get

$$N_0 \left(\frac{N_0}{2K_0} + \frac{d-2}{2L} \right) = 0 \, .$$

For asymptotically massless vector fields we should have $N_0 = 0$ [23]. Moving to a higher order in ϕ we get

$$\begin{aligned} N_1 &= 0 \\ N_2 &= \frac{L}{\lambda - 1} \mathcal{M}_2^2 \, . \end{aligned}$$

Clearly a breakdown occurs for $\lambda = 1$ and we have the remainder term

$$\mathcal{H}_{\text{rem}} = \frac{1}{2} \mathcal{M}_2^2 \phi^2 A^i A_i + \mathcal{O}(\phi^3, A^4) \qquad \text{for } \lambda = 1 \, .$$

Proceeding to the next levels increases the number of invariants dramatically, making the complete analysis extremely tedious. On level four however, the resulting equations can be solved by setting to zero the coefficients of all the terms except for $F_{ij}F^{ij}$ [23]. This leads to the equation

$$\mathcal{H}_F^{(4)} = \left(\frac{d-4}{4L} G_0 - \frac{1}{4} K_0 \right) F_{ij} F^{ij} = 0 \, ,$$

where G_0 is the constant part of the coefficient of the $F_{ij}F^{ij}$ counterterm. The equation breaks down for $d = 4$ leading to a logarithmic divergence of Γ and a contribution to the conformal anomaly [23]. Thus, up to level four, the anomaly contributions from the vector sector are

$$\mathcal{A} = \begin{cases} -\frac{1}{2} L \mathcal{M}_2^2 \phi^2 A_i A^i & (\lambda = 1) \\ \frac{1}{4} L K_0 F_{ij} F^{ij} & (d = 4) \, . \end{cases}$$

5 Non-relativistic Holography

Despite the auspicious realization of a specific theory for a holographic description of a physical system, many things remain to be achieved. Most importantly, and regardless of many attempts, the promise of managing to describe a strongly correlated system that is experimentally tangible is yet to be fulfilled. Notwithstanding the contribution to a qualitative understanding of real-time dynamics and transport properties of the quark-gluon plasma in QCD, no holographic dual matching the precise microscopic details of any such system has emerged.

In search of a possible dual of a strongly correlated system that can be engineered in the laboratory, attention was drawn to non-relativistic systems. This field provides a plethora of strongly correlated systems which can be experimentally studied in detail. Moreover, some of these are of extraordinary technological interest. The focus would of course be on non-relativistic conformal field theories arising from these systems that would potentially have a gravity dual. So, the ambition is to develop a holographic approach to the non-relativistic theories that would describe condensed matter systems.

In particular, the question might be stated as which field theories with Galilean scaling symmetry ([25] and references therein and thereto) have a holographic dual. Along the lines of generalizing the Poincaré algebra to the conformal algebra, one could extend the Galilean algebra to the so called Schrödinger algebra.

An example of a system (at least conjectured) to realize this Schrödinger symmetry is fermions at unitarity. To achieve the desired scale invariance one can fine tune the interactions (with an external magnetic field for example) to obtain a massless two-body boundstate. This effectively leads to an infinite scattering length, therefore to a strongly correlated system.

As we described in Chapter 3 the "original" AdS/CFT [10, 14, 15] correspondence maps relativistic conformal field theories holographically to gravitational dynamics in a higher dimensional asymptotically AdS spacetime. So, by pushing the rules of gauge-gravity duality beyond the case considered by Maldacena, the idea is to get a gravity dual of a non-relativistic field theory. The gravity dual of non-relativistic CFTs was proposed almost simultaneously by Son [26] on one hand and independently by Balasubramanian and McGreevy [27] on the other. The goal is to demand respect of the Schrödinger algebra which is obtained from the relativistic conformal algebra by reducing along a light-cone. The procedure resembles light-cone quantization, where for a fixed light-cone momentum only a Galilean subgroup of the Lorentz group is manifest.

Contrary to the approach of [10], the proposals of Balasubramanian, McGreevy and Son have a different starting point. The fact that the conformal group in the context of field theories coincides with the isometries of AdS spacetime in one extra dimension was known before the statement of the AdS/CFT conjecture. However, Maldacena, even though he did not prove the duality, constructed it in a very consistent and convincing manner directly through careful considerations in a string theory context. The setting in these proposals is somewhat different. Here, the gauge/gravity duality is a given and the starting point. Not bothering with a string theory construction, the writers accept the idea that non-relativistic CFTs do indeed have a holographic gravitational dual. They directly provide the gravitational duals focusing only on symmetry considerations. Embedding in string theory followed in [28, 29, 30]. Following the structure of Chapter 3 we first discuss the so-called Schrödinger group and then the corresponding geometry. We then present the bulk theory and briefly outline an example of a string theory embedding.

5.1 Schrödinger Group

The Schrödinger group essentially is the Galilean group which includes the spatial translations P_i, rotations M_{ij}, Galilean boosts K_i, and time translations H, with the addition of a dilatation operator D and a particle number (or conserved rest mass) operator N. Particle numbers can be conserved and this is related to the fact that we cannot have pairs of particles-antiparticles creation in a non-relativistic theory. This does not exclude the presence of antiparticles in a non-relativistic theory as long as it is invariant under $t \to -t$. But this will not be the case in what we will consider here.

Scale invariance can be realized in a number of ways. We will take advantage of the freedom of the relative scale dimension of time and space, called the "dynamical critical exponent" and which we denote as z. Assuming spatial isotropy, we can in general have the scaling action

$$(t, \vec{x}) \to (\lambda^z t, \lambda \vec{x}) \qquad \lambda \in \mathbb{R}\,, \tag{5.1}$$

where z can in principle take any positive value but for the Schrödinger case, as we will see, it is $z = 2$. The dynamical critical exponent was first introduced as an anisotropic space and time scaling of the renormalization group [31] and we will discuss its meaning in the following.

The relevant algebra is given for completeness:

$$
\begin{aligned}
&[M_{ij}, N] = [M_{ij}, D] = 0, \quad [M_{ij}, P_k] = i\left(\delta_{ik}P_j - \delta_{jk}P_i\right), \\
&[M_{ij}, K_k] = i\left(\delta_{ik}K_j - \delta_{jk}K_i\right), \\
&[M_{ij}, M_{kl}] = i\left(\delta_{ik}M_{jk} - \delta_{jk}M_{il} + \delta_{il}M_{kj} - \delta_{jl}M_{ki}\right) \\
&[P_i, P_j] = [K_i, K_j] = 0, \quad [K_i, P_j] = i\delta_{ij}N, \quad [D, P_i] = iP_i\,, \\
&[D, K_i] = (1-z)\,iK_i, \quad [H, N] = [H, P_i] = [H, M_{ij}] = 0, \quad [H, K_i] = -iP_i\,, \\
&[D, H] = izH, \quad [D, N] = i\,(2-z)\,N\,.
\end{aligned}
$$

Where $i, j = 1, \ldots, d$ label space dimensions. The last commutator indicates that the case of $z = 2$ is rather special. At this value of z, the dilatations commute with the number operator. Therefore, both D and N can be diagonalized, so representations of the Schrödinger algebra can be in general labeled by the scaling dimension Δ and one more number ℓ. For fermions at unitarity this is precisely the fermion (particle) number. This commutator is the mathematical expression of the fact that mass is dimensionless at $z = 2$ (maintaining $\hbar = 1$). This is why the free Schrödinger equation can be scale invariant with this particular time and space scaling. Moreover, the $z = 2$ case allows for an extra "special conformal" generator, C, to be added to the algebra with non-trivial commutators

$$[D, C] = -2iC\,, \qquad [H, C] = -iD\,.$$

It is however noteworthy that apart from the free Schrödinger theory there are known examples of interacting theories which respect the Schrödinger symmetry at a quantum level and are known as non-relativistic conformal field theories [25].

The Dynamical Critical Exponent

Before we discuss the holographic construction of the non-relativistic conformal field theories it is worth mentioning a few interesting facts about z and its physical consequences. We will start by providing a somewhat crude but rather intuitive way of looking at z. By making the traditional identifications for energy we see that

$$E \sim \frac{\partial}{\partial t} \sim \lambda^{-z}\,, \tag{5.2}$$

based on the rescaling described by (5.1). Similarly for the momentum one would have

$$p \sim \frac{\partial}{\partial x} \sim \lambda^{-1}\,. \tag{5.3}$$

Meanwhile, if we were to examine a free Hamiltonian we could insist that it will depend on the momentum to an arbitrary power #. That is

$$H \sim p^{\#}.$$

We immediately see from (5.2), (5.3) that this arbitrary number # has to be equal to z. So, at least at a superficial level, the dynamical critical exponent can be viewed as the power of the momentum that appears in the Hamiltonian of a system. This reasoning is in full agreement with the Schrödinger equation ($z = 2$). In the same spirit, it also agrees with the relativistic equation that relates energy and momentum, $E = \sqrt{p^2 + m^2}$ for $z = 1$. We can keep this in mind when we encounter the gravity dual of a NRCFT for a general z. We should expect that the metric will turn into that of an AdS spacetime when we set $z = 1$.

Perhaps the most important aspect of z is that it has the following physical consequence: it determines the critical dimension of interactions. To address this, we will closely follow [32]. In this discussion, time and all frequencies that will appear will be Euclidean, because the real time description of actions which are non-analytic in frequencies is quite subtle.

Suppose we have a free field theory of the form

$$S = \int^{\{\Lambda,\Lambda\}} \frac{d^{d-1}k\, d\omega}{(2\pi)^d} \left(r + k^2 + |\omega|^{2/d}\right) |\Phi(\omega, k)|^2$$

where Λ is the UV cutoff for both frequencies and momenta and Φ is an N component vector. Firstly, we integrate out modes with momenta and energies between some lower cutoff Λ' and the original cutoff Λ. Due to the anisotropic rescaling between time and space, the trick is to lower the energy and momentum cutoffs by different amounts:

$$\Lambda'_k = e^{-l}\Lambda, \quad \Lambda'_\omega = e^{-zl}\Lambda, \quad \text{for some } l > 0\,.$$

The action then becomes

$$S = \int^{\{\Lambda'_k,\Lambda'_\omega\}} \frac{d^{d-1}k\, d\omega}{(2\pi)^d} \left(r + k^2 + |\omega|^{2/d}\right) |\Phi(\omega, k)|^2 + \text{const}\,.$$

The second step according to the renormalization procedure is to rescale the momenta, energies and the field Φ in order to restore the action to its original form with a rescaled value of the "coupling" r. If we let

$$k' = e^{l}k, \quad \omega' = e^{zl}\omega, \quad \Phi'(\omega', k') = e^{-(z+d+1)l/2}\Phi(\omega, k)\,, \tag{5.4}$$

then the action becomes

$$S = \int^{\{\Lambda'_k,\Lambda'_\omega\}} \frac{d^{d-1}k\, d\omega}{(2\pi)^d} \left(re^{2l} + k'^2 + |\omega'|^{2/d}\right) |\Phi'(\omega', k')|^2 + \text{const}\,.$$

This shows that the theory can be renormalized to lower energies and momenta by the rescalings (5.4). If we add a quartic interaction

$$S_{int.} = \int^{1/\Lambda} d^{d-1}x\, d\tau\, u \left(\Phi^2\right)^2,$$

it would be of interest to know whether this interaction becomes stronger or weaker as we flow to lower energies. From (5.4), noting that the Fourier transform implies $\Phi'(\tau', x') = e^{(z+d-3)l/2}\Phi(\tau, x)$, we have

$$u' = e^{(5-z-d)l} u\,.$$

So, the coupling u becomes weaker at low energies (irrelevant) if

$$d > d_c = 5 - z\,.$$

For $z = 1$ we recover the result that the critical spacetime dimension of relativistic Φ^4 theory is $d = 4$. Interestingly though, for $z > 1$ we observe that the critical dimension is lowered. This fact was first noted in [31] and implies that quantum critical points are increasingly tractable by perturbative methods. In [32], one can find an incomplete but useful list of systems that are described by different values of z with brief explanations.

5.2 Holographic Construction

The goal would be to be able to study strongly coupled Galilean-invariant conformal field theories using the gauge/gravity duality. In order to do this we need to realize the Schrödinger algebra geometrically. However, we now have two symmetry generators which may be diagonalized simultaneously and whose eigenvalues label nonequivalent representations (in the usual AdS case, there is only the scaling dimension). This leads to pushing the AdS/CFT "rules" slightly: since it is impossible to arrange for the whole algebra of a d dimensional Schrödinger invariant field theory to act as the isometries of a $d+1$ dimensional spacetime we must consider a candidate dual in $d+2$ dimensions! Such geometries where explicitly constructed in [26, 27]. The metric is (following the notation of [27]):

$$ds^2 = L^2 \left(-\frac{dt^2}{r^{2z}} + \frac{d\vec{x}^2 + 2d\xi dt}{r^2} + \frac{dr^2}{r^2} \right) \tag{5.5}$$

which is invariant under the anisotropic scale transformation

$$(r, \vec{x}, t, \xi) \to \left(\lambda r, \lambda \vec{x}, \lambda^z t, \lambda^{2-z} \xi \right).$$

Here, $\vec{x}$ is a d-vector and the generators of the Schrödinger algebra are geometrically given by

$$\begin{aligned}
&P_i = -i\partial_i, \quad H = -i\partial_t, \quad M_{ij} = -i\left(x^i\partial_j - x^j\partial_i\right),\\
&K_i = -i\left(-t\partial_i + x^i\partial_\xi\right), \quad D = -i\left(zt\partial_t + x^i\partial_i + (2-z)\,\xi\partial_\xi + r\partial_r\right), \\
&N = -i\partial_\xi\,.
\end{aligned} \tag{5.6}$$

The last identification is perhaps the least obvious, but rather important. The particle number is given by the momentum in the ξ direction. It is common in the AdS/CFT correspondence for global symmetries in field theory to appear as extra dimensions in the gravitational dual. The fact that the ξ direction is null ($|\partial_\xi|^2 = 0$) and that the N generator arises in the commutator of two spacetime symmetries is rather unusual. We also note once more that for the special case where $z = 2$ the dilatations commute with the number operator. It is worth mentioning that the metric (5.5) is not invariant under time reversal, but it is under the combined operation

$$t \to -t, \qquad \xi \to -\xi\,.$$

Given the interpretation of the ξ-momentum as rest mass we can interpret this combination $\mathcal{CT}$ as charge conjugation and time reversal.

What is important here is that in systems of physical interest the number operator (i.e. the spectrum of masses) is quantized. This tells us that ξ in the bulk description must be periodic:

$$\xi \sim \xi + 2\pi L_\xi\,. \tag{5.7}$$

This identification introduces a mass scale. Unless $z = 2$ we can see from (5.6) that dilatations will not preserve the length L_ξ and hence are no longer isometries of the background. It is therefore impossible to have a scale invariant Galilean theory with a nontrivial discrete mass/particle number spectrum for $z \neq 2$.

If we compactify ξ, boost invariance remains unbroken precisely because the ξ direction is null; this follows from the commutator $\left[\hat{N}, \hat{K}\right] = 0$ in the Schrödinger algebra [29]. Having a circle becoming very small in our geometry may render the calculations unreliable. This is because the identification of the null direction leads to a zero proper length, the supergravity regime cannot be trusted. Fortunately, when considering a sector with large non-zero light-cone momentum (along our null direction ξ) one finds regions in the geometry where the circle has a non-zero size, so that the calculations can be trusted [30]. This potential conical singularity for $r \to \infty$ which suggests unreliability of the metric in the IR, turns out to be unphysical. More specifically, the singularity goes away as soon as we turn on finite temperature. This is, physically, a perfectly reasonable situation: we will always have some finite T in a realistic cold-atom system, and thus an IR regulator. In [30] it is argued that by considering in a finite number density resolves potential problems and it is obviously a physically sensible thing to do. In [29] it is mentioned that even in the $T \to 0$ limit the dynamics will resolve the singularity in a way already familiar from the study of null orbifolds of flat space [33, 34, 35, 36].

Causal Structure and Global Coordinates

Another important characteristic, is that the metric (5.5) has the causal structure that naturally reproduces the Galilean light-cone of the field theory. The causal structure of a non-relativistic theory is intrinsically degenerate. The metric (5.5) shares this degeneracy and therefore is consistent with the boundary Galilean invariance. This a crucial ingredient in the gauge/gravity duality. A bulk theory with a well behaved causal structure cannot be holographically dual to a non-relativistic theory or we would end up having inconsistencies when calculating correlation functions. In our case this is ensured by the sign of g_{tt}, as all points at some fixed $t = t_0$ share the same causal future and past [28, 32].

However, it should be made clear that the spacetime in question is causal in the sense that it doesn't have closed timelike curves. Moreover, if the sign of g_{tt} is reversed while keeping the orientation of ξ fixed, the space can become unstable to modes with large particle number [37]. Also, by reversing the sign we lose non-relativistic causality. This is due to the fact that the $dtd\xi$ term grows at the same rate as $d\vec{x}$ when going towards the boundary $r \to 0$. In order for the lightcones to flatten the g_{tt} term must be negative. For $z \neq 1, 2$ a different sign can lead to geodesic incompleteness at the boundary and the so-called pp singularities [37]. In [38] the "Schrödinger" metrics (for $z = 2$) are shown to have a global, geodesically complete coordinate system and that spacetimes with $z > 2$ admit no global timelike Killing vector fields. This means that the global metric will necessarily be time-dependent.

Starting from the fact that the $z = 2$ algebra has the central element $P_- = \partial_\xi$, we are looking for new coordinates

$$(t, r, \vec{x}, \xi) \to \left(T, R, \vec{X}, V\right)$$

in which $H + C$ and P_- are simultaneously diagonal,

$$H + C = \partial_T, \qquad P_- = \partial_V \,.$$

This is accomplished by the transformation

$$\begin{aligned} t = \tan T, \quad r = \frac{R}{\cos T}, \quad \vec{x} = \frac{\vec{X}}{\cos T}\,, \\ \xi = V + \frac{1}{2}\left(R^2 + \vec{X}\right)\tan T\,. \end{aligned}$$

This leads to the following form of the metric

$$\begin{aligned} ds^2 &= -\left[\frac{1}{R^4} + \left(1 + \frac{\vec{X}}{R^2}\right)\right] dT^2 + \frac{1}{R^2}\left(-2dTdV + dR^2 + d\vec{X}\right) \\ &= -\frac{dT^2}{R^2} + \frac{1}{R^2}\left[-2dTdV - \left(R^2 + \vec{X}^2\right) dT^2 + dR^2 + d\vec{X}^2\right] . \end{aligned}$$

A few comments are in order. First of all, it is geodesically complete. What is most fortunate about the above transformation is that it is considerably more simple than the AdS counterpart. For example, in the AdS case, instead of the relation $t = \tan T$ used here, the Poincaré to global time transformation is

$$\tan T = \frac{2t}{1 + r^2 + \vec{y} - t^2} \, .$$

A relevant and remarkable feature of the global metric is that the difference between the Poincaré metric is only in a single term: the coefficient of dT^2. Again, comparing between the two, the AdS metric is substantially different when expressed in Poincaré or global coordinates.

5.3 The Bulk Action

According to [27] the metric (5.5) is sourced by the following stress tensor

$$T_{ab} = -\Lambda g_{ab} - \mathcal{E}\delta_a^0\delta_b^0 g_{00} \, .$$

It consists of a negative cosmological constant Λ and a pressureless "dust" of constant density $\mathcal{E} = \left(2z^2 + z - 3\right) L^{-2}$ for $d = 3$. A negative cosmological constant is a basic ingredient and as we shall see, so is a massive vector field.

Without further delay, we give the action that can be used to engineer the metric (5.5):

$$S = \int d^{d+3}x \, \sqrt{-g} \left(-\frac{1}{4}F^2 + \frac{1}{2}|D\Phi|^2 - V\left(|\Phi|^2\right) \right) ,$$

where $D_a\Phi \equiv \left(\partial_a + ieA_a\right)\Phi$, with a Mexican-hat potential

$$V\left(|\Phi|^2\right) = \mathsf{g}\left(|\Phi|^2 - \upsilon^2\right)^2 + \Lambda \, .$$

This produces the correct dust stress tensor for arbitrary g as long as the gauge field mass is $m_A^2 = e^2\upsilon^2 = \frac{z(z+d)}{L^2}$.

Similarly, Son [26] uses the same ingredients to generate the solution (5.5). He suggests the action (in his notation):

$$S = \int d^{d+2}x \, dz \, \sqrt{-g} \left(\frac{1}{2}R\right) - \Lambda - \frac{1}{4}H_{\mu\nu}H^{\mu\nu} - \frac{m^2}{2}C_\mu C^\mu \, ,$$

where $H_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$ and of course C_μ is the gauge field. The desired metric* together with

$$C^- = 1$$

*In Son's notation: $ds = z^{-2}\left[-2z^{-2}(dx^+)^2 + -2dx^+dx^- + dx^i dx^i + dz^2\right]$.

and the following choices

$$\Lambda = -\frac{1}{2}\left(d+1\right)\left(d+2\right), \quad m^2 = 2\left(d+2\right)$$

is indeed a solution to the coupled Einstein and Proca equations.

Turning back to [27] we can observe that due to the non-stationary form of the metric, the stress tensor for an electric field in the radial direction F_{rt} has only a 00 component.

Therefore, we consider a nonzero current j^{ξ} of the form† $j^{\xi} = \rho_0 r^{\alpha}$. Then we calculate (starting from Maxwell's equation in the bulk):

$$\begin{aligned} \frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g}F^{ab}\right) &= j^b \Rightarrow \\ A_t &= \frac{\rho_0}{\left(\alpha-2\right)\left(\alpha-d-2\right)}r^{\alpha-2}. \end{aligned}$$

Thus, the ansatz $A_t \propto r^z$ solves the equations of motion, provided that

$$\Lambda = -\frac{1}{2}\left(d+1\right)\left(d+2\right), \quad m^2 = z\left(z+d\right) .$$

5.4 String Theory Embedding

As already mentioned, the string theory embedding was achieved by [28, 30, 29]. We will present the procedure followed by [29] and mention similarities with the procedure followed by [28]. It is somewhat beyond the scope of this work to thoroughly discuss this issue, so we will simply mention some basic aspects of it.

In the work of Herzog, Rangamani and Ross [28] and independently by Adams, Balasubramanian and McGreevy [29] the same tool is employed to achieve a string theory embedding. Namely, the Null Melvin Twist (Appendix C), which is a solution generating technique for IIB supergravity.

Null Melvin Twist

Interestingly, some solutions with the desired general characteristics have already appeared using the Null Melvin Twist [39, 40]. For $T = 0$ one of the appearing solutions yield a Schrödinger spacetime for $z = 3$. These theories, which have some part of the symmetry group broken, are described as dual to "dipole theories" which are also discussed [30]. The algorithm of the NMT is given here and details can be found in Appendix C:

1. Choose a translationally invariant direction (e.g. y) and boost along it by γ

†Because of the term $dtd\xi$ in the metric, this choice will produve the desired result.

2. T-dualize along y
3. Re-diagonalize isometry generators by shifting $d\phi \to d\phi + \alpha dy$
4. T-dualize along y again
5. Boost back along y by $-\gamma$.

We follow [29] and then discuss the [28] case which is similar. The starting point will be the extremal D3-brane, which is a IIB supergravity solution with the following metric:

$$ds^2 = \frac{1}{h}\left(-d\tau^2 + d\vec{x}^2\right) + h\left(d\rho^2 + \rho^2 ds^2_{S^5}\right)$$

where $h^2 = 1 + \frac{R^4}{\rho^4}$ is the D3 harmonic function, and self dual flux form

$$F^{(5)} = \frac{1}{r^5} d\tau \wedge dy \wedge dx_1 \wedge dx_2 \wedge dr + \Omega_5 d\theta \wedge d\phi \wedge d\psi \wedge d\mu \wedge d\chi\,,$$

with $\Omega_5 = 1/8 \cos\theta \cos\mu \sin^3\mu$. We take dy to lie along the worldvolume and $d\phi$ along S^5 and choose–without loss of generality–coordinates such that $y = x_3$. A convenient choice for $d\phi$ is given by the Hopf fibration $S^1 \to S^5 \to \mathbb{P}^2$ (see Appendix C), with the metric

$$ds^2_{S^5} = ds^2_{\mathbb{P}^2} + (d\chi + \mathcal{A})^2$$

with χ being a local coordinate on the Hopf fiber and $\mathcal{A}$ is the 1-form potential for the Kähler form on $\mathbb{P}^2$ so that $J_{\mathbb{P}^2} = d\mathcal{A}$. We take $d\phi = d\chi$. Both dy and $d\phi$ act freely.

The result is (ignoring $ds^2_{\mathbb{P}^2}$ for now)

$$\begin{aligned} ds^2 &= \frac{1}{h}\Big[-d\tau^2\left(1+\beta^2\rho^2\right) + dy^2\left(1-\beta^2\rho^2\right) + 2d\tau dy\left(\beta^2\rho^2\right)\Big] + h\rho^2(d\chi+\mathcal{A})^2 \\ B &= 2\beta\rho^2\,(d\chi + \mathcal{A}) \wedge (d\tau + dy) \\ \Phi &= \Phi_0\,, \end{aligned}$$

with $\beta = \alpha c$ (see Appendix C). Nothing happend to the five-form, since T-dualizing takes $d\Omega^5$, the top form on the sphere, to $dy \wedge d\Omega^5$, so that the twist $d\phi \to d\phi + \beta dy$ acts trivially. So,

$$F_5 = (1 + \star)\,\Omega_5 d\theta \wedge d\phi \wedge d\mu \wedge d\chi\,.$$

To make the Schrödinger nature of the solution transparent we take a few more steps. We change the coordinates according to

$$t = \frac{y + \tau}{\sqrt{2}}$$

$$\xi = \frac{y - \tau}{\sqrt{2}}$$

so that the background becomes

$$\begin{aligned} ds^2 &= h^{-1}\left(\beta^2\rho^2 dt^2 + 2dtd\xi\right) + h\rho^2\left(d\chi + \mathcal{A}\right)^2 \\ B &= 2\beta\rho^2\left(d\chi + \mathcal{A}\right)\wedge dt \\ \Phi &= \Phi_0 \,. \end{aligned}$$

Adding back everything we obtain

$$\begin{aligned} ds^2 &= h^{-1}\left(-\beta^2\rho^2 dt^2 + 2dtd\xi + d\vec{x}^2\right) + h\left(d\rho^2 + \rho^2 ds^2_{S^5}\right) \\ B &= 2\beta\rho^2\left(d\chi + \mathcal{A}\right)\wedge dt \\ \Phi &= \Phi_0 \,. \end{aligned}$$

As a final step, we take the near-horizon limit $h \to R^2/\rho^2$ and switching to the global radial coordinate $r = R^2/\rho$, which makes $h = r^2/R^2$ the solution becomes

$$\begin{aligned} ds^2 &= \frac{R^2}{r^2}\left(\frac{2\Delta^2}{r^2}dt^2 + 2dtd\xi\right) dr^2 + R^2 ds^2_{S^5} \\ B &= 2\sqrt{2}\frac{\Delta R^2}{r^2}\left(d\chi + \mathcal{A}\right)\wedge dt \\ \Phi &= \Phi_0 \,, \end{aligned}$$

with $\Delta = \beta R^2$. After compactifying on S^5, we recover the desired Schrödinger geometry with $d = 2$ and $z = 2$.

In [28] essentially they perform the same procedure and obtain effectively the same result in a slightly different notation and in changed coordinates. We mention their result here for completeness (in their notation)

$$\begin{aligned} ds^2 &= r^2\left(-2dudv - r^2du^2 + d\vec{x}^2\right) + \frac{dr^2}{r^2} + \left(d\psi + A\right)^2 + d\Sigma_4^2 \\ F_{(5)} &= 2\left(1+\star\right)d\psi\wedge J\wedge J \\ B_{(2)} &= r^2 du\wedge\left(d\psi + A\right) \end{aligned} \tag{5.8}$$

in light-cone coordinates

$$u = \beta(t+y)\,, \qquad v = \frac{1}{2\beta}(t-y)\,, \tag{5.9}$$

where now, $\beta = \frac{1}{2}\alpha e^{\gamma}$ with γ being the same as before.

However, they also consider non-extremal D3-branes and repeat the NMT. So, the starting point is the planar Schwarzschild-AdS black hole times S^5 with a $F_{(5)}$ flux

$$ds^2 = r^2\left(-f(r)dt^2 + dy^2 + d\vec{x}^2\right) + \frac{1}{r^2}\left(\frac{dr^2}{f(r)} + r^2 d\Omega_5^2\right),$$

where again the S^5 will be written as a fibration over $\mathbb{CP}^2$. Now the solution generating machine leads to the string frame metric

$$\begin{aligned} ds^2_{\rm str} &= r^2\left[-\frac{\beta^2 r^2 f(r)}{k(r)}(dt+dy)^2 - \frac{f(r)}{k(r)}dt^2 + \frac{dy^2}{k(r)} + d\vec{x}^2\right] + \frac{dr^2}{r^2 f(r)} \\ &\quad + \frac{(d\psi + A)^2}{k(r)} + d\Sigma_4^2 \\ e^\phi &= \frac{1}{\sqrt{k(r)}} \\ F_{(5)} &= dC_{(4)} = 2(1+\star)d\psi\wedge J\wedge J \\ B_{(2)} &= \frac{r^2\beta}{k(r)}\left(f(r)dt + dy\right)\wedge(d\psi + A)\,, \end{aligned} \tag{5.10}$$

with

$$f(r) = 1 - \frac{r_+^4}{r^4}\,, \qquad k(r) = 1 + \beta^2 r^2(1-f(r)) = 1 + \frac{\beta^2 r_+^4}{r^2}\,, \tag{5.11}$$

with the obvious notation r_+ for the horizon of the solution. The parameter β appearing in this case is an independent physical parameter. The lack of extremality has broken the boost symmetry. Therefore, we cannot boost along the ty plane and set this parameter equal to unity, as we could normally do in the extremal case.

5.5 The 5d effective Lagrangian

The five dimensional Lagrangian is developed also in [28]. The starting point is the Lagrangian of Son [26] and how that is related to the ten dimensional IIB theory. The effective Lagrangian is repeated here

$$S = \int d^{d+2}x\, dr\sqrt{-g}\left(R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu\right). \tag{5.12}$$

If one starts from IIB supergravity and Kaluza-Klein (Appendix D for a discussion of KK compactifications) reduce the solution (5.8) on an undeformed S^5 the procedure is straightforward and the result is the desired metric in five dimensions. The two-form however has different behavior compared to the case where a $\mathbb{CP}^2$ fibration is used, because it depends on the coordinates of S^5. Such a mode of the two-form produces a massive vector transforming in the **15** of $\mathsf{SO}(6)$.

If a KK reduction is performed on (5.10)we obtain

$$\begin{aligned} ds^2_{\rm E} &= r^2k(r)^{-2/3}\left[-\beta^2r^2f(r)\left(dt+dy\right)^2-f(r)dt^2+dy^2+k(r)d\vec{x}^2\right] \\ &\quad +k(r)^{1/3}\frac{dr^2}{r^2f(r)} \\ &= r^2k(r)^{-2/3}\left[\left(\frac{1-f(r)}{4\beta^2}-r^2f(r)\right)du^2+\frac{\beta^2r_+^4}{r^4}dv^2-\left(1+f(r)\right)dudv\right] \\ &\quad +k(r)^{1/3}\left(r^2d\vec{x}^2+\frac{dr^2}{r^2f(r)}\right) \end{aligned} \tag{5.13}$$

where the light-cone coordinates (5.9) were used in the second line, and with massive gauge field and scalar

$$\begin{aligned} A &= \frac{r^2\beta}{k(r)}\left(f(r)dt+dy\right) \\ &= \frac{r^2}{k(r)}\left(\frac{1+f(r)}{2}du-\frac{\beta^2r_+^4}{r^4}dv\right), \\ e^\phi &= \frac{1}{\sqrt{k(r)}}, \end{aligned}$$

for the same $f(r)$ and $k(r)$ in (5.11). In these light-cone coordinates the solution asymptotically approaches the extremal solution (5.5), with β however remaining a physical parameter, in the sense that the full metric depends on β.

The metric (5.13) is a solution to

$$S_{(5)}=\frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left(R-\frac{4}{3}\partial_\mu\phi\partial^\mu\phi-\frac{1}{4}e^{-8\phi/3}F_{\mu\nu}F^{\mu\nu}-4A_\mu A^\mu-V(\phi)\right),$$

with scalar potential

$$V(\phi)=4e^{2\phi/3}(e^{2\phi}-4)\,.$$

The black hole geometry is *not* a solution to (5.12). The scalar that appears is because of the non-vanishing dilaton of the black hole geometry and melvinization causes the fibration over $\mathbb{CP}^2$ to be squashed. The squashing can be intuitively ascribed to the distortion of the asymptotics of spacetime as a feature of the NMT.

Although the writers of [28] determine this action and use it in their work, they add it with a disclaimer. They introduce it, even though they have no argument that the five-dimensional action describes a consistent truncation of the full ten-dimensional theory. The mention that some modes that transform non-trivially under $\mathsf{SO}(6)$ which are turned on in their ansatz may couple to other Kaluza-Klein harmonics which they neglect. In favor of the ansatz however, they mention that, by construction the black hole solution uplifts to IIB supergravity.

6 Hamilton-Jacobi and non-relativistic holography

In the previous chapters we laid the ground so that we can finally attempt to apply the holographic renormalization procedure, based on the Hamilton-Jacobi theory, to the non-relativistic version of gauge/gravity duality. Despite the extensive literature on renormalization group flows in AdS/CFT, this framework has not yet been implemented to the non-relativistic case. We shall make some formal developments and in the process, exclude some possibilities.

6.1 Non-relativistic case

For the original case, in the context of AdS/CFT, the holographic renormalization procedure using the Hamilton-Jacobi theory was outlined in Chapter 4. In Chapter 5 we discussed some aspects of the more recent developments of the gauge/gravity dualities that include non-relativistic theories. We saw in section 5.3 that the basic ingredients of the bulk action in the newly proposed duality–in both formulations [26, 27]–include a negative cosmological constant and a massive gauge field.

So, as a first step, we would directly apply the method outlined in Chapter 4. In order to do so, we should first notice that the idea we used to split the on-shell action as in (4.6) now becomes a bit more subtle. This is because of different scaling for space and time. Thus, the scaling procedure we used throughout Chapter 4 changes and depends on the dynamical critical exponent z. In particular, what makes this entire procedure interesting, is the fact that because of the form of the Schrödinger metric ($z = 2$), all the time componets of the equations stemming from the Hamilton-Jacobi equation should "jump" to the next level equation.

It is interesting to consider what we could expect to obtain from this procedure. In [30] there is a discussion on the proposals of [26, 27] for non-relativistic holography bearing in mind that a discrete light cone quantization (DLCQ) of a field theory gives a non-relativistic system. Based on the known example of performing a DLCQ on the M5 brane theory [41], it is discussed that the DLCQ of a relativistic conformal theory with a gravity dual, is suggestive of performing the DLCQ of AdS space. It is also noted that subtleties occur when performing a DLCQ of gravitational backgrounds and the gravity approximation cannot be naively applied.

In the present case, it would be interesting if we could retrieve some structure resembling the DLCQ limit of a theory as we approach the boundary. We will see that it is not so easy to organize things in such a way as to make this structure transparent.

6.2 Application

In order to apply the method we massage the original metric to bring it into a convenient form. Starting from

$$ds^2 = L^2 \left(-\frac{dt^2}{r^{2z}} + \frac{2dtd\xi + d\vec{x}^2 + dr^2}{r^2} \right)$$

we make the following transformation

$$\frac{dr^2}{r^2} = d\tilde{r}^2 \Rightarrow \frac{dr}{r} = d\tilde{r} \Rightarrow \tilde{r} = \ln r \Rightarrow r = e^{\tilde{r}} \, .$$

The substitution in (5.5) is then straightforward

$$ds^2 = d\tilde{r}^2 - e^{-2z\tilde{r}} dt^2 + e^{-2\tilde{r}} \left(2d\xi dt + d\vec{x}^2 \right)$$

but now the boundary is at $\tilde{r} \to -\infty$ and we implicitly take $L = 1$. From now on we will write $\tilde{r}$ as r. Now, the metric effectively has the desired form of $ds^2 = dr^2 + g_{ij}(x, r)dx^i dx^j$ where $i, j = 1, 2 \ldots, d+2$. It is also in the desired gauge in the sense that the lapse and shift functions have the values $N = 1$ and $N^i = 0$. We also have $\sqrt{g} = (e^{-r})^{d+2}$ as opposed to $\sqrt{\tilde{g}} = r^{-(d+3)}$ for the original metric (5.5).

Before moving on we will rewrite the Hamilton constraint for an action of the general form which we derived in section 4.2:

$$S = \int d^{d+1}x \, \sqrt{\tilde{g}} \, n \left(-\frac{1}{4}R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) + \frac{1}{4} K(\phi) F_{\mu\nu} F^{\mu\nu} \right. \\ \left. + \frac{1}{2} \mathcal{M}^2(\phi) A_\mu A^\mu \right) .$$

We temporarily assume a metric of the form

$$ds^2 = N^2 dr^2 + g_{ij}(x, r) dx^i dx^j$$

and the inverse metric is

$$g^{\mu\nu} = \begin{pmatrix} 1/N^2 & 0 \\ 0 & g^{ij} \end{pmatrix} .$$

For the action we have chosen the variation yields the Hamilton constraint

$$\mathcal{H} = \frac{1}{4}R + \frac{1}{2}\left(\partial_r\phi\right)^2 + \frac{1}{2}M^2A_r^2 + \frac{1}{2}K(\phi)g^{ij}F_{ri}F_{rj} - \frac{1}{2}g^{ij}\partial_i\phi\partial_j\phi \\ - \frac{1}{2}M^2A_iA^i - \frac{1}{4}K(\phi)F_{ij}F^{ij} - V(\phi)$$

However, in our case, the inverse metric of (5.5) is

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & e^{2r} & 0 \\ 0 & e^{2r} & -1 & 0 \\ 0 & 0 & 0 & e^{2r}\mathbf{1}_{d\times d} \end{pmatrix}.$$

This, unfortunately, renders the procedure problematic from the start, because according to [27] in order to get the desired stress tensor we need to have an electric field in the r direction. Therefore, we consider a nonzero current j^ξ of the form $j^\xi = \rho_0 r^\alpha$. This yields (starting from Maxwell's equation in the bulk and using the original form of the metric):

$$\begin{aligned} \frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g}F^{ab}\right) &= j^b \Rightarrow \\ A_t &= \frac{\rho_0}{(\alpha-2)(\alpha-d-2)}r^{\alpha-2}. \end{aligned}$$

The important thing here is that the gauge field has a time component only. Given the inverse metric, we see that all the remaining Maxwell related terms in the variation of the action with respect to the lapse function vanish. Moreover, the expected interesting feature where the tt terms would appear in the next higher level of the Hamilton-Jacobi equation is gone.

If we choose to ignore this, we can continue and see how the treatment of the scalar potential looks like. Our potential has the form:

$$\begin{aligned} V(\phi) &= \left(gv^2+\Lambda\right) - 2gv^2\phi^2 + g\phi^4. \\ c &\equiv gv^2+\Lambda. \end{aligned}$$

We assume that $U(\phi)$ to be:

$$U(\phi) = \lambda + \frac{1}{2}\lambda_1\phi^2 + \frac{1}{4}\lambda_2\phi^4.$$

Using the equation for the superpotential, as discussed in Chapter 4

$$\frac{d}{d-1}U^2 - \frac{1}{2}\left(\partial_\phi U\right)^2 = V(\phi)\,, \tag{6.1}$$

we have explicitly

$$\frac{d}{d-1}\left(\lambda^2+\frac{\lambda_1^2}{4}\phi^4+\frac{1}{16}\lambda_2^2\phi^8+\lambda\lambda_1\phi^2+\frac{\lambda\lambda_2}{2}\phi^4+\frac{\lambda_1\lambda_2}{4}\phi^6\right)-\\ \left(\lambda_1^2\phi^2+\lambda_2^2\phi^6-2\lambda_1\lambda_2\phi^4\right)=c-2gv^2\phi^2+g\phi^4.$$

The next step is to determine each coefficient order by order. So,

$$\frac{d\lambda^2}{d-1}=c\Rightarrow$$
$$\lambda=\pm\sqrt{\frac{d-1}{d}c}\,,$$

$$\begin{aligned}\left(\frac{d}{d-1}\lambda-\lambda_1\right)\lambda_1\phi^2 &= -2gv^2\phi^2\Rightarrow\\ \lambda_1 &= \frac{\frac{d\lambda}{d-1}\pm\sqrt{\left(\frac{d\lambda}{d-1}\right)^2-8gv^2}}{2\frac{d\lambda}{d-1}}\end{aligned}$$

and

$$\begin{aligned}g &= \frac{\lambda_1^2}{4}+\frac{\lambda\lambda_2}{2}+2\lambda_1\lambda_2\Rightarrow\\ \lambda_2 &= \frac{g-\lambda_1^2/4}{\lambda/2+2\lambda_1}.\end{aligned}$$

In search of a different metric that would have the same desired characteristic, namely Schrödinger invariance, one could look for version with non-zero temperature and/or non-zero chemical potential. An apparently relevant case is the derived metric (**??**) of [28], which we rewrite here for convenience:

$$\begin{aligned}ds_{\mathrm{E}}^2 &= r^2k(r)^{-2/3}\left[-\beta^2r^2f(r)\left(dt+dy\right)^2-f(r)dt^2+dy^2+k(r)d\vec{x}^2\right]\\ &\quad +k(r)^{1/3}\frac{dr^2}{r^2f(r)}\end{aligned}$$

with

$$f(r)=1-\frac{r_+^4}{r^4},\quad k(r)=1+\beta^2r^2(1-f(r))=1+\frac{\beta^2r_+^4}{r^2}$$

and with massive vector and scalar

$$A=\frac{r^2\beta}{k(r)}\left(f(r)dt+dy\right),\qquad e^{\phi}=\frac{1}{\sqrt{k(r)}}\,.$$

According to [28] this is a solution of the equations of motion from the $5-d$ effective action:

$$S=\frac{1}{16\pi G_5}\int d^5x\,\sqrt{-g}\left(R-\frac{4}{3}\partial_\mu\phi\partial^\mu\phi-\frac{1}{4}e^{-8\phi/3}F_{\mu\nu}F^{\mu\nu}-4A_\mu A^\mu-V(\phi)\right)$$

with a scalar potential

$$V(\phi) = 4e^{2\phi/3}\left(e^{2\phi} - 4\right).$$

Before even discussing how to split the action based on power divergences (i.e. looking at the metric asymptotics) we can simply look at the zeroth order and treat the scalar potential like we did previously. Ideally, we would like an analytic solution of 6.1. In principle, this equation does not necessarily have an analytic solution. We can however, try the obvious ansatz

$$U(\phi) = ae^{c\phi} - be^{d\phi}.$$

We will now use the 5-dimensional version of the potential equation, as found in [21]

$$V(\phi) = \frac{1}{3}U^2 - \frac{1}{2}U'^2$$

giving

$$\left(\frac{1}{3} - \frac{1}{2}c^2\right) a^2e^{2c\phi} + b^2e^{2d\phi}\left(\frac{1}{3} - \frac{1}{2}d^2\right) + 2ab\left(\frac{cd}{2} - \frac{1}{3}\right)e^{(c+d)\phi} = 4e^{8\phi/3} - 16e^{2\phi/3}.$$

The obvious problem–the last term in the left hand side– is taken care of for $cd = 2/3$. Inserting that back we get

$$\left(\frac{1}{3} - \frac{1}{2}c^2\right) a^2e^{2c\phi} + be^{4/3c\phi}\left(\frac{1}{3} - \frac{1}{2}\left(\frac{2}{3c}\right)^2\right) = 4e^{8\phi/3} - 16e^{2\phi/3}$$

from where it is obvious that not both powers of the exponential can be correct for a single choice of c. We need to use Taylor expansion. The potential has the profile shown in Fig. 6.1

The Taylor expansion is

$$V(\phi) = -12 + \frac{32}{3}\phi^2 + \frac{320}{27}\phi^3 + \frac{224}{27}\phi^4 + \mathcal{O}(\phi^5).$$

Following the same steps as we did for the original action we assume a potential*

$$U(\phi) = \lambda + \frac{1}{2}\lambda_1\phi^2 + \frac{1}{3!}\lambda_2\phi^3 + \frac{1}{4!}\lambda_3\phi^4.$$

This results in an imaginary constant term for the potential $U(\phi)$. The interpretation of this is not so clear. We therefore abort this procedure and turn to a more general way to determine the leading term.

*although stopping at ϕ^3 also maintains a local minimum around $\phi = 0$

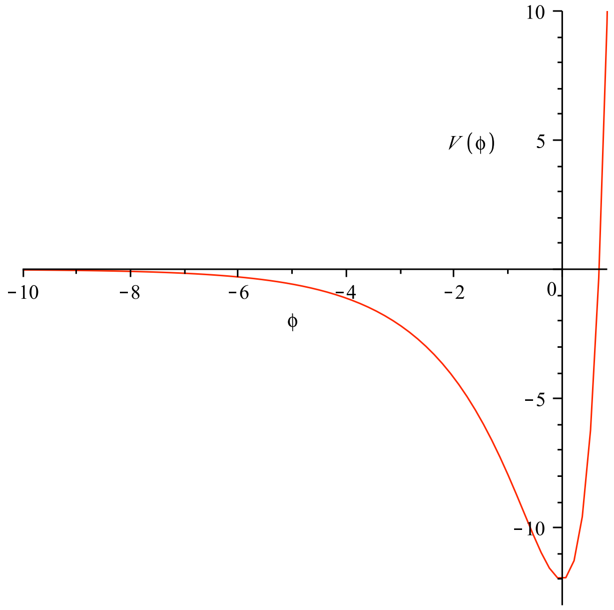


Fig. 6.1: Scalar potential for $5-d$ effective action

6.3 Domain wall solutions

The general way to determine the relation between the $V(\phi)$ and $U(\phi)$ is inspired by supergravity but does not require supersymmetry, and it is outlined in [42]. The motivation in [42] is phenomenological and they present a solution generating technique involving branes. In our case, we have no reason to include branes. In fact, it would make it more complicated to remain consistent with the required symmetries. The fact that breaking the dilatation symmetry results in a local action that consists only of the potential $U(\phi)$ is something that simplifies the analysis to some extent.

On the other hand, we must reconsider the form of the metric trying to respect as many of the symmetries of the group, but not dilatation invariance. This results in assuming a metric of the following form:

$$ds^2 = dr^2 - e^{2b(r)}dt^2 + e^{2a(r)}\left(2d\xi dt + d\vec{x}^2\right)$$

where $b(r)$ and $a(r)$ are some functions of r that have to be determined. We further assume a general action of gravity, scalars and a Maxwell term. We use the conventions of [42] for the metric (by adding the factor of two before the functions $a(r)$ and $b(r)$) and in the action, so that we can compare the results. So, the action looks like:

$$S = \int_M d^4x\, dr \sqrt{|\det g_{\mu\nu}|}\left[-\frac{1}{4}R + \frac{1}{2}(\partial\phi)^2 - V(\phi) - \frac{1}{4}F^2\right],$$

where M is the whole of spacetime. Rotational invariance allows for only two components of the gauge field to be non-zero. This makes it pointless to include a mass term for the gauge field for this class of metrics. It is also the reason our fields and metric only have an r dependence. We can also gauge away one component and we choose to retain the time component of the gauge field which will be a function of the radial coordinate, $f(r)$, only.

We can obtain the Ricci tensor:

$$R_{rr} = -4\left(a'' + a'^2\right), \qquad R_{ij} \overset{i,j\neq t}{=} -e^{2a(r)}\left(a'' + a'^2\right),$$
$$R_{tt} = e^{2b(r)}\left(2a' + b'' + 2b'^2\right),$$

where we use primes to denote d/dr, and R_{ij} includes the off diagonal entries $R_{t\xi}$. The equations of motion now read:

$$\begin{aligned} 4a'\phi' + \phi'' &= \frac{\partial V(\phi)}{\partial \phi} \\ a'' &= -\frac{2}{3}\phi'^2 \\ a'^2 &= -\frac{1}{3}V(\phi) + \frac{1}{6}\phi'^2, \end{aligned} \tag{6.2}$$

which are identical to the equations in [42]. In addition, we have one more equation for the tt component,

$$4a'' + 8a'^2 + 2V(\phi) + \phi' = b'' + 2b'^2 - \frac{f^2}{2\pi}$$

and Maxwell's equations yield

$$\frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g}F^{ab}\right) = e^{b(r)-2a(r)}\left(f\left(2a' + b'\right) + f'\right).$$

So, we are left with a set of non-linear equations. The purpose of this method is to reduce the system (6.2) to three decoupled first order ordinary differential equations two of which are separable. This method becomes non-trivial when more than just one scalar are considered. One of the differential equations has ϕ as the independent variable. For several scalars it would become a difficult partial differential equation.

The idea is to assume that the potential $V(\phi)$ has the special form [42]:

$$V(\phi) = \frac{1}{8}\left(\frac{\partial U(\phi)}{\partial \phi}\right)^2 - \frac{1}{3}U(\phi)^2$$

and verify that a solution to

$$\phi' = \frac{1}{2}\frac{\partial U}{\partial \phi}, \qquad a' = -\frac{1}{3}U(\phi)$$

is also a solution to (6.2), which is true. For our purpose we need to generalize the above approach to d dimensions and then proceed to solve the remaining equations so as to define $b(r)$ and $f(r)$. This way we can relate the characteristics of the desired class of metrics to the potential U and finally deduce the leading term in the local action that is related with power divergences.

7 | Discussion

The extension of gauge/gravity duality in order to describe non-relativistic systems is a promising endeavour. However, a lot remains to be done. There are several subtleties involved with various aspects of non-relativistic holography that remain unclear. Perhaps the most intriguing aspect of this new correspondence is the one extra dimension of the bulk description (in addition to the usual radial direction). The null ξ direction which seems to be associated to conserved rest mass results in a variety of interesting features of the theory.

In the present thesis we discussed the holographic renormalization procedure, based on the Hamilton-Jacobi theory, starting from the basic idea of ordinary renormalization. The connection of the radial direction and the energy scale of the boundary theory in "traditional" AdS/CFT correspondence is a key element in approaching the problem of renormalizing the theory. Using tools from general relativity applied to AdS/CFT, we presented a method of holographic renormalization.

We discussed some aspects of the recently conjectured non-relativistic holographic duality and then proceeded to bring the machinery of holographic renormalization to the non-relativistic framework. The anisotropic scaling of time and spatial directions makes the synthesis of the non-relativistic holographic renormalization procedure fascinating in its own right. The suspicion was that it would be possible to regain some structure resembling the discrete light cone quantization of a field theory in the appropriate limit.

There are several issues that need to be resolved. The main problem in the analysis is that the null direction since it renders the existing approach insufficient in maintaining information of the t components of various elements. Although, several deformations of the metric were attempted, no promising results emerged. Finally, a generalized description was adopted in order to determine the leading term in the renormalization procedure. This approach remains incomplete, but it appears adequate to provide some structure with the desired properties. However, it remains an open problem for further research.

Acknowledgements

I would like to express my gratitude to my advisor, Jan de Boer for his help and guidance throughout this last year. I would also like to thank Kyriakos Papadodimas for his support and endless discussions on physics and beyond.

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Appendices

A Dimensional Regularization

Here we outline the idea of dimensional regularization. In a nutshell, what we do is to compute a Feynman diagram as an analytic function of arbitrary dimensionality, d. For sufficiently small d, any loop-momentum integral will converge. The final expression for any observable quantity should have a well-defined limit as $d \to 4$.

So for a loop momentum integral, we generalize its expression for d dimensions

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2+m^2)^2} = \int \frac{d\Omega_d}{(2\pi)^d} \cdot \int dp \frac{p^{d-1}}{(p^2-m^2)^2} \,.$$

The area of a d-dimensional unit sphere is

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \,,$$

where Γ is the gamma function.

We push this idea further to consider non-integer dimensionality, and write the dimensions as $d = 4 - \epsilon$. So, we obtain

$$\lim_{\epsilon \to 0^+} \int \frac{dp}{(2\pi)^{4-\epsilon}} \frac{2\pi^{(4-\epsilon)/2}}{\Gamma\left(\frac{4-\epsilon}{2}\right)} \frac{p^{3-\epsilon}}{(p^2+m^2)^2} \,.$$

B | Anti-de Sitter space

The maximally symmetric d-dimensional spacetime with negative curvature ($\kappa = \frac{R}{d(d-1)} < 0$) is known as anti-de Sitter (AdS) space. It is the vacuum solution of Einstein's field equation with an attractive (negative) cosmological constant Λ:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}R &= \frac{1}{2}\Lambda g_{\mu\nu} \\ R &= \frac{d}{2-d}\Lambda \\ R_{\mu\nu} &= \frac{\Lambda}{2-d}\, g_{\mu\nu} \,. \end{aligned}$$

So, in these spacetimes the Ricci tensor is proportional to the metric tensor (Einstein spacetimes). Maximal symmetry in addition, suggests that

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)}(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma})\,.$$

Of course, in different coordinate systems the intrinsic properties must remain the same. Calculations however, may be substantially simplified with the right choice of coordinates. Moreover, certain characteristics can become more transparent.

In order to describe AdS space in global coordinates we must follow a series of transformations of the coordinates of (3.2). These are:

$$x_0 = R\cosh\rho\cos\tau\,, \quad x_{d+1} = R\cosh\rho\sin\tau\,, \quad x_i = R\sinh\rho\Omega_i$$

where $i = 1, \ldots, d$, $\rho \geq 0$, $0 \leq \tau \leq 2\pi$, Ω_i are the coordinates on S^{d-1} and satisfy $\sum_{i=1}^{d} \Omega_i^2 = 1$ and R is the AdS radius. Then

$$ds^2 = R^2\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_i^2\right)$$

where $d\Omega_i^2$ is the metric of the S^{d-1}. Near $\rho \simeq 0$ the metric becomes

$$ds^2_{\rho\to 0} = R^2\left(-d\tau^2 + d\rho^2 + d\Omega_i^2\right).$$

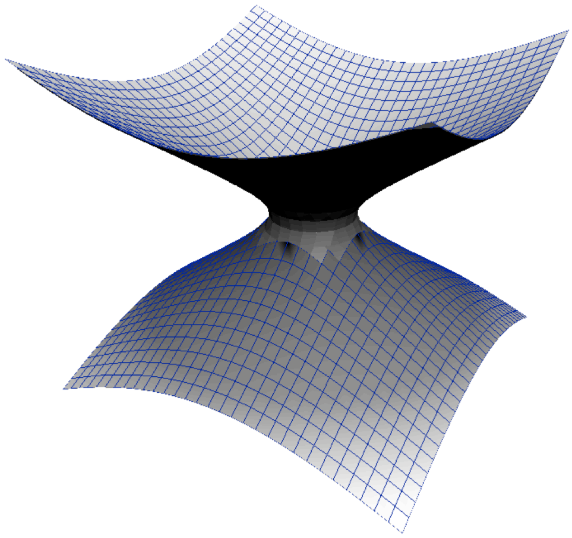

Fig. B.1: AdS_{1+1} embedded in three dimensions

Another convenient choice of coordinates is the so-called Poincaré coordinates. This coordinate system can be introduced by first defining the light cone coordinates:

$$
\begin{aligned}
u &\equiv \frac{(x_0 - x_d)}{R^2}\,, \\
v &\equiv \frac{(x_0 + x_d)}{R^2}\,.
\end{aligned}
$$

Redefining the other coordinates as

$$
\begin{aligned}
X^i &\equiv \frac{x_i}{Ru} \qquad \text{(spacelike)} \\
t &\equiv \frac{x_{d+1}}{Ru} \qquad \text{(timelike)}
\end{aligned}
$$

and so (3.1) takes the form

$$R^4 uv + R^2 u^2 (t^2 - \vec{x}^2) = R^2$$

with $\vec{x} = \sum_{i=1}^{d} x_i^2$. From this we find

$$
\begin{aligned}
x_0 &= \frac{1}{2u}(1 + u^2(R^2 + \vec{x}^2 - t^2) \\
x_d &= \frac{1}{2u}(1 + u^2(-R^2 + \vec{x}^2 - t^2) \\
x_i &= RuX^i \qquad i = 1, \ldots, d-1 \\
x_{d+1} &= Rut\,.
\end{aligned}
$$

It is now useful to change to the coordinate $z \equiv \frac{1}{u}$. In this way the Poincaré coordinates z, $\vec{x}$, t are defined as

$$\begin{aligned} x_0 &= \frac{1}{2z}(z^2 + R^2 + \vec{x}^2 - t^2) \\ x_d &= \frac{1}{2z}(z^2 - R^2 + \vec{x}^2 - t^2) \\ x_i &= \frac{RX^i}{z} \qquad i = 1, \ldots, d-1 \\ x_{d+1} &= \frac{Rt}{z} \, . \end{aligned}$$

In terms of these coordinates the AdS metric takes the form

$$ds^2 = \frac{R^2}{z^2}(dz^2 + d\vec{x}^2 - dt^2) \, .$$

The coordinate z behaves as a radial coordinate and divides the AdS space in two regions. The first chart is the region $z > 0$ and corresponds to the one half of the hyperboloid. In global coordinates this region can be obtained by imposing the condition $\sinh \rho \Omega_i < 1$. The other half of the hyperboloid $x_0 < x_d$ corresponds to $z < 0$, or in global coordinates $\sinh \rho \Omega_i > 1$. As we see in Fig. B.2 the hyperplane $x_0 = x_d$ cuts the entire AdS space.

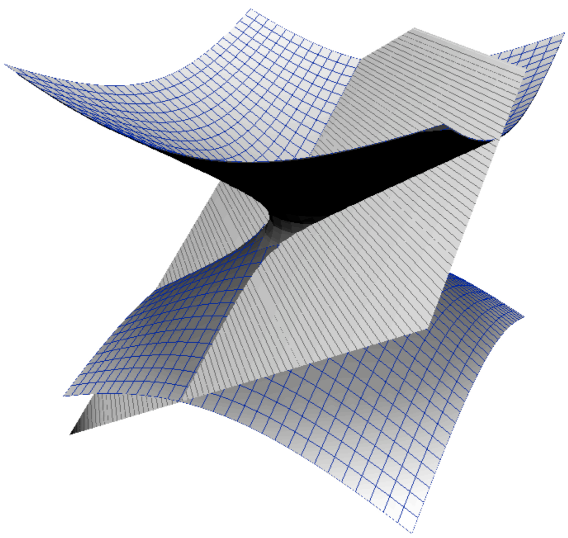

Fig. B.2: AdS hyperboloid intersected by the hyperplane $x_0 = x_{d-1}$.

C | Null Melvin Twist

Here we present the Null Melvin Twist and some mathematical notions relevant to Chapter 5.4

C.1 Complex Projective Space

Complex projective space, $\mathbb{CP}^n$ sometimes denoted $\mathbb{P}^n$, is a compact manifold with n complex dimensions. It can be constructed by taking $\mathbb{C}^{n+1}/\{0\}$, that is a set of $(z^1, z^2, \ldots, z^{n+1}) \neq (0, 0. \ldots, 0)$ and making the identifications

$$\left(z^1, z^2, \ldots, z^{n+1}\right) \sim \left(\lambda z^1, \lambda z^2, \ldots, \lambda z^{n+1}\right),$$

for any nonzero complex λ. These are homogeneous coordinates in the traditional sense of projective geometry. Thus, lines in $\mathbb{C}^{n+1}$ correspond to points in $\mathbb{CP}^n$. One may also regard $\mathbb{CP}^n$ as a quotient of the unit $2n+1$ sphere in $\mathbb{C}^{n+1}$ under the action of $\mathsf{U}(1)$:

$\mathbb{CP}^n = S^{2n+1}/\mathsf{U}(1)\,.$

This is because every line in $\mathbb{C}^{n+1}$ intersects the unit sphere in a circle. By first projecting to the unit sphere and then identifying under the natural action of $\mathsf{U}(1)$ one obtains $\mathbb{C}^{n+1}$. For $n = 1$ this construction yields the classical Hopf bundle.

C.2 Hopf Fibration

As mentioned in Chapter 5.4 convenient choice is to realize S^5 as a Hopf fibration over $\mathbb{P}^2$.

The round metric on $\mathbb{P}^n$ and S^{2n+1} can be expressed in terms of invariant 1-forms of $\mathsf{SU}(N)$. In particular, for $\mathsf{SU}(3)$ we have

$$\begin{aligned}
\sigma_1 &= \frac{1}{2}\left(d\theta\cos\psi + d\phi\sin\theta\sin\psi\right) \\
\sigma_2 &= \frac{1}{2}\left(d\theta\sin\psi - d\phi\cos\psi\sin\theta\right) \\
\sigma_3 &= \frac{1}{2}\left(d\psi + d\phi\cos\theta\right).
\end{aligned}$$

In terms of the 1-forms, the metrics on $\mathbb{P}^2$ and S^5 can be written as

$$\begin{aligned}
ds^2_{\mathbb{P}^2} &= d\mu^2 + \sin^2\mu\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\cos^2\mu\right) \\
ds^2_{S^5} &= ds^2_{\mathbb{P}^2} + \left(d\chi + \sigma_3\sin^2\mu\right)
\end{aligned}$$

where χ is the local coordinate on the Hopf fiber and $\mathcal{A} = \sin^2\mu\sigma_3 = x^{1/2}\sin^2\mu\,(d\psi + d\phi\cos\theta)$ is the 1-form potential for the Kähler form on $\mathbb{P}^2$ ($d\chi + \mathcal{A}$ is the vertical 1-form along the Hopf fibration).

C.3 Buscher Rules and Conventions

$$
\begin{aligned}
g'_{yy} &= \frac{1}{g_{yy}}\,, \quad g'_{ay} = \tfrac{B_{ay}}{g_{yy}}\,, \quad g'_{ab} = g_{ab} - \frac{g_{ay}g_{yb} + B_{ay}B_{yb}}{g_{yy}} \\
\Phi' = \Phi - \frac{1}{2}\ln g_{yy}\,, &\quad B'_{ay} = \tfrac{g_{ay}}{g_{yy}}\,, \quad B'_{ab} = B_{ab} - \frac{g_{ay}B_{yb} + B_{ay}g_{yb}}{g_{yy}}\,.
\end{aligned}
$$

C.4 Algorithm Steps

We directly apply the Null Melvin Twist on the case of interest following [29], we start off by using the black D3-brane solution

$$ds^2 = h^{-1}\left(-d\tau^2 f + dy^2 + d\vec{x}^2\right) + h\left[\frac{d\rho^2}{f} + \rho^2\left(ds^2_{\mathbb{P}^2} + (d\phi + \mathcal{A})^2\right)\right],$$

where $h^2 = 1 + R^4/\rho^4$ is the D3 harmonic function and $f = 1 + g = 1 - \rho_H^4/\rho^4$ is the emblackening factor. Since in the following steps the terms $d\vec{x}^2$, $d\rho^2$ and $ds^2_{\mathbb{P}^2}$ will remain intact so we will not carry them around, but they will be reintroduced when the calculation is complete. Thus, the metric we will work on looks like

$$ds^2 = h^{-1}\left(-d\tau^2 f + dy^2\right) + h\rho^2\,(d\phi + \mathcal{A})^2\ .$$

- We now boost by γ, so that $\tau \to c\tau - sy$ with $c = \cosh\gamma$ and $c^2 - s^2 = 1$,

$$ds^2 = h^{-1}\left[-d\tau^2\left(1 + gc^2\right) + dy^2\left(1 - gs^2\right) + 2d\tau dy\, gcs\right] + h\rho^2\,(d\chi + \mathcal{A})^2\,.$$

- Then, we T-dualize along the dy isometry using the Buscher rules

$$
\begin{aligned}
ds^2 &= -d\tau^2\frac{f}{h\,(1 - gs^2)} + h\left[\rho^2\,(d\chi + \mathcal{A})^2 + dy^2\frac{1}{1 - gs^2}\right] \\
B &= 2dy \wedge d\tau\left(\frac{-gcs}{1 - gs^2}\right) \\
\Phi &= \Phi_0 - \frac{1}{2}\ln\left(\frac{1 - gs^2}{h}\right).
\end{aligned}
$$

- Shift $d\chi \to d\chi + \alpha dy$, where α has dimension $[\mathrm{L}]^{-1}$,

$$ds^2 = -d\tau^2 \frac{f}{h\left(1-gs^2\right)} + h\left[\rho^2\left(d\chi+\mathcal{A}\right)^2 + dy^2\frac{1+\rho^2\alpha^2\left(1-gs^2\right)^2}{1-gs^2} \right.$$
$$\left. + 2dy\,\alpha\rho^2\left(d\chi+\mathcal{A}\right)^2\right].$$

- T-dualize back along dy

$$\begin{aligned}
ds^2 &= \frac{-d\tau^2}{h\left(1-gs^2\right)}\left[f - \frac{g^2c^2s^2}{1+\rho^2\alpha^2\left(1-gs^2\right)}\right] \\
&+2\frac{dyd\tau}{h}\left[\frac{gcs}{1+\rho^2\alpha^2\left(1-gs^2\right)}\right] + \frac{dy^2}{h}\left[\frac{1-gs^2}{1+\rho^2\alpha^2\left(1-gs^2\right)}\right] \\
&+h\rho^2\left(d\chi+\mathcal{A}\right)^2\left[1+\rho^2\alpha^2\left(1-gs^2\right)\right]^{-1} \\
B &= \frac{\alpha\rho^2}{1+\rho^2\alpha^2\left(1-gs^2\right)}\left(d\chi+\mathcal{A}\right)\wedge\left[gcs\,d\tau + \left(1-gs^2\right)dy\right] \\
\Phi &= \Phi_0 - \frac{1}{2}\ln\left(\frac{1-gs^2}{h}\right).
\end{aligned}$$

- Boost back by $-\gamma$ and take the limit $\alpha \to 0$ with $\alpha c = \beta$ held fixed ($\beta \sim [\mathrm{L}]^{-1}$). We also reintroduce the terms that were suppressed in the following steps since they remained unchanged

$$\begin{aligned}
ds^2 &= \frac{1}{hK}\left[-d\tau^2\left(1+\beta^2\rho^2\right)f + dy^2\left(1-\beta^2\rho^2 f\right) + 2d\tau dy\,\beta^2\rho^2 f\right] \\
&+h^{-1}d\vec{x} + h\left[\frac{d\rho^2}{f} + \rho^2 ds^2_{\mathbb{P}^2} + \frac{\rho^2}{K}\left(d\chi+\mathcal{A}\right)^2\right] \\
B &= \frac{2\beta\rho^2}{K}\left(d\chi+\mathcal{A}\right)\wedge\left(f d\tau + dy\right) \\
\Phi &= \Phi_0 - \frac{1}{2}\ln K\,.
\end{aligned}$$

- The final step is to take the near-horizon limit in order to compare with the solutions of [26, 27]. So, $h \to R^2/\rho^2$. We switch variables to the global radial coordinate

$$\frac{r}{R} = \frac{R}{\rho}\,,$$

where the boundary now lies at $r = 0$ and the horizon at $r_H = R^2/R_H$. Using the parameter $\Delta = \beta R^2$ we have

$$\beta^2\rho^2 = \frac{\Delta^2}{r^2}\,,\qquad h = \frac{r^2}{R^2}\,,\qquad f = 1 - \frac{r^4}{r_H^4}\,,\qquad K = 1 + \frac{\Delta^2 r^2}{r_H^4}$$

and the metric becomes

$$
\begin{aligned}
ds^2 &= \frac{R^2}{r^2 K}\left[-d\tau^2\left(1+\frac{\Delta^2}{r^2}\right)f + dy^2\left(1-\frac{\Delta^2}{r^2}f\right) + 2d\tau dy\,\frac{\Delta^2}{r^2}f\right] \\
&\quad + K d\vec{x} + K\frac{dr^2}{f} + r^2\left[K ds^2_{\mathbb{P}^2} + (d\chi + \mathcal{A})^2\right] \\
B &= 2\Delta\frac{R^2}{r^2 K}\,(d\chi + \mathcal{A}) \wedge (f d\tau + dy) \\
\Phi &= \Phi_0 - \frac{1}{2}\ln K\,.
\end{aligned}
$$

K varies smoothly between the boundary and the horizon between 1 and $1 + \Delta^2/r_H^2$. An important fact is that as $f \to 0$ and $B_t \to 0$, the surface $r = r_H$ remains a non-singular null horizon. Null geodesics which span the horizon have a perpendicular timelike Killing vector ∂_τ near the horizon. This means that we have a non-rotating black hole. This is interesting because the geometry is not static but only stationary* and therefore we might have expected a Killing horizon outside a black hole.

*We call stationary, any metric that possesses a Killing vector that is timelike near infinity. A metric is called static if it possesses a timelike Killing vector that is orthogonal to a family of hypersurfaces. Physically, by stationary we mean that something is "doing the same thing at every time" while static we mean "doing nothing at all".

D | Kaluza-Klein Compactifications

The Kaluza-Klein theory was first formulated as an attempt to unify gravity and electromagnetism. In 1921 Kaluza[43] proposed that gravitation and electromagnetism could be unified in a theory of five-dimensional Riemannian geometry. Later, in 1926, Oscar Klein [44] suggested that the fourth spatial dimension is curled up in a circle of small radius and took the original idea further. In this settting, a particle moving a short distance along that dimension would return to its initial position. When we have a spacetime with such compact dimensions we talk about compactification. In the 1970's this approach was revived by Scherk and Schwarz and by Cremmer and Scherk, and as extra true dimensions became necessary for a variety of theories, Kaluza-Klein compactification evolved into a commonly used tool for dimensional reduction. A classical review on the subject, from a modern point o view, and in context with supergravity is by Duff, Nillson and Pope[45].

Let us start describing the mechanism directly in D-dimensional spacetime $(D = d + 1)$. We consider the case with x^d being periodic, i.e.

$$x^d = x^d + 2\pi R\,,$$

and with the remaining dimensions, x^μ for $\mu = 0, \ldots, d$ noncompact. This is known as *toroidal compactification.* The d-dimensional metric then separates into $g_{\mu\nu}$, $g_{\mu d}$ and g_{dd} which are effectively, the four-dimensional metric, a vector (the gauge field) and a scalar.

Then the metric ansatz* is

$$ds^2 = g^D_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + g_{dd} \left(dx^d + A_\mu dx^\mu \right)^2 ,$$

or in block form

$$g_{MN}(x^\mu, g_{dd}) = \begin{pmatrix} g_{\mu\nu} & A_\mu \\ A_\mu & g_{dd} \end{pmatrix} .$$

The fields $g_{\mu\nu}$, g_{dd} and A_μ can only depend on the noncompact coordinates and in d-dimensional actions indexes are raised and lowered with $g_{\mu\nu}$ only. The action then becomes

$$S_D = \int dx^D \sqrt{-g} R(g_{(D)})\,.$$

*where, in an obvious notation, capital Latin letters refer to D dimensions and Greek indexes run over noncompact dimensions $0, \ldots, d-1$.

Then, noticing that the Ricci scalar "splits" into a d dimensional part R_d and a part with $F_{\mu\nu}$, then by varying the action with respect to A_μ one obtains Maxwell's equations ($\mathrm{d}F = 0$ and $\mathrm{d}F = 0$) and by varying with respect to the d-dimensional metric recovers Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{\Lambda^2} T_{\mu\nu} \, ,$$

where $T_{\mu\nu}$ is the energy-momentum tensor and is equal to $T_{\mu\nu} = F^{\mu\nu} F^{\kappa\sigma} g_{\nu\sigma} - {}^1\!/\!{}_4 g^{\mu\kappa} |F|^2$.

The metric ansatz is the most general metric invariant under translations of x^d. This allows d-dimensional reparameterizations $x'^\mu(x^\nu)$ and

$$x'^d = x^d + \lambda(x^\mu) \, .$$

This leads to

$$A'_\mu = A_\mu - \partial_\mu \lambda \, .$$

So, gauge transformations arise as part of the higher dimensional coordinate invariance. This is the so-called *Kaluza-Klein* mechanism.

Let us consider a massless scalar ϕ in D dimensions with $g_{dd} = 1$ for simplicity. Then, the momentum in the periodic dimension is quantized, $p_d = {}^n\!/\!{}_R$. We can expand the x^d dependence of ϕ in a complete set

$$\phi(x^M) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) \exp\left(\frac{inx^d}{R}\right) .$$

The D-dimensional wave equation becomes

$$\partial_\mu \partial^\mu \phi_n(x^\mu) = \frac{n^2}{R^2} \phi_n(x^\mu) \, .$$

The modes of the D-dimensional field become an infinite tower of d dimensional fields labeled by n. The d-dimensional mass-squared is non-zero for all fields with non-vanishing p_d

$$-p^\mu p_\mu = \frac{n^2}{R^2} \, .$$

Let us now discuss what we have seen so far in more modern terms. To make things concrete, we now restrict ourselves in five dimensions. We effectively start with the five-dimensional Einstein-Hilbert action $S = \int d^5x \sqrt{g} R$ and instead of assuming that the ground state of this system is five-dimensional Minkowski space (M^5), we take the ground state to be the product of four-dimensional Minkowski M^4 space and the unit circle S^1, $M^4 \times S^1$. We choose to do this, although it is difficult to classically to decide which of the two spaces is appropriate. One assumes that the radius of the circle is microscopically small (e.g. the order of Planck scale) and this accounts for the fact the existence of the extra dimension is not observed.

In this picture the physical spectrum is determined by studying small oscillations around this ground state.

When we examine the symmetries of the ground state $M^4 \times S^1$, we find that we have four-dimensional Poincaré symmetry acting on M^4 and a $\mathsf{U}(1)$ symmetry on S^1. These symmetries appear as gauge symmetries in the four-dimensional space. The massless modes that emerge turn out to be a spin-two graviton and a spin-one photon.

So, in principle, if we choose an appropriate higher-dimensional manifold X we can exploit its symmetries to construct an effective four-dimensional world with the desired gauge symmetries. So, the generalization of the ansatz for a higher-dimensional compact space would correspond to an ansatz of the form

$$g_{MN}(x^\alpha, \phi^k) = \begin{pmatrix} g_{\mu\nu}(x^\alpha) & \sum_a A^a_\mu(x^\alpha) K^a_i(\phi^k) \\ \sum_a A^a_\mu(x^\alpha) K^a_i(\phi^k) & \gamma_{ij}(\phi^k) \end{pmatrix}.$$

This is for an n-dimensional compact manifold X where ϕ_i, $i = 1, \ldots, n$ are the coordinates X. We have assumed generators of the symmetry group of X, T^a, $a = 1, \ldots, N$. The symmetry generator on the ϕ_i acts as $\phi_i \to \phi_i + K^a_i(\phi)$, where $K^a_i(\phi)$ is a Killing vector associated with the symmetry T^a. By Killing vector we mean that the ground-state metric on X has a vanishing Lie derivative with respect to K^a_i, $\pounds_K \gamma_{ij} = 0$. So, $A^a_\mu(x^\alpha)$ are the massless gauge fields of the symmetry group of X and γ_{ij} is the metric tensor of the space X. In this way, it is possible to obtain arbitrary Abelian or non-Abelian gauge group as components of a gravitational field in $4 + n$ dimensions.

Although a realistic unified theory does not arise by this considerations alone (nor by any other known to date) there are some interesting conclusion that can be drawn, just by symmetry considerations on the compact manifold. The gauge group we would like to obtain is obviously $\mathsf{SU}(3) \times \mathsf{SU}(2) \times \mathsf{U}(1)$. So the symmetry group G of the compact space X must contain the Standard Model group at least as a subgroup. In [46] we find a clear exposition of the reasoning that follows.

For any symmetry group G the space of lowest dimension is a homogeneous space G/H, where H is a maximal subgroup of G. In the case of $\mathsf{G} = \mathsf{SU}(3) \times \mathsf{SU}(2) \times \mathsf{U}(1)$ the subgroup with the largest possible dimension that is suitable is $\mathsf{SU}(2) \times \mathsf{U}(1) \times \mathsf{U}(1)$. Any larger subgroup would no longer be a symmetry of the group G/H. The dimensionality that emerges for $\mathsf{SU}(3) \times \mathsf{SU}(2) \times \mathsf{U}(1)/\mathsf{SU}(2) \times \mathsf{U}(1) \times \mathsf{U}(1)$ is therefore $(8+3+1) - (3+1+1) = 7^\dagger$. This is a remarkable result, since it suggests that the dimensionality of the space $M^4 \times X$ is necessarily at least eleven. This agrees with the result in supergravity and string theory. If we were to consider higher than eleven dimensions we would have to include a massless particle of spin higher than two. But there are

†Since the dimensionality of G/H is determined by the dimension of G minus the dimension of H.

several reasons why higher than two spin massless particles coupled to gravity do not exist. So, eleven dimensions are not only just enough to include the Standard Model, but the also upper threshold if we take into consideration field theory reasoning.

As a more concrete example of this we can consider the following. We already saw that $\mathsf{U}(1)$ symmetry is already obtained if we consider the circle S^1. The lowest dimensional space with $\mathsf{SU}(2)$ symmetry is the ordinary two-dimensional sphere S^2. For the $\mathsf{SU}(3)$ symmetry the lowest dimensional space is the complex projective space $\mathbb{CP}^2$ (Appendix C) which has real dimension four. So, the space $\mathbb{CP}^2 \times S^2 \times S^1$ has the desired $\mathsf{SU}(3) \times \mathsf{SU}(2) \times \mathsf{U}(1)$ symmetry and has $4+2+1=7$ dimensions. Although it is shown in [46] that the proper group is possible to achieve, there are still aspects of the Standard Model that are unattainable by KK compactification. Also, there are several issues that emerge (e.g. the instability of the KK ground state [47] and the fact that it is impossible to obtain chiral fermions) if the Kaluza-Klein theory is approached as an isolated framework of obtaining a unified four-dimensional theory. As a tool however, it remains rather popular for compactifications in the context of string theory or supergravity.

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