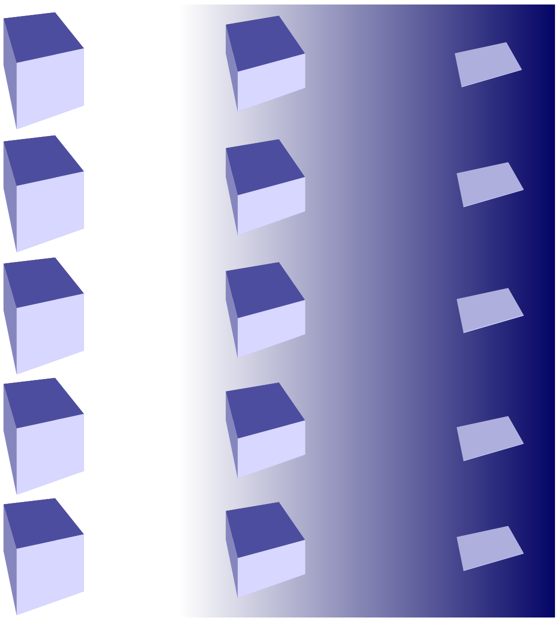
Noncritical String Theory



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Master's thesis

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Abstract

In strings theory, a critical dimension, D_c is required to yield consistent theories. For bosonic strings $D_c = 26$ and for superstrings $D_c = 10$. These numbers arise naturally from the theory itself. Less familiar are noncritical string theories, theories with $D \neq D_c$. These theories emerge when background fields are included to the theory, in particular linear dilaton backgrounds. We will study quintessence-driven cosmologies and show an analogy between them and string theories in a timelike linear dilaton theory. We will also present a set of exact solutions for the linear dilaton-tachyon profile system that gives rise to a bubble of nothing. Generalizing this setting induces a dimension-changing bubble, which can also be solved exactly at one-loop order. Eventually, we will consider transitions from one theory to another. In this way, noncritical string theories can be connected to the familiar web of critical string theories. Surprisingly, transitions from superstring theories can yield pure bosonic theories. Our main focus will be bosonic strings.

Preface

Foreword

When I started writing my thesis, I was immediately confronted with a tremendous abundancy of background material on string theory. Even though string theory is relatively new, already a lot of books and an enormous amount of articles have been written on the subject, and the level of difficulty varies greatly. Some books were very advanced, others were much more comprehensible but somewhat limited in detail. I found it was a challenging task to restrict my focus only on those subjects relevant to the scope of this thesis, because it is very easy to get lost in all the fascinating features and idea's that are indissoluble connected to string theory.

At first I had some doubt whether I should go into a lot of detail with calculations or not. But gradually, I found that I gained most satisfaction out of explaining most of the intermediate steps needed to complete a calculation, whenever I thought they contributed to the clarity of the subject. Whenever a calculation would become to detailed or advanced, I decided to give a reference to the book or article the problem can be found in.

I wrote this thesis on the level of graduate students, with preliminary knowledge on quantum field theory, general relativity and string theory. Only chapter 3 involves some knowledge on algebra and representations. It will not be used later on. With this in mind, I still tried to make the thesis as self-contained as possible, reviewing some general facts where needed.

Just as many people before me, I too found out that writing a thesis is by far the most challenging and difficult task of completing my studies as a physics student. Nevertheless, I've experienced this as a very educative and enjoyable time, and during this period I found out that my interest in doing theoretical research has grown considerably.

Acknowledgments

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complicated subjects completely comprehensible.

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Contents

Notation and conventions

M	A two-dimensional manifold, denoted as the surface of an arbitrary world-sheet.
∂M	The boundary of M.
D	Number of spacetime dimensions.
d	Number of spatial dimensions. $D = d + 1$.
$\eta_{\mu\nu}$	Minkowski metric, or flat metric. $\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$
c	Speed of light. $c = 1$.
$\hbar$	Planck's constant. $\hbar = 1$.
l	String length scale. $l = \sqrt{2\alpha'}$.
α'	Regge slope parameter.
T	String tension. $T = \frac{1}{2\pi\alpha'}$.
x^μ	Spacetime coordinates. $\mu = 0, 1, \ldots, D-1$.
x^i	Space coordinates. $i = 1, \ldots, D-1$.
$X^\mu(\tau, \sigma)$	Spacetime embedding functions of a string.
$P^\mu(\tau, \sigma)$	Momentum conjugate to $X^\mu(\tau, \sigma)$. $P^\mu(\tau, \sigma) = T\partial_\tau X^\mu(\tau, \sigma)$.
$x^\pm$	Spacetime lightcone coordinates. $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$
x^I	Transverse coordinates. $I = 2, \ldots, D-1$.

(τ, σ)	World-sheet coordinates of a string. $(\tau, \sigma) = (\sigma^0, \sigma^1)$.
σ^2	Euclidean world-sheet time. $\sigma^0 = i\sigma^2$.
$\sigma^\pm$	World-sheet lightcone coordinates. $\sigma^\pm = \sigma^0 \pm \sigma^1$. Another convention is $\rho^\pm = -\sigma^\mp$.

Introduction

Introduction and motivation

For many decades already, string theory has been proposed to be the 'theory of everything'. Various problems and ideas lead physicists to believe that picturing elementary particles as one-dimensional objects, called strings, could very well account for (a lot of) these problems. At a very early stage, however, it became clear that string theory, described in four dimensions, lead to inconsistent theories. In order to solve this problem, it was necessary to describe string theory in an arbitrary number of dimensions, D, and then determine this number by hand. For the simplest case, bosonic string theory, various calculations suggested that this number should be $D = 26$. Since our world clearly does not consist of pure bosonic particles, there was also need for a fermionic version. Superstring theory turned out to be this theory. For superstring theory, it was found that the number of dimensions that would give consistent theories should be $D = 10$. Since only these numbers give consistent theories, and they arise so naturally from the theory itself, they are refered to as the critical dimension. Later on even, it was understood that different superstring theories are all limits of an eleven-dimensional theory, namely the theory of supergravity.

One could wonder if a theory that requires that many more dimensions than we observe in nature is still a realistic one. But on the other hand, if nature is constructed in such a way that only four dimensions are noticeable at large scales, this should not really have to be a problem. A way to 'effectively remove' these extra dimensions is by means of compatification. Compactifying dimensions means that these dimensions are no longer infinite in extend, but finite, and they are (highly) curled up. Periodicity is also allowed, so that one could picture compacitified dimensions as very small circles. In this way, compactified dimensions are no longer observable at lenght scales, large compared to the radius of these circles. Therefore, in bosonic string theories, the extra dimensions can be viewed as a 22-dimensional compactified sphere or torus, an for superstring this is a six- (or seven) dimensional sphere or torus. So with this approach, the problem of extra dimensions had therefore been solved.

A way to determine the critical dimension is by calculating the β function, that comes from rescaling the world-sheet. One important feature of string theory is that the world-sheet is scale invariant, and therefore the β function should vanish. For bosonic strings in flat spacetime this then simply comes down to saying $D = 26$, and for superstring $D = 10$. However, more general theories arise if one includes background fields to the theory, such as a curved spacetime, a Kalb-Ramond field, or a scalar dilaton field. Eventhough (apart from curved spacetime) these background fields have not yet been observed in nature, they play an important rôle in string theory, and it is very useful to consider them. One important feature of including background fields is that the condition for vanishing β functions also changes. Depending on the actual form of the background fields, vanishing β functions are now able to render consistent string theories outside of the critical dimension. These theories are called noncritical string theories.

The simplest example of a noncritical string theory is the linear dilaton background, a theory where spacetime is flat, and the dilaton has linear dependence on the spacetime coordinates. The influence of this background becomes apparent when a tachyon profile is also taken into account. Interactions with tachyons can be described by a low energy effective action. The equation of motion for this action is called the on-shell tachyon condition, and its solution is called the tachyon profile. When the on-shell tachyon condition is solved for a theory with linear dilaton background, the solution becomes an exponent of the spatial coordinates. The linear dilaton and tachyon profile can then be included to the world-sheet action, which is then called a Liouville theory. The tachyon, which couples to the world-sheet as a potential, starts to act as a barrier which becomes inpenetrable for strings. When a tachyon profile grows (obtains a vacuum expectation value), this is usually called a tachyon condensate. Strings that come in contact with such a barrier are reflected off, or they are pushed outwards if the barrier is dynamical. So, it is clear that including a linear dilaton background to the theory can have tremendous effects on the strings living in this background.

Even though the structure of such theories a quite simple, it is still extremely difficult to find solutions for these theories, due to the exponential dependence of the tachyon profile. The only hope for obtaining correct result would be to find exact solutions for these theories. In this thesis, we will present exact solutions for some of these theories. We will see that when a tachyon condenses along the null direction X^+, all quantum corrections to this theory vanish. So, in fact, the solutions at the classical level are the exact solutions for this theory. For these solutions, we will see that the tachyon barrier can be seen as 'a bubble of nothing', absence of spacetime itself. Strings that come in contact with this barrier are expelled from this region.

As a first application, we will study the analogy between string theories with a (timelike) linear dilaton background and quintessence-driven cosmologies. It turns out that the action of cosmologies with quintessence has exactly the same form as the low energy effective action for massless closed strings. Comparing the two theories, we find that the tree-level potential of this string theory gives rise to an equation of state at the border between accelerating and decelerating cosmologies. Time-dependent backgrounds in string theory have always been hard to solve. By comparing quintessent cosmologies and string theories, we will be able to find solutions for strings in time-dependent backgrounds.

Aside from tachyon condensation in the null direction, we will also consider a theory where the tachyon has oscillatory dependence on more coordinates. In this setting, the theories turn out to be exact at one loop order, still simple enough to be calulated. When we impose dependence on more coordinates $X_2, \ldots, X_n$, we will see that only strings that do not oscillate in these directions are able to penetrate the (tachyon) bubble interior. All other strings will be pushed outwards and get frozen into their excited states. This effectively means that strings inside the bubble interior start out in a D-dimensional theory, but end up in a $(D - n)$-dimensional theory. These processes are called dynamical dimension changing solutions, and these theories can be described for bosonic strings, as well as superstrings. It is even possible for strings to start out in one theory, but end up in another theory. In this case we call these processes transitions. Even though noncritical string theories are consistent internally, it has always been difficult to link them to the familiar web of theories in the critical dimension. We will see that we are able to link them in the setting that we will use in this thesis. A surprising result is that there are even transitions possible where superstring theories turn into pure bosonic string theories, a relation that has not been achieved before.

Outline

In chapter 1 we will first argue why there is need for string theory at all. Then we'll treat the basic principles of string theory. We start with the classical point particle and discuss its analogy with a classical (bosonic) string moving in spacetime. There can be open and closed strings. The classical equations of motion of the string will be derived. Thereafter we will quantize the string and analyse its spectrum.

In chapter 2 we will discuss scale transformations. In physics, the concept of symmetries has grown extremely important for constructing theories. Symmetries in string theory, in particular scaling symmetries, are important, because the whole theory is built under the assumption that a rescaling leaves the theory invariant. In addition to this we will also discuss coupling parameters and β functions. We will show that in order to have consistent theories, strings require a so-called critical dimension to live in.

In chapter 3 we shall discuss conformal field theory. CFT is a very extended subject, and our goal is not to discuss all details. Rather we will derive some basic results, to give a global understanding of the field. Some main subjects will be conformal transformations, primary fields, operator product expansions and Virasoro operators.

In chapter 4 we look at vertex operators. We will discuss interacting strings and show that a rescaling of the world-sheet can actually deform the theory in such a way that it can be described in a completely different way, making use of operator-state correspondence. It is argued that string states can be represented by vertex operators, attached to the world-sheet. We discuss various vertex operators, such as the tachyon vertex operator and the massless vertex operator. Both can be studied for the open and closed string case. Scattering amplitudes can be calculated and we shall do so for some simple examples. Finally we will derive some results from varying some parameters in the world-sheet action, which will later be used extensively.

In chapter 5 we will discuss strings in the vicinity of backgrounds. So far we only looked at flat spacetime, but as we will see later on, including backgrounds to the theory can have tremendous effects on the theory. We will incorporate some aspects from general relativity into string theory and see that such an extension of the theory makes good sense. Thereafter, we will also include an antisymmetric tensor and a dilaton as backgrouds into string theory. Here we see a close analogy with some scale symmetries discussed in chapter 2.

In chapter 6 for the first time, string theory is considered from a completely different point of view. Instead of describing the physics from the world-sheet point of view, an effective spacetime action is introduced. This spacetime action describes the effective low energy physics of the theory. Switching over to the effective action allows one to analyse different aspects of the theory.

In chapter 7 we will be looking at backgrounds, involving the dilaton. First we will discuss the constant dilaton, and show that this simple model actually provides us with a tool for constructing a UV finite theory of quantum gravity, a result which no other theory has yet provided. Subsequently, we will discuss the linear dilaton background. This theory is still simple and exactly solvable, and it turns out that the linear dilaton is even capable to alter the number of dimensions of spacetime the string lives in.

In chapter 8 we will study quintessence driven cosmologies, theories that resemble the behaviour of cosmologies with a cosmological constant. We will examine different solutions, which depend on the equations of state of these cosmologies. Finally we will discuss these solutions in terms of Penrose diagrams.

In chapter 9 we will compare string theory in the vicinity of a timelike linear dilaton background, with quintessence driven cosmologies. We show that the solutions of a quintessence driven cosmology are really the same as those of a timelike linear dilaton theory. We will analyse massless and massive modes in these theories and determine under which conditions these are stable against perturbations of background fields.

In chapter 10 will go into more detail and talk about a tachyon-dilaton model. We will give a world-sheet description of this theory and find a solution that is exactly solvable, even at the quantum level. When a tachyon profile is added to this theory, we see particular solutions give rise to a spacetime-destroying "bubble of nothing", bouncing off all material that it encounters. Finally, we will give a more general low energy effective action for this theory.

In chapter 11 will generalize the tachyon condensation along the null direction. We will consider a theory where the tachyon also has oscillatory dependence on more coordinates and see that this results in dimension-changing exact solutions. Quantum corrections terminate at one-loop order, so they are still easy enough to be solved. We will also consider transitions between different string theories.

Part I

Theoretical frame-work

Chapter 1

Basic principles on string theory

1.1 Why string theory?

For many centuries theoretical physicists have been trying to unify physical theories. Very often a theory gives a valid description up to a certain limit, but as soon as the limit is crossed, the theory breaks down. For example, the dynamics of moving particles is well understood and neatly described by *Newton's laws of physics* in the limit where velocities are small compared to the speed of light c. Switching to a frame which has a velocity $\overrightarrow{v}$ relative to the initial frame, simply comes down to adding or subtracting this velocity to velocities of particles described in the initial frame. But as we know, c (which in terms of SI units is $2.99792458 \times 10^8 ms^{-1}$) is constant, no matter what frame an observer is in. This immediately leads to an inconsistency in the theory, which we now know is solved by *Einstein's law of special relativity.*

Another example is the classical description of black body radiation. Both *Rayleigh-Jeans law* and *Wien's approximation* for black body radiation give an accurate description for only part of spectrum of the radiation that is emitted. This problem was attacked by *Max Planck*, who proposed that the radiation energy E is quantized, $E = h\nu$, where h is Planck's constant ($6,62606896 \times 10^{-34} Js$ in SI units) and ν is the frequency of the radiation. This proposal eventually led to the theory of quantum mechanics.

In the early 20^{th} century there were two major developments in theoretical physics. First of all, Einstein developed his theory of *general relativity.* This theory covers dynamics of particles with arbitrary velocities, in the presence (or absence) of gravity, and is mainly applied to big scales. It completely solved the inconsistencies that arose in Newton's theory of dynamics, and dramatically changed our view on the structure of space and time.

Secondly, a number of different ideas and results eventually led to the theory of *quantum mechanics*, drastically changing our view on physics on small scales. Later on, the theory of quantum mechanics for particles was extended to *quantum field theory* in order to cope with interactions of many particle systems and relativistic quantum mechanics. Quantum field theory eventually developed into the *Standard model for elementary particles* and is now globally accepted as the theory for all known particle interactions, supported by an overwhelming amount of evidence. Together, these two theories (general relativity and quantum mechanics) are able to describe all (known) physics.

A full understanding of the fundamental laws of physics, however, is not attained until the two theories are unified to one *grand unified theory* (GUT). A problem appears, however, when we try to embed these theories into each other. We might try to quantize the theory of gravity, for example. But if we try to write down a perturbative field theory for gravitation, we run into all sorts of uncontrollable infinities. Ultraviolet divergences that arise when one works in perturbation theory grow worse at each order, so apparently these two theories can not easily be merged into each other.

In the late 1960's a theory called *string theory* was developed to solve the problem for strong nuclear forces.[1] In this theory fundamental particles are suggested to be one-dimensional objects, instead of point particles. However, in this theory a lot of technical problems arose (such as unwanted tachyons, unwanted massless spin-two particles and unwanted extra dimensions). When finally *quantum chromo dynamics* (QCD) turned out to be the correct theory for strong nuclear forces, the need for string theory seemed to have gone down the drain.

String theory made a remarkable comeback in 1974, when it was proposed to identify the massless spin-two particle with the graviton, making it a theory for quantum gravity! Since then string theory has evolved tremendously. The theory is not yet fully understood, but physicists all around the world are working very hard, trying to complete the theory. It seems that string theory is one of the most promising candidates for unifying general relativity and quantum mechanics. Superstrings and higher dimensional objects, called *D-branes*, were discovered and ever since it's discovery, string theory has turned out to be an extraordinary fascinating theory. The theory can be used to describe the expanding universe, as well as elementary particle physics. Even though we are still a long way removed from understanding it's complete description, string theory has provided us with very promising and surprising results. The dream of unifying all physics into one fundamental theory therefore seems to be within our grasp.

[1] for more information on the development of string theory, see [2].

1.2 Relativistic point particle

1.2.1 Point particle action

From Einstein's theory of general relativity, we know that mass and energy bend spacetime itself. A relativistic point particle moving in spacetime follows a path, or *worldline* along a geodesic, where x^0 is taken as the time-direction, and $\Sigma = x^i, i = 1, 2, 3$ as the spatial directions (see figure 1.1).[2] A geodesic can be seen as a path through a curved space, such that the distance is minimal. In other words, it is the path of a free particle in a curved spacetime.

A very useful tool for describing physics is the *action principle.* An action S is the spacetime integral of the Lagrangian density $\mathcal{L}$, and can be varied in its arguments (which can be coordinates, conjugate momenta, fields or derivatives of the fields). When one demands the action to be invariant under such a variation, this leads to restrictions for these arguments, also know as the *equations of motion* (EOM).

Since an action extremizes the path length of a particle, it is a logical choice to set the length of a particles worldline equal to the action. A line element in a curved space, described by a metric $g_{\mu\nu}(X)$ is given by

$$\begin{aligned} ds^2 &= g_{\mu\nu}(X) dX^\mu(\tau) dX^\nu(\tau) && (1.1a)\\ &\equiv dX_\mu(\tau) dX^\mu(\tau), && (1.1b) \end{aligned}$$

where $\mu, \nu = 0, \ldots, 3$. Since for timelike trajectories $dX_\mu(\tau)dX^\mu(\tau)$ is always negative, we can introduce a minus-sign, to make sure that ds is real for time-like paths. When we parameterise the worldline of a particle with mass m by τ, the action can be written as

$$\begin{aligned} S_{pp} &= -m \int ds && (1.2a)\\ &= -m \int d\tau \left[-g_{\mu\nu}(X) \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} \right]^{1/2} && (1.2b)\\ &= -m \int d\tau \left[-\dot{X}_\mu \dot{X}^\mu \right]^{1/2}, && (1.2c) \end{aligned}$$

where the dot represents a derivative with respect to τ. The action is invariant under reparametrizations, so if we let $\tau \to f(\tau)$, the action does not change.

1.2.2 Auxiliary field

Even though we found the correct expression for the action of a relativistic point particle, we immediately see that it can not be used for massless

[2] Appendix B gives a detailed description on this geodesics and curved spaces.

particles. Furthermore, the square root in the action also complicates deriving its equations of motion. These problems can, however, be omitted by introducing an auxiliary field $e(\tau)$. Now consider the following action,

$$\tilde{S}_{pp} = \frac{1}{2}\int d\tau \left[\frac{\dot{X}_\mu \dot{X}^\mu}{e(\tau)} - e(\tau)m^2\right]. \tag{1.3}$$

It is not hard to show (see [2]) that this action is equivalent to (1.2). Imposing reparametrization invariance here as well, implies that $e(\tau)$ should transform accordingly. When we know how $e(\tau)$ transforms, we can reparameterize in such a way that we can set $e(\tau) = 1$. This then brings the action into the form

$$\tilde{S}_{pp} = \frac{1}{2}\int d\tau \left[\dot{X}^2 - m^2\right], \tag{1.4}$$

where $\dot{X}^2 = \dot{X}_\mu \dot{X}^\mu$.

For a free particle, the metric $g_{\mu\nu}(X)$ just becomes the Minkowski metric $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$. We can then easily derive the equations of motion for X^μ and $e(\tau)$, to find

$$\ddot{X}^\mu = 0, \tag{1.5a}$$

$$\dot{X}^2 + m^2 = 0. \tag{1.5b}$$

These are of course the correct equations for the point particle. (1.5a) is just the condition that the particles moves in straight lines. (1.5b) is just the mass-shell condition $p^2 = -m^2$, when we realize that $p^\mu = \dot{X}^\mu$ is the momentum conjugate of X^μ. So we see that with the action principle, we obtain the same physical constrainst as with Einstein's theory of (special) relativity.

1.3 Relativistic bosonic strings

1.3.1 Polyakov action

A relativistic string, moving through spacetime, can be described in a very similar fashion as the relativistic point particle. We will start out with the description of a bosonic string, the simplest example of a string. A string is a one-dimensional object, moving in D spacetime dimensions. So instead of a worldline, it now carves out a *world-sheet* in spacetime, parameterized by *two* parameters (τ, σ), or equivalently (σ^0, σ^1). τ can be thought of as the time-direction along the world-sheet, and σ as the spatial direction.

Strings can be open or closed. The has the consequence that the world-sheet can have two different topologies, namely a cylinder for the closed string, and a sheet with two boundaries at the endpoints of the string. We

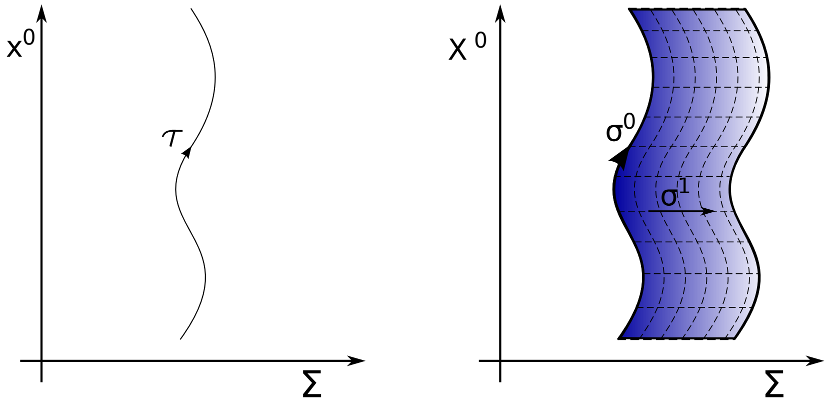

Figure 1.1: *Left: a point particle carves out a worldline in spacetime, parameterised by one parameter τ. Right: A string carves out a world-sheet, parameterised by two parameters σ^0 and σ^1. x^0 and X^0 are the time-directions and Σ stands for the spatial part of the spacetime diagrams.*

will use the convention that for open strings, σ lies in the interval $\sigma \in [0, \pi]$, and that for closed string, σ lies in $\sigma \in [0, 2\pi]$. Moreover, we will leave the number of spatial dimensions, $d = D - 1$, arbitrary for the string. In figure 1.1 we have drawn a spacetime diagram for the world-sheet of an open string, moving in D spacetime dimensions. X^0 is again the time-direction and $\Sigma = X^i, i = 1, \ldots, D-1$ are the spatial directions.

In analogy with the relativistic point particle, we also want to make use of the action principle in string theory. Instead of describing a minimal path length, we now describe a minimal *area* in this spacetime. We start out with describing a bosonic string in *flat spacetime*, since this is the easiest example. In flat spacetime, we will be using the Minkowski metric $\eta_{\mu\nu} = diag(-1, 1, \ldots, 1)$. It can be shown that the correct form of the world-sheet, known as the *Polyakov action*, is

$$S_P = -\frac{1}{4\pi\alpha'} \int_M d^2\sigma \sqrt{-h(\sigma)} h^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu, \tag{1.6}$$

where h_{ab} is the *world-sheet metric*, $h = \det(h_{ab})$, and the constant α' is called the *Regge slope parameter*. Here h_{ab} plays the same rôle as the auxiliary field in the point particle case.

It is important to realize that from the spacetime point of view, the coordinates $X^\mu(\sigma^0, \sigma^1)$, $\mu = 0, \ldots, D-1$ are just the spacetime coordinates of points on the world-sheet. However, from the world-sheet's point of view, $X^\mu(\sigma^0, \sigma^1)$ are *spacetime embedding coordinates*. These are D two-dimensional fields that live on the world-sheet.

1.3.2 World-sheet symmetries

The world-sheet action satisfies three important symmetries. These symmetries are needed in order to have consistent theories. They are,

- *Poincaré invariance.* These are symmetries under which we let the X^μ fields vary as

$$\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu, \qquad a_{\mu\nu} = -a_{\nu\mu}, \tag{1.7a}$$
$$\delta h^{ab} = 0. \tag{1.7b}$$

  The constants $a^\mu{}_\nu$ are infinitesimal Lorentz transformations, and the constants b^μ are translations in spacetime.

- *Diffeomorphism (diff) invariance*, or also called *reparametrization invariance.* These are symmetries under which we reparameterize the world-sheet, switching over to new coordinates $\tilde{\sigma}^a = \tilde{\sigma}^a(\sigma^0, \sigma^1)$. So we let

$$\sigma^a \to \tilde{\sigma}^a(\sigma^0, \sigma^1), \tag{1.8a}$$
$$\text{and} \qquad h_{ab}(\sigma) = \frac{\partial \tilde{\sigma}^c}{\partial \sigma^a}\frac{\partial \tilde{\sigma}^d}{\partial \sigma^b} h_{cd}(\tilde{\sigma}). \tag{1.8b}$$

  Intuitively this makes sense, because the physics (thus the action), should not depend on the way that we parameterize the world-sheet.

- *Weyl invariance*, or *rescaling invariance.* Weyl invariance specifically, is a symmetry of the action under which we locally rescale the world-sheet metric by an overall factor. So we let

$$\delta h_{ab} \to e^{2\omega(\sigma^0,\sigma^1)} h_{ab}, \tag{1.9a}$$
$$\delta X^\mu = 0. \tag{1.9b}$$

Rescaling transformations in general are known as *conformal transformations.* This is why Weyl invariance is also referred to as *conformal invariance.*

Only in two dimensions can an action of the form (1.6) be Weyl-invariant. This can be seen as follows. If we rescale the metric in (1.6), the determinant will rescale as

$$\left[\prod_{n=1}^{\delta} e^{2\omega(\sigma^0,\sigma^1)}\right] h = e^{4\omega(\sigma^0,\sigma^1)} h, \tag{1.10}$$

where δ is the number of dimensions, which for the world-sheet is $\delta = 2$. Taking the square root, this yields a rescaling factor of $e^{2\omega(\sigma^0,\sigma^1)}$. Furthermore, $h^{ab} = (h^{-1})_{ab}$ will rescale with a factor $e^{-2\omega(\sigma^0,\sigma^1)}$. So, in two dimensions these factors exactly cancel each other out, and therefore rescaling is a symmetry of the action. This extra symmetry is what makes string theory much more attractive to work with than a theory of higher dimensional objects (also known as *p-branes*).

These symmetries have nice consequenses. As we shall see shortly, they will even be able to determine the number of spacetime dimensions D strings can live in!

1.4 Solutions of the bosonic string

1.4.1 Choosing a world-sheet gauge

In the previous section we saw that the world-sheet action has three symmetries. Physically, this means that performing a symmetry transformation yields exactly the same theory. So it is just a different description of the same theory. These symmetries are also called *gauge symmetries*. By simply choosing one gauge, we say that we *gauge-fix* the theory. In order to solve the equations of motion for strings, we can gauge-fix the world-sheet action, just as we did with the point particle. By knowing how the world-sheet metric transforms, we can put it in a convenient form.

First of all, since the world-sheet metric is a symmetric tensor, we find $h_{01} = h_{10}$, so there are only three independent components. Next, we can use reparametrization invariance to choose two components of h, leaving us with only one independent component. And finally we use the rescaling invariance to completely fix the world-sheet metric. One of the most convenient choices is the gauge in which the world-sheet metric has Minkowski signature, so

$$h_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.11}$$

In some cases a Euclidean signature is more convenient. In most calculations it should be clear what signature is used.

1.4.2 Constraints for embedding coordinates

To solve the X^μ field equations is a rather lengthly and detailed derivation. We will not give a full derivation here, but merely give a short sketch of what steps can be taken to obtain the solutions. There are three sets of equations that constrain the X^μ fields. The first set are equations of motion, coming from varying the action with respect to the X^μ fields. The second set are equations of motion, coming from varying the action with respect to the world-sheet metric. And the third set are boundary conditions, which are imposed by Poincaré invariance. The second set equations make up the so-called *energy-stress tensor*, who's definition is

$$T^{ab}(\sigma) = \frac{4\pi}{\sqrt{-h(\sigma)}} \frac{\delta}{\delta h_{ab}(\sigma)} S. \tag{1.12}$$

From its definition, it follows that T^{ab} is a symmetric tensor. Conformal invariance implies $T^{ab} = 0$, so this puts extra constraints on the X^μ fields. Moreover, one can check that the energy-stress tensor T^{ab} is conserved, meaning that

$$\partial_a T^{ab} = \partial_a T^{ba} = 0. \tag{1.13}$$

Next, one can use Poincaré invariance of the action to determine boundary conditions for the strings. At this point we should make a distinction between closed and open string.

- For *closed strings*, boundary conditions imply embedding coordinates X^μ to be periodic in σ, with period 2π, so

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi). \tag{1.14}$$

- For *open strings*, we find that there are two possible boundary conditions, namely *Neumann boundary conditions* and *Dirichlet boundary conditions*.

  - *Neumann boundary conditions* tell us that

$$\partial_\sigma X_\mu \equiv X'_\mu = 0 \qquad \text{for} \qquad \sigma = \{0, \pi\}, \tag{1.15}$$

  so no momentum is flowing through the endpoints of the string.

  - *Dirichlet boundary conditions*, however, tell us that

$$\delta X^\mu = 0 \qquad \text{for} \qquad \sigma = \{0, \pi\}, \tag{1.16a}$$

$$\text{so} \qquad X^\mu\big|_{\sigma=0} = X_0^\mu \quad \text{and} \quad X^\mu\big|_{\sigma=\pi} = X_\pi^\mu, \tag{1.16b}$$

  where X_0^μ and X_π^μ are constants. What these boundary conditions tell us is that the endpoint of the string are fixed in some (say p) directions. The modern interpretation of this seemingly strange condition is that the constants X_0^μ and X_π^μ represent the positions of (higher dimensional) objects, called D*p*-*branes*. D*p*-branes are a fascinating feature of string theory, but we will not be needing them in the course of this thesis.

1.4.3 Solutions for embedding coordinates

To find the explicit forms of the solutions of the embedding coordinates $X^\mu(\tau, \sigma)$, one usually switches over to world-sheet *lightcone coordinates* $\sigma^\pm = \sigma^0 \pm \sigma^1$. After having switched over to lightcone coordinates and working out the equations of motion, we finally end up with the solutions for the X^μ fields that satify all of the constraints.

- Solutions for the closed string X^μ fields turn out to be a superposition of *right-moving fields* $X_R^\mu(\tau - \sigma)$, and *left-moving fields* $X_L^\mu(\tau + \sigma)$. We find

$$X^\mu_{closed}(\tau,\sigma) = X_R^\mu(\tau-\sigma) + X_L^\mu(\tau+\sigma), \quad \text{with} \tag{1.17a}$$

$$X_R^\mu(\tau,\sigma) = \frac{1}{2}x^\mu + \frac{1}{2}l^2 p^\mu(\tau-\sigma) + \frac{i}{2}l\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau-\sigma)}, \quad \text{and} \tag{1.17b}$$

$$X_L^\mu(\tau,\sigma) = \frac{1}{2}x^\mu + \frac{1}{2}l^2 p^\mu(\tau+\sigma) + \frac{i}{2}l\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)}, \tag{1.17c}$$

  where the constant x^μ is the string's center of mass, p^μ is the string's total momentum and l is the string length scale, related to the Regge slope parameter by $\alpha' = \frac{1}{2}l^2$. Furthermore, α_n^μ and $\tilde{\alpha}_n^\mu$ are called *right-movers* and *left-movers* respectively. They obey the equality

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^* \qquad \text{and} \qquad \tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^*, \tag{1.18}$$

  in order for the X^μ fields to be real. These are also called the *oscillator modes* of the string. It turns out that on the world-sheet, every right-mover α_n^μ is always accompanied by its left-mover $\tilde{\alpha}_n^\mu$, and vice versa.

- Solutions for the open string X^μ fields with Neumann boundary conditions are written as

$$X^\mu_{open,N} = x^\mu + l^2 p^\mu \tau + il\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma, \tag{1.19}$$

  where x^μ, p^μ, l and α_n^μ have the same interpretation as in the closed string case. Also, the equality $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$ holds, to render X^μ real.

- Solutions for the open string X^μ fields with Dirichlet boundary conditions are written as

$$X^\mu_{open,D} = x^\mu + l^2 \tilde{p}^\mu \sigma + il\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\sin n\sigma. \tag{1.20}$$

  The parameters still have the same interpretation, with one exception. Namely, $\tilde{p}^\mu$ does not longer have the interpretation of momentum anymore. Again, the equality $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$ holds, to render X^μ real.

For later purposes, it is convenient to define $\alpha_0^\mu = lp^\mu$ for open strings, and $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{2}lp^\mu$ for closed strings.

All these solutions for the X^μ fields are *classical* solutions. However, we want to include quantum mechanics into our theory as well. How this is achieved is discussed in the next section.

1.5 Quantizing the relativistic string

1.5.1 Commutation relations

When a theory is quantized, we let observables like position x^μ, momentum p^μ, angular momentum L^μ, etc. become *operators* that can act on states of a *Hilbert space*. Furthermore, if we have a classical theory, we can calculate *Poisson brackets* of two observables A and B, $\{A, B\}$. Then, if we quantize the theory, we substitute the *commutator* $[A, B]$ for the Poisson brackets. For example, a classical relativistic point particle has observables x^μ and p^μ, but if we quantize the theory, these become operators, with commutation relations $[x^\mu(\tau), p^\nu(\tau)] = i\eta^{\mu\nu}$ (and other commutators zero).

In string theory, we follow the same procedure. First we promote X^μ and its conjugate momentum

$$P^\mu = T\dot{X}^\mu \tag{1.21a}$$
$$= \frac{1}{2\pi\alpha'}\partial_\tau X^\mu \tag{1.21b}$$

to operators and impose their commutation relations

$$[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] = i\delta(\sigma'-\sigma)\eta^{\mu\nu}, \tag{1.22a}$$
$$[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = 0, \tag{1.22b}$$
$$[P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] = 0. \tag{1.22c}$$

One can show that by working this out, for open strings this leads to the conditions

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \tag{1.23a}$$
$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}, \tag{1.23b}$$

and when we consider closed strings, we also obtain the relations

$$[\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{m+n,0}\eta^{\mu\nu}, \tag{1.24a}$$
$$[\tilde{\alpha}^\mu_m, \alpha^\nu_n] = 0. \tag{1.24b}$$

First of all, it's nice to notice that when the string has no oscillations[3], we obtain the same result as for the point particle case. Secondly, if we look at the oscillator modes and rescale them as $\alpha^\mu_m \to \frac{a^\mu}{\sqrt{m}}$, we see that they actually represent the modes of a harmonic oscillator! We already know to interpret the oscillator modes of a harmonic oscillator. With a harmonic oscillator we introduce a ground state with momentum k^μ. We can act on this ground state with raising operators α^ν_{-m}, $m > 0$ to obtain excited states.

[3] So all the α^μ_n (and $\tilde{\alpha}^\mu_n$) vanish and the only degrees of freedom are the string's center of mass position and momentum.

Moreover, lowering operators α_m^ν, $m > 0$ lower excited states and annihilate the ground state. Since we now have an infinite set of lowering operators, the ground state is infinitly degenerate. For open strings, the ground state $|0, k^\mu\rangle$ is defined by

$$\alpha_m^\nu |0, k^\mu\rangle = 0, \qquad \text{for} \qquad m > 0, \tag{1.25a}$$
$$p^\nu |0, k^\mu\rangle = k^\nu |0, k^\mu\rangle, \tag{1.25b}$$

and for closed strings it is defined by

$$\tilde{\alpha}_m^\rho \alpha_m^\nu |0, 0, k^\mu\rangle = 0, \qquad \text{for} \qquad m > 0, \tag{1.26a}$$
$$p^\nu |0, 0, k^\mu\rangle = k^\nu |0, 0, k^\mu\rangle. \tag{1.26b}$$

By acting on the groundstate with modes that have $m < 0$, we get excited states.

So physically, a quantized (bosonic) string is an one-dimensional object that lives in D spacetime dimensions. It has waves propagating on its world-sheet, traveling at the speed of light. The waves are excited states of a harmonic oscillator. For closed strings, left-moving parts are always accompanied by their right-moving parts, so these waved propagate in opposite directions.

1.5.2 Mass levels

To determine the mass of a particle, we can use the *on-shell condition*, or *mass-shell condition* $M^2 = -p^2$, where M is the particle's mass and p^μ is its momentum. In string theory, we also determine the string's mass by applying this condition. The question is, however, can we calculate p^2 for a string? It turns out that this is done by using so-called *Virasorro-operators* L_m, or specifically L_0. These operators can be constructed from the oscillator modes, and they can be deduced from *conformal field theory* (CFT).[4] The mass-squared becomes an operator, and it is written as

$$M^2 = \frac{2}{\alpha'} \left[\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n - 1 \right], \qquad \text{for open strings,} \tag{1.27a}$$
$$M^2 = \frac{2}{\alpha'} \left[\sum_{n=1}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) - 2 \right], \qquad \text{for closed strings.} \tag{1.27b}$$

As we already saw, the m^{th} level excitation of a string is created by acting on the ground state with a mode α_{-m}^μ for open strings, and $\tilde{\alpha}_{-m}^\rho \tilde{\alpha}_{-m}^\mu$ for closed strings, with $m = 0, 1, \cdots$. So, we can apply the mass operator to a m^{th} level state and, making use of the commutation relations for the modes,

[4] We will investigate CFT at a basic level in chapter 3. For more details, the reader is referred to [12].

see that it contributes a discrete value to the strings mass! So, the string mass splits up into different levels.

- The first mass-level is a state with $m = 0$. These are states that have no oscillator modes acting on the ground state. Strangly enough, as can be seen from (1.27), the mass-squared of these states is negative! We see that $M^2 = -\frac{2}{\alpha'}$ for open strings and $M^2 = -\frac{4}{\alpha'}$ for closed strings. These string states are called *tachyons* and are not thought to represent actual particles. We will come back to them later on. Tachyons will play an important rôle throughout this thesis.

- The next mass-level to consider are modes with $m = 1$. These particles have mass-level $M^2 = 0$, so they are massless. These massless (vector) states are called *photons*. In the case of closed strings we can even identify such states with quantum particles of gravitation, called *gravitons*.[5]

Of course we can consider oscillator modes with $m > 1$. These states give rise to massive particles $M^2 > 0$. But it turns out that the masses of these states are really big compared to particles that we observe in nature, so they take a lot of energy to be created. For most purposes we do not need to consider them.

In the next chapter we will further investigate the property that the string world-sheet is scale invariant. When we combine this scale invariance with properties of the energy-stress tensor, we will find that in order to have consistent theories, strings require a critical dimension to live in.

[5]for further reference on this subject for example see [2], [11] or [14].

Chapter 2

Scale transformations and interactions

2.1 Couplings

2.1.1 Interacting theories

In physics, a very important concept is interaction and interaction strength. The reason that we are able to describe physics at all is that particles and fields interact with each other. For example, gravitation, mass, electric charge and spin are all quantities that can have interactions, and it is through this interaction that we are able to do measurements and experiments. Nowadays, most physics is described in terms of *Lagrangians* $\mathcal{L}$ and *actions* S. In quantum field theory, the interaction of particles is described by a Lagrangian containing fields, kinetic parts as well as interacting parts. In Maxwell's theory of electromagnetism, charged particles couple to electromagnetic fields, and in general relativity, *energy-momentum tensor* fields couple to the curvature of spacetime itself.

In order to tell how strong interactions between fields are, we need to compare it with something. In the Lagrangian formalism, a coupling parameter g determines the strength of interacting fields with respect to the (free) kinetic part of the theory, or two sectors of the interacting part of the theory. For example, a *gauge coupling* g in a non-Abelian gauge theory appears in the Lagrangian density as

$$\mathcal{L} = \frac{1}{4g^2} Tr\, F^a_{\mu\nu} F^{a\ \mu\nu}, \tag{2.1}$$

where $F^a_{\mu\nu}$ is a gauge field tensor and the trace runs over the index a.

There are two important regimes for the coupling parameter where one can work in, namely

- $g \ll 1$, *called weak coupling.* In this regime, perturbation theory is a good way to describe to interactions up to a certain order in g. Including higher orders in g comes down to giving a more accurate description of the interaction.

- $g \gg 1$, *called strong coupling.* Perturbation theory no longer works, and one has to find another way to calculate interactions. For example, one can try to find exact solutions, so that perturbation is no longer necessary.

2.1.2 Running couplings and renormalization

In quantum field theory, looking at shorter distances amounts to going to higher energy scales. At short distances 'virtual particles' go off the mass shell. Such processes renormalize the coupling, making it dependent on the energy scale μ, so $g \to g(\mu)$. This dependence of the coupling on the energy scale is called *running of the coupling* and is described by the renormalization group.

Renormalization is a process of rescaling a theory, and therefore will involve conformal transformations. The importance of renormalization in theoretical physics has grown considerably over the years. One of its major applications is found in condensed matter theory, where it is described within the framework of the so-called *renormalization group* (RG). In this framework, a lattice, with lattice-spacing a and coupling g is considered. Then, a Fourier transformation is performed, going over to momentum space. The lattice permits certain frequencies, namely high frequency modes (or so-called *fast modes*), corresponding to short distances, and low frequency modes (so-called *slow modes*), corresponding to long distances.

To renormalize this theory, one first writes down an effective action that can be used to describe slow mode interactions only. The next step is to integrate out all fast modes. After this integration, a rescaling is performed on the relevant parameters, such as the modes and the coupling. This completes the renormalization. We end up with a description of the same theory, on a different scale. So starting out with a theory of fast and slow modes and coupling g, we perform a renormalization transformation and end up with a theory of slow modes and coupling g'. If a theory is invariant under a scale transformation, it is called *conformal invariant.* In that case the theory is at a *fixed point* (more information on this subject can be found in [1]).

2.1.3 Renormalization β functions

Renormalization and conformal invariance also play a big rôle in quantum field theory and string theory. In quantum field theory, the running of a

Quantum field theory		*String theory*
coupling g	$\longleftrightarrow$	world-sheet action S
energy scale μ	$\longleftrightarrow$	world-sheet metric h_{ab}
$\beta(g) = \frac{1}{\mu}\frac{\partial g}{\partial \mu}$	$\longleftrightarrow$	$\beta \simeq T^a{}_a \simeq \frac{h_{ab}}{\sqrt{-h}}\frac{\delta S}{\delta h_{ab}}$

Table 2.1: *analogy between renormalization functions in quantum field theory and string theory.*

coupling parameter is described by what is called the *renormalization β function*,

$$\beta(g) = \mu\frac{\partial g(\mu)}{\partial \mu} = \frac{\partial g(\mu)}{\partial \ln \mu}. \tag{2.2}$$

As can be seen from (2.2), when a theory is scale invariant, the β function should vanishes. When a β function does not vanish, it is said to have a *conformal anomaly.*

One big difference between quantum field theory and string theory is the coupling. In quantum field theory, the coupling parameter depends on the energy scale. In string theory however, (as we will see in chapter 6) the coupling will become a spacetime dependent function, namely the exponent of a scalar field, $e^{\Phi(X)}$.

Another big difference between quantum field theory and string theory is the meaning of the β function. As we said before, in quantum field theory it is the dependence of the *coupling* on the *energy scale*. In string theory, however, it is the dependence of the *action* on the local *world-sheet scale*.

We can perform a conformal transformation, locally rescaling the world-sheet metric (see (1.9a)). Conformal invariance is a fundamental symmetry of string theory. This means that (at least classically) there is no conformal anomaly when we rescale the world-sheet metric, and therefore these β functions should vanish. Since the variation of the world-sheet action, with respect to the metric is proportional to the energy-stress tensor T^{ab}, we can see the analogy between β functions in quantum field theory and in string theory (see table 2.1). The importance of β functions will become clear in subsequent sections.

2.2 Weyl invariance and Weyl anomaly

2.2.1 Weyl invariance

Weyl transformations locally rescale the world-sheet by an overall factor. As we saw in section 1.3, only in two dimensions can a theory be conformal invariant. Let's investigate the property of Weyl invariance a bit further. An infinitesimal Weyl transformation (see (1.9a)) says that locally

$$h^{ab} \longrightarrow h^{ab} + \delta h^{ab} \tag{2.3a}$$
$$= h^{ab} + 2h^{ab}\delta\omega. \tag{2.3b}$$

For the action to be Weyl invariant, we need the variation of the action with respect to the metric h^{ab} to vanish, so

$$\frac{\delta S}{\delta h_{ab}} = 0. \tag{2.4}$$

According to *Noether's theorem* the energy-stress tensor is written as (1.12). From this we see that the energy-stress tensor and the claim for Weyl invariance are closely related. Actually the claim for Weyl invariance can be put in form

$$T^a{}_a(\sigma) = \frac{4\pi}{\sqrt{-h(\sigma)}} \left[\frac{\delta}{\delta h_{ab}(\sigma)} S\right] h_{ab}(\sigma) \tag{2.5a}$$
$$= 0. \tag{2.5b}$$

In other words, Weyl invariance implies the energy-stress tensor to be traceless.

2.2.2 Path integral approach

However, this derivation is only true classically. In order to include quantum corrections, we need a different way of introducing the energy-stress tensor. Therefore, consider the path integral

$$Z \simeq \int [dX] \exp(-S[X, h]). \tag{2.6}$$

This path integral counts the number of field configurations on the worldsheet. However, there is a huge set of configurations that only differ by a Weyl or diff transformation, and therefore render the same action. Therefore, there is an enormous overcounting of field configurations in this path integral. What we really want is the path integral, divided by the volume of the Weyl × diff symmetry group, $V_{diff \times Weyl}$. So we need to gauge fix the path integral.

The way to do this is to choose a path trough the volume of the symmetry group, in such a way that each slice that represents the same theory is only crossed once. A way to carry out such a gauge fixing is known as the *Faddeev-Popov procedure*, which is explained nicely in [11]. We will not go into detail here, but simply state that it comes down to introducing two new (anticommuting) Grassmann *ghost fields*, c^a and b_{ab} and writing down an action for these fields of the form,

$$S_g = \frac{1}{2\pi}\int d^2\sigma\,\sqrt{-h}b_{ab}\nabla^a c^b, \tag{2.7}$$

where h^{ab} is now some fixed metric. Then the gauge fixed Polyakov path integral is locally written as

$$Z = \int [dX\,db\,dc]\exp(-S[X,b,c,h]) \tag{2.8a}$$

$$= \int [dX\,db\,dc]\exp(-S[X,h]-S_g). \tag{2.8b}$$

This gauge fixed path integral can be used for calculating correlation functions $\langle\ldots\rangle_h$, where

$$\langle\ldots\rangle_h \equiv \int [dX\,db\,dc]\exp(-S[X,b,c,h])\ldots . \tag{2.9}$$

Most of the time we will omit the discussion on ghosts in this thesis for the sake of simplicity, but formally they need to be included.

With this correlation function, we are able to deduce another expression for the energy-stress tensor. The way to do this is to vary this correlation function with respect to h_{ab}. When we do so and rewrite (1.12) as

$$\frac{\delta S}{\delta h_{ab}(\sigma)} = \frac{\sqrt{-h(\sigma)}}{4\pi}T^{ab}(\sigma), \tag{2.10}$$

we see that this variation can be written as[1]

$$\delta\langle\ldots\rangle_h = -\frac{1}{4\pi}\int d^2\sigma\,\sqrt{-h(\sigma)}\delta h_{ab}(\sigma)\left\langle T^{ab}(\sigma)\ldots\right\rangle_h. \tag{2.11}$$

So, the energy-stress tensor can be written as the infinitesimal variation of the path integral with respect to the metric.

This result is derived for general variations of the world-sheet metric. However, we are interested in the specific case where the variation was a Weyl variation. Therefore we can substitute (2.3b), i.e. $\delta h_{ab} = 2h_{ab}\delta\omega$. Performing a Weyl transformation and using (2.11), the expression for T^{ab} becomes

$$\delta_W\langle\ldots\rangle_h = -\frac{1}{2\pi}\int \mathrm{d}^2\sigma\,\sqrt{-h(\sigma)}\delta\omega(\sigma)\langle T^a{}_a(\sigma)\ldots\rangle_h. \tag{2.12}$$

[1] From a mathematical point of view, the path integral (or partition function) (2.8b) can be seen as a functional $Z\big[S[h_{ab}(\sigma)]\big]$. Varying this functional with respect to h_{ab} requires applying the chainrule for functionals. With this procedure, we also need to integrate over the parameters $\sigma^a, a=0,1$. This explains the extra surface integral.

2.2.3 Critical dimension

As we said before, demanding our theory to be Weyl-invariant now requires that the energy-stress tensor is traceless. So classically

$$T^a_{\ a} \overset{classically}{=} 0. \tag{2.13}$$

However, quantum effects can contribute to the trace of the energy-stress tensor, causing a conformal anomaly, or *Weyl anomaly* to occur. We have already investigated this property in chapter 3. One can show that (see [11]) the Weyl anomaly is equal to

$$\langle T^a_{\ a} \rangle \overset{QM}{=} -\frac{c}{12}R, \tag{2.14}$$

where c is called the central charge (which will be introduced in chapter 3) and R is the world-sheet Ricci scalar.[2] The only way to obtain a consistent theory is if the total central charge is $c = 0$.

The central charge is made up of two components, namely the X^μ bosonic field contributions c^X and the ghost contributions c^g. It can be shown the central charge for the ghost fields is $c^g = -26$. Furthermore, every bosonic spacetime coordinate field X^μ contributes an amount of $+1$ to the central charge. Therefore, we find that the total central charge is

$$c = c^X + c^g = D - 26. \tag{2.15}$$

So, Weyl invariance can only be achieved for $D = D_c = 26$, where D is the number of spacetime fields X^μ, and therefore equal to the number of dimensions. This is the famous result that bosonic strings can only live in 26 dimensions.[3] A bosonic string theory in $D_c = 26$ is called a *critical (bosonic) string theory* and for a critical string, D_c is called the *critical dimension.* One can also consider string theories with fermions, called *superstring theories.* It turns out that the critical dimension for superstrings in flat spacetime is $D_c = 10$, as can be showed in quite a similar way.

In the derivation above we considered a string theory in flat spacetime. In a little while we shall include other backgrounds in our theory and see that this has a big influence on the condition for Weyl invariance. As we will see, the number of spacetime dimensions the strings lives in will be able to deviate from the critical dimension!

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[2] The Ricci scalar basically tells you how much curvature there is locally. For a non-flat world-sheet R generally is non-zero. Also see section B.

[3] Note however, that have just looked at flat spacetime here! The metric is equal to the Minkowski metric $\eta_{\mu\nu}$ and no other background fields are involved.

2.3 Fields and target space

When looking at string theory, it's important to be aware of the similarities and differences between string theory and ordinary quantum field theory for point particles. For one, *a classical point particle* carves out a *worldline*, which can be parameterised by a parameter τ, and can be set equal to the particle's *proper time* (see section 1.2). We can write down an action for a point particle, from which we can derive it's equations of motion. A particle in *quantum field theory* is described by quantum fields $\phi(x^\mu), \mu = 0, \ldots, 3$, living in $D = 4$ dimensions. A theory of these particles can be described by an action, involving the fields and derivatives of these fields. The four dimensions quantum fields live in, are equal to our spacetime. When we want to describe particles interacting with each other, we write down an action, involving the fields and derivatives, plus higher order terms and corresponding couplings.

This is not the case in string theory. First of all, strings are one-dimensional objects embedded in D spacetime dimensions. These strings are not generated by fields, as in quantum field theory. It is, however, possible to define a field theory for strings. Such a theory is called *String field theory.*[4] You could say that string theory relates to string field theory as a point particle does to quantum field theory.

In contrast to the point particle, a string has infinitely many more internal degrees of freedom, the oscillator modes, or *string modes.* These string modes can be seen as waves propagating on the world-sheet at the speed of light. When we quantize the string, the string modes actually become quantum modes of a two dimensional quantum field theory. Since a string propagates in D spacetime dimensions, there are D such fields $X^\mu(\tau, \sigma), \mu = 0, \ldots, D-1$. From the world-sheet's point of view, the fields X^μ can be seen as the coordinates of a manifold, called *target space.* In the case of string theory this is just equal to spacetime.

So particles in quantum field theory are described by quantum fields, living in $D = 4$ dimensions. Interactions can be described by putting higher order terms in the Lagrangian. Particles in string theory are described by different modes of a string propagating in spacetime. From the world-sheet's point of view, a string mode is some configuration of D two dimensional quantum fields, which are embedded in spacetime.

In the next chapter we will study conformal field theory. Since the two-dimensional string world-sheet is conformal invariant, it is not a bad idea to

[4] In string field theory there are fields $\Phi[X(\sigma)]$ which create and annihilate strings in a certain configuration. However, such fields do not even live in ordinary spacetime anymore, but in some sort of 'stringy' target space. It is possible to write down an action for open string field theory, but it turns out to be very complicated, if not impossible, to write down an action for closed string fields. This subject is, however, far beyond the scope of this thesis.

investigate this property to full extent. Conformal field theory provides us with a good way of achieving this. After having studied the mathematical framework of conformal field theory, we will incorparate it into a theory of interacting strings.

Chapter 3

Conformal field theory

3.1 Complex coordinates

3.1.1 Wick rotation

Conformal field theory is a very important tool for string theory. Interactions between strings and other strings, or backgrounds can be very hard to describe if one would try to parameterize the theory, and apply correct boundary conditions. An important idea that has been put forward is that interactions (for example the emission or absorption of strings) should be transformed (rescaled) in such a way that they become ‘pointlike’ operators on the world-sheet. Since the world-sheet is conformal invariant, such transformations leave the theory unchanged. After having applied these transformations, it becomes very interesting to see how these operators behave in the vicinity of each other. When two operators approach each other on the world-sheet, the quantum effects become apparent. A very convenient setting in which to study these sorts of interactions is called *conformal field theory* (CFT). CFT is a very extensive subject, so we will just cover some bacis properties and give a global overview on the subject.

It turns out to be convenient to study interacting strings in a Euclidean framework, $h_{ab} = \delta_{ab}$. A Euclidean metric can easily be obtained by performing a *Wick rotation* on one of the world-sheet coordinates (usually the propertime coordinate).[1] So we let

$$w = \sigma^0 + \sigma^1 \tag{3.1a}$$

$$= \sigma^1 + i\sigma^2, \tag{3.1b}$$

$$\text{and} \qquad \bar{w} = \sigma^1 - i\sigma^2. \tag{3.1c}$$

It is easily checked that with these coordinates, the world-sheet metric indeed has a Euclidean signature.

[1] One has to be careful that when performing a Wick rotation, no poles are crossed. In most examples, however, this is not the case.

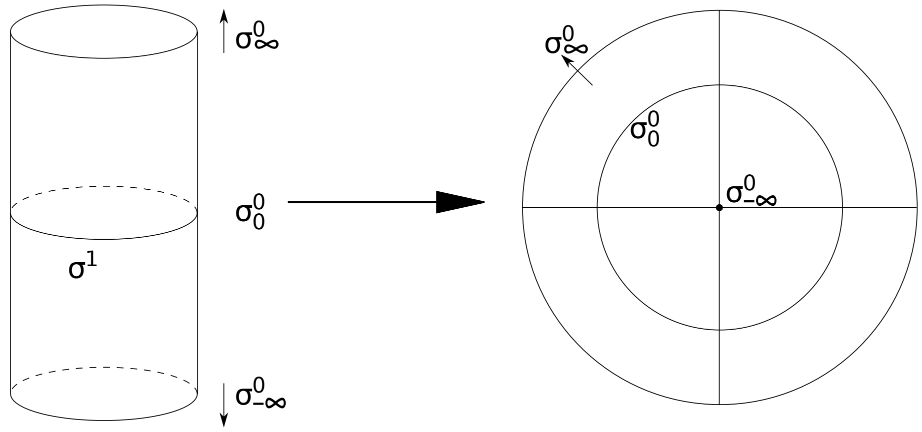

Figure 3.1: *Left: The world-sheet of a closed string is a cylinder in space-time. Right: The conformal mapping to complex coordinates. In the complex plane, the origin corresponds to the string's proper time at minus infinity,* $\sigma^0_{-\infty}$*. Points at* $|z| = \infty$ *correspond to the string's proper time at plus infinity,* σ^0_∞

3.1.2 Conformal transformation

The next convenient choice is to perform a conformal transformation on these coordinates, such that

$$z = e^{-i\sigma^1+\sigma^2} \qquad (= e^{-iw}) \qquad = z^1 + iz^2, \tag{3.2a}$$

$$\bar{z} = e^{i\sigma^1+\sigma^2} \qquad (= e^{i\bar{w}}) \qquad = z^1 - iz^2. \tag{3.2b}$$

Both open and closed strings can be studied in these coordinates. However, since closed strings are most easily described in this setting, we will focus on them for the remainder of this chapter.

As we discussed before, the world-sheet of a closed string is a cylinder, where the string's proper time σ^0 runs from $-\infty$ to $+\infty$, and the σ^1 coordinate runs from 0 to 2π. In our new z coordinates, however, the string's proper time runs radially outwards, with $z = 0$ corresponding to $\sigma^0 \to -\infty$, and $|z| \to +\infty$ to $\sigma^0 \to +\infty$. The σ^1 coordinates at a fixed time σ^2 are represented by circles around the origin. See figure 3.1.

Derivatives on the complex plane are defined by

$$\partial \equiv \partial_z = \frac{1}{2}(\partial_{z^1} - i\partial_{z^2}), \tag{3.3a}$$

$$\bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_{z^1} + i\partial_{z^2}), \tag{3.3b}$$

and in general, vectors with lower and upper indices transform as

$$(V^1, V^2) \to (V^z, V^{\bar{z}}), \tag{3.4a}$$

$$(V^z, V^{\bar{z}}) = \frac{1}{2}(V^1 + iV^2, V^1 - iV^2), \tag{3.4b}$$

and

$$(V_1, V_2) \to (V_z, V_{\bar{z}}), \tag{3.4c}$$

$$(V_z, V_{\bar{z}}) = \frac{1}{2}(V_1 - iV_2, V_1 + iV_2). \tag{3.4d}$$

With these new coordinates, the metric also transforms. By working out the details, we find $g_{zz} = g_{\bar{z}\bar{z}} = 0$ and $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$. Therefore, the inverse of the metric gives $g^{zz} = g^{\bar{z}\bar{z}} = 0$ and $g^{z\bar{z}} = g^{\bar{z}z} = 2$.[2]

3.2 Operator-product expansions

3.2.1 Currents and charges

On the world-sheet, one often encounters *conserverd currents* J^a, implying that $\partial_a J^a = 0$, and *conserved charges* Q, implying that $\frac{d}{dt}Q = 0$. A conserved current induces a conserved charge. By applying *Gauss' divergence theorem*, one can show that the conserved charge on the world-sheet, induced by a conserved current equals

$$Q = \frac{1}{2\pi}\int_0^{2\pi} d\sigma^1 J^0. \tag{3.5}$$

Next, we can switch to complex coordinates $(w, \bar{w})$, and Wick rotate $J^0 = iJ^2$. In this case, the integral splits up into a *holomorfic part* (only dependent on w) and a *anti-holomorphic part* (only dependent on $\bar{w}$). Then, we can switch to the conformally transformed coordinates $(z, \bar{z})$ and see that the integrals turn into contour integrals around the origin.

We know from *Noether's theorem* that every symmetry of an action gives rise to a conserved current. We will use this to study the conserved current for conformal symmetry. Fist of all, consider the energy-stress tensor on the world-sheet, T^{ab}. We already discussed that the energy-stress tensor itself is conserved, meaning $\partial_a T^{ab} = 0$. Then, moving to the complex plane, it can be shown that this leads to the conditions

$$\partial_{\bar{z}} T_{zz} = 0, \qquad \to \qquad T_{zz} = T(z), \tag{3.6a}$$

$$\partial_z T_{\bar{z}\bar{z}} = 0, \qquad \to \qquad T_{\bar{z}\bar{z}} = \bar{T}(\bar{z}), \tag{3.6b}$$

so $T(z)$ is holomorfic, and $\bar{T}(\bar{z})$ is anti-holomorfic.

This derivation works quite similar for the current of conformal symmetry, $J_a(\epsilon) = T_{ab}\epsilon^b$, where ϵ^b is an infinitesimal conformal transformation.

[2] Be cautious that raising or lowering indices can lead to counter intuitive results!

If $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ represent infinitesimal conformal transformations on the complex plane, the current $J_a(\epsilon)$ also splits up into a holomorfic and anti-holomorfic part,

$$J_z = T(z)\epsilon(z), \tag{3.7a}$$
$$J_{\bar{z}} = \bar{T}(\bar{z})\bar{\epsilon}(\bar{z}). \tag{3.7b}$$

In the following we will show how this can be used to generate conformal transformations of fields, living on the complex plane.

3.2.2 Generators of conformal transformations

In quantum field theory one often needs the product of two (or more) operators, for example when calculating correlation functions. But such a product only makes sense if it is *time-ordered*.[3] Since the world-sheet's proper time coordinate of a (closed) string in conformal complex coordinates runs radially outwards, we impose *radial ordering* for operator products, i.e.

$$RA(z,\bar{z})B(w,\bar{w}) = \begin{cases} A(z,\bar{z})B(w,\bar{w}) & \text{for } |z| > |w| \\ B(w,\bar{w})A(z,\bar{z}) & \text{for } |z| < |w| \end{cases}. \tag{3.8}$$

Now we turn our attention back to the conserved charge for conformal transformations, Q_ϵ. Such a charge can generate (infinitesimal) conformal transformations. The quantum version of an infinitesimal conformal transformation of a field $\phi(z,\bar{z})$ is given by the commutator with Q_ϵ, i.e.

$$\begin{aligned} \delta_\epsilon\phi(z,\bar{z}) &= [Q_\epsilon, \phi(z,\bar{z})] && (3.9a) \\ &= \frac{1}{2\pi i}\oint du\,\epsilon(u)\left[T(u)\phi(z,\bar{z}) - \phi(z,\bar{z})T(u)\right], && (3.9b) \end{aligned}$$

where we have just written down the holomorphic part here. The anti-holomorphic part is written is a similar way. But as we just discussed with radial ordering, the first term only makes sense for $|u| > |z|$, and the second term only makes sense $|z| > |u|$. Since these contour integral are integrated along contours around the origin, we can deform these contours in such a way that we end up with just one contour around the point z, γ_z (see [12] for a more detailed derivation). So, therefore the variation of a field $\phi(z,\bar{z})$ can finally be written as

$$\begin{aligned} \delta_\epsilon\phi(z,\bar{z}) &= [Q_\epsilon, \phi(z,\bar{z})] && (3.10a) \\ &= \frac{1}{2\pi i}\oint_{\gamma_z} du\,\epsilon(u) R\left(T(u)\phi(z,\bar{z})\right). && (3.10b) \end{aligned}$$

[3] Time ordering makes sure that the operators are put in the correct order. If this is not the case, one can have a situation where the expectation values of the product blow up, and would therefore be undefined.

3.2.3 Operator-product expansions

We will now try to evaluate $\delta_\epsilon\phi$ in (3.10b), the quantum version of the variation of a field $\phi(z,\bar{z})$. Therefore, we first need to know how a field globally transforms under a conformal transformation. It turns out that fields transform in a very similar way as tensors do under general coordinate transformations. If we start with a field $\phi(z,\bar{z})$, and we perform a conformal transformation, letting $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$, the field globally transforms as

$$\phi(z,\bar{z}) = \Big(\frac{\partial f(z)}{\partial z}\Big)^h \Big(\frac{\partial \bar{f}(\bar{z})}{\partial \bar{z}}\Big)^{\bar{h}} \phi(f(z),\bar{f}(\bar{z})). \tag{3.11}$$

Here h and $\bar{h}$ are called the *conformal weights* of the field. They are real, but need not be integers. A field that transform according to (3.11) is also called a *primary*, or *conformal field* of weight $(h,\bar{h})$.

Since (3.10b) is just the holomorphic part, we expect a conformal transformation just to transform z. Clasiccaly, this means that an infinitesimal conformal transformation would transform $\phi(z,\bar{z})$ as

$$\delta_\epsilon\phi(z,\bar{z}) = h\partial_z\epsilon(z)\phi(z,\bar{z}) + \epsilon(z)\partial_z\phi(z,\bar{z}). \tag{3.12}$$

Now that we know how fields classically transform under conformal transformations, we will turn our attention back to the quantum version of such transformations. First of all, when we deformed the contour in (3.10b), we assumed that the product $R(T(u)\phi(z,\bar{z}))$ is analytic in the neighbourhood of the point z. When this is the case, we can expand the product in a *Laurent series*,

$$R(T(u)\phi(z,\bar{z})) = \sum_n (u-z)^n O_n(z,\bar{z}), \tag{3.13}$$

where the O_n are usually operators. The idea of expanding operators near each other in a Laurent series is known as *operator-product expansion* (OPE).

After we have expanded the product in a Laurent series, we can substitute it back into (3.10b). If we then compare this result with the classical version (3.12), we see we obtain the correct result for the operator product expension if[4]

$$R(T(u)\phi(z,\bar{z})) = \frac{h}{(u-z)^2}\phi(z,\bar{z}) + \frac{1}{u-z}\partial_z\phi(z,\bar{z}). \tag{3.14}$$

We can now take this equation, (3.14), (plus its anti-holomorphic version) as the definition of what we mean by a conformal field of weight h, in the quantum regime. In other words, if the (radial ordered) operator-product expansion of the energy-stress tensor $T(u)$ with a field $\phi(z,\bar{z})$ has the form of (3.14), this field is called a conformal field of weight h (and analogious for the anti-holomorphic part).

[4] There are of course more terms in the expansion, but these have no poles at $u = z$. Therefore they do not contribute to the integral and we will leave them out.

3.2.4 Free bosons and OPE's

Now that we have found a way to describe conformal fields in the quantum regime, we can look at some explicit examples. We can also use the operator-product expansion technique to investigate other OPE's. First of all, since we are interested in applying CFT to string theory eventually, let's consider the action for free boson fields, Φ^i. On the cylinder, in Minkowski spacetime this action is just equal to the Polyakov action, (1.6). However, if we switch to complex coordinates on the cylinder, $(w, \bar{w})$, this action turns into

$$S = \frac{1}{2\pi\alpha'} \int_M dw d\bar{w} \sum_{i=1}^{D} \partial_w \Phi^i(w, \bar{w}) \partial_{\bar{w}} \Phi^i(w, \bar{w}), \tag{3.15}$$

where we have made use of the fact that $dzd\bar{z} = 2d^2\sigma$. Classically, this implies that the energy-stress tensor is written as

$$T(w) = -\frac{1}{2} \sum_{i=1}^{D} \partial_w \Phi^i(w) \partial_w \Phi^i(w), \tag{3.16a}$$

$$\bar{T}(\bar{w}) = -\frac{1}{2} \sum_{i=1}^{D} \partial_{\bar{w}} \bar{\Phi}^i(\bar{w}) \partial_{\bar{w}} \bar{\Phi}^i(\bar{w}). \tag{3.16b}$$

Next, we can use this action and then apply the correct boundary conditions to find the solutions of the equations of motion of the boson fields. After having found them, and having switched over to complex coordinates on the plane, $(z, \bar{z})$, we find that the solutions of the bosons are

$$\Phi^i(z, \bar{z}) = q^i - i\big[p^i \log(z) + p^i \log(\bar{z})\big] + i \sum_{n \neq 0} \frac{1}{n} \big[\alpha_n^i z^n + \tilde{\alpha}_n^i \bar{z}^n\big], \tag{3.17}$$

where (q^i, p^i) play the same rôle as (x^μ, p^μ) in foregoing, and the operators α_n^i and $\tilde{\alpha}_n^i$ are again oscillator modes of a harmonic oscillator. This time we have excluded the string lenghtscale l, since in this derivation we do not need to interperate the operators as representing position or momentum.

Next, we want to find the OPE of two boson fields $\Phi^i(z, \bar{z})$ and $\Phi^j(u, \bar{u})$. In the foregoing, we already discussed that these fields need to be radially ordered, in order to give correct results. However, if we multiply two boson fields, we run into another ordering ambiguity! To see this, note that the operators q^i and p^i, as well as the oscillator modes α_n^i, do not commute. Therefore, when we multiply these operators, we need to define the correct order to put them in. The ordering that we will use is called *normal ordering*.

Let's say that we have two operators L and R and we define normal ordering of the two operators such that R is always to the right of L. Then we may write this as

$$:RL: \; =: LR: \; = LR, \tag{3.18}$$

where the colons stand for normal ordering. First of all, we can see that operators between colons behave as classical objects (within colons, they commute). Secondly, note that this implies

$$RL =: RL: + [R, L]. \tag{3.19}$$

For the operators q^i, p^i, we define normal ordering such that p^i is always to the right of q^i. And for the oscillator modes α_n^i and $\tilde{\alpha}_n^i$, we define normal ordering such that the annihilations operators (modes with $n > 0$) are always to the right of the creation operators (modes with $n < 0$). With these definitions we are now ready to derive OPE's for various fields and operators. We will not perform the actual calculations here, but merely state their results.

- The OPE for two bosonic fields $\Phi^i(z, \bar{z})$ and $\Phi^j(u, \bar{u})$ becomes

$$R\big(\Phi^i(z,\bar{z})\Phi^j(u,\bar{u})\big) =: \Phi^i(z,\bar{z})\Phi^j(u,\bar{u}): -\delta^{ij}\big[\log(z-u)+\log(\bar{z}-\bar{u})\big]. \tag{3.20}$$

  We are usually interested in OPE's, in the limit $z \to u$. In this limit, the classical part (so, the part within normal ordering signs) always yields a constant. This part does not contribute in contour integrals, so it is usually left out. Also, in OPE's, one always implies radial ordering. For this reason, is the radial ordering symbol is also usually left out. Therefore the OPE (3.20) can be written as

$$\Phi^i(z,\bar{z})\Phi^j(u,\bar{u}) = -\delta^{ij}\big[\log(z-u) + \log(\bar{z}-\bar{u})\big]. \tag{3.21}$$

- By differentiation of (3.20), one can also calculate the OPE of the derivatives of the fields, $\partial\Phi^i(z,\bar{z})\partial\Phi^j(u,\bar{u})$. In this case, we obtain

$$\partial\Phi^i(z,\bar{z})\partial\Phi^j(u,\bar{u}) = -\frac{\delta^{ij}}{(z-u)^2}, \tag{3.22}$$

  and equivalent for the anti-holomorphic part.

- Recall that the definition of a conformal field $\phi(z, \bar{z})$ of weight h involved the OPE of the energy-stress tensor with the field, $T(u)\phi(z, \bar{z})$. Since in complex coordinates the energy-stress tensor classically is written as (3.16), we see that we have to be careful when we want to write down the quantum version. Making use of (3.22), we see that we can define a nice, non-singular version of the energy-stress tensor in the quantum regime as

$$T(z) \equiv -\frac{1}{2}: \sum_{i=1}^{D} \partial\Phi^i(z)\partial\Phi^i(z): \tag{3.23a}$$

$$= -\frac{1}{2}\sum_{i=1}^{D} \lim_{z\to u}\left[\partial\Phi^i(z)\partial\Phi^i(u) + \frac{\delta^{ii}}{(z-u)^2}\right], \tag{3.23b}$$

and equivalently for the anti-holomorphic part. With this definition it is possible to calculate the OPE of $T(z)$ with $\partial\Phi^i(u)$. If we work out the calculation, we find that it gives

$$T(z)\partial\Phi^i(u) = \frac{1}{(z-u)^2}\partial\Phi^i(u) + \frac{1}{z-u}\partial^2\Phi^i(u). \tag{3.24}$$

Comparing this result with (3.14), we can conclude that $\partial\Phi^i(u)$ is a conformal field of weight 1. And of course, the anti-holomorphic part is calculated in the same fashion.

- The last example of an OPE that we will discuss is the OPE of $T(z)$ with itself. By working out the details, one can show that it yields

$$T(z)T(u) = \frac{c/2}{(z-u)^4} + \frac{2}{(z-u)^2}T(u) + \frac{1}{z-u}\partial_u T(u), \tag{3.25}$$

where the constant $c = D$, in this case equal to the number of bosons. This example shows that $T(z)$ is not a conformal field. It would have been if the first term was absent. In that case its conformal weight would have been $h = 2$, equal to its classical value. This first term is caused by the quantum effects and is called the *conformal anomaly.*

These OPE's were just a few examples in order to illustrate how they can be obtained. In string theory, when one wants to study the quantum effects of fields on the world-sheet near each other, OPE's are a very useful tool.

3.3 Virasoro operators

3.3.1 Virasoro algebra

In the foregoing we have looked at a current for a conformal symmetry, $J_z = T(z)\epsilon(z)$ (and similar for the anti-holomorphic part). The function $\epsilon(z)$ is an arbitrary holomorphic function. Therefore, we are able to expand it in modes. The expansion will in general depend on the surface that the function is expanded on. In this case, we are expanding on the complex plane, and we shall include the point ∞, so that surface is in fact the *Riemann sphere.*

It can be shown (see [12]) that the Laurent modes generate transformations of the form $z \to z - z^{n+1}$, if $\epsilon(z) = z^{n+1}$. So in fact, we obtain an infinite set of currents $J^n(z) = T(z)z^{n+1}$. Just as before, each of these currents has its own conserved charge, which we will now denote by L_n. We also saw that in the quantum regime, these charges become generators of these transformations and they are written as

$$L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}T(z), \tag{3.26}$$

where we integrate the contour around the origin. If we invert this relation, we obtain

$$T(z) = \sum_n z^{-n-2} L_n. \tag{3.27}$$

Again, this can similarly be repeted for the anti-holomorphic part.

It turns out that the generators L_n are elements of an algebra, called the *Virasoro algebra*. The elements themselves are called *Virasoro operators*. By calculating the commutator of two generators, we find that the Virasoro algebra obeys

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}. \tag{3.28}$$

Classically, the last term is absent and indeed the commutator of two elements is itself again an element of the algebra. But in the quantum algebra, we now obtain an extra term, including c (introduced in (3.25)).

Formally, such a constant term is not allowed in an algebra, but we can solve this problem. We should interpret c not as a number, but as an element of the algebra that commutes with all other elements. In that case, the operator c has a constant value in any representation of the algebra, and this value is equal to the number c. A constant term that arises as in the algebra (3.28) is often called a *central charge*.

The Virasoro operators can also be used to show that they obey important conditions, called *Virasoro constraints*. Classically, the Virasoro constraints say $L_m = \bar{L}_m = 0$, for all m. One particular important Virasoro constraint is $L_0 = \bar{L}_0 = 0$. If we consider an action of free bosonic fields, which have solutions of the form (3.17), the classical Virasoro constraints for L_0 and $\bar{L}_0$ are written as

$$L_0 \equiv \sum_{n=0}^{\infty} \alpha^{\mu}_{-n}\alpha_{n\mu} = 0, \tag{3.29a}$$

$$\bar{L}_0 \equiv \sum_{n=0}^{\infty} \tilde{\alpha}^{\mu}_{-n}\tilde{\alpha}_{n\mu} = 0, \tag{3.29b}$$

where we have used the fact that $\alpha^{\mu}_0 = \tilde{\alpha}^{\mu}_0 = \frac{1}{2}lp^{\mu}$. Now, one can show that the Hamiltonian of the system classically is equal to the sum of these two Virasoro operators, so we find

$$H = L_0 + \bar{L}_0 = 0. \tag{3.30}$$

This result will be used later on in the thesis.

3.3.2 Operator-state correspondence

Now that we have found the Virasoro algebra, we want to study its representations. Even though this algebra has many representations, we will just consider the so-called *unitary highest weight representations.* A representation of the Virasoro algebra is called *unitary* if all operators L_n are realized as operators acting on a Hilbert space, and obey the condition $L_n^\dagger = L_{-n}$. A representation is a *highest weight representation* if it contains a state with a smallest value of L_0.[5]

If $|h, \bar{h}\rangle$ is a *highest weight state*, and has eigenvalues

$$L_0|h, \bar{h}\rangle = h|h, \bar{h}\rangle, \tag{3.31a}$$

$$\bar{L}_0|h, \bar{h}\rangle = \bar{h}|h, \bar{h}\rangle, \tag{3.31b}$$

then it is annihilated by all generators L_n and $\bar{L}_n$, with $n > 0$, so

$$L_n|h, \bar{h}\rangle = \bar{L}_n|h, \bar{h}\rangle = 0, \qquad \text{for } n > 0. \tag{3.32}$$

Beside highest weight states, we can also define the groundstate, $|0\rangle$, of this system. The groundstate can be defined as the state that respects the maximum number of symmetries. Therefore, it must be annihilated by the maximum number of conserved charges, L_n and $\bar{L}_n$. Classically, this amounts to all n, but due to the central charge term in the quantum algebra, this is not possible. It turns out that the maximal symmetry is[6]

$$L_n|0\rangle = \bar{L}_n|0\rangle = 0, \qquad \text{for } n \geq -1, \tag{3.33}$$

and its Hermitean conjugate $\langle 0|$ satisfies

$$\langle 0|L_n = \langle 0|\bar{L}_n = 0, \qquad \text{for } n \leq 1. \tag{3.34}$$

We are now ready to introduce the so-called *operator-state correspondence* (OSC). As we will now show, one can find a connection between a local operator (or conformal field) $\phi(z, \bar{z})$ of weight $(h, \bar{h})$ and a highest weight state $|h, \bar{h}\rangle$. If we define a state $|h, \bar{h}\rangle$ as

$$|h, \bar{h}\rangle = \lim_{z,\bar{z}\to 0} \phi(z, \bar{z})|0\rangle, \tag{3.35}$$

one can show that this state indeed satisfies

$$L_n|h, \bar{h}\rangle = \bar{L}_n|h, \bar{h}\rangle = 0, \qquad \text{for } n > 0, \tag{3.36}$$

so that the condition for a highest weight state is indeed satisfied. It is also not difficult to show that

$$\lim_{z,\bar{z}\to 0}[L_0, \phi(z, \bar{z})] = \lim_{z,\bar{z}\to 0} h\phi(z, \bar{z}), \tag{3.37a}$$

$$\lim_{z,\bar{z}\to 0}[\bar{L}_0, \phi(z, \bar{z})] = \lim_{z,\bar{z}\to 0} \bar{h}\phi(z, \bar{z}), \tag{3.37b}$$

[5] Physically this can be interpreted as a ground state of the Hilbert space, or the state with the lowest energy. Since $H = L_0 + \bar{L}_0$, and this should of course be bounded from below.

[6] For more details see [12].

so that if we let this condition act on the vacuum, we find that the state $|h,\bar{h}\rangle$ indeed has the correct eigenvalues $(h,\bar{h})$. Such a relation can also be obtained outside the limit $(z,\bar{z})\to 0$. It can be shown that the operators L_{-1} and $\bar{L}_{-1}$ generate translations on the cylinder. This means that if we want to find a local operator on the complex plane at a point $(u,\bar{u})$, we define

$$\phi(u,\bar{u})|0\rangle = e^{uL_{-1}+\bar{u}\bar{L}_{-1}}|h,\bar{h}\rangle. \tag{3.38}$$

So, apparently we are able to find a one-to-one correspondence between (highest weight) states and local operators (fields) on the world-sheet.[7]

Interactions between strings can be very difficult to describe. We could, of course try to put higher order terms in the world-sheet action, but that would only describe interactions of the fields on the world-sheet, and therefore a more complicated theory on the world-sheet. But that's still a theory of a single string, and no interacting strings! This is where OSC comes in. OSC in string theory means replacing string states on the world-sheet by local operators. How to describe interaction strings is treated in the next chapter.

[7] We haven't actually given a formal proof of the one-to-one correspondence here. For more details, the reader is refered to [12].

Chapter 4

Vertex operators and amplitudes

4.1 Operator-state correspondence

4.1.1 Applying CFT to string theory

Up till now we have just studied some basic properties of the (bosonic) string and the world-sheet action. But if string theory is supposed to be the potential 'theory of everything', it should be able to properly describe interactions as well. As we mentioned in the foregoing, interactions between strings can not simply be build in by putting higher order terms in the world-sheet Lagrangian. Parameterising the theory and applying the correct boundary conditions also isn't a real practicle idea, because the level of difficulty for these calculations gets out of hand real quick. So we need to come up with something completely different.

Fortunately, we already have such a tool to our disposal. Conformal field theory is the correct way to describe interactions. In particular we will make use of the OSC.

4.1.2 Introducing the vertex operator

Let's focus on the scattering amplitude of one string emitting another string. As we know, we can apply a conformal transformation to the world-sheet, locally rescaling it. It's even possible to apply such a transformation that the emitted string becomes a puncture in the world-sheet! First of all we should divide between open and closed interacting strings. The world-sheet of an *open string* emitting another open string will become a disk with punctures on the boundary. The world-sheet of a *closed string* emitting another closed string will become a sphere with punctures on the surface, since there are no boundaries for closed strings. See figure 4.1 and 4.2.

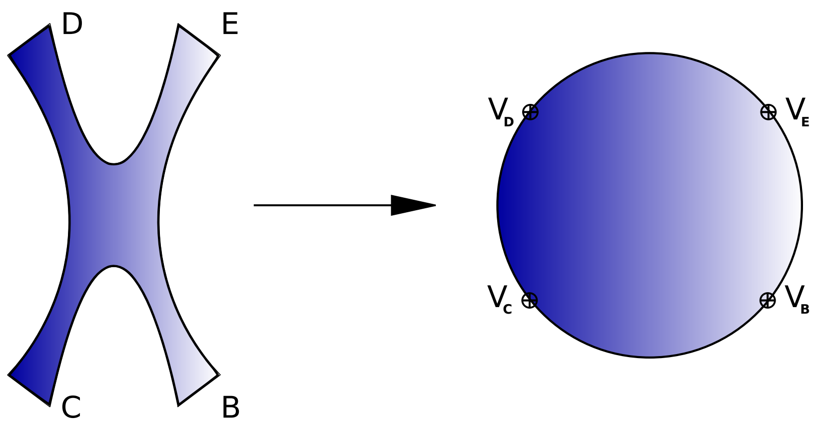

Figure 4.1: *Open string scattering. Left: open strings with sources at* $X^0 = \pm\infty$*. Right: A conformally equivalent picture, a disk with four vertex operators attached to the boundary. We should note that we have used a Euclidean signature for the world-sheet here. A Minkowski signature would yield a strip with two vertex operators attached to the boundary.*

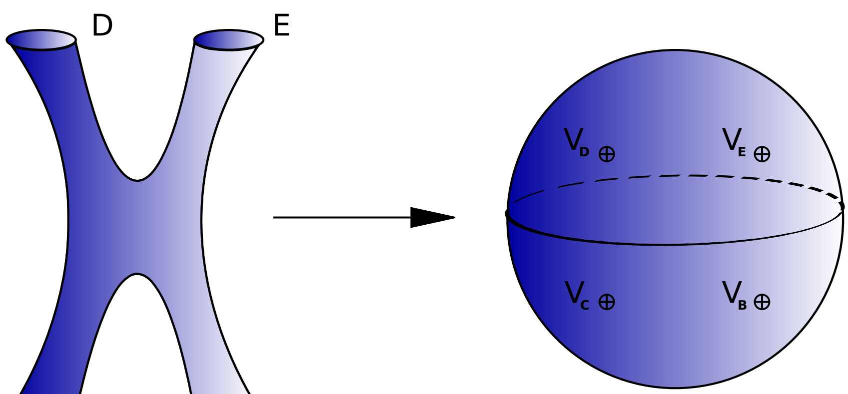

Figure 4.2: *Closed string scattering. Left: closed strings with sources at* $X^0 = \pm\infty$*. Right: A conformally equivalent picture, a sphere with four vertex operators attached. Here we also used Euclidean signature. A Minkowski signature would yield a cylinder with two vertex operators attached to the surface.*

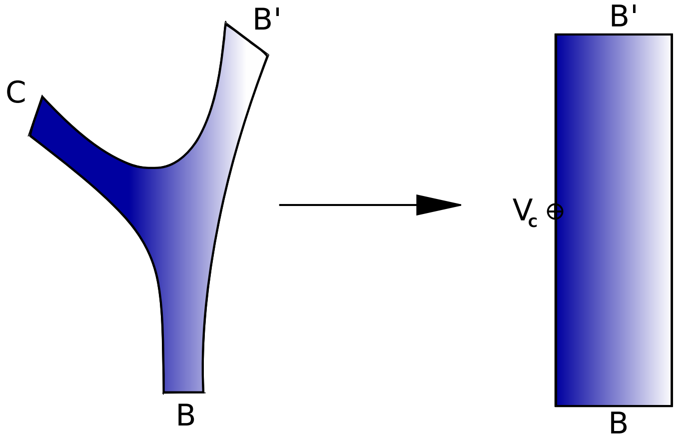

Figure 4.3: *Left: Emission of one open string. Right: Conformally equivalent picture, the emitted string has become a vertex operator on the world-sheet. This time we used a Minkowski signature, and therefore obtained a stip with one vertex operator attached to the boundary.*

A nice example, illustrating how to deal with string interactions and scatterings, is calculating the probability amplitude A of interacting strings. Let's, for the moment, turn our attention to the simplest open string tree-diagram, one open string emitting another (see figure 4.3). We know from quantum mechanics that if we want to calculate the probability amplitude, we take the inner product of the in-state and out-state:

$$A_{QM} = \langle \psi_{in} \mid \psi_{out} \rangle. \tag{4.1}$$

But this doesn't hold for interacting strings. The difference is that when a string in the in-state $|B\rangle$ emits another string $|C\rangle$, the out-state $|B'\rangle$ is no longer the same as the in-state. Instead, what we can do is introduce an operator $V(\tau, \sigma)$ that turns the out-state into the in-state, and then take the inner product,

$$A_{ST} = \langle B \mid V(\tau, \sigma) \mid B' \rangle. \tag{4.2}$$

So in fact we have substituted an operator for a string state! These operators are called *vertex operators*, and here we have made explicit use of the OSC.

The fact that a theory of interacting strings can equivalently be described by a theory of operators on the world-sheet is one of the true elegant features of string theory. In the next section we will calculate some amplitudes, using these vertex operators.

4.2 Tachyon tree-diagrams for open strings

4.2.1 Vertex operators for open strings

Let's first find the correct expression for a vertex operator. Since each different string state has it's own vertex operator, we need to consider each case seperately. We begin with a simple case, the *tachyon state* which has negative mass squared ($M^2 = -k^2 = -\frac{2}{\alpha'}$).[1] As was said in chapter 1, the definition of an open string tachyon state with momentum k^μ is the groundstate of a string, $|0, k^\mu\rangle$, which obeys

$$p^\nu|0, k^\mu\rangle = k^\nu|0, k^\mu\rangle, \tag{4.3a}$$
$$\alpha^\nu_m|0, k^\mu\rangle = 0, \quad \text{for} \quad m > 0. \tag{4.3b}$$

Using this and recalling the commutation relations for the operators x^μ and p^μ, (1.23a), we can calculate the momentum of the following state

$$p^\nu|\psi\rangle = p^\nu e^{ik_{1\rho}X^\rho}|0, k_2^\mu\rangle \tag{4.4a}$$
$$= \left\{\left[p^\nu, e^{ik_{1\rho}X^\rho}\right] + e^{ik_{1\rho}X^\rho}p^\nu\right\}|0, k_2^\mu\rangle \tag{4.4b}$$
$$= (k_1^\nu + k_2^\nu)e^{ik_{1\rho}X^\rho}|0, k_2^\mu\rangle \tag{4.4c}$$
$$= (k_1^\nu + k_2^\nu)|\psi\rangle. \tag{4.4d}$$

In other words, the state $|\psi\rangle$ has momentum $k_1^\mu + k_2^\mu$. Again, by making use of these commutation relations, it is not hard to show that $\alpha^\nu_m|\psi\rangle = 0$, for $m > 0$. So in fact we have proven that $|\psi\rangle$ is the groundstate, i.e.

$$e^{ik_{1\rho}X^\rho}|0, k_2^\mu\rangle = |0, k_1^\mu + k_2^\mu\rangle. \tag{4.5}$$

Therefore it changes the momentum of a tachyon state from k_2 to $k_1 + k_2$. In other words, the operator $e^{ik_{1\rho}X^\rho}$ has exactly the property that we are looking for in a vertex operator! It's conjugate expression is

$$\langle 0, k_2^\mu|e^{-ik_{1\rho}X^\rho} = \langle 0, k_1^\mu + k_2^\mu|, \tag{4.6}$$

so that the condition for momentum conservation of a tachyon state becomes

$$\langle 0, k_1|0, k_2\rangle = \mathcal{N}\delta^D(k_1 - k_2), \tag{4.7}$$

and $\mathcal{N}$ is just a normalization constant, it has no further significance.

The oscillator modes do not commute, so when we write the exponential as a power series there is an ordering ambiguity. We can solve this problem by using *normal ordering*. Normal ordering means that we write all annihilations operators to the right and all creation operators to the left. We write the normal ordered version of an operator $\mathcal{F}$ as $:\mathcal{F}:$. Therefore, the

[1]In the remainder of this section we will set $\alpha' = 1$ for simplicity.

normal ordered form of an open string tachyon vertex operator, located at (τ_1, σ_1), can be written as

$$V_{tachyon,o}(\tau_1,\sigma_1) = g_o \int_{\partial M} ds : e^{ik\cdot X(\tau,\sigma)} : \delta^{(2)}(\tau_1,\sigma_1) \tag{4.8a}$$
$$= g_o : e^{ik^\mu X_\mu(\tau_1,\sigma_1)} : , \tag{4.8b}$$

where we have integrated along the boundary of the world-sheet. Furthermore, we have introduced an *open string coupling* g_o. It comes from the fact that when we add a vertex operator, it couples to the world-sheet with interaction strength g_o. We can always set the open string coupling equal to $g_o = 1$, by redefining the fields X^μ and will do so in the following. In a section 4.3 we will go into some more detail about the string coupling.

4.2.2 Emitting one open tachyon state

We will now calculate the simplest scattering amplitude for strings, namely the amplitude of one open string tachyon emitting another open string tachyon, $A_{1\leftrightarrow 2}$ (see figure 4.3). For this example we will impose Neumann boundary conditions for the open string, so therefore we can substitute the solution (1.19) for the X^μ fields. We find that the vertex operator for a tachyon state becomes

$$\begin{aligned} V_{tachyon,o}(\tau,\sigma) &= : e^{ik^\mu X_\mu(\tau,\sigma)} : \\ &= : e^{ik^\mu (x_\mu + l^2 p_\mu \tau + il\sum_{n\neq 0}\frac{1}{n}\alpha_{n\mu}e^{-in\tau}\cos n\sigma)} : \\ &= e^{ik^\mu (x_\mu + l^2 p_\mu \tau)} \\ &\quad \times e^{-lk^\mu \sum_{n<0}\frac{1}{n}\alpha_{n\mu}e^{-in\tau}\cos n\sigma} \times e^{-lk^\mu \sum_{n>0}\frac{1}{n}\alpha_{n\mu}e^{-in\tau}\cos n\sigma} . \end{aligned} \tag{4.9}$$

Since we can choose either side of the world-sheet ($\sigma = 0$ or $\sigma = \pi$) as the boundary where the vertex operator is located, we make the convenient choice $\sigma = 0$. Furthermore, the probability amplitude is invariant under a translation along the τ-direction, so we can use this to set $\tau = 0$. If we assign incoming momenta with a plus sign and outgoing momenta with a minus sign, the probability amplitude will become

$$A_{1\leftrightarrow 2} = \langle B \mid V_C(0,0) \mid B' \rangle \tag{4.10a}$$
$$= \langle 0, k_1 \mid e^{-ik_2^\mu x_\mu} \times e^{lk^\mu \sum_{n<0}\frac{1}{n}\alpha_{n\mu}} \times e^{lk^\mu \sum_{n>0}\frac{1}{n}\alpha_{n\mu}} \mid 0, -k_3 \rangle \tag{4.10b}$$
$$= \langle 0, k_1 + k_2 \mid 0, -k_3 \rangle \tag{4.10c}$$
$$= \mathcal{N}\delta^D(k_1 + k_2 + k_3), \tag{4.10d}$$

which is really nothing more than saying that the momentum of the tachyons is conserved! If we take it one stap further and calculate two tachyons scattering off each other, things become a little more interesting.

4.2.3 2-tachyon open string scattering

What was done for the emission (or absorption) of one single string state can be repeated for two of states as well. Since it is an instructive exercise, we will go through a detailed calculation of the 2-tachyon open string scattering. This process is depicted in figure 4.1

First of all, we will again assume Neumann boundary conditions, so we will use the solutions (1.19) for the X^μ fields. Next, we should notice that this time we have to insert two vertex operators, $V_C(\tau_C, \sigma_C)$ and $V_D(\tau_D, \sigma_D)$. Without loss of generality, we can put both vertex operators on the same side, so we choose $\sigma_C = \sigma_D = 0$. Furthermore, we can use translational invariance to put the τ parameter of one vertex operator to zero, but then we have to integrate over the other one. Therefore, the simplest form the scattering amplitude $A_{2\leftrightarrow 2}$ can be brought into is

$$A_{2\leftrightarrow 2} = \int_0^\infty d\tau \, \langle B \mid V_C(0,0) V_D(\tau,0) \mid E\rangle \tag{4.11a}$$

$$= \int_0^\infty d\tau \, \langle 0, k_1| : e^{-ik_2^\mu X_\mu(0,0)} : : e^{-ik_3^\nu X_\nu(\tau,0)} : |0, -k_4\rangle \tag{4.11b}$$

$$= \int_0^\infty d\tau \, \langle 0, k_1| e^{-ik_2^\mu x_\mu} \exp\left\{ l k_2^\mu \sum_{n>0} \frac{\alpha_{n\mu}}{n} \right\} \tag{4.11c}$$

$$\times \exp\left\{ l k_3^\nu \sum_{m<0} \frac{\alpha_{m\nu}}{m} e^{-im\tau} \right\} e^{-ik_3^\nu (x_\nu + l^2 p_\nu \tau)} |0, -k_4\rangle. \tag{4.11d}$$

We have assigned the incoming momenta k_1 and k_2 with plus signs and the outgoing momenta $-k_3$ and $-k_4$ with minus signs.

In order to solve this, we need to know how to rearrange exponents of noncommuting operators. First of all, when two operators A and B do not commute, the equality $e^{A+B} = e^A e^B$ does no longer hold. Therefore, we need to make use of the *Baker-Campbell-Hausdorff* formula (BCH)

$$e^A e^B = e^B e^A e^{[A,B]}, \tag{4.12}$$

and can also be put into the form

$$e^{t(A+B)} = e^{tA} e^{tB} \times e^{-\frac{t^2}{2}[A,B]} \times e^{\frac{t^3}{6}(2[B,[A,B]+[A,[A,B]])} \times \ldots . \tag{4.13}$$

Secondly, we know that the operators x^μ and p^μ have commutation relations (1.23a). If we now apply BCH to (4.11) and use the fact that $k^2 = 2$ for open string tachyons, one can show that

$$e^{ik_\rho (x^\rho + l^2 p^\rho \tau)} = e^{ik_\rho x^\rho} e^{i(l^2 k_\rho p^\rho + 1)\tau}. \tag{4.14}$$

Now that we know how to rearrange the exponents, we let the exponent involving p^ρ act on the outgoing state $|0, -k_4\rangle$. Next, we let the operators

$e^{ik_\rho x^\rho}$ act on the incoming and outgoing states. Together, this yields

$$A_{2\leftrightarrow 2} = \int_0^\infty d\tau\, e^{i(l^2 k_3^\nu k_{4\nu}+1)\tau} \langle 0, k_1 + k_2 | \exp\{l k_2^\mu \sum_{n>0} \frac{\alpha_{n\mu}}{n}\} \times \exp\{l k_3^\nu \sum_{m<0} \frac{\alpha_{m\nu}}{m} e^{-im\tau}\} |0, -k_3 - k_4\rangle. \quad (4.15)$$

A good idea next, is to switch the order of the modes with $n > 0$ and $m < 0$. Therefore we need to make use of BCH again. In that case, the modes annihilate the ingoing and outgoing states, so that we only end up with the commutator terms. So, by making use of BCH and (1.23b) we find that

$$\left[l k_2^\mu \sum_{n>0} \frac{\alpha_{n\mu}}{n}, l k_3^\nu \sum_{m<0} \frac{\alpha_{m\nu}}{m} e^{-im\tau}\right] = l^2 k_2^\mu k_3^\nu \sum_{\substack{n>0\\m<0}} \frac{e^{-im\tau}}{mn} [\alpha_{n\mu}, \alpha_{m\nu}] \quad (4.16a)$$

$$= l^2 k_2^\mu k_3^\nu \sum_{\substack{n>0\\m<0}} \frac{e^{-im\tau}}{mn} n \delta_{m+n} \eta_{\mu\nu} \quad (4.16b)$$

$$= l^2 k_2^\mu k_{3\mu} \sum_{n,p>0} \frac{e^{ip\tau}}{-p} \delta_{p,n} \quad (4.16c)$$

$$= -l^2 k_2^\mu k_{3\mu} \sum_{n>0} \frac{e^{in\tau}}{n} \quad (4.16d)$$

$$= l^2 k_2^\mu k_{3\mu} \ln(1 - e^{i\tau}). \quad (4.16e)$$

We can now plug this back into (4.15) and use the orthogonality of the incoming and outgoing state, to obtain the expression

$$A_{2\leftrightarrow 2} = \int_0^\infty d\tau\, e^{i(l^2 k_3\cdot k_4+1)\tau} (1 - e^{i\tau})^{l^2 k_2\cdot k_3} \mathcal{N} \delta^{(D)}(k_1 + k_2 + k_3 + k_4), \quad (4.17)$$

Again, the delta function means conservation of momentum and we will leave it out from here on. We would like to express the outgoing momenta k_3 and k_4 in terms of incoming momenta k_1 and k_2. Therefore, we use the fact that

$$(k_i + k_j)^2 = k_i^2 + k_j^2 + 2k_i \cdot k_j \quad (4.18a)$$

$$= 4 + 2k_i \cdot k_j \quad (4.18b)$$

to show that $k_3 \cdot k_4 = k_1 \cdot k_2$. Now we do a Wick-rotation $i\tau \to -\tau$ and substitute $x = e^{-\tau}$, so $d\tau = -\frac{dx}{x}$. This can be used to put the amplitude into the form

$$A_{2\leftrightarrow 2} = \int_0^1 dx\, x^{l^2 k_1\cdot k_2} (1 - x)^{l^2 k_2\cdot k_3}. \quad (4.19)$$

Eventhough this is the final result, there is yet another useful form that the amplitude can be put into. Let's switch to (rescaled) Mandelstam variables, $s = -l^2(k_1 + k_2)^2$ and $t = -l^2(k_2 + k_3)^2$. Substituting these into the scattering amplitude yields

$$A_{2\leftrightarrow 2} = \int_0^1 dx\, x^{-s/(2l^2)-2}(1-x)^{-t/(2l^2)-2} \tag{4.20a}$$

$$= \int_0^1 dx\, x^{a-1}(1-x)^{b-1}, \tag{4.20b}$$

where we have defined $a = -s/(2l^2) - 1$ and $b = -t/(2l^2) - 1$. The form of the scattering amplitude, (4.20b), is equal to the Euler beta-function $B(a, b)$, which brings the scattering amplitude, in terms of Γ-functions, into its final form

$$A = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{4.21}$$

As you can see this function is nicely symmetric in a and b.

4.2.4 Vertex operators for excited states

Up till now, we discussed vertex operators for open string *tachyon states.* We could of course also consider vertex operators of *excited* open string states. The first excited state is the open string photon, a massless state. An open string photon state has a polarization η_μ, so we need a vertex operator that can be contracted with this polarization in order to obtain a scalar. It turns out that the correct form of the open string photon vertex operator is

$$V_{photon,o}(\tau_1, \sigma_1) = g_o \int_{\partial M} ds : \left[\eta_\mu \dot{X}^\mu(\tau,\sigma)\right] e^{ik\cdot X(\tau,\sigma)} : \delta^{(2)}(\tau_1, \sigma_1) \tag{4.22a}$$

$$= \; : \left[\eta_\mu \dot{X}^\mu(\tau_1,\sigma_1)\right] e^{ik\cdot X(\tau_1,\sigma_1)} : , \tag{4.22b}$$

where the dot again means a derivative with respect to τ. This vertex operator can be used to calculate scattering amplitudes and other quantities in the same fashion as the tachyon case. The photon vertex operator for open strings satisfies

$$M^2 = -k^2 = 0, \tag{4.23}$$

and it is Weyl-invariant if it has polarization

$$k^\mu \eta_\mu = 0. \tag{4.24}$$

Of course we can take this one step further and look at strings at the first massive level and so on, but in practice most calculations are done only for the lightest string states only. As we mentioned before, it requires a huge amount of energy to create massive string states in comparison to particles that we observe in nature. Therefore, considering vertex operators for light string only should be a good approximation to describe interactions.

4.3 Tachyon tree-diagrams for closed strings

4.3.1 Closed string tachyon vertex operators

We showed that the topology of the world-sheet for closed strings is completely different from that of the open string. Interactions cannot be described by punctures on the boundary, but will become punctures on the interior of the world-sheet. Since we cannot set $\sigma = 0$ or $\sigma = \pi$, and cannot fix τ either, we have to integrate over the entire world-sheet. Therefore vertex operators for closed string tachyons are written as

$$V_{tachyon,c} = 2g_s \int_M d^2\sigma \sqrt{h} e^{ik\cdot X}, \tag{4.25}$$

where we have now introduced a *closed string coupling constant* g_s, which is obtained when a closed string is added to the world-sheet. Since we now have to integrate over the entire world-sheet, we have to include the world-sheet metric $h_{\mu\nu}$ as well. For the moment, we have also assumed a general world-sheet metric, and have therefore dropped the minus sign in the square root. As soon as a Minkowski signature is implied, the minus sign is reintroduced again. Switching to complex coordinates, the vertex operator can be written as

$$V_{tachyon,c} = g_s \int d^2z :e^{ik\cdot X}: . \tag{4.26}$$

There turns out to be a relation between the open and closed string coupling, namely $g_s = g_o^2$. Why this is the case can be seen as follows. Adding an exteral closed string (or *closed string source*) to the world-sheet, comes down to making a puncture in the conformally equivalent compactified space M. In other words, the topology of the world-sheet changes because a *boundary* is added. Adding another closed string to the world-sheet, due to a *loop interaction* comes down to adding a *handle* to the world-sheet.

In the open string case, we can add an exteral open string (or *open string source*) to the world-sheet. When we do so, the topology of the world-sheet doesn't change, but we do add two *corners* to the world-sheet. And adding an open string to the world-sheet due to a loop interaction means that we are adding a *boundary* to the world-sheet.

There is a famous topological invariant that can be used to describe these sorts of topological changes of a manifold M, called the *Euler characteristic* $\chi(M)$. The Euler characteristic is a quantity that depends on the above mentioned topological concepts, such as *handles, boundaries* and also *cross-caps* (to be discussed in chapter 7). When amplitudes are calculated by means of a path integral, one can show that the Euler characteristic of M determines the power of the string couplings g_s and g_o in the path integral.

So adding strings comes down to changing the topology of world-sheets. This, in turn, determines the power of the string coupling in the path integral. In [11], equal interaction processes for open and closed strings are compared, and in this way it is shown that the string couplings are indeed related through the equality $g_s = g_o^2$. In chapter 7 we will work this out in more detail and show that the coupling in string theory actually becomes a spacetime dependent function.

4.3.2 Closed massless string vertex operator

The vertex operator for a *massless closed string* is a bit more complicated. Unlike the open string case, we cannot use just one derivative to contract with the photon polarization to obtain a scalar. Therefore, we have to use two derivatives. Moreover, operators which are included in the path integral (2.8b) need to respect the local diff×Weyl symmetry of the system. A nice way to achieve this effect is to introduce *renormalized operators* $[\mathcal{F}]_r$. These operators have the property that they are automatically diff-invariant. Weyl-invariance needs to be checked by hand though. See appendix A for more details on renormalized operators.

By using renormalized operators, the most general vertex operator for massless closed strings (at fixed momentum) that is diff-invariant is given by

$$V_{massless,c} = \frac{g_s}{\alpha'} \int d^2\sigma\, \sqrt{h}\big\{(h^{ab}s_{\mu\nu} + i\epsilon^{ab}a_{\mu\nu})[\partial_a X^\mu \partial_b X^\nu e^{ik\cdot X}]_r + \alpha' \phi R[e^{ik\cdot X}]_r\big\}. \quad (4.27)$$

Here $s_{\mu\nu}$ is a symmetric matrix, $a_{\mu\nu}$ is an antisymmetric matrix, ϕ is a constant and R is the world-sheet Ricci scalar. Furthermore ϵ^{ab} is an antisymmetric tensor, normalized such that $\sqrt{-h}\epsilon^{12} = 1$. In the most general case, one has to sum over all different momenta, but we will just be looking at fixed momentum now.

Later on, we will find that $s_{\mu\nu}$, $a_{\mu\nu}$ and ϕ play an important rôle when we introduce background fields to the theory. It can be shown that the symmetric matrix $s_{\mu\nu}$ actually represents the polarization of a graviton state (see [2], [11], or [14] for further reference on this subject). We can also see that the term including ϕ is one order higher in α' than the other terms. This is because it is lacking the factors coming from the fields X^μ. It can be shown that the matrices have polarizations

$$k^\mu s_{\mu\nu} = \quad k^\mu s_{\nu\mu} = 0, \quad (4.28a)$$

$$k^\mu a_{\mu\nu} = -k^\mu a_{\nu\mu} = 0. \quad (4.28b)$$

Since we have used renormalized operators in (4.27), diff-invariance is automatically satisfied. But Weyl invariance needs to be checked out by hand. Therefore, we apply a Weyl transformation to this vertex operator. When we do so, the Ricci scalar transforms, as well as the renormalized operators (see appendix A how renormalized operators transform under Weyl transformations). The variation of the renormalized operators introduces several derivatives with respect to the variation $\delta\omega$. Also a renormalization parameter γ is obtained and, by choosing the proper renormalization, can take abritrary values. Working out the details, one can show that the variation of the vertex operator to first order in $\delta\omega$ is equal to

$$\delta_W V_{massless,c} = \frac{g_s}{2}\int d^2\sigma\,\sqrt{h}\delta\omega\big\{(h^{ab}S_{\mu\nu} + i\epsilon^{ab}A_{\mu\nu})[\partial_a X^\mu \partial_b X^\nu e^{ik\cdot X}]_r + \alpha' F R[e^{ik\cdot X}]_r\big\}, \quad (4.29)$$

where $S_{\mu\nu}$, $A_{\mu\nu}$ and F are defined by

$$S_{\mu\nu} = -k^2 s_{\mu\nu} + k_\nu k^\omega s_{\mu\omega} + k_\mu k^\omega s_{\nu\omega} - (1+\gamma)k_\mu k_\nu s^\omega{}_\omega + 4k_\mu k_\nu \phi, \quad (4.30a)$$

$$A_{\mu\nu} = -k^2 a_{\mu\nu} + k_\nu k^\omega a_{\mu\omega} - k_\mu k^\omega a_{\nu\omega}, \quad (4.30b)$$

$$F = (\gamma - 1)k^2\phi + \frac{1}{2}\gamma k^\mu k^\nu s_{\mu\nu} - \frac{1}{4}\gamma(1+\gamma)k^2 s^\nu{}_\nu. \quad (4.30c)$$

In this case the renormalization parameter $\gamma = -\frac{2}{3}$, but sometimes it is convenient to explicitly keep it for other calculations. We will not have need for it in the course of this thesis though.

In the next chapter we will be looking at strings with nontrivial backgrounds. It turns out that the form of the massless closed string vertex operator, (4.27), will be playing a very important rôle in establishing the equations of motion for strings with backgrounds. This will be essential for finding a low energy effective action for closed strings (introduced in chapter 6), one of the building blocks for noncritical string theory.

Chapter 5

Strings with backgrounds

5.1 Strings in curved spacetime

5.1.1 Nonlinear sigma model

Up till now we have only considered string in flat spacetime, in absence of any other backgrounds. More realistic models will include a curved spacetime (related to gravitation). Moreover, an antisymmetric tensor (closely related to electromagnetic fields) and a dilaton (a scalar background field) can be added to the theory.[1] In a couple of calculations and derivations to come, we will be using techniques involving curvature. Readers who like to review some general facts and definitions in curved space mathematics can turn to appendix B.

First we will try to incorporate a curved spacetime into string theory. Recall that the world-sheet action of a string moving in D flat dimensions is given by the Polyakov action (1.6), where the world-sheet metric h^{ab} has the signature $(-,+)$. In Minkowski spacetime, the flat metric $\eta_{\mu\nu}$ has signature $(-,+,+,\ldots,+)$. We know from general relativity that if we want to describe a theory in curved spacetime, we replace the flat metric $\eta_{\mu\nu}$ by a more general metric $G_{\mu\nu}(X)$, which describes a curved spacetime. We could try to do the same thing in string theory. This would imply that the same replacement on the Polyakov action yields

$$S_\sigma = \frac{1}{4\pi\alpha'} \int_M d^2\sigma \sqrt{h} h^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu. \tag{5.1}$$

This theory is known as the *nonlinear sigma model*. One could wonder, though, whether we are allowed to makes such a replacement at all. As we will show in the following, we are.

[1]We should note that these fields have not yet been observed in nature though.

5.1.2 Coherent background of gravitons

We know that the graviton is a state of a string itself. So writing down a curved spacetime metric should somehow be equivalent to a background of graviton states. In other words, we want to show that a curved spacetime really is really the same thing as a coherent background of graviton states. Therefore, consider a curved spacetime metric that is close to flat spacetime, so

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + \chi_{\mu\nu}(X), \tag{5.2}$$

with $|\chi_{\mu\nu}(X)| \ll 1$. If we write down the world-sheet path integral and expand around $\eta_{\mu\nu}$ we obtain

$$\begin{aligned} \exp(-S_\sigma) &= \exp\left[-\frac{1}{4\pi\alpha'}\int_M d^2\sigma\,\sqrt{h}h^{ab}(\eta_{\mu\nu}+\chi_{\mu\nu})\partial_a X^\mu \partial_b X^\nu\right] \\ &= \exp(-S_P)\left[\sum_{k=0}^{\infty} -\frac{1}{4\pi\alpha' k!}\int_M d^2\sigma\,\sqrt{h}h^{ab}\chi_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\right]^k \\ &\approx \exp(-S_P)\left[1-\frac{1}{4\pi\alpha'}\int_M d^2\sigma\,\sqrt{h}h^{ab}\chi_{\mu\nu}\partial_a X^\mu \partial_b X^\nu)\right]. \end{aligned} \tag{5.3}$$

Comparing our result with the vertex operator for massless closed strings, (4.27), we see that to first order, χ is equal to the vertex operator for the graviton state if we make the identification

$$\chi_{\mu\nu}(X) = -4\pi g_s e^{ik\cdot X} s_{\mu\nu}. \tag{5.4}$$

In other words, a very small fluctuation around flat spacetime gives rise to the vertex operator of a graviton, so it creates a graviton state!

So a remarkable thing has happened here. Replacing the flat spacetime metric $\eta_{\mu\nu}$ by a general metric $G_{\mu\nu}(X)$ comes down to exponentiating the graviton vertex operator, creating a coherent background of gravitons. Therefore we are indeed allowed to replace the flat metric by a general metric $G_{\mu\nu}(X)$ when we want to describe strings moving through a curved spacetime.

5.2 Other background fields and β functions

5.2.1 β functions up tp first order in background fields

The idea of including backgrounds into the string world-sheet action can be generalized. If we take a look at (4.27) we see that we can also include more backgrounds for massless strings. Namely an antisymmetric tensor field $B_{\mu\nu}(X)$, which is called the *Kalb-Ramond field*, and a scalar field $\Phi(X)$, called the *dilaton*. Following the example of curved spacetime, we can also incorporate these other backgrounds to the theory. This leads to

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_M d^2\sigma\,\sqrt{h}\Big[\Big(h^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X)\Big)\,\partial_a X^\mu \partial_b X^\nu + \alpha' R\Phi(X)\Big]. \quad (5.5)$$

It can be shown (see [11]) that this action still respectes the gauge symmetries. Therefore, this action is also Weyl-invariant. In chapter 2 we showed that this implies the β function to vanish. Since this action can now be varied to *three* variables $G_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$, we expect three β functions to arise.

Solving the β functions to full generality is hard, if not impossible. The best approach is to consider very small fluctuations of the background fields again. When we do this, we can calculate the β functions to first order, second order, etc. We will start with the simplest approximation for the β functions.

Let's consider the case in where $B_{\mu\nu}(X)$ and $\Phi(X)$ are small, Just as in the case of the curved spacetime background. Writing the action as $S_\sigma = S_P - V_1 + \ldots$, we find that we can identify

$$G_{\mu\nu}(X) = \eta_{\mu\nu} - 4\pi g_s s_{\mu\nu} e^{ik\cdot X}, \quad (5.6a)$$

$$B_{\mu\nu}(X) = -4\pi g_s a_{\mu\nu} e^{ik\cdot X}, \quad (5.6b)$$

$$\Phi(X) = -4\pi g_s \phi e^{ik\cdot X}. \quad (5.6c)$$

To first order in $\chi_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$, the Weyl variation of the action with background fields is given by (4.29). For convenience, we can set the renormalization parameter $\gamma = 0$, by choosing a proper renormalization.

So, when we perform a Weyl variation of the vertex operator $V_{massless,c}$, we know how the background fields enter this variation. It's not so hard now to relate this variation (4.29) to our prior expression of $T^a_{\ a}$ (2.12) in the path integral approach. Comparing the two, will tell us exactly what the Weyl anomaly looks like! We find that

$$T^a_{\ a} = -\frac{1}{2\alpha'}\left(\beta^G_{\mu\nu} h^{ab} + i\beta^B_{\mu\nu}\epsilon^{ab}\right)\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta^\Phi R, \quad (5.7)$$

where

$$\beta^G_{\mu\nu} \approx -\frac{\alpha'}{2}\left(\partial^2\chi_{\mu\nu} - \partial_\nu\partial^\omega\chi_{\omega\nu} - \partial_\mu\partial^\omega\chi_{\omega\nu} + \partial_\mu\partial_\nu\chi^\omega_{\ \omega}\right) + 2\alpha'\partial_\mu\partial_\nu\Phi, \quad (5.8a)$$

$$\beta^B_{\mu\nu} \approx -\frac{\alpha'}{2}\partial^\omega H_{\omega\mu\nu}, \quad (5.8b)$$

$$\beta^\Phi \approx \frac{D-26}{6} - \frac{\alpha'}{2}\partial^2\Phi. \quad (5.8c)$$

We have used the fact that

$$\partial^\omega \chi_{\mu\nu} = -4\pi g_s i k^\omega e^{ik\cdot X} s_{\mu\nu}, \tag{5.9a}$$

$$\partial^\omega \Phi = -4\pi g_s i k^\omega e^{ik\cdot X} \phi, \tag{5.9b}$$

and

$$H_{\omega\mu\nu} = \partial_\omega B_{\mu\nu} + \partial_\mu B_{\nu\omega} + \partial_\nu B_{\omega\mu}. \tag{5.9c}$$

$H_{\omega\mu\nu}$ can be seen as a generalization of the electromagnetic field tensor $F_{\mu\nu}$.

Weyl invariance implies the energy-stress tensor to be traceless. This now leads to the condition

$$\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0. \tag{5.10}$$

So we see that the presence of background fields, yield more complicated restrictions than simply saying $D = 26$. However, it is not untill the next order of the β functions that we find a nice interpretation of these restrictions.

5.2.2 β functions up to first order in α'

As we said before, we only considered the variation to first order in $\chi_{\mu\nu}(X)$, $B_{\mu\nu}(X)$ and $\Phi(X)$. Of course we can take this one step further and consider variations to second order in the background fields. One can show [11] that in this case, the β functions yield

$$\beta^G_{\mu\nu} = \alpha' \left(\mathbf{R}_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu\lambda\omega} H_\nu{}^{\lambda\omega} \right) + O(\alpha'^2), \tag{5.11a}$$

$$\beta^B_{\mu\nu} = \alpha' \left(-\frac{1}{2} \nabla^\omega H_{\omega\mu\nu} + \nabla^\omega \Phi H_{\omega\mu\nu} \right) + O(\alpha'^2), \tag{5.11b}$$

$$\beta^\Phi = \frac{D-26}{6} + \alpha' \left(-\frac{1}{2} \nabla^2 \Phi + \nabla_\omega \Phi \nabla^\omega \Phi - \frac{1}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right) + O(\alpha'^2). \tag{5.11c}$$

The terms in (5.11) are now made covariant, and furthermore $\mathbf{R}_{\mu\nu}$ is the Ricci tensor for spacetime, instead of the world-sheet Ricci tensor R_{ab}. Of course, Weyl invariance again implies (5.10).

We can make two important observations to make here. The first one is that, eventhough we worked up to second order in background fields, we have found the exact expressions for the β functions up to *first order in α'*. Every order in α' corresponds to a different energy level of the theory. The higher the order in α', the higher the energy level. So we could say that the conditions (5.11) represent the *low energy limit* for this theory.

The second observation is a remarkable feature that arises in string theory. By demanding the β functions to vanish, we obtain actual equations of motion in spacetime! Notice that the equation $\beta^G_{\mu\nu} = 0$ resembles Einstein's equation (B.21), with source terms coming from $\Phi(X)$ and $B_{\mu\nu}(X)$. Also notice that the equation $\beta^B_{\mu\nu} = 0$ is a generalization of Maxwell's equation.

Therefore it surely looks like these equations are very sensible equations and we are still on the right track. We will come back to this absolutely non-trivial result in chapter 6.

Chapter 6

Low energy effective action

6.1 The string metric

6.1.1 Equations of motion

In the previous chapter we have derived the expressions for the β functions, and shown that these have to vanish in order for our theory to be Weyl invariant. This in turn gave us some useful equations, namely (5.10). You could say that these equations describe the low energy physics of the theory. When we derived these equations, we started from the world-sheet action for the massless closed string S_σ, (5.5).

However, there is another way to obtain the same equations (and therefore physics). Up to now we have only worked with the *world-sheet action*, and as discussed before, this action can be seen as a two dimensional interacting field theory. We also know that excitations of the string (except the photon), create massive string modes and take a lot of energy to be created. Since in nature, we haven't encountered these massive particles yet, it is not such a bad idea to consider just the lightest modes of the theory and approximate this by an effective action. Consider the following *spacetime action* $\mathbf{S}$,

$$\mathbf{S} = \frac{1}{2\kappa_0^2}\int d^D x\,\sqrt{-G^{(S)}}e^{-2\Phi}\Big[-\frac{2(D-26)}{3\alpha'} + \mathbf{R}^{(S)} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi + O(\alpha')\Big], \quad (6.1)$$

where $G^{(S)}_{\mu\nu}$ and $\mathbf{R}^{(S)}$ are now the *spacetime metric* and *spacetime Ricci scalar* respectively.[1] It is not immediately obvious, but this action actually describes the same low energy behaviour as the world-sheet action for the massless closed string S_σ! In order to prove this, we need to vary this action

[1] The label S just means that we are working in the string frame, but this will be explained shortly.

in all it's arguments. First we do an infinitesimal coordinate transformation

$$X^\lambda \longrightarrow X^\lambda + \xi^\lambda(X). \tag{6.2}$$

With this variation, the other fields in the Lagrangian vary as

$$G_{\mu\nu}(X) \longrightarrow G_{\mu\nu}(X) + \delta G_{\mu\nu}(X), \quad \text{with} \quad \delta G_{\mu\nu}(X) = \xi^\rho(X)\partial_\rho G_{\mu\nu}(X), \tag{6.3a}$$

$$B_{\mu\nu}(X) \longrightarrow B_{\mu\nu}(X) + \delta B_{\mu\nu}(X), \quad \text{with} \quad \delta B_{\mu\nu}(X) = \xi^\rho(X)\partial_\rho B_{\mu\nu}(X), \tag{6.3b}$$

$$\Phi(X) \longrightarrow \Phi(X) + \delta\Phi(X), \quad \text{with} \quad \delta\Phi(X) = \xi^\rho(X)\partial_\rho \Phi(X). \tag{6.3c}$$

Now we substitute this into (6.1) and consider all variations $\delta G_{\mu\nu}(X)$, $\delta B_{\mu\nu}(X)$ and $\delta\Phi(X)$, up to first order in $\xi^\lambda(X)$. This is however, a very tricky and lengthly calculation.[2] It will be helpful to make use of the following equalities

$$\delta\sqrt{-G} = -\frac{1}{2}\sqrt{-G}G_{\mu\nu}\delta G^{\mu\nu}, \tag{6.4a}$$

$$\delta G^{\mu\nu} = -G^{\nu\rho}G^{\mu\lambda}\delta G_{\lambda\rho}, \tag{6.4b}$$

$$\delta(e^{-2\Phi}) = -2e^{-2\Phi}\delta\Phi, \tag{6.4c}$$

$$\delta\mathbf{R} = \mathbf{R}_{\mu\nu}\delta G^{\mu\nu} + \nabla^\mu\nabla^\nu\delta G_{\mu\nu} - G^{\mu\nu}\nabla^2\delta G_{\mu\nu}. \tag{6.4d}$$

To keep the calculation up to first order, one has to make repeated use of the fact that

$$\delta\mathbf{S}[AB] = -\frac{1}{2\kappa_0^2\alpha'}\int d^Dx\ [(A+\delta A)(B+\delta B) - AB] \tag{6.5a}$$

$$= -\frac{1}{2\kappa_0^2\alpha'}\int d^Dx\,[A\delta B + (\delta A)B]. \tag{6.5b}$$

If all the terms are properly taken into account, the variation of (6.1) finally becomes

$$\delta\mathbf{S} = -\frac{1}{2\kappa_0^2\alpha'}\int d^Dx\,\sqrt{-G}e^{-2\Phi}\Big[\delta G_{\mu\nu}\beta^{G\mu\nu} + \delta B_{\mu\nu}\beta^{B\mu\nu} + (2\delta\Phi - \tfrac{1}{2}G^{\mu\nu}\delta G_{\mu\nu})(\beta^{G\omega}_{\ \ \omega} - 4\beta^\Phi)\Big], \tag{6.6}$$

[2]See [11] for further reference.

where $\beta^{G\mu\nu}$, $\beta^{B\mu\nu}$ and β^{Φ} are the β functions, defined in (5.11).

Form this it immediately follows that if we want $\delta\mathbf{S}$ to vanish, all separate terms have to vanish, and therefore all the β functions have to be zero. So in fact, the equations of motion of the spacetime action produce the exact same world-sheet results, (5.10), and therefore describe the same low energy limit of this theory! This action is known as the *low energy effective action.* The spacetime metric used in this action is known as the *string metric*, and is often written as $G^{(S)}_{\mu\nu}(X)$.

One has to keep in mind that, eventhough effective actions can provide us with useful and sometimes new features of a theory, they do not always provide us with the correct answers. When this is the case, one needs to include higher order terms in the action in order to obtain correct answers.

6.1.2 Spacetime dependent coupling

In section 2.1.3, we said that in quantum field theory, a coupling appears in a Lagrangian as a factor $\frac{1}{g^2}$. Now that we have found a spacetime string action, we can also consider this coupling. Comparing the coupling from quantum field theory with the string action, (6.1), we see that we can make the identification $g_s = e^{\Phi(X)}$. In other words, the *string coupling* is a spacetime dependent function! Therefore, we can say that, at least in the low energy limit, the string coupling is equal to the exponent of the dilaton field.

6.2 Link between β functions and EOM

In the previous section we showed that the claim for conformal invariance on the world-sheet somehow was similar to some equations of motion (EOM) for the spacetime action. It is absolutely not trivial that this should be the case. One way to look at this is the following.

If we write down a path integral for the world-sheet action (5.5), and we integrate out the fields X^μ, we end up with a path integral that looks very similar to that of the low energy effective action (6.1). So

$$e^{-S[G,B,\Phi]} = \int \mathcal{D}X \, e^{-S[X,G,B,\Phi]}. \tag{6.7}$$

If we now vary the left as well as the right hand side with respect to the metric $G_{\mu\nu}$, we get

$$\frac{\delta S}{\delta G_{\mu\nu}} e^{-S[G,B,\Phi]} = \int \mathcal{D}X \, V_{graviton}(Z_0) e^{-S[X,G,B,\Phi]}. \tag{6.8}$$

Recall that in section 5.1 the origin of the graviton vertex operator is made explicit. Z_0 is the point on the complex plain where the vertex operator arises.

Now if we take a look at this equation, we see that both the left-hand side and that right-hand side are equal to zero. First of all, the right hand side is zero, because a one-point function is always zero,[3]

$$\int \mathcal{D}X\, V_{graviton}(Z_0) e^{-S[X,G,B,\Phi]} = \langle V_{graviton}(Z_0)\rangle = 0. \qquad (6.9)$$

And second of all, the left hand side being zero exactly corresponds to the equations of motion found for the spacetime action.

So, we see that this derivation implies a connection between spacetime equations of motion and vanishing β functions.

6.3 The Einstein metric

6.3.1 Effective action in the Einstein frame

See appendix B for how to write down an action for curved spacetime in general relativity. In the absence of the matter part and cosmological constant, it is written slightly different, namely

$$\mathbf{S}_H[G^{(E)}] = \int d^D x\, k\sqrt{-G^{(E)}}\,\mathbf{R}^{(E)}, \qquad (6.10)$$

where $G^{(E)}_{\mu\nu}$ means the spacetime metric in the *Einstein frame*, which will be explained in a short while and $\mathbf{R}^{(E)}$ is the Ricci scalar constructed from $G^{(E)}_{\mu\nu}$.[4]

The effective action (6.1) has the same Einstein-Hilbert term, $\sqrt{-G^{(S)}}\mathbf{R}$, although multiplied with an factor $e^{-2\Phi}$. We can, however, perform a few simple transformations in order to put the Hilbert term in exactly the same form as (6.10). Let

$$G^{(E)}_{\mu\nu}(x) = e^{2\omega(x)} G^{(S)}_{\mu\nu}(x), \qquad (6.11)$$

which relates $G^{(E)}_{\mu\nu}$ and $G^{(S)}_{\mu\nu}$ by an overall rescaling of the metric, thus a Weyl transformation. Since the Ricci scalar is constructed from the metric, it also transforms, according to (see [11])

$$\mathbf{R}^{(E)} = \exp(-2\omega)\left[\mathbf{R}^{(S)} - 2(D-1)\nabla^2_E\omega - (D-2)(D-1)\partial_\mu\omega\partial^\mu\omega\right], \qquad (6.12)$$

where $\mathbf{R}^{(E)}$ is the Ricci scalar, constructed from the Einstein metric $G^{(E)}_{\mu\nu}$ and $\mathbf{R}^{(S)}$ is the Ricci scalar, constructed from the string metric $G^{(S)}_{\mu\nu}$. Also,

[3]The graviton vertex operator is translation invariant and scale invariant. Translation invariance implies that the vertex operator is constant. Then scale invariance requires this constant to be zero.

[4]$k = (16\pi G_N)^{-1} \approx 2.95 \times 10^{36} GeV^2$ in units where $c = 1$.

indices are raised with $G^{(E)}_{\mu\nu}(X)$. Next define

$$\omega = -\frac{2\bar{\Phi}}{D-2}, \tag{6.13a}$$

$$\text{with} \qquad \bar{\Phi} = \Phi - \Phi_0, \tag{6.13b}$$

such that $\bar{\Phi}$ has vanishing expectation value. Putting this together, we can rewrite the spacetime action (6.1) as

$$\mathbf{S} = \frac{1}{2\kappa^2}\int d^DX\,\sqrt{-G^{(E)}}\Bigg[-\frac{2(D-26)}{3\alpha'}e^{4\bar{\Phi}/(D-2)} + \mathbf{R}^{(E)} - \frac{1}{12}e^{-8\bar{\Phi}/(D-2)}H_{\mu\nu\lambda}\tilde{H}^{\mu\nu\lambda} - \frac{4}{D-2}\partial_\mu\bar{\Phi}\tilde{\partial}^\mu\bar{\Phi} + O(\alpha')\Bigg], \tag{6.14}$$

where the tildes mean that the indices are raised with $G^{(E)}_{\mu\nu}(X)$. As you can see the Hilbert term $\sqrt{-G^{(E)}}\mathbf{R}$ has been put in the exact same form as the Hilbert action (6.10). When the spacetime action has been put in this form, the metric $G^{(E)}_{\mu\nu}(X)$ is referred to as the *Einstein metric*. In this case,

$$\kappa = \kappa_0 e^{\Phi_0}\left(= \frac{1}{\sqrt{2k}}\right) \tag{6.15}$$

is the observed gravitational coupling constant.[5]

6.3.2 Utility of the Einstein frame

When we want to calculate actual physical quantities and compare them with experimental data, we use the Einstein metric. This is due to the fact that when we observe experiments, the measurements experience the Einstein metric. However, a string moving through spacetime experiences the string metric. The string metric does have nicer symmetries though, but there is no preferred metric, and both shall be used later on.

As we shall see, effective actions play a major rôle in string theory. In this chapter we only considered the low energy effective action for massless (closed) strings, since we assume that this low energy limit gives a good approximation of the physics we observe. Eventhough tachyons are not thought to represent actual physical particles, it is useful to consider their interactions with other strings. We could therefore, of course, also consider a low energy effective action for tachyons. We will do so in the next chapter, where tachyons have interactions with strings in a dilaton background.

[5] In $D = 4$ this has the value $4.11 \times 10^{-19} GeV^{-1}$.

Part II

Applications on noncritical string theory

Chapter 7

Away from the critical dimension

7.1 Constant dilaton

7.1.1 Constant dilaton action

So far we have considered strings in general backgrounds, different from flat spacetime. However, we have not yet given any concrete examples. As a warm-up we'll consider the simplest case for the dilaton, the *constant dilaton*. For this derivation, it's useful to switch to complex coordinates. Doing so and looking at the massless world-sheet action (5.5), we see that the dilaton enters the theory as

$$S_\Phi = \frac{1}{4\pi}\int_M d^2z\,\sqrt{h}\Phi(X)R, \tag{7.1}$$

where R is the Ricci scalar from the two dimensional world-sheet. Let's focus on the case where is constant, $\Phi(X) = \Phi_0$.[1] First of all, when the dilaton $\Phi(X)$ is constant, the integrant can locally be written as a total derivative. Secondly, one can show that in the case of a constant dilaton, the action is invariant under variations of the metric ($h_{ab} \to h_{ab} + \delta h_{ab}$).[2] This means that the value of the integral only depends on the global topology of the world-sheet and therefore does not contribute to classical field equations.

7.1.2 Euler characteristic

A nice feature about this action is that it actually is a very famous topological invariant, named the *Euler characteristic* of M, $\chi(M)$ (for more

[1]Notice that we haven't said anything about the topology of the world-sheet.

[2]This is due to the fact that when we vary the action with respect to the metric, we obtain the Einstein equations in two dimensions, multiplied with δh_{ab}. In two dimensions, the Einstein equations are always equal to zero. N.B. This tells us that there can not be gravity in $D = 2$.

information on this subject see [2], [11] or [10]). So

$$S_\Phi = \frac{\Phi_0}{4\pi}\int_M d^2z\,\sqrt{h}R \tag{7.2a}$$

$$= \Phi_0\chi(M) \tag{7.2b}$$

$$= \Phi_0(2-2n_h-n_b-n_c), \tag{7.2c}$$

where n_h is the number of handles (also called *genus* g), n_b the number of boundaries, and n_c the number of cross-caps of M.[3] For example, a sphere has no handles, no boundaries and no cross-caps. The simplest case, therefore, is the Euler characteristic for a sphere, $\chi(sphere) = 2$. There are two cases with $\chi(M) = 1$, namely the disk, which has one boundary, and the projective plane, which has one cross-cap. A projective plane can be constructed by taking a disk and identifying opposite points on the boundary as equivalent. Furthermore there are four topologies that have Euler characteristic $\chi(M) = 0$, namely a *torus* (one handle, no boundaries or cross-caps), an *annulus* (two boundaries, no handles or cross-caps), the Moebius strip (one boundary and one cross-cap), and finally a *Klein bottle* (two cross-caps, no handles or boundaries. More complicated topologies allow for negative Euler characteristics, but the main idea should be clear now.

7.1.3 UV finite quantum gravity

The simple idea we just discussed, actually has much deeper consequences, and this is one of the example where string theory shows it's true power. As we know from quantum field theory, interactions can be described by Feynman diagrams. Each order in the perturbation theory is determined by the number of loops in the diagram. When amplitudes are calculated, ultraviolet divergences appear, which need to be dealt with. Now consider a theory of closed, oriented strings (in $D = 10$, superstring theories with these properties are type II or heterotic theories). Since the theory is closed, the world-sheet does not have any boundaries. Also, the theory must be free from cross-caps, because a cross-cap would render the theory unorientable. So the topology of the world-sheet for these theories is completely determined by the genus n_h. This also means that when we calculate the partition function, the sum over different metrics just turns into a sum over

[3] A cross-cap can be obtained by diametrically identifying points on opposite sides of boundaries, as is done in the same manner with the Moebius strip. Another definition is that we consider the complex plane, we can cut a hole with radius slightly less than one and identify z and $-\frac{1}{\bar{z}}$ to be equivalent.

different number of handles of the theory, so

$$Z_{\Phi=\Phi_0} = \int [dX] \exp(S_\Phi) \tag{7.3a}$$

$$= \sum_{topologies} \exp[\Phi_0(2-2n_h)] \tag{7.3b}$$

$$= \sum_{topologies} g_s^{2-2n_h}, \tag{7.3c}$$

where g_s is the string coupling.

Furthermore, on can say that interactions in string theory are smeared out in spacetime, softening the short-distance divergencies that arise in quantum field theory. Some very technical calculations (see [11]) show that at each order the amplitudes are free of ultraviolet divergences. This means that these theories (type II and heterotic theories) are actually ultraviolet finite theories of quantum gravity. So far, this is the only theory that has achieved this result!

7.2 Linear dilaton background

The second simplest case for the dilaton is the case where the dilaton $\Phi(X)$ is a linear function of X_μ. Or more specifically,

$$G^{(S)}_{\mu\nu}(X) = \eta_{\mu\nu}, \quad B_{\mu\nu}(X) = 0, \quad \Phi(X) = V_\mu X^\mu. \tag{7.4}$$

This theory is called the *linear dilaton background*, a theory in which spacetime is equal to Minkowski space, there is no antisymmetric background field, and the dilaton is linear in X^μ. The linear dilaton background is called *spacelike* if $V_\mu V^\mu > 0$ and *timelike* if $V_\mu V^\mu < 0$. In the literature, one often encounters the quantity q, with $-q^2 = V_\mu V^\mu$. Due to the simple structure of the theory, the linear dilaton background is a very useful model for studying string theories with backgrounds and, moreover, it is one of the most important ingredients for studying noncritical string theory.

One way to determine the critical dimension for a string in *flat Minkowski spacetime* was to look at the β functions (5.11). In that case $\beta^\Phi = 0$ simply says $D = 26$. In the case for strings with a *linear dilaton background*, the β functions $\beta^G_{\mu\nu}$ and $\beta^B_{\mu\nu}$ become trivial, but β^Φ yields

$$\frac{D-26}{6} + \alpha' V_\mu V^\mu = 0, \tag{7.5a}$$

$$\text{or} \qquad D = 26 - 6\alpha' V_\mu V^\mu, \tag{7.5b}$$

where we have used (5.11), so we considered the β functions up to first order in α'. But looking more closely at the β functions, we see that in the case of the linear dilaton background $\mathbf{R}_{\mu\nu}, H_{\mu\lambda\omega}, \nabla_\mu\nabla_\nu\Phi(X)$ and $\nabla^2\Phi(X)$

and all higher order derivatives vanish! So the result (7.5) actually is exact. As a side remark, note that this also leads to the equality

$$q = \left(\frac{D-26}{6\alpha'}\right)^{1/2}. \tag{7.6}$$

We can write down an expression for the energy-stress tensor of the conformal field theory in presence of a linear dilaton background. Going to the complex plane, the energy-momentum tensor is written as

$$T(z) = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu: + V_\mu \partial^2 X^\mu, \tag{7.7a}$$

$$\bar{T}(\bar{z}) = -\frac{1}{\alpha'} :\bar{\partial} X^\mu \bar{\partial} X_\mu: + V_\mu \bar{\partial}^2 X^\mu. \tag{7.7b}$$

In this way one can work out (see [11]) that the central charge indeed satisfies

$$c = \tilde{c} = D + 6\alpha' V_\mu V^\mu, \tag{7.8}$$

which actually is an exact result, in perfect agreement with the condition (7.5) for cancellation of the Weyl anomaly.

If the dilaton was set to zero, the world-sheet action would just reduce to the flat world-sheet Polyakov action (1.6), and indeed, (7.5) would reduce to the well know condition that $D = 26$. But, depending on whether V^μ is timelike, spacelike or null, D can now, in principle, take any value! Therefore string theories exist where $D \neq D_c$! These theories are called *noncritical string theories*, which will be the main focus for the remainder of this thesis.

- When $D < D_c$ a theory is called *subcritical*,
- When $D > D_c$ a theory is called *supercritical*.

This applies to bosonic, as well as superstring theories.

7.3 Tachyon profile

7.3.1 On-shell tachyon condition

As we saw in section 6 we can look for an effective action that produces the same physical behaviour as the world-sheet action does in the low energy limit. Said somewhat differently, we look for actions that effectively describe the corresponding (low energy) conformal field theory. One can look for all sorts of low energy effective actions in a lot of different processes. For example, closed strings, open strings, tachyons, massless string states, tree-level amplitudes, loop amplitudes, and so on.

When we look at the scattering amplitude of a massless string and two string tachyons, for example (see [11] for a detailed calculation), we find that this process is described by the corresponding low energy effective action $\mathbf{S} = \mathbf{S}_{massless} + \mathbf{S}_{\mathcal{T}}$, where $\mathbf{S}_{massless}$ is the massless closed string action (6.1), and $\mathbf{S}_{\mathcal{T}}$ is the *closed string tachyon action*,[4]

$$\mathbf{S}_{\mathcal{T}} = -\frac{1}{2}\int d^D x\,\sqrt{-G^{(S)}}e^{-2\Phi}\left(G^{(S)\mu\nu}\partial_\mu\mathcal{T}(x)\partial_\nu\mathcal{T}(x) - \frac{4}{\alpha'}\mathcal{T}^2(x)\right). \quad (7.9)$$

Now that we have this tachyon effective action, we can simply plug in the linear dilaton background. Subsequently we can vary this action, letting

$$\mathcal{T}(x) \longrightarrow \mathcal{T}(x) + \delta\mathcal{T}(x). \quad (7.10)$$

Working this out, we get

$$\delta\mathbf{S}_{\mathcal{T}} = -\frac{1}{2}\int d^D x\,e^{-2V_\rho x^\rho}\left(2\eta^{\mu\nu}\partial_\mu\mathcal{T}(x)\partial_\nu(\delta\mathcal{T}(x)) - \frac{8}{\alpha'}\mathcal{T}(x)\delta\mathcal{T}(x)\right) \quad (7.11a)$$

$$= -\int d^D x\,e^{-2V_\rho x^\rho}\left(2V^\mu\partial_\mu\mathcal{T}(x) - \partial_\mu\partial^\mu\mathcal{T}(x) - \frac{4}{\alpha'}\mathcal{T}(x)\right)\delta\mathcal{T}(x) \quad (7.11b)$$

and end up with the linearized tachyon field equation,

$$-\partial_\mu\partial^\mu\mathcal{T}(x) + 2V^\mu\partial_\mu\mathcal{T}(x) - \frac{4}{\alpha'}\mathcal{T}(x) = 0. \quad (7.12)$$

This equation is the condition for Weyl invariance of the linear dilaton energy-momentum tensor (7.7) and it ensures that the tachyon momentum is *on-shell*. The solution to this equation, the tachyon field, or *tachyon profile*, is

$$\mathcal{T}(x) = \mu^2\exp(B_\rho x^\rho), \quad (7.13a)$$

$$\text{with} \qquad (B-V)^2 = \frac{2-D}{6\alpha'}, \quad (7.13b)$$

where μ^2 is a parameter that determines the interaction strength.

[4]We need to be cautious with writing down an effective action for interactions between tachyons and massless string states though. An effective action gives a good approximation of interactions for low energy processes, meaning that we only look at light (massless) fields. It takes a lot of energy to create massive fields, so therefore they are omitted in this action. A tachyon, on the other hand, can have arbitrary large negative mass squared, which comes down to perturbating the theory around an unstable point in the vacuum. If the tachyon is given just a little bit of energy, it will start rolling down its potential to $\rightarrow -\infty$, and, in principle, is capable to create excited string states in interactions. Therefore, we should restrict this tachyon effective action to processes where time scales are such, that excited strings are not created.

7.3.2 Liouville field theory

A particular simple solution is obtained when we look at a linear dilaton background that is only dependent on one direction (the 1-direction, for example). In that case

$$V_\mu = \delta^1_\mu \left(\frac{26-D}{6\alpha'}\right)^{1/2}. \tag{7.14}$$

Writing out the solution for the on-shell condition, (7.13b), we obtain

$$\begin{aligned}
0 &= B_\mu B^\mu - 2V_\mu B^\mu + V_\mu V^\mu - \frac{2-D}{6\alpha'} && (7.15a)\\
&= (B_1)^2 + \tilde{B}^2 - 2B_1\left(\frac{26-D}{6\alpha'}\right)^{1/2} + \frac{26-D}{6\alpha'} - \frac{2-D}{6\alpha'} && (7.15b)\\
&= (B_1)^2 - 2B_1\left(\frac{26-D}{6\alpha'}\right)^{1/2} + \frac{4}{\alpha'} + \tilde{B}^2, && (7.15c)\\
\text{with} \qquad & \tilde{B}^2 = -(B_0)^2 + \sum_{k=2}^{D-1}(B_k)^2. && (7.15d)
\end{aligned}$$

We can solve this quadratic equation for B_1. If we do so, we end up with the most general solution for the tachyon profile,

$$\begin{aligned}
& B_1 = \alpha_- \qquad \vee \qquad B_1 = \alpha_+, && (7.16a)\\
\text{with} \qquad & \alpha_\pm = \left(\frac{26-D}{6\alpha'}\right)^{1/2} \pm \left(\frac{2-D}{6\alpha'} + \tilde{B}^2\right)^{1/2}. && (7.16b)
\end{aligned}$$

Depending on whether the tachyon profile is *timelike* (i.e. $B_\mu B^\mu < 0$), *spacelike* (i.e. $B_\mu B^\mu > 0$), or *null* (i.e. $B_\mu B^\mu = 0$), the tachyon can become a real exponential.

Let's now, for simplicity assume that $\tilde{B}^2 = 0$. In that case, we end up

$$B_1 = \alpha_\pm = \left(\frac{26-D}{6\alpha'}\right)^{1/2} \pm \left(\frac{2-D}{6\alpha'}\right)^{1/2}. \tag{7.17}$$

For $D > 2$, B_1 becomes complex and the tachyon oscillates. However, for $D \leq 2$ the tachyon profile becomes a real exponential, diverging at $x^1 \to +\infty$. We should keep in mind that the tachyon will have nonlinear corrections, but as it turns out they do not effect the qualitative form of the background. However, only the solution with α_- does not lead to a non-singular background (see [11] for reference). Therefore the tachyon profile can be written as

$$\mathcal{T}(x) = \mathcal{T}_0 \exp(\alpha_- x^1), \tag{7.18}$$

where $\mathcal{T}_0 = \alpha' \mu^2$ is again the interaction strength.

We can add this tachyon field to the world-sheet action. When we do so, the tachyon field become a potential on the world-sheet. Therefore, when considering world-sheets, the tachyon profile is sometimes referred to as a tachyon potential. In this linear dilaton background, the tachyon profile $\mathcal{T}(x)$ starts to act as a sort of barrier. This can be seen when we look at a world-sheet action, including a linear dilaton and tachyon background,

$$S_\sigma = \frac{1}{4\pi\alpha'} \int_M d^2\sigma \sqrt{h} \left[h^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + \alpha' R V_1 x^1 + \mathcal{T}_0 \exp(\alpha_- x^1) \right]. \tag{7.19}$$

In this theory, there is an asymptotic region $x^1 \to -\infty$, where the tachyon goes to zero, and is said to be weak. Here strings can interact freely with each other. But when x^1 becomes very large positive, the tachyon begins to dominate the linear dilaton in the action. This means that in this region the path integral is suppressed because of an effective tachyon potential, meaning that it becomes very hard for strings to penetrate this region. Strings propagating in a region where $x^1 \gg 1$ can interact with each other, but bounce off the potential barrier, back into asymptotic region! A theory of the form (7.19) is called a *Liouville field theory*, and the barrier is known as a *Liouville wall*.

The tachyon barrier acts as a sort of elastic wall, so when strings start to feel the barrier, they still travel a distance ΔL before they reflect off. One could say that the wall has thickness ΔL. In a lot of calculations one uses the approximation $\Delta L \to 0$. This approximation is called the *thin wall approximation*.

The exponential in the action renders the theory hard to solve due to quantum corrections. In chapter 10, we will again take a look at the tachyon dilaton theory. There, however, we will choose such a framework in which all quantum corrections vanish. The advantage of this framework is that in that case, the classical solutions become equal to the quantum solutions, rendering the results exact! Before we go there, we will first show a connection between a string theory with a linear dilaton background, and a theory of expanding cosmologies driven by quintessence.[5]

[5] To be explained in the next chapter.

Chapter 8

Quintessence-driven cosmologies

8.1 Quintessent cosmologies

8.1.1 Quintessence

As is often the case in string theory, results can have very close connections with cosmological features. The linear dilaton background that we studied in the previous chapter seems to be one of those cases. The authors in [6] have written a few articles, extensively studying string theory in the noncritical framework, making use of a linear dilaton background. As it turns out, timelike linear dilaton theories are really the same as expanding FRW cosmologies, driven by *quintessence*. In order to show this, we shall first discuss the principle of quintessence.

When Einstein derived his theory of general relativity and applied it to our universe, he noticed that his equations (B.19) would not allow a static solution of our universe, unless an extra constant term was added, namely the *cosmological constant* Λ. Even though including a cosmological constant leads to a static solution, this solution is an unstable equilibrium. A slight expansion of the universe will result in an accelerating expanding universe. And vice versa, a slight contractment of the universe will result in a continuing contracting universe.

However, soon after the introduction of this static solution, observations by *Edwin Hubble* showed that our universe actually is expanding. Therefore, there didn't seem to be any need for a cosmological constant anymore. Einstein discarded it, calling it "the biggest blunder of his life".

Ironically, the cosmological constant made it's comeback when observations in the late $1990's$ showed that the expansion of the universe is accelerating. Reintroducing a very small positive cosmological constant could account for this observation.[1] A positive cosmological constant causes a negative pressure to the universe and is also referred to as *dark energy.*

There are however, more models that could explain an accelerating universe, quintessence for example. Quintessence in physics is one or more scalar background fields $\phi^i(X)$, which can be added to the Lagrangian for the cosmology. It has a kinetic part and a potential part $\mathcal{V}(\phi)$, which is proportional to the exponential of the scalar field(s), and mimics the behaviour of the cosmological constant. Including quintessence, results in a hypothetical form of dark energy, postulated to explain observations of an expanding universe. Quintessent cosmologies are defined through the equation of state

$$w \equiv \frac{p_Q}{\rho_Q}, \tag{8.1}$$

where p_Q is the quintessence pressure and ρ_Q is the quintessence energy density. In general, w can be some complicated spacetime dependent function, but we will assume that our cosmology model has a constant equation of state w. One can show that accelerating expanding cosmologies satisfy w

$$-1 \leq w < w_{crit}, \tag{8.2}$$

where w_{crit} is a critical value for the equation of state. In $D = 4$ one finds $w_{crit} = -\frac{1}{3}$, but in general, w_{crit} will depend on the number of spacetime dimensions D, as we will show in the forthcoming.

The simplest way to generate quintessence models is to introduce just one real scalar field, which enters the action with a kinetic part and an exponential part,

$$\mathcal{V}(\phi) = c\exp(\gamma\phi), \tag{8.3}$$

where $c, \gamma > 0$ and γ determines the equation of state w.

8.1.2 FRW cosmologies in D dimensions

Our universe is made up of galaxies. When going to big enough length scales, the universe becomes homogenous and isotropic. This means that the universe can be seen as a *perfect fluid*, where the galaxies are the fluids particles. Moreover, since our universe is expanding, the spatial coordinates have time-dependence, which can be described by a *cosmological scale factor* $a(t)$. A system that meets these conditions can be described by Einstein's field equations (see appendix B). It's solution, called the *Friedmann-Robertson-walker cosmology* (FRW) reads

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^{D-2}d\Omega_{D-2}\right), \tag{8.4}$$

[1] $\Lambda \approx 10^{-120}$ in Planck units.

where k describes the spatial curvature, which can be -1, 0 or $+1$. Furthermore t is the called *FRW time*, and Ω_D represent the angle coordinates of a sphere in D dimensions with radius r.

For the derivation of the equations of motion for quintessent cosmological backgrounds in D dimensions, we will be working in the Einstein frame, discussed in chapter 6. Furthermore, we will restrict our discussion to the case where spacelike hyper surfaces are flat. So, in other words, $k = 0$. Going over to a Cartesian coordinate system, the metric for a spatially flat FRW cosmology is now simply given by

$$ds^2 = -dt^2 + \sum_{i=1}^{D-1} a^2(t) dx^i dx^i. \tag{8.5}$$

When $\ddot{a}(t) \left(= \frac{d^2 a}{dt^2}\right) > 0$, the expansion of the universe is accelerating, and when $\ddot{a}(t) < 0$, the expansion of the universe is decelerating.

Present day observations show that distant galaxies are redshifted, because of a Doppler effect due to their motion. The further a galaxy lies, the more redshifted is becomes. This means that the further galaxies are apart, the faster they drift apart. It should be noticed that this expansion is an expansion of spacetime itself, causing the galaxies to expand along with it. This phenomenon is described by *Hubble's law*, which reads

$$v(t) = H(t)d, \tag{8.6}$$

where $v(t)$ is the velocity of a galaxies at a distance d from the observer, and $H(t)$ is the Hubble parameter. The Hubble parameter can be directly measured[2], but is can also be calculated in terms of the scale factor $a(t)$, in which case it reads

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}. \tag{8.7}$$

It can be a very useful quantity in cosmology calculations. Let us try to investigate this a bit further.

Consider the Einstein equations (B.20), in the case of the FRW metric (8.5). Furthermore, take the energy-momentum tensor to be of the form (B.16), and recall that we assumed the curvature of spatial slices, k, to be zero. When we work out the Einstein equations in this setting, in D dimensions, we obtain two very famous equation, namely

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{2}{(D-1)(D-2)}\kappa^2 \rho, \tag{8.8a}$$

$$\text{and} \qquad \frac{\ddot{a}}{a} = -\frac{D-3+w(D-1)}{(D-1)(D-2)}\kappa^2 \rho, \tag{8.8b}$$

[2]Present day observations made by WMAP in 2008, show that the Hubble parameter is about $71.9^{+2.6}_{-2.7}\ kms^{-1}Mpc^{-1}$.

which are called the *Friedmann equations*. Indeed, for $D = 4$ (and reintroducing k), they reduce to the well known results

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\kappa^2\rho - \frac{k}{a^2}, \tag{8.9a}$$

and

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\kappa^2\left(\rho + 3p\right). \tag{8.9b}$$

We can immediately read off the Hubble parameter from (8.8a).

8.1.3 Determining the critical equation of state

In the following we derive the critical equation of state for a quintessence-driven cosmology. We will make use of the explicit form of the quintessent Lagrangian density. As we said before, the matter part of the theory enters the Lagrangian density as a kinetic part and a exponential part, thus

$$\mathcal{L}_\phi = \frac{1}{\kappa^2}\sqrt{-G^{(E)}}\left[\frac{1}{2}G^{(E)\mu\nu}\partial_\mu\phi\partial_\nu\phi - c\exp(\gamma\phi)\right], \tag{8.10}$$

where κ is again the gravitational coupling constant. With the action $S[\mathcal{L}_\phi]$, we are able to calculate the energy-momentum tensor $T_{\mu\nu}$, associated with this field ϕ. It reads

$$\begin{aligned} T_{\mu\nu} &\equiv \frac{-2}{\sqrt{-G^{(E)}}}\frac{\delta S}{\delta G^{(E)\mu\nu}} && (8.11a)\\ &= \partial_\mu\phi\partial_\nu\phi + G^{(E)}_{\mu\nu}\mathcal{L}_\phi. && (8.11b)\end{aligned}$$

If we now assume the scalar field ϕ to be isotropic and homogeneous, we can make the identifications

$$\rho_Q = T_{00} = \frac{1}{2}\dot{\phi}^2 + \mathcal{V}(\phi) \tag{8.12a}$$

$$p_Q = T_{ii} = \frac{1}{2}\dot{\phi}^2 - \mathcal{V}(\phi). \tag{8.12b}$$

At this point we will make explicit use of the fact that w is constant. By making this assumption, one can show that $\dot{\phi}^2$, H^2 and $\mathcal{V}$ all scale as t^{-2}. This, in turn, implies that $\phi(t)$ and $a(t)$ can be put into the form

$$\phi(t) = \lambda\ln\left(\frac{t}{t_1}\right), \tag{8.13a}$$

$$a(t) = a_0\left(\frac{t}{t_0}\right)^\alpha, \tag{8.13b}$$

for some constants t_0, t_1, α and λ.[3] If we make use of these explicit expressions, we can substitute them into the constraint equations (8.8) and

[3]See [6] for the technical details of this derivation.

the equation of state (8.1). Then, with some technical tricks we can finally determine the coefficients λ, α and γ. We will skip the details, but simply state that they yield

$$\alpha = \frac{2}{(1+w)(D-1)}, \tag{8.14a}$$

$$\gamma^2 = \frac{2(D-1)(w+1)}{D-2}, \tag{8.14b}$$

$$\lambda\gamma = -2. \tag{8.14c}$$

It is now easy to see that the cosmological scale factor $a(t)$ accelerates as a function of FRW time for $\alpha > 1$. Taking a close look at (8.14a), we see that this restricts w to

$$-1 \leq w < w_{crit}, \tag{8.15a}$$

$$\text{where} \qquad w_{crit} = -\frac{D-3}{D-1}. \tag{8.15b}$$

We have been able now, to determine the critical equation of state for a quintessence-driven cosmology in D dimensions. Substituting $D = 4$ yields the well-known result $w_{crit} = -\frac{1}{3}$.

8.2 Global structures in quintessent cosmologies

8.2.1 Global structures

So basically, if we construct a cosmology with quintessence and adopt the ansatz that the equation of state is constant, we find that this cosmology accelerates, if and only if w is bounded from above by w_{crit}. This upper bound, in turn, is dependent on the number of spacetime dimensions D. Notice that for $D \geq 4$, w_{crit} is always negative and that for large D, the range for w for an accelerating cosmology becomes asymptotically small.

There are three interesting cases that can be considered, $w < w_{crit}$, $w = w_{crit}$ and $w > w_{crit}$. In all three cases the spatial slice $t = 0$ defines an initial singularity. However, the behaviour at $t \to +\infty$ will depend on the equation of state.

The value of w determines the global structure of the cosmology. We can investigate this in greater detail by applying a coordinate transformation which puts the FRW metric (8.5) in a conformally flat form. Let's introduce a new coordinate $\bar{t}$,

$$\bar{t} \equiv \left(\frac{\xi}{a_0} t_0^{\frac{2}{(D-1)(1+w)}}\right) t^{\frac{1}{\xi}}, \tag{8.16a}$$

$$\text{where} \qquad \xi = \frac{(D-1)(1+w)}{(D-1)w + (D-3)}. \tag{8.16b}$$

With this new coordinate, the FRW metric can be written as

$$ds^2 = \omega(\bar{t})^2 \left[-d\bar{t}^2 + \sum_{i=1}^{D-1} dx^i dx^i \right] \tag{8.17a}$$

$$= \omega(\bar{t})^2 \left[-d\bar{t}^2 + dr^2 + r^{D-2} d\Omega_{D-2}^2 \right], \tag{8.17b}$$

where Ω_D are coordinates of a D dimensional sphere, with radius r. So, with this redefinition of t, we have shown that (8.5) in this setting actually is globally conformally equivalent to flat spacetime! Furthermore, $\omega(\bar{t})$ is given by

$$\omega(\bar{t}) \equiv l \left[\frac{\bar{t}}{\xi} \right]^{\Delta}, \tag{8.18a}$$

$$\text{with} \qquad \Delta = \frac{2}{(D-1)w + (D-3)}, \tag{8.18b}$$

$$\text{and} \qquad l = a_0 \left(\frac{a_0}{t_0} \right)^{\Delta}. \tag{8.18c}$$

We should now examine the two different cases, namely accelerating cosmologies and decelerating cosmologies. It then becomes clear that they have very different global structures. We find that

- for *accelerating* cosmologies ($-1 \leq w < w_{crit}$), both ξ and Δ are negative. Looking closely at (8.16), we see that this implies the range for $\bar{t}$ to be $\bar{t} \in (-\infty, 0)$. The initial singularity lies at $\bar{t} = -\infty$ and the infinite future lies at $\bar{t} = 0$.

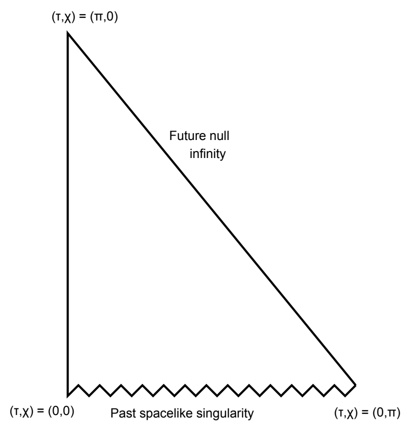

Figure 8.1: *Penrose diagram for a decelerating cosmology, $w > w_{crit}$. The initial singularity is spacelike, and the future boundary is null*

- For *decelerating* cosmologies ($w > w_{crit}$), ξ and Δ are positive. It follows that the range for $\bar{t}$ becomes $\bar{t} \in (0, \infty)$. Now the initial singularity lies at $\bar{t} = 0$ and the infinite future is located at $\bar{t} = +\infty$.

8.2.2 Penrose diagrams

The different global structures can be made more explicit when one switches to the use of *Penrose diagrams*. Penrose diagrams can be constructed by ignoring the $(D-2)$-sphere in (8.17b). Next, a coordinate transformation is applied to conformally compactify the two remaining coordinates $\bar{t}$ and r,

$$r \equiv \frac{\sin\chi}{\cos\chi + \cos\tau}, \qquad \bar{t} = \frac{\sin\tau}{\cos\chi + \cos\tau}. \tag{8.19}$$

In these coordinates, the FRW metric (8.17b) is written as

$$ds^2 = \frac{l^2}{4} \frac{[\frac{1}{2|\xi|}\sin|\tau|]^{2\Delta}}{\left[\cos(\frac{\chi+\tau}{2})\cos(\frac{\chi-\tau}{2})\right]^{2+2\Delta}} \Big(-d\tau^2 + d\chi^2 \Big). \tag{8.20}$$

By careful examination, it can be shown that in these new coordinates τ and χ, the range becomes

- *for an accelerating cosmology*: $\tau \in [-\pi, 0], \qquad \chi \in [0, \pi + \tau]$,
- *for an decelerating cosmology*: $\tau \in [0, +\pi], \qquad \chi \in [0, \pi - \tau]$.

The nice thing about Penrose diagrams is that they compactify a multi-dimensional spacetime into a two dimensional picture, in such a way that

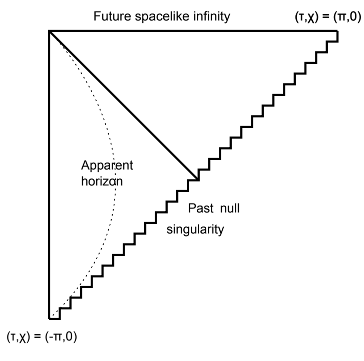

Figure 8.2: *Penrose diagram for an accelerating cosmology, $-1 < w < w_{crit}$. Now the initial singularity is null, and the future infinity is spacelike. The future spacelike boundary is obscured by a horizon.*

the global structure can be written off very easily. If we look at the Penrose diagram of the decelerating cosmology ($w > w_{crit}$), we see that the spatial slice $t = 0$ is a spacelike Big-Bang singularity, and future infinity $t \to +\infty$ is null. This is depicted in figure 8.1 on page 74.

Figure 8.2 on page 75 shows the Penrose diagram of an accelerating cosmology ($-1 < w < w_{crit}$). The hyper surface at FRW time $t = 0$ is null and future infinity is spacelike. Furthermore static observers see an apparent horizon at a distance[4]

$$L_H = t\xi. \tag{8.21}$$

This horizon recedes at a fixed proper speed and approaches the speed of light as $w \uparrow w_{crit}$.

Then finally there's the case with $w = w_{crit}$. This case can be seen as a hybrid of the two. It has a null initial singularity and a null future infinity. In fact, this quintessent model actually is conformally equivalent to Minkowski space. This case is depicted in figure 8.3.

In the next chapter we will show that there is a nice analogy between quintessence-driven cosmologies and string theory. This, in turn, can be used to derive some useful results for this string theory, based on the cosmological results we studied in this chapter.

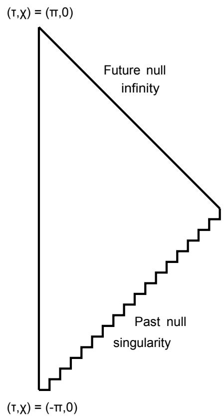

Figure 8.3: *Penrose diagram for a cosmology with a critical equation of state,* $w = w_{crit}$. *This diagram is globally conformally equivalent to Minkowski space*

[4]See [6] for reference.

Chapter 9

String theory with cosmological behaviour

9.1 Linear dilaton as quintessent cosmologies

9.1.1 Comparing the two theories

In the foregoing we derived the global solutions for a quintessence driven cosmology. We showed that there were actually three distinct possibilities, namely, accelerating expanding cosmologies, with $-1 \leq w < w_{crit}$, decelerating cosmologies, with $w > w_{crit}$ and the limit case, where $w = w_{crit}$. We showed that the limit solution actually is globally conformally equivalent to Minkowski space. In this chapter, we will show that there is a very close relation between quintessence driven cosmologies and string theories with a linear dilaton.

The starting point for deriving this relation will be the spacetime effective action for the massless closed string (6.14). We can simplify things a bit by assuming that the fundamental strings are invariant under reversal of orientation. The procedure where a projection is made onto strings that are invariant under reversal of orientation is called an *orientifold projection*. When such a projection is done, there is no antisymmetric field $B_{\mu\nu}(X)$, and therefore the term involving $H_{\mu\nu\lambda}$ in the spacetime action vanishes. In addition to this, we can rescale $\bar{\Phi}(X) \to \frac{1}{2}\sqrt{D-2}\,\phi(X)$. Working this out, we obtain

$$\mathbf{S} = \frac{1}{2\kappa^2}\int d^D X\,\sqrt{-G^{(E)}}\left[-\frac{2(D-26)}{3\alpha'}e^{\frac{2\phi}{\sqrt{D-2}}} + R^{(E)} - (\partial\phi)^2\right], \qquad (9.1)$$

where $(\partial\phi)^2 = G^{(E)\mu\nu}\partial_\mu\phi\partial_\nu\phi$.

It can immediately be seen that the Lagrangian density in (9.1) has exactly the same form as the Lagrangian density for a quintessence driven cosmology, (8.10), with coefficients

$$\gamma = \frac{2}{\sqrt{D-2}}, \tag{9.2a}$$

$$c = \frac{D-26}{3\alpha'} = 2q^2, \tag{9.2b}$$

provided that the (rescaled) dilaton ϕ satisfies $(\partial\phi)^2 < 0$ (is timelike). So we see that a quintessence driven cosmology is really the same thing as the low energy limit of a string theory with closed massless strings and the presence of a dilaton field. We can use this fact to derive some useful equations, which were originally used to solve for the quintessence driven cosmology.

Let's start by making an important observation. From (8.14b) it follows that, with the expression for γ in (9.2), the equation of state for a string theory with (timelike) dilaton is

$$w = -1 + \frac{(D-2)\gamma^2}{2(D-1)} \tag{9.3a}$$

$$= -\frac{D-3}{D-1} \tag{9.3b}$$

$$= w_{crit}. \tag{9.3c}$$

In other words, the action (9.1) yields an equation of state, right at the transition between an accelerating and a decelerating cosmology. We have already seen that this limit case is conformally equivalent to Minkowski space.

Thinking about this, we see that this is in perfect agreement with what we already knew for a (timelike) linear dilaton background! After all, Such a theory is defined to have a target space that is equivalent to Minkowski space. And of course, for a linear dilaton background we also assumed that the range of the coordinates X^μ is infinite. So in fact, we could say that this string theory gives rise to an equation of state $w = w_{crit}$, at the boundary between accelerating and decelerating cosmologies, driven by quintessence.

9.1.2 Fixing the scale factor

Next, we will determine the explicit forms for the dilaton $\Phi(X)$ and the scale factor $a(t)$, making use of the quintessent cosmology solutions. We already showed that for the critical case $\gamma = \frac{2}{\sqrt{D-2}}$. One can also show that the other coefficients (recall (8.14)) for the field and scale factor are $\alpha = 1$ and $\lambda = -\sqrt{D-2}$. Plugging these results into equation (8.13), we find that the dilaton and scale factor satisfy

$$\Phi(X) = \Phi_0 - \frac{D-2}{2} \ln\left(\frac{t}{t_0}\right), \tag{9.4a}$$

$$a(t) = \frac{a_0}{t_0} t, \tag{9.4b}$$

where we switched back to the original dilaton field $\Phi(X)$.

The time coordinate t, used here, is still equal to the FRW time, introduced in section 8.1. With this coordinate, it is not so clear that the dilaton is linear. We can obtain this result, however, by considering a more natural time coordinate t_{conf},

$$t_{conf} = \frac{(D-2)}{2q} \ln\left(\frac{t}{t_0}\right), \tag{9.5a}$$

$$t = t_0 \exp\left(\frac{2q}{D-2} t_{conf}\right) \tag{9.5b}$$

with q defined as in (7.6). With this new time coordinate, the dilaton becomes

$$\Phi(X) = \Phi_0 - q t_{conf} \tag{9.6a}$$

$$= \Phi_0 - q X^0, \tag{9.6b}$$

where we have set $t_{conf} \equiv X^0$. From this it is clear that the dilaton is indeed linear. Another advantage of switching over to t_{conf} is that the Einstein metric has now become conformally flat. This can be seen if we first recognize that

$$dt^2 = \frac{4q^2}{(D-2)^2} t^2\, dt_{conf}^2. \tag{9.7}$$

Then, using this and (9.4b), and plugging it into the FRW metric, (8.5), we find that the Einstein metric becomes

$$ds^2 = G^{(E)}_{\mu\nu} dX^\mu dX^\nu \tag{9.8a}$$

$$= \frac{a_0^2}{t_0^2} t^2 \eta_{\mu\nu} dX^\mu dX^\nu \tag{9.8b}$$

$$= a_0^2 \exp\left(\frac{4q t_{conf}}{D-2}\right) \eta_{\mu\nu} dX^\mu dX^\nu \tag{9.8c}$$

$$= \frac{4q^2}{(D-2)^2} t^2 \left(-dt_{conf}^2 + \sum_{i=1}^{D-1} dX^i dX^i\right), \tag{9.8d}$$

which shows that it is indeed conformally flat. Moreover, this sets

$$t_0 = \frac{(D-2)a_0}{2q}. \tag{9.9}$$

Now that we have chosen a coordinate system, such that the metric is conformally flat, we can set a_0 and t_0 to our convenience. We will make the choice such that *string frame* metric is equal to the Minkowski metric, so $G^{(S)}_{\mu\nu} = \eta_{\mu\nu}$. This choice completely fixes a_0 and t_0. If we recall that we defined the Einstein metric as a conformal transformation of the string metric,

$$G^{(E)}_{\mu\nu} = \exp\left(\frac{4qt_{conf}}{D-2}\right) G^{(S)}_{\mu\nu}, \tag{9.10}$$

we see that this choice implies

$$a_0^2 \exp\left(\frac{4qt_{conf}}{D-2}\right) = \exp\left(\frac{4qt_{conf}}{D-2}\right), \tag{9.11}$$

and therefore we find $a_0 = 1$,[1] so that the scale factor for this model is equivalent to

$$a(t) = \frac{2q}{D-2} t^2. \tag{9.12}$$

So, by acknowledging that a string theory with a timelike linear dilaton background is actually equivalent to a quintessence driven cosmology, we are able to derive the cosmological features of this theory and solve for all its variables. The results turn out to be in perfect agreement with what we already found earlier.

9.2 Stable modes

9.2.1 Stability

In the previous section we derived the cosmological solutions of the timelike linear dilaton background. We have, however, said nothing about what string modes are considered stable in this background. In this section we will be investigating this issue.

First of all, we can ask the question "we do we mean by stable modes?" A good way to think of stability is to see how a string mode responds to a small fluctuation of the background fields. But in time-dependent backgrounds there is no natural definition for stability. Since we are studying a linear dilaton background, there are two background fields that can be varied, namely the metric $G_{\mu\nu}(X)$ and the dilaton $\Phi(X)$. When we vary these fields, a string mode can react to this fluctuation in three different ways. It can remain constant, it can damp out or grow as time advances. When the response of a string mode to a small fluctuation, grows exponentially as time advances, this string mode is considered to be *unstable*. On the other hand, if the response of a string mode damps out, or at most remains constant when time advances, this string mode is considered to be *stable*.

[1]The authors in [6] find the result $a_0 = e^{-\frac{2\Phi_0}{D-2}}$. This difference comes from the fact that they define the Einstein metric as $e^{-\frac{4\Phi}{D-2}} G^{(S)}_{\mu\nu}$, instead of $e^{-\frac{4\Phi}{D-2}} G^{(S)}_{\mu\nu}$, used in this thesis.

We need to be careful, however. With this definition of stability, we can easily mistaken modes that are pure gauge [2] for unstable modes. Therefore, we need to look a bit more careful with our definition of stability. Once again, consider the low energy effective action for the massless closed string (6.1). We can couple string modes to backgrounds in this action. String modes can be represented by scalar fields $\sigma(X)$. Consider for example, a massless scalar field $\sigma(X)$ that couples to metric $G_{\mu\nu}(X)$. It enters the Lagrangian density as

$$\mathcal{L}_\sigma = -\frac{1}{2\kappa_0^2}\sqrt{-G^{(S)}}e^{-2\Phi}(\partial\sigma)^2 \tag{9.13a}$$

$$= -\frac{1}{2\kappa^2}\sqrt{-G^{(E)}}(\partial\sigma)^2. \tag{9.13b}$$

As is seen explicitly in the string frame case (9.13a), scalar fields that couple to background fields are suppressed by the sting coupling $g_s = e^{\Phi(X)}$. Therefore, can introduce a very convenient definition for stability of string modes in background fluctuations. A *stable mode* is one that grows *slower* than g_s^{-1} at late times, and an *unstable mode* is a mode that grows *faster* than g_s^{-1} at late times.

9.2.2 Massless modes

Let's see how this works for a massless scalar field $\sigma(X)$. First of all, we need to rescale the field $\sigma(X)$ canonically if we want the field fluctuations to represent normalizable string states. Therefore, we let

$$\tilde{\sigma}(X) \equiv e^{-\Phi}\sigma(X), \tag{9.14}$$

so that we find

$$e^{-2\Phi}(\partial\sigma)^2 = (\partial\tilde{\sigma} + 2\tilde{\sigma}\partial\Phi)^2 \tag{9.15a}$$

$$= (\partial\tilde{\sigma})^2 + \tilde{\sigma}^2(\partial\Phi)^2 + 2\tilde{\sigma}(\partial\tilde{\sigma})\cdot(\partial\Phi) \tag{9.15b}$$

$$= (\partial\tilde{\sigma})^2 - q^2\tilde{\sigma}^2 + 2\tilde{\sigma}(\partial\tilde{\sigma})\cdot\left[(\partial\Phi)_{bg} + (\partial\Phi)_{fl}\right], \tag{9.15c}$$

where we have used the fact that $(\partial\Phi)^2 = V_\mu V^\mu = -q^2$ for a linear dilaton background. As can be seen, this last term is split in two parts, namely a constant background part $(\partial_\mu\Phi)_{bg}$, and a fluctuation part $(\partial_\mu\Phi)_{fl}$. Since the background is constant, the part $(\partial_\mu\Phi)_{bg}$, contracted with $(\partial_\mu\tilde{\sigma})$ yields a total derivative and does therefore not contribute. The second part $(\partial_\mu\Phi)_{fl}$ is a fluctuating term, which represents a trilinear vertex. We discard this term. Therefore, we recover a new expression for the Lagrangian density of

[2] Such as overall rescalings of the metric, or constant shifts of massless scalars in the action (see [6] for reference.

rescaled field. It now contains a mass term $-q^2$, which couples to a trivial metric $G^{(S)}_{\mu\nu} = \eta_{\mu\nu}$,

$$\mathcal{L}_{\tilde{\sigma}} \sim -\frac{1}{2\kappa_0^2}\sqrt{-G^{(S)}}\left[(\partial\tilde{\sigma})^2 - q^2\tilde{\sigma}^2\right]. \tag{9.16}$$

For a timelike linear dilaton theory, $q^2 > 0$, so the mass term is tachyonic.

Next, we will solve the equations of motion for this Lagrangian and determine whether the solutions are stable or unstable modes. Since the scalar field $\tilde{\sigma}(X)$ is free in the spatial directions x^i, $i = 1, \ldots, D-1$, its solutions in the spacial directions are plane waves. One can easily check that the correct solutions are

$$\tilde{\sigma}(X) = \mathcal{A}e^{i\vec{k}\cdot\vec{x}\pm\tilde{\Gamma}t_{conf}}, \tag{9.17a}$$

$$\text{where} \qquad \tilde{\Gamma}^2 = q^2 - \vec{k}^2, \tag{9.17b}$$

and $\mathcal{A}$ is an arbitrary amplitude, such that $\tilde{\sigma}(X)$ is still real. To study the behaviour of the massless modes further, it is convenient to move back to the original FRW time t, and original field $\sigma(X)$. There are tree distinct possibilities, namely, *overdamped modes*, with ($q > |\vec{k}|$), *critically damped modes*, with ($q = |\vec{k}|$), and finally, *underdamped modes*, with ($q < |\vec{k}|$).

- In the case of overdamped modes, $\tilde{\Gamma}$ is real, and the solutions reduce to

$$\sigma_{over}(X) = \mathcal{A}e^{\Phi_0+i\vec{k}\cdot\vec{x}}\left(\frac{t}{t_0}\right)^{\mathcal{B}_\pm}, \tag{9.18a}$$

$$\text{where} \qquad \mathcal{B}_\pm \equiv \frac{D-2}{2q}\Gamma_\pm, \tag{9.18b}$$

$$\text{and} \qquad \Gamma_\pm \equiv \pm\sqrt{q^2 - \vec{k}^2} - q. \tag{9.18c}$$

Modes of the form (9.18a) are sometimes referred to as *pseudotachyons*.

It is interesting to see what happens at $\vec{k} = 0$. $\Gamma_\pm$ then takes two values, namely $\Gamma_+ = 0$ and $\Gamma_- = -2q$. For $\Gamma_+ = 0$, $\sigma(X)$ approaches a constant value and we say that the mode represents a *condensation* of the massless field. This is the same as saying that the field obtains a *vacuum expectation value* (vev). For $\Gamma_- = -2q$, however, the scalar field takes the form

$$\sigma_{over}(X) = \mathcal{A}e^{\Phi_0}\left(\frac{t}{t_0}\right)^{-(D-2)}. \tag{9.19}$$

So we see that in these cases, the modes damp out as time advances. Therefore, these modes are considered to be stable.

- In the critically damped case, $(q = |\vec{k}|)$, $\tilde{\Gamma} = 0$, so the scalar field just reduces to plane waves along the spacial directions, i.e.,

$$\sigma_{crit}(X) = \mathcal{A}e^{\Phi_0 + i\vec{k}\cdot\vec{x}}. \tag{9.20}$$

  Also here we see that these modes are stable.

- And finally, we can look at the underdamped modes $(q < |\vec{k}|)$. In this case $\tilde{\Gamma}$ becomes complex, and it is more convenient to switch to ω, with the property

$$\omega^2 = -\tilde{\Gamma}^2 \tag{9.21a}$$
$$= \vec{k}^2 - q^2. \tag{9.21b}$$

  It is a simple exercise to find that the underdamped modes of the scalar field in this case are written as

$$\sigma_{under}(X) = \mathcal{A}e^{\Phi_0 + i\vec{k}\cdot\vec{x}} \left(\frac{t}{t_0}\right)^{-\frac{(D-2)}{2q}(q \pm i\omega)}, \tag{9.22}$$

  and it is clear that this mode damps out at late times.

So, in all cases, the modes asymptote to zero (or at most stay constant) as $t \to \infty$. This means that when we start out with a massless scalar field σ, and properly (canonically) normalize it to give $\tilde{\sigma}(X) \equiv e^{-\Phi}\sigma(X)$, the modes are stable under fluctuations of the dilaton field. All modes have the property that $g_s\tilde{\sigma}(X) \to 0$ (or stay constant at most) at late times, in perfect correspondence with our definition of stability.

9.2.3 Massive modes

One can wonder what the effect of dilaton fluctuations would be if we had taken a *massive field* $\sigma_m(X)$, with mass m, instead of the massless field $\sigma(X)$. This would mean that, instead of (9.13a), a term of the form

$$\mathcal{L}_{\sigma_m} = -\frac{1}{2\kappa_0^2}\sqrt{-G^{(S)}}e^{-2\Phi}[(\partial\sigma_m)^2 + m^2\sigma_m^2] \tag{9.23}$$

would be added to the Lagrangian density. In that case the modes would break up into overdamped modes, with $\vec{k}^2 < q^2 - m^2$, critically damped modes, with $\vec{k}^2 = q^2 - m^2$, and underdamped modes with $\vec{k}^2 > q^2 - m^2$. Again, all these modes turn out to be stable in the sense that $g_s\tilde{\sigma}_m(X) \to 0$ at late times, as long as $m^2 \geq 0$.

But what would happen if the scalar field was a tachyon, in the sense that $m^2 < 0$? For this analysis it is convenient to focus on fields with $\vec{k}^2 < |m^2|$.

One can show that in this case, the *overdamped modes* have the exact same form as (9.18a), but now with

$$\mathcal{B}_{\pm} \equiv \frac{D-2}{2q}\Gamma_{(m)\pm}, \tag{9.24a}$$

and

$$\Gamma_{(m)\pm} \equiv \pm\sqrt{q^2 + |m^2| - \vec{k}^2} - q. \tag{9.24b}$$

We see that in this case, $\Gamma_{(m)+}$ can become positive, causing a positive exponential *growth* in the scalar field. Therefore, we see that it is possible to obtain unstable modes in this theory. These states correspond to *non-normalizable* states of the string, but they do not have the interpretation of particle excitations. Rather, they are seen as unstable modes of the vacuum, meaning that we are expanding around the wrong point in the vacuum. Just as we saw with the massless field, the tachyon can also approach a constant value at late times, namely if we consider $\Gamma_{(m)-}$, with $|m^2| = \vec{k}^2 = 0$. When this happens, the tachyon acquires an expectation value. This process is then called *tachyon condensation*.

In this chapter we have studied an effective string action with timelike dilaton, and found that this theory gave rise to a quintessence-driven cosmology, at the boundary between an accelerating and decelerating background. Moreover, we have been able to derive some of its dynamics, using corresponding cosmological solutions. And finally, we derived that most string modes are stable against fluctuations of the background in such a theory. What is strinking about this analysis is the fact that we have been able to find solutions of a string theory with a time-dependent background at all. In general, such theories are very hard to solve. In the next chapter, we will focus our attention on the world-sheet dynamics of a timelike linear dilaton background, coupled to a tachyon. We will choose such a setting that this theory becomes exactly solvable!

Chapter 10

Exact tachyon-dilaton dynamics

10.1 Exact solutions and Feynmann diagrams

10.1.1 Lightcone gauge

In chapter 7 we already considered the tachyon in a strict linear dilaton background. There we started out with the low energy effective action for a tachyon profile in the vicinity of the linear dilaton background, and derived the linearized tachyon field equation (7.12) from it. Solving the linear tachyon field equation gave us a tachyon profile of the form (7.13a), $\mathcal{T}(X) = \mu^2 \exp(B_\mu X^\mu)$. Then finally, we added this tachyon profile to the world-sheet action and were able to derive some of the dynamics of this system.

In this section we will study general solutions of a theory with a non-zero tachyon. We will be looking at solutions at the classical level and show that these are in fact also exact at the quantum level. In the next section, after having studied this non-zero tachyon, we will go to a more general setting where we let the dilaton background derivate from the strict linear dilaton background. Using our novel set of solutions, we will derive a more general effective action for tachyon-dilaton interactions.

First of all, let us go through a few basic fact about world-sheet calculus, in a slightly different way than we have already seen. It turns out to be convenient to work in world-sheet lightcone coordinates of the form

$$\rho^{\pm} = -\tau \pm \sigma \tag{10.1a}$$

$$= -\sigma^0 \pm \sigma^1. \tag{10.1b}$$

These world-sheet lightcone coordinates are closely related to the complex world-sheet coordinated z and $\bar{z}$, introduced in chapter 3. In these coordinates the energy-stress tensor can be written as

$$T_{++} = -\frac{1}{\alpha'} : \partial_+ X^\mu \partial_+ X_\mu : \; + V_\mu \partial_+^2 X^\mu, \tag{10.2a}$$

$$T_{--} = -\frac{1}{\alpha'} : \partial_- X^\mu \partial_- X_\mu : \; + V_\mu \partial_-^2 X^\mu, \tag{10.2b}$$

where

$$\partial_+ \equiv \partial_{\rho^+} = \tfrac{1}{2}(-\partial_{\sigma^0} + \partial_{\sigma^1}), \tag{10.3a}$$

$$\partial_- \equiv \partial_{\rho^-} = \tfrac{1}{2}(-\partial_{\sigma^0} - \partial_{\sigma^1}), \tag{10.3b}$$

and we can see the close analogy with (7.7). We also explained that physical string states correspond to local (vertex) operators $V(\rho^+, \rho^-)$ on the world-sheet, which need to be normal ordered. Therefore we can write

$$V(\rho^+, \rho^-) \equiv : \mathcal{T}(X) : \, , \tag{10.4}$$

where $\mathcal{T}(X)$ satisfies the momentum on-shell condition (7.12). For tachyon profiles of the form (7.13a), the on-shell condition comes down to condition (7.13b), or put in a slightly different way,

$$B^2 - 2V^\mu B_\mu + \frac{4}{\alpha'} = 0. \tag{10.5}$$

In general, for arbitrary B_μ, this will lead to nontrivial interacting theories. There is, however, a set of choices for B_μ, such that the solutions to the theory are exact and conformal to all orders in perturbation theory. This set of choices comes down to choosing the tachyon profile to be *lightlike*, or equivalently B_μ null. When make this choice for B_μ, the first term in (10.5) vanishes. If one now works out the OPE of two tachyon vertex operators, it can be seen that in the vicinity of each other they do not become singular, as they would in general. Normally in free field theories, singularities in normal-ordered operators arise when the propagators for from one free field are contracted with propagators from the other. In this case, all contractions would render terms proportional to B^2, and would therefore vanish. Furthermore, we can always perform a Lorentz boost, to put B_μ into the form

$$B_0 = B_1 \equiv \frac{\beta}{\sqrt{2}}, \tag{10.6a}$$

$$B_i = 0, \quad i \geq 2. \tag{10.6b}$$

If we then also adapt to lightcone spacetime coordinates,

$$X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^1), \tag{10.7}$$

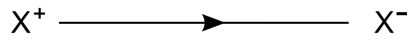

Figure 10.1: *A propagator for $X^{\pm}$, depicted as an arrow, pointing from X^+ to X^-.*

The tachyon profile couples to the world-sheet as a (normal ordered) potential. Therefore, the tachyon vertex operator can be written as

$$V(\rho^+,\rho^-) = :\mu^2 \exp(\beta X^+): \tag{10.8a}$$

$$= \mu^2 \exp(\beta X^+). \tag{10.8b}$$

As can be seen, at this point the normal ordering symbols are dropped. Normal ordering would yield self-contractions of the B_μ-fields, which all vanish.

10.1.2 Exact solutions

Now that we have written the tachyon in terms of lightcone coordinates $X^{\pm}$, let us also express the kinetic term for $X^{\pm}$ of the world-sheet Lagrangian density. One can show that in terms of $X^{\pm}$, the Lagrangian density, including the tachyon contribution, takes the form

$$\mathcal{L} = -\frac{1}{2\pi\alpha'}\left[(\partial_{\sigma^0}X^+)(\partial_{\sigma^0}X^-) - (\partial_{\sigma^1}X^+)(\partial_{\sigma^1}X^-) + \alpha'\mu^2\exp(\beta X^+)\right]. \tag{10.9}$$

Just looking at the kinetic part, we see that the X^+ fields are always coupled to the X^-fields. The propagator for the $X^{\pm}$ fields therefore is *orientated* and always has one X^+ at one end, and one X^- at the other. So a diagram for a $X^{\pm}$ propagator can be depicted as an arrow, pointing from X^+ to X^-. See figure 10.1

Now, with this Lagrangian, it is not hard to write down its equations of motion. By varying the fields, or using the Euler-Lagrange equations, we find that the equations of motion for the string are

$$\partial_+\partial_- X^i = 0, \qquad \text{for } i = 2,3,\ldots,D-1, \tag{10.10a}$$

$$\partial_+\partial_- X^+ = 0, \tag{10.10b}$$

$$\partial_+\partial_- X^- = \frac{\alpha'\beta M^2}{4}, \tag{10.10c}$$

where $M^2 \equiv \mu^2 \exp(\beta X^+)$. These equations are exactly solvable at the classical level. First of all, we note that the most general solution for (10.10b) can, at most, be a sum of a function of ρ^+ and a function of ρ^-. Secondly, by making use of some basic integral calculus, one can show that the exact solutions to these equations of motion are

$$X^+ = f_+(\rho^+) + f_-(\rho^-) \tag{10.11a}$$

$$X^- = g_+(\rho^+) + g_-(\rho^-) \tag{10.11b}$$

$$+ \frac{\alpha'\beta\mu^2}{4}\left\{\int_{\rho^+}^{\infty} dy^+ \exp[\beta f_+(y^+)]\right\}\left\{\int_{\rho^-}^{\infty} dy^- \exp[\beta f_-(y^-)]\right\}, \tag{10.11c}$$

where $f_\pm(\rho^\pm)$ and $g_\pm(\rho^\pm)$ are arbitrary functions. Next, we will argue that the exactness of the solutions extends to the quantum level, so that we have obtained a full set of solutions for the non-zero tachyon theory.

As we said before, doing world-sheet physics can really be seen as describing $2D$ quantum field theory, but then with some extra conditions on the fields X^μ. It is therefore possible to describe interactions in the same fashion as in quantum field theory. In quantum field theory, when an interaction coupling g is small, perturbation theory is a good way to describe the interactions as long as the energies don't become to large. When using perturbation theory, different contributions to interactions are given by different powers of g and correspond to a different number of loops in the diagrams. The classical limits of interactions are given by *tree diagrams*, and quantum corrections are given by loop diagrams.

Coming back to the problem at hand, all interactions at the quantum level with the non-zero tachyon depend only on X^+. This means that when we use perturbation theory to describe interactions, the corresponding diagrams can only have *outgoing* lines. In other words, it is impossible to construct diagrams with loops, because these would involve X^- dependence as well. In other words, the only diagrams possible are tree diagrams and all of these correspond to their classical limits! Since we already solved all these contributions exactly we conclude that we have obtained the complete set of solutions for this theory. See figure 10.2 for some tree-level diagrams.

One can also write down the OPE's of the $X^\pm$ fields. The structure of these OPE's is just as simple, but we will not go into that here. For further detail, the reader is referred to [6].

10.2 Bubble of nothing

10.2.1 Bubble of nothing

In this section we will give a physical interpretation of the exact solutions for the model of a non-zero tachyon and linear dilaton we found in the previous

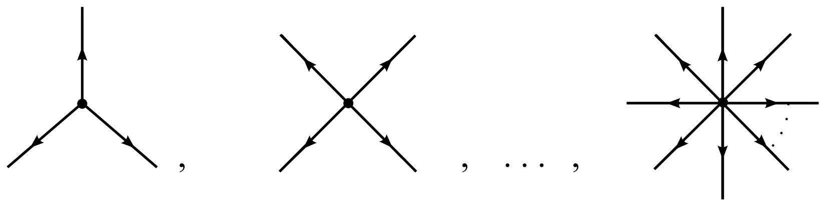

Figure 10.2: *Different tree-interaction vertices for a world-sheet with non-zero tachyon. Only diagrams with outgoing lines are possible.*

section. We will see that this solution actually gives rise to a *bubble of nothing*, a region where no particles can enter. In section 7.3 we already encounterd a model that is very closely related to this model, and there we also saw that the tachyon field became an impenetrable barrier for matter, but only in the X^1 direction. The power of this model, however, lies in the fact that we are able to solve this theory exactly at the quantum level (since there are no quantum corrections in the interactions). Later on in this section we will also describe the trajectory of a string colliding with the bubble.

Just as in section 7.3, the tachyon profile $\mathcal{T}(X) = \mu^2 \exp(\beta X^+)$ here also acts as a barrier. It can be thought of as a phase boundary in spacetime between the region where $\mathcal{T} \approx 0$ and a region where $\mathcal{T} > 0$. In the region $\mathcal{T} > 0$, where the tachyon becomes relevant, matter starts being pushed outwards, and one can say that this is the point at which the bubble is expanding. If the linear dilaton were absent, the boundary of this bubble would be moving to the left (that is, in the $-X^1$ direction) at the speed of light. No degrees of freedom can live inside this bubble at all, not even the graviton. Matter that encounters the bubble is rapidly pushed outwards, approaching the speed of light. You could say that this is an actual *absence of spacetime* itself.

So far, we have considered the null solution for the closed string tachyon description of a bubble of nothing. A full classical solution is not known explicitly, but in [6] it is suggested that in the presence of a timelike linear dilaton, a closed tachyon theory would approach the null solution, long after the nucleation of the bubble. The difference from a bubble of nothing in a trivial flat spacetime (so, no dilaton) with this model, is that the barrier wall is not properly accelerating. This is due to Hubble friction and the presence of the timelike linear dilaton. From the linearized tachyon field equation, it can be seen that when the thickness of the bubble is of order $\alpha'|\dot{\Phi}| \sim \beta^{-1}$, the drag force of the background fields stop the acceleration.

From the action for this model, it is clear that no particles can enter the high potential region, since particle wavefunctions would be suppressed by the potential barrier. But since are able to solve this model exactly, we can actually describe trajectories of particle that come in contact with the barrier wall!

10.2.2 Particle trajectories

Let us, for sake of simplicity, consider pointlike strings, which only depend on σ^0 and not on σ^1. We then use the string's conserved momenta, introduced in chapter 1,

$$P^I \equiv T\dot{X}^I, \qquad \text{for } I = 2, \ldots, D-1, \tag{10.12a}$$

$$P^+ = T\dot{X}^+, \tag{10.12b}$$

$$P^- = T\dot{X}^-, \tag{10.12c}$$

For a pointlike string, one can write the X^+ solution as

$$X^+ = \alpha' p^+(\sigma^0 - \sigma^0_0), \tag{10.13a}$$

$$P^+ = \frac{1}{2\pi} p^+, \tag{10.13b}$$

$$P^I = \frac{1}{2\pi} p^I, \tag{10.13c}$$

where σ^0_0 is a constant.[1] Then, using this solution and equation (10.10c), it's easy to show that the X^- solution becomes

$$X^- = \alpha' p_0^-(\sigma^0 - \sigma^0_0) + \frac{\mu^2}{\beta\alpha'(p^+)^2} \exp\left[\alpha'\beta p^+(\sigma^0 - \sigma^0_0)\right], \tag{10.14a}$$

$$P^- = \frac{1}{2\pi} p_0^- + \frac{\mu^2}{2\pi\alpha' p^+} \exp\left[\alpha'\beta p^+(\sigma^0 - \sigma^0_0)\right], \tag{10.14b}$$

where p_0^- is a constant of motion.

There is a relation between the constant of motion p_0^- and the other conserved momenta p^I. We will show this relation by using the fact that $P_\mu P^\mu = -2P^-P^+ + P_I^2$ and the Virasoro constraints, introduced in chapter 3. For pointlike strings, all excited modes are absent, so the only contributions come from α_0^μ and $\tilde{\alpha}_0^\mu$. Therefore, the classical Virasoro constraint for

[1] Recall that we $T = \frac{1}{2\pi\alpha'}$, and that we set $\alpha' = 1$.

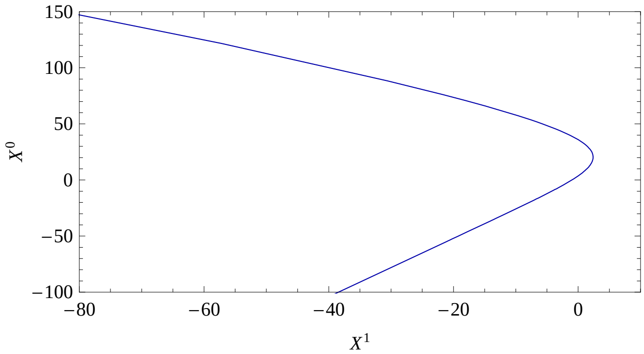

Figure 10.3: *Trajectory of a pointlike string which encounters an expanding bubble of nothing. We have set* $\alpha' = 1$ *and taken the values* $\mu^2 = 1$, $\beta = 0.1$, $p^+ = 3$ *and* $\alpha' p_I^2 = 8$.

H yields

$$\begin{aligned} H &= \alpha_0^2 + \tilde{\alpha}_0^2 && (10.15a) \\ &= \frac{1}{2}\alpha' P_\mu P^\mu + \frac{1}{2}\alpha' P_\mu P^\mu && (10.15b) \\ &= -\alpha' p_0^- p^+ + \frac{1}{2}\alpha' p_I^2 + \mu^2 \exp(\beta X^+) && (10.15c) \\ &= 0. && (10.15d) \end{aligned}$$

By taking the limit $X^+ \to -\infty$, we find that p_0^- is equal to

$$p_0^- = \frac{p_I^2}{2p^+}. \tag{10.16}$$

This is al the information we need to plot the trajectory of a pointlike string that collides with the bubble wall. The only parameters that are still free to choose are $\alpha', \beta, \mu^2, p^+$ and p_I^2. It can be more insightful to plot the trajectory, using the original coordinates X^0 and X^1. Therefore, we take (10.13a) and (10.14a) and invert relation (10.7) and use the obtained expressions for X^0 and X^1 to plot a trajectory. In figure 10.3 we plotted such a trajectory.

It is also possible to plot the particle's velocity as it collides with the bubble wall. First of all, we note that the particle's initial velocity is given

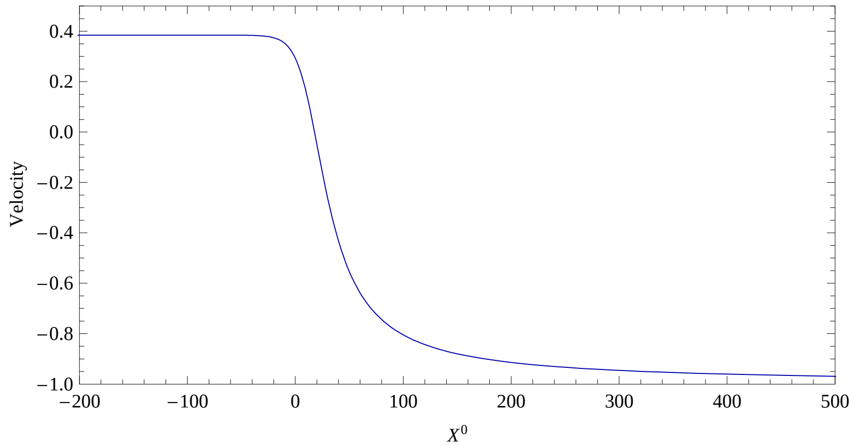

Figure 10.4: *The velocity of a pointlike string that encounters the same expanding bubble of nothing as in figure 10.3. We have assumed the same numerical values. It can be seen that the particle's velocity is reversed, and rapidly approaches the speed of light.*

by

$$v \equiv \left.\frac{\dot{X}^1}{\dot{X}^0}\right|_{initial} \tag{10.17a}$$

$$= \left.\frac{\dot{X}^+ - \dot{X}^-}{\dot{X}^+ + \dot{X}^-}\right|_{initial} \tag{10.17b}$$

$$= \left.\frac{P^+ - P^-}{P^+ + P^-}\right|_{initial} . \tag{10.17c}$$

The particle moves with this velocity until it collides with the bubble wall. There, the exponential term in P^- becomes large and starts to dominate the numerator and denominator. Therefore, the particle's velocity rapidly goes to -1. The velocity of the trajectory in figure 10.3 is given in figure 10.4.

10.3 Tachyon-dilaton low energy effective action

10.3.1 General two-derivative form

In chapter 6 we introduced the low energy effective action. We argued that such an effective action can be very useful for describing the low energy physics of a system. In chapter 7 the effective action provided us with the on-shell condition for the tachyon, and in the chapter 9 we were even able to link a timelike linear dilaton theory to a quintessence cosmology by means of the effective action.

We have derived some interesting features of string theories in the vicinity of a tachyon profile and linear dilaton background. Moreover, we found a world-sheet description of a theory with linear dilaton and tachyon field that turned out to be exactly solvable. But we haven't written down an effective action for this model yet. In this section we will derive the most general second-order derivative effective action for the tachyon-dilaton model we discussed.

First of all, recall that we only have to deal with tree-level interactions in our model.[2] This means that the dilaton dependende appears as an overall factor of $e^{-2\Phi(X)}$ in the effective action. Then, the most general two-derivative low energy effective action for a dilaton-tachyon theory is

$$S = \frac{1}{2\kappa^2}\int d^D x\,\sqrt{G}\Big[\mathcal{F}_1 R - \mathcal{F}_2(\nabla\Phi)^2 - \mathcal{F}_3(\nabla\mathcal{T})^2 - \mathcal{F}_4 - \mathcal{F}_5(\nabla\mathcal{T})\cdot(\nabla\Phi)\Big], \quad (10.18)$$

where the functions $\mathcal{F}_i$ are defined by

$$\mathcal{F}_1 \equiv e^{-2\Phi} f_1(\mathcal{T}), \quad (10.19a)$$
$$\mathcal{F}_2 \equiv -4e^{-2\Phi} f_2(\mathcal{T}), \quad (10.19b)$$
$$\mathcal{F}_3 \equiv e^{-2\Phi} f_3(\mathcal{T}), \quad (10.19c)$$
$$\mathcal{F}_4 \equiv 2e^{-2\Phi} \mathcal{V}(\mathcal{T}), \quad (10.19d)$$
$$\mathcal{F}_5 \equiv e^{-2\Phi} f_5(\mathcal{T}), \quad (10.19e)$$

and $f_i(\mathcal{T})$ and $\mathcal{V}(\mathcal{T})$ are five arbitrary functions of the tachyon field. Moreover, we have chosen an Euclidean signature for the spacetime metric here, and the prefactors are for later convenience.

We obtain the equations of motions for this model by varying this action to it's fields, the metric $G_{\mu\nu}(X)$, the dilaton $\Phi(X)$ and the tachyon $\mathcal{T}(X)$. Varying the action to the metric gives the Einstein equations, and varying to $\Phi(X)$ and $\mathcal{T}(X)$ gives two other equations of motion. Subsequently, they are,

[2] Due to the fact that we have chosen such a coordinate system that all self contractions the tachyon field would render terms proportialnal to B^2, which are equal to zero

- *Einstein equations*:

$$\begin{aligned}
&\left[\nabla^\mu\nabla^\nu - G^{\mu\nu}\nabla^2 + \frac{1}{2}G^{\mu\nu}R - R^{\mu\nu}\right]\mathcal{F}_1 \\
&+\left[\nabla^\mu\Phi\nabla^\nu\Phi - \frac{1}{2}G^{\mu\nu}(\nabla\Phi)^2\right]\mathcal{F}_2 + \left[\nabla^\mu\mathcal{T}\nabla^\nu\mathcal{T} - \frac{1}{2}G^{\mu\nu}(\nabla\mathcal{T})^2\right]\mathcal{F}_3 \\
&-\left[\frac{1}{2}G^{\mu\nu}\right]\mathcal{F}_4 + \left[\frac{1}{2}\nabla^\mu\mathcal{T}\nabla^\nu\Phi + \frac{1}{2}\nabla^\nu\mathcal{T}\nabla^\mu\Phi - \frac{1}{2}G^{\mu\nu}(\nabla\mathcal{T})\cdot(\nabla\Phi)\right]\mathcal{F}_5 \\
&= 0
\end{aligned} \tag{10.20}$$

- *Equation of motion coming from varying with respect to* $\Phi(X)$:

$$\begin{aligned}
&-2Rf_1 + 8f_2(\nabla\Phi)^2 - 8f_2'(\nabla\mathcal{T})\cdot(\nabla\Phi) - 8f_2\nabla^2\Phi \\
&\qquad + (2f_3 + f_5')(\nabla\mathcal{T})^2 + f_5\nabla^2\mathcal{T} + 4\mathcal{V} = 0.
\end{aligned} \tag{10.21}$$

- *Equation of motion coming from varying with respect to* $\mathcal{T}(X)$:

$$\begin{aligned}
&Rf_1' + (4f_2' - 2f_5)(\nabla\Phi)^2 + f_3'(\nabla\mathcal{T})^2 - 4f_3(\nabla\Phi)\cdot(\nabla\mathcal{T}) \\
&\qquad - 2\mathcal{V}' + f_5\nabla^2\Phi + 2f_3\nabla^2\mathcal{T} = 0.
\end{aligned} \tag{10.22}$$

10.3.2 Determining the final form

Without more imput, this is about as far as we can get. But what we were really looking for, was the most general low energy effective action that was able to reproduce the tachyon-dilaton theory we discussed! So, it is therefore reasonable to assume that this effective action admits a solution where

$$\mathcal{T}(X) = \mu^2\exp(\beta X^+), \tag{10.23a}$$

$$G^{(S)}_{\mu\nu}(X) = \eta_{\mu\nu}, \tag{10.23b}$$

$$\Phi(X) = -qX^0. \tag{10.23c}$$

This assumption, together with the equations of motion (10.20), (10.21) and (10.22) impose conditions on the functions $f_i(\mathcal{T})$ and $\mathcal{V}(\mathcal{T})$. First of all, the on-shell condition for the tachyon immediately lead to the condition $\beta q = \frac{2\sqrt{2}}{\alpha'}$. We will not work out the details here, but in [6] it is shown that the functions $f_i(\mathcal{T})$ meet the following conditions,

$$f_2 = f_1, \tag{10.24a}$$

$$f_3 = -\frac{1}{\mathcal{T}}f_1' - f_1'', \tag{10.24b}$$

$$\mathcal{V}(\mathcal{T}) = \frac{D-26}{3\alpha'}f_1 + \frac{4}{\alpha'}\mathcal{T}f_1', \tag{10.24c}$$

$$f_5 = 4f_1'. \tag{10.24d}$$

So, apparently it is possible to express f_2, f_3, f_5 and $\mathcal{V}(\mathcal{T})$ entirely in terms of f_1.[3]

Taking all of this together, we end up with the most general (two-derivative) effective action that produces solutions for the tachyon-dilaton model we considered,

$$S = \frac{1}{2\kappa^2}\int d^D x\,\sqrt{G}e^{-2\Phi}\Big[f_1 R + 4f_1(\nabla\Phi)^2 + \Big(\frac{1}{\mathcal{T}}f_1' + f_1''\Big)(\nabla\mathcal{T})^2 - 4f_1'(\nabla\mathcal{T})\cdot(\nabla\Phi) - \frac{2(D-26)}{3\alpha'}f_1 - \frac{8}{\alpha'}f_1'\mathcal{T}\Big]. \qquad (10.25)$$

This result holds for all spacetime dimensions D.

A question we can ask ourselves is, if we take a model where the tachyon condences in a direction other than the null direction, are the equations of motion (10.20), (10.21) and (10.22) still satisfied? It turns out that this is not the case (see [6]). This doesn't tell us, however, that our theory is wrong, or that the tachyon potential $\mathcal{V}(\mathcal{T})$ should vanish, but merely that the effective action is not complete enough to describe more general settings. A way to continue is include higher order derivatives into the effective action, in such a way that the null tachyon background is still an exact solution of the action. A nice way to obtain this result is to first notice (see (7.12)) that a null tachyon background satisfies the equation

$$(\partial_\mu\Phi)\cdot(\partial^\mu\mathcal{T}) = \frac{2}{\alpha'}\mathcal{T} \qquad (10.26)$$

everywhere. So, including a term of the form

$$\Big[(\partial_\mu\Phi)\cdot(\partial^\mu\mathcal{T}) - \frac{2}{\alpha'}\mathcal{T}\Big]^2\cdot\mathcal{F}\big[G_{\mu\nu}(X),\Phi(X),\mathcal{T}(X)\big] \qquad (10.27)$$

to the action, where $\mathcal{F}\big[G_{\mu\nu}(X),\Phi(X),\mathcal{T}(X)\big]$ is an arbitrary function of the background fields, will automatically satisfy the null tachyon - timelike linear dilaton background we discussed.

In this chapter we have investigated the tachyon-dilaton interactions further. Also, by choosing the proper coordinates, we have been able to give exact solutions for this model. We saw that we obtained a solution which can be thought of as a spacetime-destroying bubble of nothing. In this model, however, the tachyon profile only depended X^+. In the next chapter we will study a similar model, but there we will let the tachyon profile depend on more coordinates $X^2,\ldots,X^n$. We will see that the solutions to this system are still simple enough to be solved exactly, and that the solutions are able to dynamically change the number of dimensions of the theory!

[3] It is worth noticing that if the tachyon potential $\mathcal{V}(\mathcal{T})$ is not trivial (and the linear dilaton is nonvanishing), the function $f_1(\mathcal{T})$ cannot be constant. This means that in an effective action for such a model, the normalization of the Einstein term R must also be nontrivial. This is quite an important result, since various articles have approximated the normalization of the Einstein term as constant. See [6] for further reference.

Chapter 11

Dimension-changing solutions

11.1 Dimension-change for the bosonic string

11.1.1 Oscillatory dependence in the X_2 direction

In the previous chapter we studied a bosonic string theory in a timelike linear dilaton theory, in which the tachyon profile $\mathcal{T}(X^+)$ condensed along the null direction X^+. By considering the theory in this lightcone coordinate system, we were able to find exact solutions, due to the fact that all quantum corrections vanished.

There are, however, more general settings to consider. We could, for example, study a similar theory, but assume that the tachyon profile also has oscillatory dependence on more coordinates, $X_2, X_3, \ldots, X_n$.[1] This generalization is possible, as long as the on-shell tachyon condition, (7.12), is still satisfied. Let us, for the moment, focus on a tachyon profile that has oscillatory dependence on a third coordinate X_2, and consists of a superposition of perturbations. Such a profile has the form

$$\mathcal{T}(X) = \mu_0^2 \exp(\beta X^+) - \mu_k^2 \cos(kX_2) \exp(\beta_k X^+), \tag{11.1a}$$

$$\text{with} \qquad q\beta_k = \sqrt{2}\left(\frac{2}{\alpha'} - \frac{1}{2}k^2\right). \tag{11.1b}$$

As we already encountered in (10.8), the tachyon profile couples to the world-sheet as a normal ordered potential, $-\frac{1}{2\pi} :\mathcal{T}(X):$. Furthermore, we can expand this potential around the vacuum $X_2 = 0$. If we do so, and recall that we were allowed to drop the normal ordering symbols for the null

[1] Usually these 'extra' fields are written with upper indices. However, for the purposes in this chapter, it is more convenient to write them with lower indices. Moreover, since the spacetime metric is in the Minkowski frame, (recall that this was how a linear dilaton background was defined) there is no distinction between the two anyway.

coordinate X^+, we find

$$\mathcal{T}(X) = \mu_0^2 \exp(\beta X^+) - \mu_k^2 \exp(\beta_k X^+) + \frac{1}{2} k^2 \mu_k^2 \exp(\beta_k X^+) : X_2^2 : + O(k^4 X_2^4). \quad (11.2)$$

We can simplify this theory significantly by taking the wavelenght of the oscillatory part, k^{-1}, long compared to the string scale l. In other words, we can take $k \to 0$. In this limit, the part $O(k^4 X_2^4)$ vanishes, and $\beta_k \to \beta$. Furthermore, if we define $\mu^2 \equiv \alpha' k^2 \mu_k^2$, and fix $\mu'^2 \equiv \mu_0^2 - \mu_k^2$, the tachyon profile becomes

$$\mathcal{T}(X) = \frac{\mu^2}{2\alpha'} \exp(\beta X^+) : X_2^2 : + \mathcal{T}_0(X^+), \quad (11.3a)$$

$$\text{with} \quad \mathcal{T}_0(X^+) = \frac{\mu^2 X^+}{\alpha' q \sqrt{2}} \exp(\beta X^+) + \mu'^2 \exp(\beta X^+). \quad (11.3b)$$

We can intuitively interpret these solutions as follows. For $X^+ \to -\infty$ the tachyon is zero, so strings are free to propagate in all spatial directions, $X_1, \ldots, X_d$. But in the region where the tachyon becomes relevant, $X^+ \sim 0$, strings are confined to a region where the tachyon is minimal. In other words, strings are confined to $X_2 \to 0$. At late times, $X^+ \to +\infty$, it becomes impossible for strings to move in the X_2 at all, so they are frozen in at $X_2 = 0$. Initially, these strings move in D spacetime dimensions, but at late times they can only move in $(D-1)$ dimensions, and therefore the number of spacetime dimensions has effectively been reduced by one! Strings that continue to oscillate in the X_2 direction are expelled from the region where the tachyon condensate is large. They are pushed outwards, along the X^+ direction, in a very similar way that was described in chapter 10. The process where strings at late times live in a lower number dimensions than at early times is called *dynamical dimensions change*.

Of course, this is not the complete story. We have said nothing about the other terms appearing in $\mathcal{T}_0(X^+)$. If we look at them, we see that it a logical idea to interpret the term involving μ'^2 as the tachyon condensate at late times (so in $(D-1)$ spatial dimensions). This term is a conformal field, so it can be tuned to zero by setting μ'^2 to vanish. The other term, however, can not simply be tuned away. But as one can show, fortunately, a quantum effective potential that is generated upon integrating out the X_2 field, exactly cancel out this term.[2] Therefore, we are left with a clear interpretation of this theory, namely a *dimension-changing bubble*.

[2]It can be shown that this quantum effective potential turns out to contribute an amount of $\Delta V = -\frac{\mu^2}{8\pi} \beta X^+ \exp(\beta X^+)$ to the vacuum energy of the system. Using that $\beta q = \frac{2\sqrt{2}}{\alpha'}$, we see that this indeed exactly cancels the extra term in $\mathcal{T}_0$. See [7] for more details.

11.1.2 Classical world-sheet solutions

We will now take a closer look at this system. Therefore, we will consider the world-sheet action for this theory and try to find its classical solutions. Writing down the full action, we find

$$S = \frac{1}{2\pi\alpha'}\int_M d^2\sigma\Big[-(\partial_{\sigma^0}X^+)(\partial_{\sigma^0}X^-)+(\partial_{\sigma^1}X^+)(\partial_{\sigma^1}X^-) + \frac{1}{2}(\partial_{\sigma^0}X_I)(\partial_{\sigma^0}X^I)-\frac{1}{2}(\partial_{\sigma^1}X_I)(\partial_{\sigma^1}X^I)-\alpha'\mathcal{T}(X)\Big]. \quad (11.4)$$

The equations of motion for this action become

$$\partial_+\partial_-X^+ = \frac{\alpha'}{4}\partial_{X^-}\mathcal{T} = 0, \quad (11.5a)$$
$$\partial_+\partial_-X_2 = -\frac{\alpha'}{4}\partial_{X_2}\mathcal{T} \quad (11.5b)$$
$$= -\frac{1}{4}\mu^2\exp(\beta X^+)X_2, \quad (11.5c)$$
$$\partial_+\partial_-X^- = \frac{\alpha'}{4}\partial_{X^+}\mathcal{T} \quad (11.5d)$$
$$= \frac{\beta}{8}\mu^2\exp(\beta X^+)X_2^2+\frac{1}{4}\partial_{X^+}\mathcal{T}_0, \quad (11.5e)$$
$$\partial_+\partial_-X_J = 0, \qquad \text{for} \qquad J=3,\ldots,D-1, \quad (11.5f)$$

where just as before, $\partial_\pm$ are derivatives with respect to $\rho^\pm$. From these equations, we can derive some results straightaway. First of all, just as in chapter 10, we notice that if $\partial_+\partial_-$ acting on a function yields zero, the most general solution is a sum of a function that only depends on ρ^+ and a function that only depends on ρ^-, so

$$X^+ = f_+(\rho^+)+f_-(\rho_-), \quad (11.6a)$$
$$X^J = f_+^J(\rho^+)+f_-^J(\rho_-). \quad (11.6b)$$

The second result is that the equation of motion for the X_2 field, (11.5c), is exactly the equation of motion for a (scalar) field with physical mass $M(X^+)\equiv\mu\exp\left(\frac{1}{2}\beta X^+\right)$. This interpretation is of course sensible if we treat X^+ as fixed.

Since the equations of motion are nonlinear, they are difficult to solve in full generality. But we do not need to know the general solution in order to study the behaviour of particles in this background. We will try to solve the equations of motions in a simpler setting, namely when we consider pointlike strings (so no dependence on σ^1), just as in the previous chapter. Therefore, we will asume X^+ to be of the form (10.13a). With this simplification, the equation of motion for X_2 becomes

$$\ddot{X}_2 = -\omega^2(\sigma^0) X_2, \tag{11.7a}$$

$$\text{with} \qquad \omega(\sigma^0) \equiv M(\sigma^0) = \mu \exp\left[\frac{1}{2}\beta\alpha' p^+ (\sigma^0 - \sigma^0_0)\right], \tag{11.7b}$$

and the dots are again derivatives with respect to σ^0. The solutions of this equation are Bessel functions of the first and second kind (J_0 and Y_0 respectively), so we end up with

$$X_2 = A J_0 \left[\frac{2\omega(\sigma^0)}{\beta\alpha' p^+}\right] + B Y_0 \left[\frac{2\omega(\sigma^0)}{\beta\alpha' p^+}\right], \tag{11.8}$$

where A and B are just constants of motion.

11.1.3 An energy consideration

In order to understand the behaviour of this system it is important to realize that, with respect to the X_2 field, particles behave like harmonic oscillators with time-dependent frequency $\omega(\sigma^0)$. The energy of a harmonic oscillator with constant frequency can easily be calculated. One could wonder, though, if in this case the changes in frequency are slow enough for particles to adapt to these changes. In other words, does the system obey the adiabatic theorem? It turns out that this is the case. If a system is charaterized by a frequency ω, changes in the wavelength $\lambda = \omega^{-1}$ with respect to time σ^0 should vanish, so

$$\lim_{\sigma^0 \to \infty} \frac{d\lambda}{d\sigma^0} = \lim_{\sigma^0 \to \infty} \frac{-1}{\omega^2} \frac{d\omega}{d\sigma^0} = 0. \tag{11.9}$$

It is easily checked that the system meets this condition.

From this result, we can deduce that the energy in the oscillator modes grows proportionally to $\omega(\sigma^0)$. Quantum mechanics tell us that the (total) energy for a hamonic oscillator is propotional to the number of excitations, or more specifically, $E_N = (N + \frac{1}{2})\hbar\omega$. With this in mind, we can write down an expression for the total energy (in the X_2 direction) of a particle. Even though we are only considering the classical theory at this point, we write down

$$E = E_{kinetic} + E_{potential} \tag{11.10a}$$

$$= \frac{1}{2}\dot{X}_2^2 + \frac{1}{2}\omega^2(\sigma^0) X_2^2 \tag{11.10b}$$

$$\equiv \frac{\hbar\alpha'}{R} N(\sigma^0)\omega(\sigma^0), \tag{11.10c}$$

where R is the world-sheet scalar, $N(\sigma^0)$ is the number of excitations, and we have explicitly written down Planck's constant $\hbar$. Moreover, we have left out the factor of $\frac{1}{2}$, since we are considering a classical derivation.

Even though $N(\sigma^0)$, in general is some time-dependent function, it is not difficult to show that at late times, it approaches a constant, N_{final}. To see this, one has to write out the explicit forms for the Bessel functions and consider their behaviour at late times.[3] Working out the details, it can be shown that the kinetic term, as well as the potential term both scale as $\omega^{-1}(\sigma^0)$. We can substitute this back into (11.10), and use the fact that $\lim_{\sigma^0\to\infty}\omega^{-1}(\sigma^0) = 0$. Then, by comparing left and right, we see that this can only be true if $\lim_{\sigma^0\to\infty} N(\sigma^0) = N_{final}$. In other words, at late times, the number of oscillator excitations becomes constant.

This is actually a rather important result. Using this in combination with the *virial theorem*, we are able to understand the behaviour of particles in this system at late times. First of all, the virial theorem predicts that the average kinetic and potential energy of a system satisfy a certain relation. For potentials that scale with the distance r as $E_{potential} \sim r^{n+1}$, the virial theorem says that[4]

$$\langle E_{kinetic}\rangle = \frac{n+1}{2}\langle E_{potential}\rangle. \tag{11.11}$$

The potential for a harmonic oscillator scales as r^2, so this yields the relation $\langle E_{kinetic}\rangle = \langle E_{potential}\rangle$. It is therefore clear that both terms in (11.10b) approach $\frac{\hbar\alpha'}{2R}N(\sigma^0)\omega(\sigma^0)$ on average.

With this information we are now ready to derive the particle's behaviour at late times. First of all, we can take the equation of motion for X^-, and substitute the results that we derived above. In this way, we obtain

$$\begin{aligned}\ddot{X}^- &= \frac{1}{2}\beta\mu^2\exp(\beta X^+)X_2^2 && (11.12a)\\ &= \frac{1}{2}\beta\omega^2(\sigma^0)X_2^2 && (11.12b)\\ &\approx \frac{\beta\mu\alpha'\hbar N_{final}}{2R}\exp\left[\frac{1}{2}\beta X^+(\sigma)\right], && (11.12c)\end{aligned}$$

at late times. Furthermore, as we assumed before, we made use of the fact that the term $\mathcal{T}_0$ vanished all together. This then leads to the final form for X^-,

$$X^- \approx \frac{2\mu\hbar N_{final}}{\beta\alpha'(p^+)^2R}\exp\left[\frac{1}{2}\beta X^+(\sigma)\right], \tag{11.13}$$

for $\sigma^0 \to \infty$. The interpretation of the dynamics for particles in this system is as follows. At a certain time, a particle meets the bubble wall. If the particle has excited oscillator modes in the X_2 direction (so $N_{final} \neq 0$), the

[3] One can show that for large x, $J_0(x) \approx \sqrt{\frac{2}{\pi x}}\cos(x-\frac{\pi}{4})$, and $Y_0(x) \approx \sqrt{\frac{2}{\pi x}}\sin(x-\frac{\pi}{4})$. Therefore, for large x, $\left[J_0(x)+Y_0(x)\right]^2 \sim \frac{1}{x}$ and $\left[\partial_x J_0(x)+\partial_x Y_0(x)\right]^2 \sim \frac{1}{x}$. See [9] for reference.

[4] See [13] for reference.

particle is pushed outwards along the null-direction, and rapidly accelerates to the speed of light. These particles are energetically forbidden to enter the bubble interior. On the other hand, particles in their groundstates (so for $N_{final} = 0$) do not feel the bubble wall at all. In this case, the particles are able to penetrate the interior of the bubble.

Even if we allow σ^1-dependent modes $\exp(in\sigma^1)$ of the X_2 field back into our model (so we consider one-dimensional strings again), this pictures doesn't change. Each of these modes has a time-dependent frequency $\omega_n(\sigma^0)$, with[5]

$$\omega_n(\sigma^0) \equiv \left[M^2(\sigma^0) + \frac{n^2}{R^2} \right]^{\frac{1}{2}} . \tag{11.14}$$

It can again easily be checked that the adiabatic theorem is satisfied, so that the energy of these modes is again proportional to this frequency,

$$E_n \sim N_n(\sigma^0)\omega_n(\sigma^0). \tag{11.15}$$

And also, the number of oscillations $N_n(\sigma^0)$ approaches a constant at late times, so

$$\lim_{\sigma^0 \to \infty} N_n(\sigma^0) = N_{n,final}. \tag{11.16}$$

Only when $N_{n,final} = 0$, particles are energetically allowed into the bubble interior. So therefore we see that at late times, the interior of the bubble consists entirely of particles that are in their groundstates with respect to the X_2 direction.

This result is in perfect agreement with what we already derived before. Only particles that have no excitations in the X_2 direction are able to penetrate the interior of the bubble, and once they have entered the bubble, they are confined to the minimum of the tachyon profile, at $X_2 = 0$. These results indicate that this new tachyon background gives rise to a theory that starts out in D spacetime dimensions and ends in $(D-1)$ spacetime dimensions.

11.1.4 Oscillatory dependence on more coordinates

In the foregoing, we considered a tachyon profile that had oscillatory dependence on one extra coordinate, X_2. We can of course generalize this theory to a tachyon profile that has oscillatory dependence on n extra coordinates, $X_2, \ldots, X_{n+1}$. We will not repeat the entire calculation again, since the analysis is quite similar to the foregoing. Instead, we will shortly go through the derivation.

First of all, if we expand around $X_i = 0$, and assume the wavelengths of the extra fields long compared to the string scale, the tachyon profile

[5] See [7] for reference.

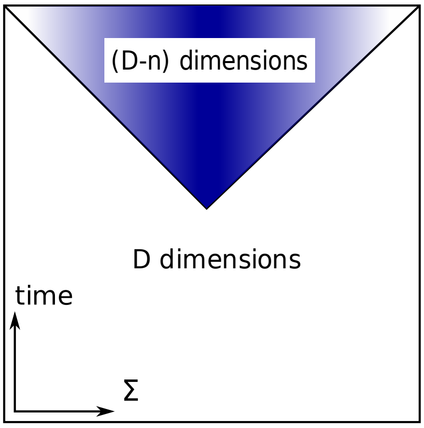

Figure 11.1: *Schematical picture of a dimension-changing bubble. Σ stands for the spatial directions of this theory. At a certain point the bubble is created. The bubble then rapidly expands, changing the interior of the bubble from a D-dimensional theory to a $(D-n)$-dimensional theory.*

simplifies to

$$\mathcal{T}(X) = \frac{\mu^2}{2\alpha'} \exp(\beta X^+) \sum_{i=2}^{n+1} :X_i^2: + \mathcal{T}_0(X^+), \tag{11.17a}$$

$$\text{with} \qquad \mathcal{T}_0(X^+) = \frac{n\mu^2 X^+}{\alpha' q\sqrt{2}} \exp(\beta X^+) + \mu'^2 \exp(\beta X^+). \tag{11.17b}$$

It can immediately be seen that either particles are confined to the minimum of the tachyon profile, $X_2 = \ldots = X_{n+1} = 0$, or they are pushed outwards, along the null-direction. Next, one can write down the Lagrangian for the world-sheet theory, and derive the equations of motion. In that case, the equation of motion for the null-field X^- becomes

$$\partial_+\partial_- X^- = \frac{\alpha'}{4}\partial_{X^+}\mathcal{T} \tag{11.18a}$$

$$= \frac{\beta}{8}\mu^2 \exp(\beta X^+) \sum_{i=2}^{n+1} X_i^2 + \frac{1}{4}\partial_{X^+}\mathcal{T}_0, \tag{11.18b}$$

where we assume the second term to vanish. Then, we can simplify the model by considering pointlike strings, to find the solutions of the extra fields, X_i. The solutions of these fields are again Bessel functions of the first and second kind, just as before. Particles behave as harmonic oscillators in these extra directions, for which the adiabatic theorem is satisfied. Then finally, the virial theorem can be used to show that at late times both the kinetic and potential energy of the oscillator modes $\omega_i(\sigma^0)$ approach an amount of $\frac{\hbar\alpha'}{2R}N_i(\sigma^0)M_i(\sigma^0)$ on average. Substituting these results into (11.18b), and solving it, we find that at late times

$$X^- \approx \sum_{i=2}^{n+1} \frac{2\mu\hbar N_{i,final}}{\beta\alpha'(p^+)^2 R} \exp\left[\frac{1}{2}\beta X^+(\sigma)\right], \tag{11.19}$$

where $N_{i,final}$ are the (constant) number of oscillator modes for the X_i fields at late times.

The interpretation of this theory is clear. Only particles that have no oscillator excitations in either of the X_i directions are able to penetrate the bubble interior. All other particles are expelled from the interior, pushed along the null-direction, approaching the speed of light. Again, this picture doesn't change when the strings do have σ^1-dependent modes of the X_i fields. Therefore, at late times, the bubble interior consists only of particles that have no oscillatory dependence in the extra directions. Moreover, since they are also confined to the tachyon minimum $X_2 = \ldots = X_{n+1} = 0$, one can say the theory has dynamically changed from a D-dimensional theory to a $(D-n)$-dimensional theory. This is schematically depicted in figure 11.1. This type of dynamical dimension-change is also called *dimension quenching*.

11.2 Quantum corrections

11.2.1 Exact solutions at one-loop order

In the foregoing, we have seen that the theory of a tachyon background that has oscillatory dependence on n extra coordinates is exactly solvable classically. In the previous chapter, where we considered a tachyon background that only depended on the null coordinate X^+, all quantum corrections on the world-sheet were absent due to the fact that vertices could only have outgoing lines. Therefore, the classical solutions were actually the exact solutions of this theory. In the theory that we consider here, not all quantum corrections on the world-sheet vanish. But remarkably enough, we will see that the perturbation series is simple enough that it can be computed exactly!

For this derivation it is useful to consider Feynmann diagrams again. Moreover, for sake of clarity we will look at the tachyon background that only depends one extra coordinate, X_2 for now. This derivation can then trivially be extended to an arbitrary number of oscillatory fields X_i. In the Feynmann diagrams we should now make a distinction between the null-fields and the oscillatory fields. The null-fields behave like *massless fields*, and we will denote them by dashed lines. Moreover, we have already shown that the propagator for null-fields is oriented, since it always connects X^+

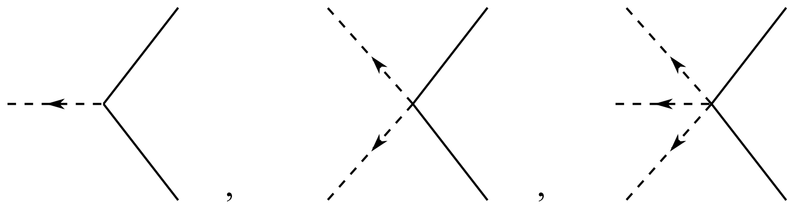

Figure 11.2: *Feynmann diagrams of zero-loop vertices. Each diagram shows a different number of dashed lines emanating from one vertex.*

fields to X^- fields. Therefore, it can be drawn as an arrow, pointing from X^+ to X^-. The oscillatory field, however, behaves like a *massive field*, and we will denote it with a solid line.

Let's now take a look at the interaction vertices for this theory. The tachyon profile only has dependence on X^+ and X_2. So, just as in the previous chapter, no loops can be formed, using massless (dashed) lines. It is also impossible to connect two vertices by a massless leg. Therefore, we see that the dashed part of an interaction vertex can only be at the tree-level. This is not the case for the massive fields, though. The form of the tachyon profile tells us that every vertex has either zero, or two solid lines passing through it. This means that two seperate solid line sigments can never be connected with either a dashed line (the tachyon has no dependence on X^-), or a solid line (since then three solid lines would emanate from these vertices). Therefore, we can conclude that every connected Feynmann diagram has either zero loops, or one loop at most.

The general structure of a diagram with zero loops can be seen as one solid line, passing through an arbitrary number of vertices, in an ordered

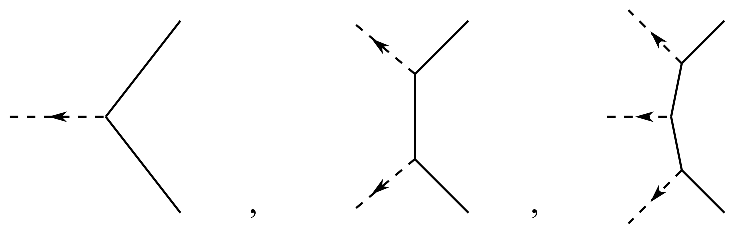

Figure 11.3: *Feynmann diagrams of zero-loop vertices. Each diagram shows a different number of vertices, with just one dashed line emanating from each of them.*

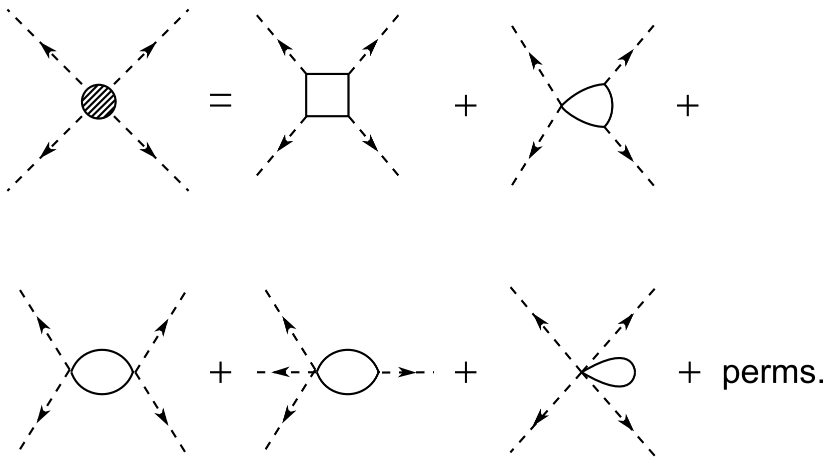

Figure 11.4: *A Feynmann diagram of a four-point interaction. Since these interactions exhaust all possibilities, these diagrams are exact at one-loop order.*

sequence. Each of these vertices has an arbitrary number of dashed lines emanating from it. Figure 11.2 shows a few Feynmann diagrams with zero loops, and a different number of dashed lines emanating from one vertex. Figure 11.3 also shows a few Feynmann diagrams with zero loops, but now with one dashed line emanating from a different number of vertices.

The general structure of a diagram with one loop can be seen as a closed solid line, with an arbitrary number of dashed lines emerging from an arbitrary number of vertices on the closed line. We have shown an example of such a diagram in figure 11.4. Here we have drawn all possibilities for a four-point diagram. This can trivially be extended to a n-point diagram.

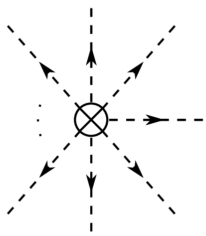

Figure 11.5: *A vertex that only depends on X^+. Such a vertex is depicted by an arbitrary number of outgoing dashed lines.*

Finally, there are counterterms in the interaction of the tachyon that only depend on X^+. The diagrams for these terms can be depicted by an arbitrary number of dashed lines emerging from one vertex, just as was shown in chapter 10. For completeness, this is shown again in figure 11.5.

The diagrams that we just discussed exhaust all possibilities! All quantum corrections terminate at one-loop order. Therefore, we can conclude that this theory is indeed exact at one-loop order, still simple enough to be calculated!

11.2.2 Dynamical readjustment

The concept of dynamical dimension-changing bubbles is a fascinating feature of string theory. The fact that these theories are exactly solvable is even more striking. But there is one important aspect that we have been overlooking. If a theory changes from a D-dimensional to a $(D-n)$-dimensional theory, doesn't the central charge of the theory change as well? If this is the case, then the theory is no longer consistent anymore. Fortunately, it turns out that the *total* central charge of the theory doesn't change. The central charge coming from the fields, indeed decreases by an amount of n, but this difference is compensated by an increase in central charge coming from the dilaton! In other words, comparing the theory at $X^+ \to -\infty$ to the theory at $X^+ \to +\infty$, we see that a *central charge transfer* takes place from the bosonic fields to the dilaton field.

To determine the change in central charge from the dilaton, one looks at the effect of the one-loop diagrams, discussed in the foregoing. It turns out that the effect of these one-loop diagrams vanishes for most of the fields involved in the theory. The only fields that are effected for $X^+ \to +\infty$ are the string fram metric $G_{\mu\nu}(X)$ and the dilaton $\Phi(X)$. To see how these fields are effected, one considers the renormalization of the string frame metric and the dilaton. This is the so-called dynamical readjustment of the metric and dilaton gradient. We will not perform the calculations here, but simply state the results. For more details, the reader is refered to [7].

We will consider dimension-change for n coordinates again. Furthermore, we will denote renormalized fields with a hat. Then, after renormalization, the string frame metric appears as

$$\hat{G}^{++} = 0, \tag{11.20a}$$

$$\hat{G}^{--} = -\frac{n\alpha'\beta^2}{24}, \tag{11.20b}$$

$$\hat{G}^{+-} = \hat{G}^{-+} = -1, \tag{11.20c}$$

where we have switched over to the lightcone coordinate system. All other components are unrenormalized. Moreover, the renormalization of the linear dilaton, $\Phi(X) = V_\mu X^\mu$ appears as

$$\hat{V}_- = -\frac{q}{\sqrt{2}}, \tag{11.21a}$$

$$\hat{V}_+ = -\frac{q}{\sqrt{2}} + \frac{n\beta}{12}. \tag{11.21b}$$

With this information, it is easy to compute the central charge of the dilaton at late times. Recall from chapter 7 that the total central charge of a linear dilaton theory was equal to

$$c = c^X + c^\Phi + c^g \tag{11.22a}$$

$$= D + 6\alpha' G_{\mu\nu} V^\mu V^\nu - 26. \tag{11.22b}$$

Therefore, we see that at late times the central charge contribution of the dilaton becomes

$$c^\Phi = 6\alpha' \hat{G}_{\mu\nu} \hat{V}^\mu \hat{V}^\nu \tag{11.23a}$$

$$= -6\alpha' q^2 + \frac{nq\beta\alpha'}{\sqrt{2}} - \frac{n\alpha'^2 q^2 \beta^2}{8} \tag{11.23b}$$

$$= -(D-26) + n, \tag{11.23c}$$

where in the last step, we made use of the fact that $q^2 = \frac{D-26}{6\alpha'}$ and $q\beta = \frac{2\sqrt{2}}{\alpha'}$. From this it is indeed clear that even though the central charge from the bosonic fields decreases by an amount of n, this difference is picked up by the dilaton field, so that the total central charge of this theory is conserved.

This is actually a very important result. We see that the theory is excatly solvable and that no central charge is lost in the process. This means that we can start in an arbitrary number of spacetime dimensions D and eventually return to a critical dimension by simply choosing $n = (D-26)$ (for the bosonic case). We can even return to a subcritical string theory. In [7] and [8], a different number of superstring theories are linked to each other by practically the same mechanism. We will have a short discussion on this subject in the final section of this chapter.

11.3 Dimension-change for superstrings

11.3.1 Superstring theories

Up till now we have only considered bosonic string theories. These theories, however, are not capable of generating theories containing fermionic particles. Therefore, bosonic string theories are not thought to be realistic theories for the universe that we observe. When supersymmetry is taken into account, string theories emerge that do contain fermions. These string theories are called *superstring theories.*

A lot of the material that we have covered for bosonic strings is also applicable to superstrings. We already mentioned that practically an equivalent calculation for a traceless energy-stress tensor on the world-sheet leads to a critical dimension for superstrings, $D_c = 10$. There is a big difference between bosonic string theories and superstring theories though. Bosonic string theory really only distincts between open and closed strings. This is not the case in superstring theory. There turn out to be a great deal of distinct superstring theories. These superstring theories are devided in type I, type IIA, type IIB, heterotic $SO(32)$ and $E_8 \times E_8$ theories. All these theories are interconnected by a web of dualities, such as T-dualities or c-dualities. A surprising result is that all these theories turn out to be different limits of one 11-dimensional theory of supergravity. Less familiar are so-called type $0A$ or type $0B$ string theories. For these theories the world-sheet is supersymmetric, but the spacetime spectrum is not and does not contain fermions. Its groundstate contains a tachyon, so that this theory resembles bosonic string theory a lot. For more detail on superstring theory, the reader is refered to [2], but most other books on string theory also cover this subject.

11.3.2 Transitions among various string theories

The null-tachyon dilaton system that we discussed in the previous sections can also be applied to superstring theories. These derivations are very extensively described in [7] and [8], but are to technical for the scope of this thesis. Therefore, we will just give a very global picture of the applications in this field of research.

The main difference between the superstring theories is the amount of supersymmetry they contain, or the orientation of the strings. Therefore, it is intuitively not so surprising that if superstrings encounter an expanding tachyon bubble, strings that do enter the interior of the bubble might have a different amount of supersymmetry or orientation in the lower dimensional theory. Therefore, not only the dimension of the theory is able to change, but the theory itself can change all together. This is why we also refer to these processes as *transitions*. In [8], a distinction is made between three types of transitions, namely

- *Stable transitions*: No perturbation of the solution can destroy or alter the final state qualitatively,

- *Natural transitions*: No instability can destabalize the solution without breaking additional symmetry,

- *Tuned transitions*: The initial conditions of an unstable mode must be fine-tuned to preserve the qualitative nature of the final state.

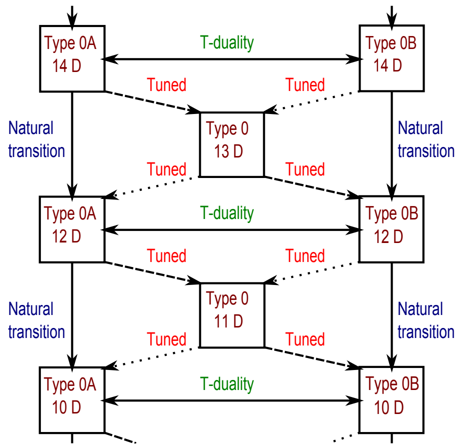

Figure 11.6: *A schematic picture of dimension quenching transitions among type* 0 *string theories. Horizontal arrows correspond to normal* T*-dualities, vertical arrows correspond to natural transistions and diagonal arrows correspond to tuned transitions. The stable endpoint of this lattice is a two-dimensional type* $0A$ *or type* $0B$ *theory.*

With these definitions, we can study various transitions among string theories. In figure 11.6, we showed a small part of a semi-infinite lattice of transitions among type 0 theories. Here we see that these theories can undergo transitions from a D-dimensional theory to a $(D-2)$- or $(D-1)$-dimensional theory. In this particular example, a stable endpoint is a two-dimensional type $0A$ or type $0B$ theory.

Another example is a theory where one starts with a type $0A$ or type $0B$ theory, but eventually ends up with a type IIA or IIB theory respectively, in the critical dimension $D_c = 10$. A fascinating feature of these sort of transitions is that in this way, we are now able to link an infinite number of noncritical string theories to the well-known web of supersymmetric string theories in $D_c = 10$. Before, these theories were thought to be completely disconnected, so this is a very important result.

Maybe even more striking, is the fact that other type of transitions are able to link type 0 theories to pure bosonic theories! In this way, type $0A$ or type $0B$ theories in D dimensions can change into bosonic string theories in D dimensions, or in $(D-1)$ dimensions. This means that we have now actually found a connection between purely bosonic string theories and super string theories! Even though there is much more to say about dimension quenching and transitions among noncritical superstring theories, we will end our discussion here. Readers who are interessed in more information on this subject are refered to the articles of Simeon Hellerman and Ian Swanson.

Chapter 12

Summary

String theory provides us with a surprising fact that these theories are only consistent in a specific number of dimensions. In contrast to theories of one-dimensional particles (quantum field theory, for example), this number arises from the theory itself, instead of being put in by hand. In this thesis we have studied various aspects of string theory, in regard to the number of spacetime dimensions. We restricted our attention mainly to pure bosonic strings. In this first part we set the stage for studying string theory in an arbitrary number of spacetime dimensions and the second part we studied various applications of noncritical string theories.

We started part 1 by explaining why there is need for a critical dimension in string theory. We investigated the classical behaviour of bosonic strings in flat spacetime, in the absence of background fields. Strings in such a theory are described by the Polyakov action, an action for a world-sheet that is carved out by a string, in an arbitrary number of spacetime dimensions. We showed that this action has three important symmetries, namely Poincaré invariance, diff (or reparametrization) invariance and Weyl (or rescaling) invariance. Then, we considered the energy-stress tensor for this action. We derived that the condition for Weyl invariance on the world-sheet, translates to the condition that the energy-stress tensor is traceless. Since rescaling the world-sheet can be described by a renormalization β function, we argued that this tracelessness of the energy-stress tensor is equivalent to saying that the β function should vanish. Then finally, we saw that by taking quantum effects into account, the condition for a vanishing β function gave rise to the critical dimension D_c of the theory. These are the famous numbers $D_c = 26$ for bosonic strings, and $D_c = 10$ for superstrings.

After determining the critical dimension for bosonic strings in flat spacetime with no background fields, we considered the most general world-sheet action that respects diff invariance. This action describes a theory of strings living in a spacetime containing three different background fields, namely a curved spacetime metric $G_{\mu\nu}(X)$, an antisymmetric tensor, called the Kalb-

Ramond field $B_{\mu\nu}(X)$, and a (scalar) dilaton field $\Phi(X)$. Since the claim for Weyl invariance on the world-sheet is still effectual, the energy-stress tensor still needs to be traceless. Or in other words, the β function still needs to vanish. Since there are now three different background fields, the β function consists of three contributions, namely $\beta^G_{\mu\nu}$, $\beta^B_{\mu\nu}$, and β^Φ, which all need to vanish seperately. We found that these β functions are difficult to solve exactly, so we derived them up to first order in α', equivalent to the low energy limit of the theory. A result that was of major importance for the course of this thesis was the fact that the condition, $\beta^\Phi = 0$, told us that the usual numbers for the critical dimension ($D_c = 26$ or $D_c = 10$) could be altered by choosing suitable corresponding background fields. In other words, by considering a string theory with background fields, the number of spacetime dimensions of this theory is allowed to deviate from the critical dimension! This is how we introduced noncritical string theories.

In part 2, we started out by introducing the linear dilaton background, one of the simplest settings for studying noncritical string theory. In this background, the spacetime metric is equal to the Minkowski metric $\eta_{\mu\nu}$, the Kalb-Ramond field is absent, and the dilaton is linear in the spacetime coordinates X^μ. The number of spacetime dimensions in this theory is now altered by an amount proportional to the square of the gradient of the dilaton, $V_\mu V^\mu$, which is always a constant for the linear dilaton background. This result is actually exact, since all higher order derivatives vanish. It turned out to be very interesting to study a tachyon profile in the vicinity of a linear dilaton background. To this end, we considered the low energy effective action for a tachyon. Solving the equation of motion for this action, the so-called tachyon on-shell condition, gave rise to the tachyon profile. When we added this tachyon to the world-sheet action for strings in a linear dilaton background, we obtained a so-called Liouville field theory. Such a theory typically shows the behaviour of a potential barrier, growing exponentially with distance, in time, or both, depending on the choice of the dilaton. These theories are typically very hard to solve exactly.

Another aspect of noncritical string theory that we studied is cosmological behaviour. There seems to be a very close analogy between cosmologies, driven by quintessence (i.e. a scalar field that enters the cosmology action with a kinetic part, and an exponential potential part) and string theories with a timelike linear dilaton background. By comparing the two theories, we found that the tree-level potential of the string theory gives rise to an equation of state at the boundary between accelerating and decelerating cosmologies. This analogy inspired us to look for cosmological solutions of this string theory. We also investigated what string modes are stable against perturbations of the background, and found that the only unstable modes of this theory are tachyonic modes, modes with negative mass-squared. A striking aspect of this analysis is we have actually been able to find solutions of strings in time-dependent backgrounds, a problem that generally is very hard to solve!

In chapters 10 and 11, we have mainly been concerned with finding exact solutions for theories with linear dilaton and tachyon backgrounds. The exact solutions that we obtained were either 'bubbles of nothing', or 'dimension-changing bubbles'. The starting point for finding exact solutions for the tachyon-dilaton theory is the Liouville theory that we already encountered in the foregoing. Now, however, we made the important assumption that the tachyon profile is 'null'. In this case, this assumption comes down to imposing the d'Alembertian of the tachyon profile, $\partial_\mu \partial^\mu \mathcal{T}(X)$, to be zero. By adapting to lightcone coordinates, $X^{\pm}$, we discovered that with this choice, the tachyon profile only depends on X^{+}. The consequence of this is that interactions on the world-sheet with the tachyon terminate at the tree-level. In other words, all quantum corrections in this theory are absent, so that the classical theory is actually the exact theory for this model! We solved the equations of motion of the world-sheet action and made a simplification by considering pointlike strings. With this simplification, we were able to plot trajectories of strings that encounter these bubbles. We discovered that all such particles are pushed outwards, rapidly accelerating to the speed of light. From this analysis it became clear that no particles whatsoever are able to penetrate the bubble interior, not even the graviton. Therefore, the bubble indeed can be seen as a spacetime-destroying bubble of nothing.

In the final chapter of this thesis, we studied the null-tachyon dilaton theory again, but now assumed the tachyon to have oscillatory dependence on n extra coordinates. By considering this theory in the limit where wavelenghts of these extra fields are long compared to the string scale, we found that we obtained a theory that actually resembles the bubble solution in the foregoing alot. Strings that have oscillations in at least one of the extra directions are expelled from the bubble interior, very much like we described in the foregoing. However, there is one major difference: strings that have no oscillations in the extra directions are able to penetrate the bubble, but in this bubble interior they are confined to the region where the tachyon is at its minimum with respect to the extra directions. Energetically, they are forbidden to have any oscillations in the extra directions, so therefore they have to be in their groundstates. The interpretation of this theory therefore is clear. Instead of a bubble of nothing, this solution is a dimension-changing bubble! A theory in this setting starts out as a D-dimensional theory, but ends up as a $(D-n)$-dimensional theory. A beautiful feature of this theory is the fact that it is still exactly solvable. In contrast to the foregoing, not all quantum corrections vanish, but they turn out to terminate at one-loop order. This result could then be used to show that the total central charge of the theory is conserved, transfered from the bosonic fields to the dilaton.

Finally, we argued that this theory can also be considered for superstring theories. Solutions that arise in these theories are again dimension-changing solutions, but as it turns out, these superstring theories can change all together! There are various transitions possible from one theory to another. In this frame-work it has even become possible to establish a connection between supercritical string theories and critical theories, a result that had not been achieved before. Moreover, there are even transitions that connect superstring theories to pure bosonic string theories!

It is clear that the study of noncritical string theory almost offers an overabundance of new and insightful information about string theory, which can help us gain a better perspective of the field. Even though noncritical string theory is not so well-known among string theorists, it is a truely elegant frame-work and it may deserve more attention in the future.

Appendix A

Renormalized operators

In we start out with an operator $\mathcal{F}$, which enters a path integral, and we want it to respect diff×Weyl invariance, we can renormalize this operator. A renormalized operator $[\mathcal{F}]_r$ is defined as

$$[\mathcal{F}]_r = \exp\left[\frac{1}{2}\int d^2\sigma d^2\sigma' \Delta(\sigma,\sigma')\frac{\delta}{\delta X^\mu(\sigma)}\frac{\delta}{\delta X_\mu(\sigma')}\right]\mathcal{F}, \tag{A.1a}$$

$$\text{where} \quad \Delta(\sigma,\sigma') = \frac{\alpha'}{2}\ln \mathrm{d}^2(\sigma,\sigma') \tag{A.1b}$$

and $\mathrm{d}(\sigma,\sigma')$ is the geodesic distance between the points σ and σ'.[1] This expression instructs us to sum over all possibile ways to contract pairs in $\mathcal{F}$, making use of $\Delta(\sigma,\sigma')$. As can be checked (see [11]), the renormalized operator automatically satisfies diff-invariance.

The Weyl-invariance needs to be checked by hand. Since a Weyl variation is a variation in the world-sheet metric (recall chapter 2), we obtain two contributions when varying a renormalized operator. The first contribution comes from the explicit metric-dependence of the operator $\mathcal{F}$. The second comes from the metric-dependence of $\Delta(\sigma,\sigma')$. Therefore, the Weyl variation of $[\mathcal{F}]_r$ can be written as

$$\delta_W[\mathcal{F}]_r = [\delta_W\mathcal{F}]_r + \frac{1}{2}\int d^2\sigma d^2\sigma' \delta_W\Delta(\sigma,\sigma')\frac{\delta}{\delta X^\mu(\sigma)}\frac{\delta}{\delta X_\mu(\sigma')}[\mathcal{F}]_r. \tag{A.2}$$

Next, we consider the case where the distance between the two points σ and σ' is small. In that case it can be derived that

$$\mathrm{d}^2(\sigma,\sigma') \approx \exp\left[2\omega(\sigma)\right](\sigma-\sigma')^2, \tag{A.3}$$

which implies

$$\Delta(\sigma,\sigma') \approx \alpha'\omega(\sigma) + \frac{\alpha'}{2}\ln(\sigma-\sigma')^2. \tag{A.4}$$

[1] Geodesic distances are introduced in appendix B.

In the limit where $\sigma' \to \sigma$, the Weyl variation is non-singular. This means that the Weyl variation of $\Delta(\sigma,\sigma')$ can be written as

$$\delta_W \Delta(\sigma,\sigma') = \alpha' \delta\omega(\sigma). \tag{A.5}$$

Now this result can be used to check the Weyl-invariance of the renormalized operator $[\mathcal{F}]_r$ explicitly, by plugging it back into (A.2) and working out the integrant.

Another way to look at this, is that when we perform a Weyl variation to a renormalized operator, we obtain a condition that ensures Weyl-invariance. For example, in [11] it is shown that performing a Weyl variation to a renormalized closed tachyon vertex operator, leads to the well known result for the tachyon's momentum

$$k^2 = -M^2 = \frac{4}{\alpha'}. \tag{A.6}$$

In some cases (depending on the actual form of the renormalized operator), one needs to work in higher order of the variation of the geodesic distance, and derivatives need to be included. One can show (see [11]) that to first and second order in derivatives,

$$\partial_a \delta_W \Delta(\sigma,\sigma')\Big|_{\sigma'=\sigma} = \frac{1}{2}\alpha' \partial_a \delta\omega(\sigma), \tag{A.7a}$$

$$\partial_a \partial'_b \delta_W \Delta(\sigma,\sigma')\Big|_{\sigma'=\sigma} = \frac{1+\gamma}{2}\alpha' \nabla_a \partial_b \delta\omega(\sigma), \tag{A.7b}$$

$$\partial_a \partial_b \delta_W \Delta(\sigma,\sigma')\Big|_{\sigma'=\sigma} = -\frac{\gamma}{2}\alpha' \nabla_a \partial_b \delta\omega(\sigma), \tag{A.7c}$$

where ∂'_b means the derivative with respect to the coordinates σ', and ∇_a is a covariant derivative.[2] Furthermore, γ is a renormalization parameter and can be chosen at will, by choosing the proper renormalization.

[2]For details on a covariant derivative, see appendix B.

Appendix B

Curvature

B.1 Path length and proper time

In flat Minkowski spacetime the *path length* or *geodesic length* of a point particle carving out a worldline $x^\mu(\lambda)$, parameterised by λ, is defined by

$$s = \int d\lambda \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}, \tag{B.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric $diag(-, +, \ldots, +)$. It is related to the particles *proper time* by

$$\tau = \int d\lambda \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}, \tag{B.2}$$

which will be positive for timelike paths. In curved spacetime, the curvature is determined by a spacetime dependent metric $g_{\mu\nu}(x)$. The proper time of a particle moving in curved spacetime can now be written down in exactly the same way as in flat spacetime, but now replacing the Minkowski metric $\eta_{\mu\nu}$ with the curved metric $g_{\mu\nu}$.

$$\tau = \int d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}. \tag{B.3}$$

B.2 Christoffel connection

If we want to use derivatives in curved manifolds, it is very useful to introduce a *covariant derivative* ∇_μ, which unlike the partial derivative ∂_μ, is independent of the coordinate system used. The covariant derivatives of a vector V^ν, and a covector ω_ν are defined by

$$V^\nu{}_{;\mu} \equiv \nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho, \tag{B.4a}$$

$$\omega_{\nu;\mu} \equiv \nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho. \tag{B.4b}$$

Here $\Gamma^{\lambda}_{\mu\nu}$ are known as the *Christoffel connections*, which in terms of the metric and its derivatives are written as

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right). \tag{B.5}$$

An important property of the covariant derivative is that $\nabla_\rho g^{\mu\nu} = \nabla_\rho g_{\mu\nu} = 0$, so locally the covariant derivative of the metric always vanishes. Furthermore the covariant derivative along a path $x^\mu(\lambda)$ can be written as

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda} \nabla_\mu. \tag{B.6}$$

One equation that is very important in curved space calculations is the *geodesic equation.* One way to think of this is as an equation of motion telling you what *straight lines* in curved space are, or equivalently, the shortest path between two points in this curved space.[1] The geodesic equation is written as

$$\frac{D}{d\lambda} \frac{dx^\mu}{d\lambda} = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0. \tag{B.7}$$

As can be seen, this equation reduces to the usual equation of motion for a particle moving in flat spacetime, which can be seen when we choose $g_{\mu\nu} = \eta_{\mu\nu}$.

B.3 Curvature tensors and scalars

Another very useful quantity is the *Riemann curvature tensor* $R^{\rho}{}_{\sigma\mu\nu}$. This tensor locally gives a description of the amount of curvature of the manifold we are describing. One way to locally describe the amount of curvature is by means of *parallel transporting* a vector. A vector can be transported in such a way that it is parallel to its original direction, along a path in a curved space (a process known as parallel transport). Then, if we choose such a path that we return to the starting point, the vector can actually deviate from the vector at the beginning. It we transport a vector V^μ in an infinitesimally closed loop in a curved space, the amount of change of the vector δV^μ will be proportional to the Riemann curvature tensor.

The Riemann curvature tensor is defined as

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma} \tag{B.8}$$

and is antisymmetric in the last two indices. Sometimes it is useful to consider contractions of the Riemann curvature tensor with itself. One particular interesting contraction is the following,

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}. \tag{B.9}$$

[1] For example, a meteorite that passes the Sun closely, seems to be deflected by the Suns gravity and therefore describes a curved path. But it actually satisfies the geodesic equation in a space, curved by the Suns gravity.

The remaining tensor is known as the *Ricci tensor*, and in terms of the Christoffel connection it is a symmetric tensor, so $R_{\mu\nu} = R_{\nu\mu}$. We can even carry this one step further and contract the Ricci tensor with itself. We then end up with what is known as the *Ricci scalar* R.

$$R = R^{\lambda}{}_{\lambda} = g^{\mu\nu} R_{\mu\nu}. \tag{B.10}$$

A nice property of the Ricci scalar is that it is independent of the coordinate system used. For example, the Ricci scalar for Euclidean spacetime is zero,

$$R_{Euclidean} = 0 \tag{B.11}$$

and the Ricci scalar for a two-sphere S^2 of radius r is

$$R_{S^2} = \frac{2}{r^2}, \tag{B.12}$$

as is shown in [4].

B.4 Riemann normal coordinates

When calculating curvature tensors or scalars, one often ends up with a very large number of terms. This can makes calculations very hard. There are, however, some simplifications that can be made, even without loss of generality. One of these simplifications is switching to what is called *Riemann normal coordinates* in point p, or RNC for short. It is always possible locally to choose a coordinate system where the metric in a point p is equal to the Minkowski metric, and the first order derivative of the metric at p vanishes. Second and higher order derivatives however, need not vanish. So, RNC at p satisfy

$$g_{\mu\nu}(p) = \eta_{\mu\nu}, \tag{B.13a}$$
$$\partial_\rho g_{\mu\nu}(p) = 0. \tag{B.13b}$$

A lot of expressions simplify considerably when switched to RNC. The Christoffel connection for example, which is made up of first order derivatives of the metric, vanishes. Therefore covariant derivatives turn into ordinary partial derivatives. The Riemann curvature tensor becomes a lot simpler and therefore it's much easier to calculate the Ricci tensor and Ricci scalar. And since the Ricci scalar is independent of the coordinate system used, the result will be completely general. However, one needs to be cautious though. RNC can only be applied to one point p!

B.5 Einstein's equations

Now with the curvature quantities defined above, one can define a *Einstein tensor*

$$\mathbf{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{B.14}$$

Using the Bianchi identity[2], one can show that

$$\nabla^{\mu} \mathbf{G}_{\mu\nu} = 0. \tag{B.15}$$

So far we just considered how curvature can be described in terms of the metric $g_{\mu\nu}$. Up till the early 20^{th} century physicists believed that our universe was a $3+1$ dimensional space, where masses attract each other through gravitational forces. Einstein, however, was the first to realize that mass and energy somehow bend spacetime itself, causing objects to follow paths in this curved spacetime. He was able to relate the energy and mass present in a region of space, to the amount of curvature it caused. He therefore introduced an *energy-momentum tensor* $T_{\mu\nu}$.[3] The form of an energy-momentum tensor is dependent on the theory that is considered.

An example of an energy-momentum tensor the one for the universe. In cosmology, our universe is considered as a homogeneous isotropic fluid, also referred to as *perfect fluid.* Galaxies in the universe are the "particles" that make up this fluid. When this assumption is made, the energy-momentum tensor, in terms of the energy density ρ and pressure p in a region of space, can (locally) be written as

$$T^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \tag{B.16}$$

The energy-momentum tensor is conserved. In a curved space this leads to the condition

$$\nabla^{\mu} T_{\mu\nu} = 0. \tag{B.17}$$

Now, *Einstein's principle of equivalence* tells us that there is no way for observers to distinct between uniform acceleration and the presence of a gravitational field. Although the actual derivation is rather subtle, intuitively it is a logical idea to somehow relate the energy-momentum tensor to Ricci curvature tensor. Since the energy-momentum tensor is conserved, we have to look for a combination of the Ricci tensor that is also conserved.

[2] $\nabla_{\lambda} R_{\rho\sigma\mu\nu} + \nabla_{\rho} R_{\sigma\lambda\mu\nu} + \nabla_{\sigma} R_{\lambda\rho\mu\nu} = 0$.

[3] This basically tells you how much energy, mass and momentum is present in a part of space.

But we already had such combination, namely the Einstein tensor, (B.14). We therefore impose that

$$\mathbf{G}_{\mu\nu} = \kappa T_{\mu\nu}, \tag{B.18}$$

where κ is the gravitational coupling constant. By working out the details, one finds[4]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu}, \tag{B.19}$$

where G_N is the gravitational constant.[5] These famous equations are known as the *Einstein equations*. If we take the trace of (B.19), we find that $R = -\frac{8\pi G_N}{c^4} T$. Plugging this back into (B.19), we end up with a slightly different version of the Einstein equations, namely

$$R_{\mu\nu} = \frac{8\pi G_N}{c^4} \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right). \tag{B.20}$$

In a vacuum, $T_{\mu\nu} = 0$, and as can be seen from (B.20), the Einstein equations then simplify to

$$R_{\mu\nu} = 0. \tag{B.21}$$

Even though this looks like a simple set of functions, they can still be very hard to solve.

The Einstein equations can also be describes in the Lagrangian formalism. If a curved space is described by a metric $g_{\mu\nu}$ and a matter-part of the theory $\mathcal{L}_M$, and we include the cosmological constant Λ, the action for this theory, known as the *Einstein-Hilbert action*, is

$$S_H[g] = \int d^4x \sqrt{-g} \, [k(R - 2\Lambda) + \mathcal{L}_M], \tag{B.22}$$

with $k = \frac{1}{2}\kappa^2$, and we also assumed $D = 4$ dimensions.

[4]See [4] of [3] for reference.

[5]$G_N = (6.67428 \pm 0.00067) \times 10^{-11} m^3 kg^{-1} s^{-2}$.

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