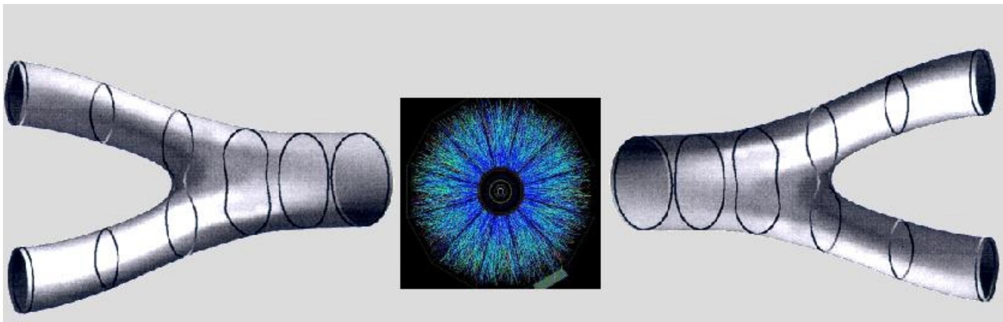

Gluon scattering amplitudes from weak to strong coupling and back

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Master's Thesis

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To my parents, Andrzej and Lucyna

Abstract

In this thesis we review recent developments in studies of gluon scattering amplitudes in planar, maximally Supersymmetric Yang-Mills theory using string theory methods. In 2007, Alday and Maldacena managed to compute a gluon scattering amplitude at strong coupling using the AdS/CFT correspondence. They showed that, in this regime, amplitudes are dual to Wilson loops along light-like contours, which eventually boils down to finding a minimal surface in Anti-de Sitter space. Surprisingly, it turned out that this equivalence between gluon amplitudes and light-like Wilson loops continues all the way down to the weak regime of the theory. After introducing sufficient background material, we present both results (at strong and weak coupling) for four gluons amplitude, and discuss the symmetry responsible for the duality at this level. Finally, we sketch some interesting results obtained in studies of amplitudes for six gluons and highlight the main difficulties there. At the end, we summarise and pose important questions which require further investigation, and might lead to interesting new paths of research.

Contents

Acknowledgments

I would like to thank to my supervisor, prof. Jan de Boer, for guidance and many enlightening discussions about physics and my various concerns. Thank you Jan! It has been a pleasure to be your student. I hope I have managed to grasp at least a bit of your fantastic physical intuition and approach to science.
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1

Introduction

1.1 How to read this thesis

We assume, that the reader will be either a master student, whose thesis topic is somewhat related to this field, or a PhD student who would like to perform further research in the field of gluon scattering amplitudes at strong coupling. As one would expect, some previous knowledge of a non-abelian quantum field theory and supersymmetry is essential in order to follow the subject. Later on in the discussion, familiarity with string theory, D-branes and the AdS/CFT correspondence will prove very useful to the reader. Of course, full appreciation of the subject would require tracking back to the deepest sources of all of the aforementioned topics, which we guess is impossible, especially on a master's level. That is why, we will elaborate a bit on some basic necessary tools to reach the final goal, a string theory computation of gluon scattering amplitudes. Furthermore, the reader is strongly encouraged to follow references to the relevant literature in this field.

In the chapter about the Alday-Maldacena prescription for the amplitudes we review their computation quite extensively. Therefore, any advanced researcher interested in details, may easily skip the introductory part.

1.2 History and motivation

In the early 1990's, 't Hooft and Susskind studied a paradox posed by Hawking about loss of information when a body falls into a black hole. They realised that theories of quantum gravity could be formulated in a holographic way. Namely, all of the information contained in a volume of space could be projected and fully described by a "hologram" living on the boundary of this volume. Their argument is known as the "holographic principle"[1]. This surprising result inspired physicists searching for a theory of quantum

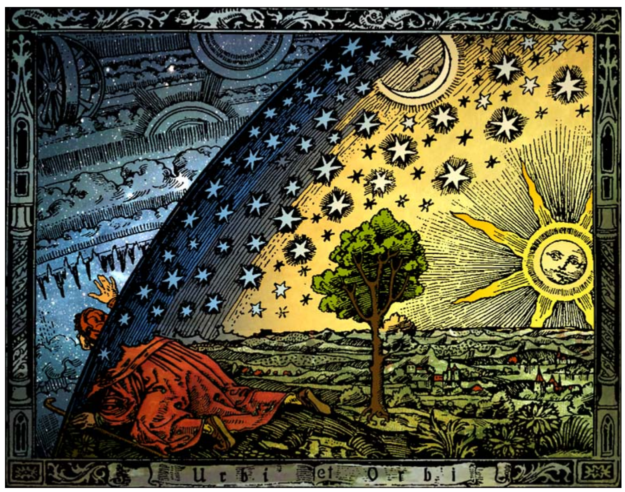

Figure 1.1: *The idea of 't Hooft and Susskind about the holographic Universe with all information on the boundary prepared the ground for the first holographic models. Picture of Camille Flammarion from 1888 shows the Universe with a boundary.*

gravity and they started studying all kinds of models based on holography. One of them was a string theorist, Juan Maldacena. In 1998, he published the famous proposal [35] suggesting the equivalence between gauge theories in four dimensions, living on a boundary of a five dimensional space (see 1.2). As an example, he conjectured the duality (equality) between string theory in Anti-de Sitter (AdS) space [2] (quantum gravity) and the maximally supersymmetric, conformal Yang-Mills theory in four dimensions living on

[1]For a nice review of the holographic principle see [8]

[2]Precisely superstring theory with $AdS_5 \times S^5$ background compactified on S^5 to five dimensions

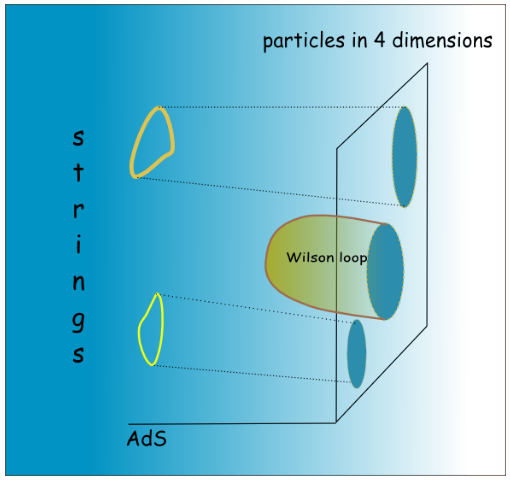

Figure 1.2: *Maldacena's model postulates that strings living in the five dimensional AdS space are equivalent to particles living at the four dimensional boundary. Wilson loops are being interpreted as open stings with end points on the boundary, while their world-sheet is stretching into the interior of AdS space ("bulk").*

the boundary of AdS. We will refer to this example as the AdS/CFT duality. The theory with gravity, described by type IIB superstring theory in ten dimensions is endowed with two parameters, the radius R of AdS, and the string coupling g_s. On the gauge theory side, the $\mathcal{N}$=4 Super-Yang-Mills theory with non-abelian, $SU(N)$ gauge group, also has two parameters: Yang-Mills coupling g_{YM} and the number of colours N. As we can expect, in order to claim an equivalence between two systems, one has to prove that observables computed in both theories are the same. That is why, to compare correlation functions in the dual systems, Maldacena proposed an identification between the four parameters:

$$g_s = g_{YM}^2, \qquad R^4 = 4\pi g_{YM}^2 N(\alpha')^2, \tag{1.1}$$

where α' is the historical Regge slope related to the string length as $l_s = \sqrt{\alpha'}$. The form of the conjecture described above is usually referred to as the "strong form", as it is supposed to hold for all values of the Yang-Mills coupling and all N.

People started to test this proposal in many possible ways. However, as it often happens in physics, things turned out to be more complicated than they appeared at first. The main difficulty was due to the lack of a quantisation method for string theory on a general curved background (here $AdS_5 \times S^5$). Therefore, the first attempts to compute something in AdS/CFT required studying its more treatable limits (see 1.3). It was known before [49] that in non-abelian gauge theories, in the so-called 't Hooft limit, perturbative expansion in g_{YM} could be reorganized into a form similar to string theory's perturbative series in $1/N$. Precisely, defining the 't Hooft coupling

$$\lambda = g_{YM}^2 N \tag{1.2}$$

and taking the number of colours to be very large, $N \to \infty$, while keeping λ fixed, only diagrams that can be drawn on a plane contributed (hence this limit is often known as planar)[3]. On the AdS side, since

$$\lambda = g_{YM}^2 N = g_s N \Rightarrow g_s = \frac{\lambda}{N}, \tag{1.3}$$

the large N limit made string theory weakly coupled, so it could be studied perturbatively as a series in g_s. Nevertheless, this was still not enough for reasonable computations. Hence, Maldacena proposed taking the limit of large 't Hooft coupling on the gauge theory side (expand in $\lambda^{-1/2}$), which would correspond to the classical type IIB supergravity solutions in AdS, and α' string expansion. It turned out that the dual theories had the same underlying symmetry, the $SU(2,2|4)$ conformal group, and studying its representations allowed for a precise mapping between observables on both sides of the conjecture. That was a great success of Maldacena's proposal. Surprisingly, this limit served as a new powerful tool, such that strong coupling physics of non-abelian gauge theories -extremely hard to treat analytically- could be approached from a weakly coupled string theory in AdS space. One of such applications was the original motivation for this thesis. Namely, how to use AdS/CFT to understand strong coupling gluon collisions.

The key link between string theory and gluon amplitudes is encoded in Wilson loops: an exponent of the integral from a gauge field A^μ along a contour C

$$W[C] = \langle e^{\oint_C A^\mu x_\mu} \rangle. \tag{1.4}$$

In the seventies, people formulated QCD with large number of colours in terms of Wilson loops, which played the role of Green's functions. In 1998, Maldacena realised that the dominant (classical saddle point) part of the

[3]we will explain this limit later in the part about gauge theories

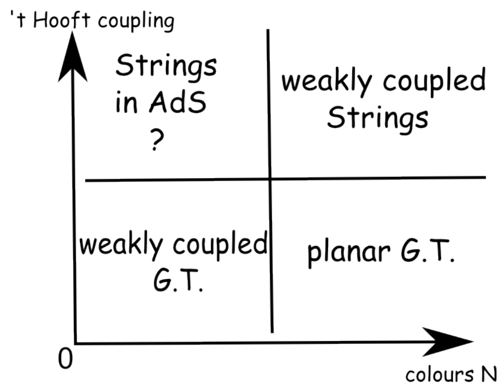

Figure 1.3: *Treatable limits of AdS/CFT. Later we will compare a proposal postulated at weak coupling λ (bottom left corner) with a string theory computation which corresponds to strong λ (upper right part).*

expectation value of a Wilson loop can be obtained from the AdS/CFT correspondence. Namely, as an exponent of the area of an open string worldsheet (minimal surface) stretching into AdS, attached to C on the boundary (see 1.2). His prescription was tested for the simplest contours, like circles or parallel lines, and seemed to give reasonable results. At the same time, the holographic treatment of the gluon scattering amplitudes became possible. However, due to the lack of any analytical predictions for strong coupling physics, the correctness of the string theory approach was impossible to verify. The picture changed in 2005, when Bern, Dixon and Smirnov [7] discovered iterative properties of Maximally-Helicity-Violating (MHV) gluon amplitudes in maximally Supersymmetric Yang-Mills. Based on their results, they proposed a general structure of the amplitudes (so-called BDS ansatz) depending on functions $f(\lambda)$ and $g(\lambda)$, which could be written for both strong and weak values of the 't Hooft coupling λ (see (3.2), (3.4)); this was the perfect moment for AdS/CFT Wilson loops to be applied. Together with Alday [1], Maldacena explained how to find a minimal surface for a particular gluon configuration and compared it with BDS. The results matched perfectly for four gluons. In this thesis we tried to generalise their result to 6 particles. This turned out to be quite a non-trivial task and nobody has found it yet. Fortunately, many new things appeared along the way. Firstly, Korchemsky and his collaborators [22] noticed the potential duality between Wilson loops and MHV amplitudes. This surprising correspondence survived an explicit test of numerical calculation for 6 gluons.

Moreover, many interesting questions appeared during intermediate steps of the research, so they are going to be discussed throughout this thesis and possible ways for solving these problems will be suggested.

1.3 Plan

- Part one will review basic ingredients of gauge theories which will serve as a background for studying the string theory results. First, we will introduce the $\mathcal{N}$=4 theory and its planar limit. Then, basic facts about MHV amplitudes, super-Ward identities and the BDS ansatz will be reviewed. At the end, we will familiarise ourselves with Wilson loops and their supersymmetric extensions.

- Part two will be dedicated to string theory tools needed in the final gluon scattering amplitude computation. We will introduce the simplest notion of the bosonic string and their actions. Finally, we will show the T-duality from the perspective of the string's world-sheet.

- The last part will present the results for gluon scattering amplitudes, from weak to strong coupling, with a strong emphasis on the Alday-Maldacena computation. In addition, the equivalence between the 4-point MHV gluon amplitude and the Wilson loop on the four light-like intervals will be shown. Also, the results for 6-gluons amplitudes which spot the failure of BDS will be presented. At the end we will conclude, pose some open questions and try to sketch a possible follow up of this work.

- The appendices will hopefully clarify more technical parts of the thesis.

Part I

Gauge Theory

2

The $\mathcal{N}$=4 Super-Yang-Mills theory

We start by introducing the gauge ingredient of the AdS/CFT duality, the $\mathcal{N}$=4 Supersymmetric Yang-Mills theory. This theory is just a special case of a non-abelian gauge theory, like the standard model's QCD (fr example see [39]), and shares many general features with all of them. For instance, one of the main motivations, other than AdS/CFT, for studying $\mathcal{N}$=4 SYM is the infrared (IR) structure of scattering amplitudes (similar to QCD). In both theories they suffer from soft and collinear divergences. Despite this fact they can be used to construct complete, gauge invariant correlators from unitarity [3]. Therefore, full control and understanding of the IR behaviour is essential for precise computations. It turns out that $\mathcal{N}$=4 SYM is a much more friendly ground for this analysis than QCD. The IR character in the former is governed by two functions: the "cusp anomalous dimension" and the "collinear anomalous dimension", which we will define below. The cusp anomalous dimension was calculated in the supersymmetric theory using the holographic correspondence [38]. This should already be sufficient motivation to embark on a study of the $\mathcal{N}$=4 SYM theory. As a reminder to the reader, for the last few decades the strong coupling regime of QCD has been accessible only by means of lattice gauge theories (see for e.g. [45]). Therefore any new method of probing this region is a breakthrough in our understanding of strong force at low energies. The calculation of the anomalous dimension using the holographic correspondence, is only the beginning of "miracles" in this new field [31]. The recent progress in studies of the gluon scattering amplitudes in $\mathcal{N}$=4 SYM using the spinor-helicity formalism, [30], [24], gave birth to a new, fruitful area of research. People discovered the so-called MHV gluon amplitudes which have a very specific helicity configuration. An explicit evaluation of the MHV's showed that they satisfy a very surprising iterative relation between higher and lower orders in perturbation theory. Such a relation led Bern, Dixon and Smirnov (BDS) to the ansatz [7] for the general structure of an n-point MHV amplitude. This will be reviewed here, and in the following chapters we will compare it with the strong coupling string theory result of Alday and Maldacena [1]. Now, full of excitement and curiosity, we can proceed with a more systematic introduction to the $\mathcal{N}$=4 SYM theory.

2.1 $\mathcal{N}$=4 SYM

We will start by disentangling all the abbreviations, letters and numbers present in the literature about this theory. First of all, it is defined in 4 dimensions. Then, the calligraphic $\mathcal{N}$ stands for the number of supersymmetries[1] in the theory, so there are 4 of them. Their function, as in the case of every supersymmetry, is to mix bosons and fermions. Finally, there is a non-abelian gauge group SU(N), where the capital N denotes the number of gluon colours.
This model has a few properties which make it special compared to other non-abelian theories. One is that it is the maximally supersymmetric theory without gravity. In order to understand this statement, we will look at the massless fields of the theory. There are eleven of them, all in the adjoint representation of $SU(N)$:

- a gluon $A_\mu(x)$ (the gauge field is called a gluon by analogy to QCD), occurring in two helicities: $h = +1$ and $h = -1$
- six real scalars $\phi_i(x)$ $(i = 1, \cdots, 6)$
- four Majorana gluinos, which are usually written as the sixteen-component Majorana-Weyl[2] spinor $\chi_\alpha(x)$, $\alpha = 1 \cdots, 16$. They can also appear in two helicities: h=-1/2 and h=1/2.

The four supersymmetries transform fermions into bosons and vice-versa. More precisely, their action is generated by operators called supercharges

[1] for a speed-of-light introduction to supersymmetry see [47]
[2] see Majorana condition (2.5)

Q_i, $i = 1,2,3,4$. They act upon fields as

$$Q_i(fermion) \rightarrow (boson) \qquad Q_i(boson) \rightarrow (fermion).$$

This can be formally viewed as shifting the helicities from -1 to 1 in units of a half

$$(A^-, h=-1) \leftrightarrow (\chi^-, h=-\frac{1}{2}) \leftrightarrow (\phi, h=0) \leftrightarrow (\chi^+, h=\frac{1}{2}) \leftrightarrow (A^+, h=1) \tag{2.1}$$

If there were more supersymmetries one would have to include particles with spin greater than 1. In order to unitarily quantize models with such objects, we would need to include a spin 2 graviton (see for e.g. [17]). This remark obtains a deeper meaning in the context of the AdS/CFT correspondence. There the theory of gravity in AdS space is fully determined by the holographic image living on the boundary, with no gravity at all.
The action for the theory is uniquely fixed by the number of colours N and the coupling constant g_{YM} [28]:

$$S = \int d^4x \frac{1}{2g_{YM}^2}\left[\frac{1}{2}\left(F^a_{\mu\nu}\right)^2 + \left(\partial_\mu \phi^a_i + f^{abc}A^b_\mu \phi^c_i\right)^2 + \bar{\chi}^a i\Gamma^\mu \left(\partial_\mu \chi^a + f^{abc}A^b_\mu \chi^c\right)\right.$$
$$\left. + i f^{abc}\bar{\chi}^a\Gamma^i\phi^b_i\chi^c - \sum_{i<j} f^{abc}f^{ade}\phi^b_i\phi^c_j\phi^d_i\phi^e_j + \partial_\mu \bar{c}^a\left(\partial_\mu c^a + f^{abc}A^b_\mu c^c\right) + \zeta(\partial_\mu A^a_\mu)^2\right],$$

where we introduced the ghost fixing part with ghost fields c^a and the gauge parameter ζ. The small Latin index a runs from 1 to N^2-1 and labels the number of generators T^a of SU(N). They are normalised to

$$\left[T^a, T^b\right] = i f^{abc}T^c, \qquad TrT^aT^b = \frac{1}{2}\delta^{ab}. \tag{2.2}$$

The f^{abc} are the structure constants of this Lie algebra and in addition satisfy

$$\sum_{cd} f^{acd}f^{bcd} = N\delta^{ab}. \tag{2.3}$$

Naturally, there is also the kinetic term for the gauge field. It contains the anti-symmetric tensor

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu. \tag{2.4}$$

All the fermionic fields can be written as a 16-component spinor χ satisfying the Majorana condition

$$\chi(x) = C\chi^\star(x), \tag{2.5}$$

where C is the charge conjugation matrix (see for e.g. [47]).
Finally, the Dirac matrices are generalised to the form

$$\Gamma^A = (\Gamma^\mu, \Gamma_i)\,, \tag{2.6}$$

where $\mu = 0\ldots,4$ and $i = 5\ldots9$. They are real, 16×16 matrices in the ten dimensional Majorana representation with the Weyl constraint. The trace of the two matrices is also normalised to

$$Tr\Gamma^A\Gamma^B = 16\delta^{AB}. \tag{2.7}$$

Let us pursue this further by analysing all the propagators. The most convenient way to do this is to pick up the covariant gauge fixing condition [16]

$$\partial_\mu A^a_\mu = 0, \tag{2.8}$$

and set ζ=1, so we are going to work in the Feynman gauge. With these assumptions the propagators in momentum space are:
The gluon

$$\Delta^{ab}_{\mu\nu}(p) = \quad = g^2_{YM}\delta^{ab}\tfrac{\delta_{\mu\nu}}{p^2} \tag{2.9}$$

The scalars

$$D^{ab}_{ij}(p) = \quad = g^2_{YM}\delta^{ab}\tfrac{\delta_{ij}}{p^2} \tag{2.10}$$

The fermions

$$S^{ab}(p) = \quad = g^2_{YM}\delta^{ab}\tfrac{-\gamma\cdot p}{p^2} \tag{2.11}$$

The ghosts

$$C^{ab}(p) = \quad = g^2_{YM}\delta^{ab}\tfrac{1}{p^2} \tag{2.12}$$

The passage to position space can be easily obtained from the Fourier transform. For general propagators in 2ω dimensions and a given number s the transform is defined as

$$\int \frac{d^{2\omega}p}{(2\pi)^{2\omega}}\frac{e^{ip\cdot x}}{(p^2)^s} = \frac{\Gamma(\omega - s)}{4^s\pi^\omega\Gamma(s)}\frac{1}{(x^2)^{\omega-s}}. \tag{2.13}$$

The next step is specifying all the interactions present in the theory. Following the conventions of the propagators, we have 8 of them, see Fig.2.1.
At the end of this section, we have to stress the most important property of $\mathcal{N}$=4 SYM: its conformal invariance. Naturally, what one has to check to find this property is the Beta function (see for e.g. [39]) of the theory. It was shown, [46], [20], [29], that in the framework of the $\mathcal{N}$=4 SYM, the

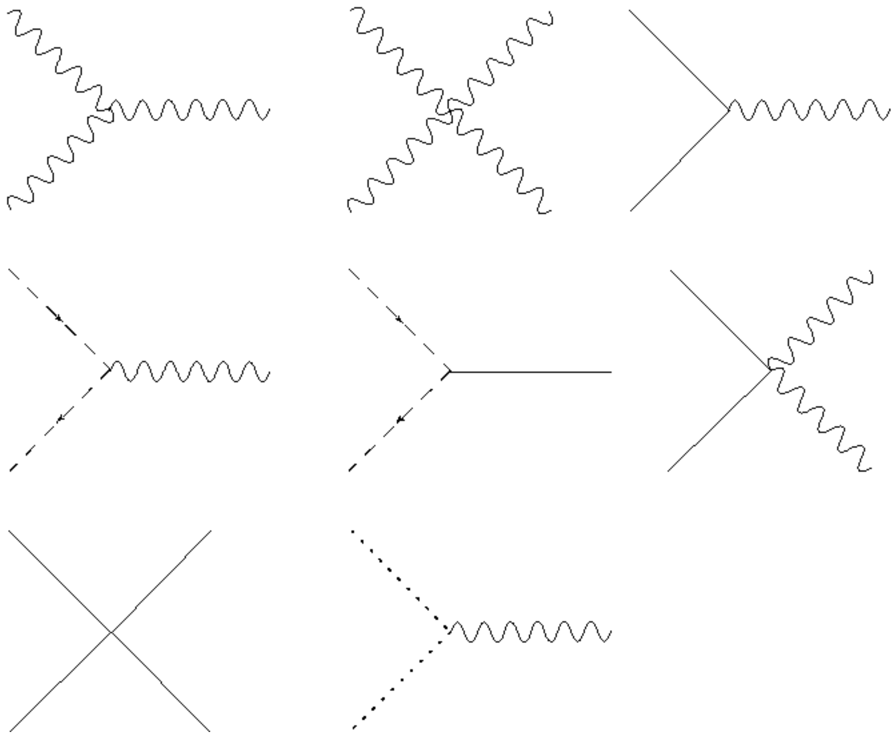

Figure 2.1: *Interactions in $\mathcal{N}$=4 SYM.*

Beta function vanishes to all orders in perturbation theory. This serves as a sufficient condition for the UV finiteness of the theory, and makes the superconformal group $SU(2,2|4)$, [18], its global symmetry, [28].
As a nice form of relaxation, we can convince ourselves about the vanishing of the Beta function (to lowest order in g^2_{YM}, [31]). The diagrams that contribute to this calculation are shown on Fig.2.1.

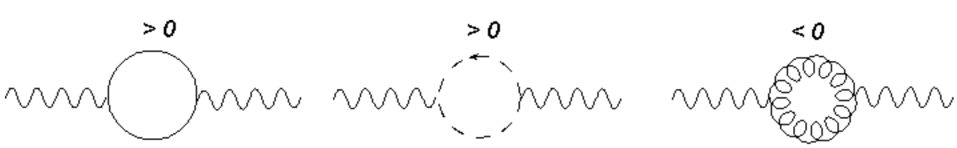

Figure 2.2: *Diagrams contributing to the vanishing of the β function (to second order in g_{YM}).*

and in the pictorial order, they give

$$\beta(g_{YM}) \equiv \mu \frac{dg_{YM}}{d\mu} = \left(6 \cdot \frac{1}{6} + 4 \cdot \frac{2}{3} - \frac{11}{3}\right) \frac{N g_{YM}^2}{4\pi} = 0, \tag{2.14}$$

where N is the number of colours and μ the renormalisation scale (see [39]).

2.2 The 't Hooft limit

In non-abelian gauge theories there exists a very special limit, in which the perturbation series gain a topological structure, [49]. Every term of the series can be classified by its topological properties in such a way that the new perturbation expansion will look exactly like that of string theory, [25]. This was first done by 't Hooft in 1973, [49], for a general class of non-abelian gauge theories. In this section we will just follow his arguments which naturally apply to $\mathcal{N}$=4 SYM. Let us assume that we have a gauge theory with a local symmetry group U(N) (it contains SU(N) as a subgroup). In addition we have a gauge (vector) field $A_{i\mu}^{j}(x)$ in the adjoint representation of this group, and the coupling constant g. If we decide to treat the number of colours N as a free parameter, we can perform the perturbative expansion in terms of $1/N$ instead of the coupling g. This can be done as follows: a single propagator carries two indices i and j. In order to keep track of them properly, we can denote an upper index by an incoming arrow and a lower index by the outgoing one. In this convention, the propagator $-\delta_{\mu\nu}/k^2$ is represented by the double line

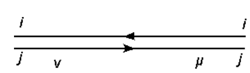

Figure 2.3: *Double-line propagator.*

Having done that, with the stripes-propagators we can construct standard Feynman diagrams for any process we want. The structure of the simplest tree level diagrams will not change too much. However, we can quickly notice that every time we have a closed loop, the number N will enter into the expression for the amplitudes, due to

$$\sum_i \delta_i^i = N, \tag{2.15}$$

For example, the three-loop diagram in Fig.2.2 (one big and two smaller) will be proportional to N^3. Realising that, we can classify all the diagrams

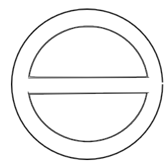

Figure 2.4: *Planar diagram proportional to N^3.*

according to the powers of N and a coupling constant g. Following 't Hooft, with every surface-diagram we associate a factor

$$r = (g^2 N)^{\frac{1}{2}V_3 + V_4} N^{2-2H-L}, \tag{2.16}$$

where V_n is the number of n-point vertices, L counts the number of loops and H is a genus of the surface on which the diagram is embedded ($H = 0$ for a sphere, $H = 1$ for a torus etc.). After labelling all of them, we group diagrams with similar powers of N. This procedure rearranges the perturbation series into the string theory form, such that surfaces with genus 0 enter at first order in perturbation theory, genus one contributes at the second, and so on. At the end we take the 't Hooft limit defined as

$$N \to \infty, \qquad \lambda \equiv g^2 N \quad fixed, \tag{2.17}$$

where λ is called the 't Hooft coupling. Non-planar diagrams are proportional to powers of 1/N and higher, and they do not contribute when $N \to \infty$. Hence, this limit is often called the planar limit.

2.3 Colour decomposition and MHVs

Our main goal in this thesis, namely gluon scattering amplitudes, is quite cumbersome in the weak coupling treatment [39],[47]. This is mostly caused by the large amount of proliferating indices at higher orders in the perturbative expansion. That is why we really need to come up with a neat and systematic solution for this problem. One strategy, which has been very fruitful in the past years, is to specify the colour quantum numbers at the very beginning. By doing so, we do not have to worry about them in the remaining computation. We will first describe this separation and then see what kind of interesting features are left thereafter.
The idea is simply based on the general structure of every amplitude for n-gluons. This is a function of kinematical variables and the quantum numbers

of particles: momenta p_i, helicities h_i and colours a_i:

$$A_n = A(p_1, \ldots, p_n, h_1, \ldots, h_n, a_1, \ldots, a_n). \tag{2.18}$$

The evaluated amplitude is a sum of traces of the group matrices T^a, with kinematical coefficients in front, called "colour-ordered amplitudes". For example, the four point amplitude can be written as

$$\begin{aligned} A_4 = &Tr\,(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\,A_4(1,2,3,4) + Tr\,(T^{a_1}T^{a_2}T^{a_4}T^{a_3})\,A_4(1,2,4,3) \\ &+Tr\,(T^{a_1}T^{a_3}T^{a_2}T^{a_4})\,A_4(1,3,2,4) + Tr\,(T^{a_1}T^{a_3}T^{a_4}T^{a_2})\,A_4(1,3,4,2) \\ &+Tr\,(T^{a_1}T^{a_4}T^{a_2}T^{a_3})\,A_4(1,4,2,4) + Tr\,(T^{a_1}T^{a_4}T^{a_3}T^{a_2})\,A_4(1,4,3,2) \end{aligned} \tag{2.19}$$

where we used the convention that the n-th number carries the kinematical properties of the n-th particle (so 1 denotes a particle with momentum p_1 and helicity h_1). From this particular example, we notice that the amplitude is a sum over all non-cyclic permutations, (σ), of particles. If we label each external particle by its helicity

$$A_n(p_1, h_1; p_2, h_2, \ldots, p_n, h_n) \equiv A_n(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}), \tag{2.20}$$

the 4-point result is easily generalised to an n-point amplitude, [32], as

$$\mathcal{A}_n = \sum_{\sigma_i noncycl.} Tr(T^a(\sigma(1)) \ldots T^a(\sigma(n)))A_n\left(\sigma(1^{h_1}), \ldots, \sigma(n^{h_n})\right). \tag{2.21}$$

This separation clearly saves us from a lot of trouble. The only thing we have to do is to find the remaining, colour-ordered amplitudes. Of course, they are the actual goal of every computation. They can be obtained in a similar way to the general amplitudes, but instead of ordinary methods we have to use colour ordered Feynman rules [32]. We will not present them here, because they are not important from the string theory point of view. Nevertheless, a very pedagogical review of this subject can be found in [30]. Among all the colour ordered amplitudes, there are a few very interesting subgroups characterised by a particular helicity configuration. The first subgroup consists of the amplitudes with: all helicities plus, all helicities minus, all helicities plus except one and all helicities minus except one. They are very specific, because the supersymmetric Ward identities present in the theory, [37], [36], force them to vanish

$$\begin{aligned} A(1^+, \ldots, n^+) &= 0 \\ A(1^-, \ldots, n^-) &= 0 \\ A(1^+, \ldots, i^-, \ldots, n^+) &= 0 \\ A(1^-, \ldots, i^+, \ldots, n^-) &= 0. \end{aligned} \tag{2.22}$$

Other, even more interesting ones, are those containing two out of n particles with different helicities from the others. They are called Maximally-Helicity-Violating. The ones with two particles carrying positive helicity are called "mostly minus" MHVs, and the ones with only two negative are known as "mostly plus" MHVs. As we have mentioned before, they have interesting iterative properties, [7], [41], [6], and we are going to discuss them in the next chapter.

3
The BDS proposal

We will begin this chapter with a brief discussion of the infrared divergent behaviour of gluon scattering amplitudes in $\mathcal{N}$=4 SYM. In this regime they can be characterised in a very simple way by two functions, $f(\lambda)$ and $g(\lambda)$. We will show the form of these functions at weak and strong 't Hooft coupling. After reviewing the IR properties we will present the BDS proposal for n-point, IR divergent, n-gluon amplitudes.

3.1 Infrared structure

As with every conformal field theory, $\mathcal{N}$=4 SYM is UV finite. However, it suffers from ordinary IR divergences, which also appear in QCD. They originate from either collinear regions, in which the transverse momenta of internal gluons vanish, or a soft-gluon exchange in which the virtual gluon energy is very small, [32]. Both of these phenomena require a regularisation, so we have to pick our favourite scheme to do it. In the $\mathcal{N}$=4 SYM case, there is a dimensional regularisation in $D = 4 - 2\epsilon_{IR}$ dimensions with $\epsilon_{IR} < 0$[1], that preserves all the supersymmetries [53]. After regularising our theory in this particular way, the parameter ϵ enters the game, and from now on

[1]For simplicity, we will drop the subscript IR form ϵ and μ in this chapter. However, there will be another ϵ_{uv} and μ_{uv} for Wilson loops, so the reader should not confuse them (at least for now).

it will be present in the results of our calculations. For instance, each of the IR divergences mentioned above will give rise to a single pole, $1/\epsilon$, at one loop, so our predictions (no matter the framework) should have a $1/\epsilon^2$ divergence at this particular level of the perturbative expansion. Naturally, at L-loop order we should expect $1/\epsilon^{2L}$ divergent behaviour. This is not the whole price we have to pay for dimensional regularisation. The scheme in $D = 4 - 2\epsilon$ dimensions breaks conformal invariance, and in order to have it back we need to expand the amplitudes in a Laurent series around $\epsilon = 0$, up to constant terms. That is why, in most of the literature on the subject, there appears the "O(ϵ)" term. It denotes terms of order ϵ and higher, which vanish when $\epsilon \to 0$. Another subtle issue we should keep in mind is that, while the coupling constant of the $\mathcal{N}$=4 SYM was conformal in 4 dimensions, after switching to $4 - 2\epsilon$ dimensions, at the IR scale, the new coupling starts to run[7]. Fortunately, the running-relation is quite simple [7]:

$$\tilde{g}_{YM} = g_{YM} \left(\frac{\mu^2}{\tilde{\mu}^2}\right)^{\epsilon} (4\pi e^{-\gamma})^{\epsilon}, \tag{3.1}$$

where μ is the IR cut-off, $\tilde{\mu}$ a particular renormalisation scale and $\gamma = -\Gamma'(1)$ stands for the Euler constant (see. App.3).
Last but not least, the IR character of gluon amplitudes is governed by only two functions [10], [33], [48]:

- the cusp anomalous dimension f(λ)
- the "collinear" anomalous dimension g(λ)

In the 't Hooft limit, their dependence on λ is known at both weak and strong coupling regimes. At weak coupling, they were computed perturbatively [7] and by using integrability [15], [5], [43]. Also, recently, as we will see in this thesis, their values have been computed at strong coupling using the AdS/CFT [1] correspondence. Summarising the results of the aforementioned work, the cusp anomalous dimension , f, is

$$\begin{aligned} f(\lambda) &= \frac{\lambda}{2\pi^2}\left(1 - \frac{\lambda}{48} + \frac{11\lambda^2}{11520} - \left(\frac{73}{1290240} + \frac{\zeta_3^2}{512\pi^6}\right)\lambda^3 + \ldots\right), \quad \lambda \to 0 \\ f(\lambda) &= \frac{\sqrt{\lambda}}{\pi}\left(1 - \frac{3\ln 2}{\sqrt{\lambda}} - \frac{K}{\lambda} + \ldots\right), \qquad \lambda \to \infty, \end{aligned} \tag{3.2}$$

where K is the "Catalan" constant:

$$K = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \backsimeq 0.9159656. \tag{3.3}$$

Whereas the collinear anomalous dimension, g is [2]:

$$
\begin{aligned}
g(\lambda) &= -\zeta_3\left(\frac{\lambda}{8\pi^2}\right)^2+\frac{2}{3}(6\zeta_5+5\zeta_2\zeta_3)\left(\frac{\lambda}{8\pi^2}\right)^3+\ldots, \qquad \lambda\to 0\\
g(\lambda) &= (1-\ln 2)\frac{\sqrt{\lambda}}{2\pi}, \qquad \lambda\to 0.
\end{aligned}
\tag{3.4}
$$

Once again, we would like to stress these two regimes. The BDS ansatz, which we will present below, was proposed for $\lambda\to 0$, whereas our main goal, gluon scattering amplitudes at strong coupling occur by default at large λ. Therefore, we present f and g in these two regions. Later, in the chapter about Wilson loops, we will see another function of the cusp anomalous dimension, which actually originates from the contour of a cusped loop. The soft anomalous dimension actually has nothing to do with an anomalous behaviour, but only controls the $1/\epsilon$ IR divergence. In the meantime, let us move to the BDS ansatz.

3.2 The Bern-Dixon-Smirnov proposal

Following the pursuit of the IR divergences, Bern, Dixon and Smirnov [7] found that MHV amplitudes at higher loops can be expressed in terms of their lower-loops counterparts. For example, the explicit computation, with the help of new-developed techniques [6], showed that the color-ordered, MHV, four-point function at two loops, $A_4^{(2)}$, divided by the same function evaluated at the tree level of the perturbation theory, $A_4^{(0)}$, exhibits a surprising, iterative structure:

$$
M_4^{(2)}(\epsilon)\equiv\frac{A_4^{(2)}}{A_4^{(0)}}=\frac{1}{2}\left(M_4^1(\epsilon)\right)^2+f^2(\epsilon)M_4^{(1)}(2\epsilon)+C^2+O(\epsilon),
\tag{3.5}
$$

where $f(\epsilon)$ is the cusp anomalous dimension at an appropriate perturbation level, whereas C is a constant. This is a remarkable result, since, from everyday experience, when people calculate higher-loop terms in any theory they only notice the enormous proliferation of complicated graphs. The work of BDS immediately attracted the attention of physicists. The avalanche of new developments, [7], [21], in this branch of gauge theory gave birth to a conjecture for the iterative relation for n-gluon, colour-ordered MHV

[2]The strong coupling result for g is actually obtained from a string theory computation, which we will see later

amplitudes evaluated at all loops. The proposal, abbreviated from the names of the authors to BDS, claims that such a general amplitude should be of the form

$$M_n^{Full} = exp[\sum_{l=1}^{\infty} \alpha^l \left(f^{(l)}(\epsilon) M_n^{(l)}(l\epsilon) + C^{(l)} + O(\epsilon) \right)], \tag{3.6}$$

where $f^{(l)}$, $M_n^{(l)}$ and $C^{(l)}$ are respectively the cusp anomalous dimension, the n-point function at l-loop order and the constant term also evaluated at the l-loop order of perturbation theory. The coefficient α guides the expansion such that if we want to know the iterative relation at 10 loops, only the terms up to α^{10} matter. At this point we can absorb the effect of the running coupling constant in $D = 4 - 2\epsilon$ dimensions into the parameter α. Since every term in the expansion will contain the coupling constant (3.1), we will get rid of all the inconvenient constants by reparametrising α as

$$\alpha = \left(\frac{\lambda \mu^{2\epsilon}}{8\pi^2} 4\pi e^{-\gamma} \right)^{\epsilon}, \tag{3.7}$$

where λ is the 't Hooft coupling in four dimensions, which we will keep fixed in the 't Hooft limit.

Beginning our analysis of the proposal, we can start with a very special and interesting feature. Precisely, neither the $f(\epsilon)$ nor the constant C depends on the number of gluons. They are universal, so we can determined them in a certain n-gluon process (at l-loop). Doing this allows us to write the iterative relation for any number of gluons. Let us check how it works in practice. The ansatz for 4-gluons (confirmed by explicit computation) has a simple form

$$M_4 = \left(A_{div}(s)\right)^2 \left(A_{div}(t)\right)^2 exp\left[\frac{f(\lambda)}{8} ln^2\left(\frac{s}{t}\right) + C \right]. \tag{3.8}$$

As we mentioned before, the divergent parts are fully determined by the anomalous dimensions f and g

$$A_{div}(s) = exp\left[-\frac{1}{8\epsilon^2} f^{-2}\left(\frac{\lambda\mu^{2\epsilon}}{s^{\epsilon}}\right) - \frac{1}{4\epsilon} g^{-1}\left(\frac{\lambda\mu^{2\epsilon}}{s^{\epsilon}}\right) \right]. \tag{3.9}$$

We will compare this prediction of the BDS ansatz with the string theory result, therefore the two important structures that we should keep in mind are:

- Finite part

$$\boxed{\frac{f(\lambda)}{8} ln^2\left(\frac{s}{t}\right)}$$

- The form of the IR divergence

$$\boxed{A_{div}(s) = exp\left[-\frac{1}{8\epsilon^2} f^{-2}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right) - \frac{1}{4\epsilon} g^{-1}\left(\frac{\lambda\mu^{2\epsilon}}{s^\epsilon}\right)\right]}$$

For later purposes, the ansatz can be "massaged" into a more useful form. Since the main feature of BDS relies on its iterative property, it is enough to focus on one-loop amplitudes. Effectively, higher order amplitudes will be polynomials of amplitudes at one-loop. Let us then consider the logarithm of MHV, M_n, at one loop order

$$\ln \mathcal{M}_n^{BDS} = Z_n + F_n^{BDS} + C_n + O(\epsilon), \tag{3.10}$$

where Z_n is the IR divergent part, F is the finite term, C is some constant and $O(\epsilon)$ vanishes when $\epsilon \to 0$. In the $\mathcal{N}$=4 theory, Z can be expressed as, [22]:

$$Z_n = -\frac{1}{4}\sum_{l=1}^{n} \alpha^l \left(\frac{f^{(l)}(\lambda)}{(l\epsilon)^2} + \frac{g^{(l)}(\lambda)}{l\epsilon}\right) \sum_{i=1}^{n} \left(-\frac{\mu_{IR}^2}{s_{i,i+1}}\right)^{l\epsilon}, \tag{3.11}$$

where $s_{i,i+1} \equiv (p_i + p_{i+1})^2$. The IR cut off $\mu_{IR} = 4\pi e^{-\gamma}\mu^2$ is a function of the dimensional regularization scale μ^2. The finite piece, F_n^{BDS}, is predicted by BDS to be

$$F_n^{BDS} = \frac{1}{2} f(\lambda)\mathcal{F}_n, \tag{3.12}$$

where $\mathcal{F}_n$, for $n = 4$, is:

$$\mathcal{F}_4 = \frac{1}{2}\ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + 4\zeta_2, \tag{3.13}$$

with Riemann's zeta function $Li_2(1) = \zeta_2 = \pi^2/6$. In the logarithm, we have used the convention for general kinematical invariants:

$$x_{i,i+j}^2 \equiv \left(p_i + \ldots + p_{i+j-1}\right)^2. \tag{3.14}$$

Finite terms for $n \geq 5$, can be written in a more systematic way:

$$\mathcal{F}_n = \frac{1}{2} \sum_{i=1}^{n} g_{n,i}, \tag{3.15}$$

where the terms in the sum are

$$g_{n,i} \equiv - \sum_{r=2}^{\lfloor \frac{n}{2} \rfloor - 1} \ln\left(\frac{x^2_{i,i+r}}{x^2_{i,i+r+1}}\right) \ln\left(\frac{x^2_{i+1,i+r+1}}{x^2_{i,i+r+1}}\right) + D_{n,i} + L_{n,i} + \frac{3}{2}\zeta_2. \tag{3.16}$$

In the above formula we used a standard "floor function", $\lfloor \frac{n}{2} \rfloor$, defined for all real numbers $x \in \mathbb{R}$ as

$$\lfloor x \rfloor = max\{n \in \mathbb{Z} | n \leq x\}. \tag{3.17}$$

So any real number takes the value of the closest lower integer (e.g. $\lfloor \frac{5}{2} \rfloor = 2$, $\lfloor \frac{7}{5} \rfloor = 1$).
The form of $D_{n,i}$ and $L_{n,i}$ depends on whether n is even or odd. For even, $n = 2m$:

$$D_{n,i} = - \sum_{r=2}^{m-2} Li_2\left(1 - \frac{x^2_{i,i+r} x^2_{i-1,i+r+1}}{x^2_{i,i+r+1} x^2_{i-1,i+r}}\right) - \frac{1}{2} Li_2\left(1 - \frac{x^2_{i,i+m-1} x^2_{i-1,i+m}}{x^2_{i,i+m} x^2_{i-1,i+m-1}}\right) \tag{3.18}$$

$$L_{n,i} = \frac{1}{4} \ln^2\left(\frac{x^2_{i,i+m}}{x^2_{i+1,i+m+1}}\right), \tag{3.19}$$

while for odd, $n = 2m + 1$:

$$D_{n,i} = - \sum_{r=2}^{m-1} Li_2\left(1 - \frac{x^2_{i,i+r} x^2_{i-1,i+r+1}}{x^2_{i,i+r+1} x^2_{i-1,i+r}}\right) \tag{3.20}$$

$$L_{n,i} = -\frac{1}{2} \ln^2\left(\frac{x^2_{i,i+m}}{x^2_{i,i+m+1}}\right) \ln^2\left(\frac{x^2_{i+1,i+m+1}}{x^2_{i+m,i+2m}}\right). \tag{3.21}$$

After this rather technical but important chapter, we shall start our journey from gauge theories to strings. We will quickly review some basic facts about Wilson loops, and then proceed with extending them to $\mathcal{N}$=4 SYM.

4
Wilson loops

Wilson loops[1] are one of the most interesting tools of gauge theories. Mathematically, they are defined as a holonomy[2] associated with a gauge field A^μ, along a contour C. Physically, if we study an infinitely massive quark in the fundamental representation of $SU(N)$, moving along the loop C, it would transform by the face factor equivalent to the Wilson loop[3]. This picture proves to be very fruitful in reconstructing the interaction potential between quark and anti-quark. In appendix 4, we demonstrated a simple example from ordinary electrodynamics on how a Wilson loop can be used to deduce the potential.
Wilson loops were also crucial in the development of string theory. Polyakov, in his pioneering paper [44], studied models with Wilson loops as "rings of glue" propagating in space-time. That led to the construction of the string actions known today. For thirty years of their existence, Wilson loops have become almost a separate field of research. Unfortunately, for the purposes of this thesis, we will not have time to study them very extensively. The only three things we will need to know about Wilson loops are:

- their perturbative analysis

[1]Named after Keneth Wilson

[2]measure, to which extent parallel transport along the closed loop preserves the information being transported [40]

[3]In quantum field theory, we can also interpret them as operators, that create excitations of the gauge field A^μ along the loop C

- supersymmetric extension to $\mathcal{N}$=4 SYM
- strong coupling interpretation using the AdS/CFT correspondence.

We will present the first two points in this chapter; the third one will be mentioned together with string theory in the next chapter.

4.1 Perturbative Wilson loops

Let us consider the theory described by an action S. The action contains a matrix gauge field $\mathbf{A}_\mu = A^a_\mu T^a$ ($a \in \{1, \dots, N^2-1\}$) where T^a are generators in the fundamental representation of the non-abelian gauge group $SU(N)$[4]. A coupling constant of the theory is g. Then, the Wilson loop W_C, along the contour C, is defined as

$$W_C = \langle \frac{1}{N} Tr \mathbf{P} e^{ig \oint_C dx^\mu A^a_\mu T^a} \rangle. \tag{4.1}$$

The curve C can be parametrized by $s \in (0, 1)$, such that $C = x(s)$. N, from $SU(N)$, will be referred to as the number of colours in the theory (e.g. QCD with a non-abelian gauge group $SU(3)$ contains 3 colours). The trace is taken over the generator matrices T^a. "P" denotes the path ordering which assures the right order of exponentiated operators. Finally, by the expectation value we mean the usual path integral

$$\langle \dots \rangle \equiv \frac{\int DA_\mu e^{-S} \dots}{\int DA_\mu e^{-S}}. \tag{4.2}$$

Defined this way loops, are gauge invariant and "well-behaved" for a large class of contours (see e.g. [39], [34]).
In order to evaluate a Wilson loop, we can study the expectation value perturbatively. If we expand the exponent up to terms of order g^2, two gauge fields under the expectation value become a propagator

$$\langle A_\mu(x) A_\nu(y) \rangle = G_{\mu\nu}(x-y) \sim \frac{\delta_{\mu\nu}}{(x-y)^2}. \tag{4.3}$$

By analogy to QCD we will call this gauge field propagator a "gluon" propagator. However, one should keep in mind that all the physical quantum

[4]Though, we will be interested in non-abelian theories, Wilson loop can be defined in the abelian theory as well. See App.4

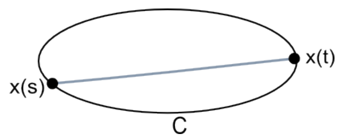

Figure 4.1: *After the expansion, parametrisation $x(s)$ describes the end points of a gluon propagator.*

numbers and properties are generalised, so that the gluon can occur in N colours etc. After inserting the propagator into (4.3), the first order perturbative Wilson loop becomes

$$W_C = 1+(ig)^2 C_F \int_0^1 ds \int_0^s dt \dot{x}^\mu(s)\dot{x}^\nu(t) \left\{G_{\mu\nu}\left(x(t)-x(s)\right)\right\}+O(g^4), \quad (4.4)$$

where C_F is the quadratic Casimir of $SU(N)$ obtained by evaluating the trace over T^a's

$$C_F = TrT^aT^a = \frac{N^2-1}{2N}. \quad (4.5)$$

Dots above x^μ denote the derivatives with respect to parameters s or t. After the expansion, $x^\mu(s)$ and $x^\nu(t)$ gain a new interpretation. Now, they parametrise the end points of the propagator along the loop C (see 4.1). Path ordering, present in (4.1), keeps the propagators' end-points in the right order. Therefore, the end point $x^\mu(s)$ is always before the $x^\mu(t)$ one. This is manifest in the integrals (4.4) concatenated in an appropriate way.
After explaining this basic step, the full perturbative Wilson loop is just a matter of generalisation. The full series of the path ordered exponent becomes

$$\mathbf{P}e^{ig\oint_C dx^\mu A_\mu} = \sum_{n=0}^{\infty}\int_0^1 ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n \frac{dx^\mu(s_n)}{ds_n} A_\mu(x(s_n)) \ldots \frac{dx^\mu}{ds_1} A_\mu(x(s_1)). \quad (4.6)$$

Next, we have to compute the trace of the T^a matrices. Then, we should carefully contract all the fields under the expectation value, and the function that is left is a Wilson loop to all orders in perturbation theory.
There is one more extremely important detail we have to take into account. With the propagator (4.3) along the loop C we quickly run into trouble.

Namely, if the end points approach each other, a Wilson loop diverges. In other words, we have the ultraviolet (UV) divergence. There are many ways to cure this problem. Here, following most of the literature, we will use dimensional regularisation in $D = 4 - 2\epsilon_{uv}$ dimensions [50]. Unlike in the IR regularisation of $\mathcal{N}$=4 SYM amplitudes, the UV cut-off ϵ_{uv} is taken to be greater than zero. We also pick a renormalisation scale $\tilde{\mu}$, and use the Fourier transform (2.13) in $D = 4-2\epsilon_{uv}$ dimensions. Finally, the regularised gluon propagator reads

$$\Delta_{\mu\nu}(x-y) = -\frac{\Gamma(1-\epsilon_{uv})(\pi\widetilde{\mu}^2)^{\epsilon_{uv}}}{4\pi^2}\frac{\delta_{\mu\nu}}{[-(y-x)^2+i0]^{\epsilon_{uv}-1}}. \tag{4.7}$$

Similar to the IR divergent amplitudes in $\mathcal{N}$=4 SYM, after regularisation, Wilson loops will depend on the UV cut-off ϵ_{uv}.

4.2 Supersymmetric Wilson loops

The supersymmetric extension of a Wilson loop to $\mathcal{N}$=4 SYM is quite a non-trivial task. First of all, we pointed out in the previous section that a Wilson loop in ordinary non-abelian gauge theory (like QCD) is the phase factor associated with a heavy quark in the fundamental representation moving along the loop. On the contrary, we saw in the chapter about $\mathcal{N}$=4 SYM, that all the fields in the theory are in the adjoint representation. The solution of this puzzle was first given by D.Gross et al. [12]. The authors generalised the standard derivation of a Wilson loop. Namely, in the theory with a matter field in the fundamental representation (like quarks), the Wilson loop is derived by writing a correlation function of the field in terms of the path integral over the quark's trajectories. As a result, one obtains the appropriate phase factor defining the Wilson loop. In the $\mathcal{N}$=4 case, the authors used W bosons from the symmetry breaking of $SU(N+1) \to SU(N) + U(1)$. They rewrote the bosonic part of the $\mathcal{N}$=4 action (terms with the gauge field A^μ and scalars Φ) in terms of A^μ, Φ and W bosons. Then, they studied W's correlation function as the path integral over W trajectories. This way they arrived at the phase factor defining the Wilson loop

$$W_C = \frac{1}{N} Tr\mathbf{P}exp\left(ig\oint_C ds\left(\dot{x}^\mu A_\mu - i\Phi_i\theta^i|\dot{x}|\right)\right). \tag{4.8}$$

Including the extra six scalars required six new coordinates y^i, one for each field Φ_i. In other words, the contour C was enlarged to ten dimensions. Therefore, their loop has a normal, four dimensional part with $A^\mu x_\mu$ (due

to the gauge invariance x_μ has to be closed) and an extra six dimensional part $\Phi^i y_i$ (y_i can be an arbitrary path in $\mathbb{R}^6$). At the end, the requirements of supersymmetry allowed for special parametrisation $y^i = \sqrt{\dot{x}^2}\theta^i$, where $\theta^2 = 1$ are coordinates on a six dimensional sphere. We should remember that physically, a Wilson loop constructed in this fashion corresponds to the phase factor with which an infinitely massive W boson will transform moving along the contour C.
This definition naturally allows the study of the perturbative expansion of the loop. We will have two propagators. One for a usual gauge field $G_{\mu\nu}(x-y)$, and the other for scalars $D_{ij}(x-y)$

$$\begin{aligned}\langle W_C\rangle = 1 - g_{YM}^2 C_F \int_0^1 ds \int_0^s dt\, [\dot{x}^\mu(s)\dot{x}^\nu(t) G_{\mu\nu}[x(s)-x(t)] \\ -|\dot{x}^i(s)||\dot{x}^j(t)| D_{ij}[x(s)-x(t)]] \,. \end{aligned} \tag{4.9}$$

The form of both propagators was defined earlier (2.9),(2.10), and their position representation is obtained by Fourier transforming (2.13) in $D = 4-2\epsilon$ dimensions. The extra feature, which distinguishes Wilson loops in $\mathcal{N}$=4 SYM, is their natural invariance under supersymmetry transformations. Let us then check what is the requirement for supersymmetry (SUSY) invariance. The bosonic fields A^μ and Φ transform under SUSY as [19]

$$\begin{aligned}\delta_\epsilon A_\mu &= \bar{\chi}\Gamma^\mu\epsilon \\ \delta_\epsilon \Phi_i &= \bar{\chi}\Gamma^i\epsilon,\end{aligned} \tag{4.10}$$

where ϵ is a 10-dimensional Majorana-Weyl spinor, while χ are gluinos. Acting with these transformations upon the Wilson loop 4.8, we arrive at the supersymmetry-invariance condition

$$\delta_\epsilon W_C = \frac{ig}{N} Tr\mathbf{P} \int ds \bar{\chi} \left(\Gamma^\mu \dot{x}^\mu(s) - i\Gamma^i\theta^i|x^i(s)|\right) \epsilon e^{ig\int dt(A_\mu \dot{x}^\mu - i\Phi_i\theta^i|\dot{x}|)} = 0. \tag{4.11}$$

Precisely, for every parameter ϵ, the following equation must hold

$$\left(\Gamma^\mu \dot{x}^\mu(s) - i\Gamma^i\theta^i|x^i(s)|\right) = 0. \tag{4.12}$$

This condition stresses an important property of supersymmetric Wilson loops. Since it clearly depends on the parametrisation $x(s)$, only a small class of loops will preserve SUSY globally. Namely, the loops along a straight line. In general, most of the loops will be supersymmetry invariant only locally. This fact will be important during the perturbative computations in the following chapters.

4.3 Gauge theories in loop space

Non-abelian gauge theories with a large number of colours can be reformulated in terms of Wilson loops. Observables, like n-point Green's functions, are replaced by averages of Wilson loops along specific contours[5]. In addition, a quantum equation of motion in an ordinary gauge theory, the Dyson-Schwinger equation (see e.g. [39]), is generalised to a so-called loop equation [51]. Later, we will see that gluon scattering amplitudes in $\mathcal{N}=4$ SYM can be linked to string theory by holographic interpretation of a Wilson loop. That is why it is useful to quickly review how observables in large N, non-abelian gauge theories are expressed by Wilson loops.
Let us start with re-writing observables in loop space form. First, the loop analog of the n-point Green's function $G_n(x_1, \ldots, x_n)$ at finite N, is the expectation value of n exponents like in a single Wilson loop

$$G_n(x_1, \ldots, x_n) \equiv \langle \frac{1}{N} Tr\mathbf{P}e^{ig \oint_{C_1} dx^\mu A_\mu}, \ldots, \frac{1}{N} Tr\mathbf{P}e^{ig \oint_{C_n} dx^\mu A_\mu} \rangle, \quad (4.13)$$

where C_n is the contour that contains the point x_n. This formula does not look too encouraging to work with, but it was proven [51] that in the limit of $N \to \infty$[6], n-point functions reduce to a surprisingly simple form

$$G_n(x_1, \ldots, x_n) \equiv \sum_C G(C) \langle \frac{1}{N} Tr\mathbf{P}e^{ig \oint_C dx^\mu A^a_\mu T^a} \rangle, \quad (4.14)$$

where now the sum runs over all possible contours C that contain points $x_1, \ldots, x_n$. The weight $G(C)$ depends on the observable, and the way to compute it for a couple of simple examples is given in [34].
In whatever way we define our observables we would like them to be dynamical objects; we need an equation which governs their dynamics. In classical mechanics, we have Euler-Lagrange equation of motion. In the theory of quantised fields there is a quantum counterpart, the Dyson-Schwinger equation (for more details see [34] or [39]). This functional relation states that the Euler-Lagrange equation of motion of a quantum field φ is also satisfied by all the Green's functions of this field, up to "commutator terms". For the action $S[\varphi]$ and functional $F[\varphi]$, the equation reads

$$\left\langle \frac{\delta S\left[\varphi\right]}{\delta \varphi\left(x\right)} F\left[\varphi\right] \right\rangle = \hbar \left\langle \frac{\delta F\left[\varphi\right]}{\delta \varphi\left(x\right)} \right\rangle . \quad (4.15)$$

[5]The material of this section is based on a book of Y.Makeenko [34].
[6]At the end, this is the same limit where we are able to test AdS/CFT

The term on the right-hand-side is a "commutator term". We kept $\hbar$ to emphasize that in the classical limit $\hbar \to 0$ we recover the E-L equation. As an explicit example, we can write down the Dyson-Schwinger equation for the simplest, Yang-Mills theory[7]. To avoid confusion, we denote a functional of the gauge field as $G[A]$

$$-\left\langle \nabla_\mu^{ab} F_{\mu\nu}^b\,(x)\, G\,[A] \right\rangle = \hbar \left\langle \frac{\delta G\,[A]}{\delta A_\nu^a\,(x)} \right\rangle . \tag{4.16}$$

As we see, on the left-hand-side there is the classical Maxwell equation, whereas the right-hand-side will give a contribution if fields in the functional approach $A_\nu^a\,(x)$.

A generalisation to Wilson loops is quite straightforward. Let us just take a functional $F[A]$ to be a Wilson loop

$$F[A] = \frac{1}{N} Tr\mathbf{P} e^{ig\oint_C dx^\mu A_\mu^a}. \tag{4.17}$$

Next, we insert $F[A]$ to (4.16), and use the functional derivative of the gauge field A_μ

$$\frac{\delta A_\mu^{ij}(y)}{\delta A_\nu^{kl}(x)} = \delta_{\mu\nu}\delta^{(d)}(x-y)\left(\delta^{il}\delta^{kj} - \frac{1}{N}\delta^{ij}\delta^{kl}\right). \tag{4.18}$$

As a result, in the limit of $N \to \infty$, we arrive at the Dyson-Schwinger equation for a Wilson loop (see [34] p.256)

$$i\langle \frac{1}{N} Tr\mathbf{P}\nabla_\mu F_{\mu\nu}(x) e^{i\oint_C d\zeta^\mu A_\mu}\rangle \;=\; g^2 N \oint_C \delta^d(x\,-\,y)\langle W_{C_{yx}} W_{C_{xy}}\rangle, \tag{4.19}$$

where W_C is a Wilson line along C. The contour C splits into two parts, C_{yx} and C_{xy}, because a functional derivative was applied at a point x of the contour, while a point y, where the gauge field "sits", runs along C. Hence, one part is a line from x to y, C_{yx}, and the other from y to x, C_{yx}. The presence of $\delta(x-y)$ on the right-hand-side implies, that a non-trivial contribution to the equation (4.19) comes from the loops, where x and y are the same points of space, but not necessarily the contour[8]. One example of a non-trivial loop is depicted in Fig.4.2. Summarising, it is possible to reformulate non-abelian gauge theories in terms of Wilson loops. From now

[7] Full Yang-Mills action should pick the contributions from ghosts and gauge-fixing terms. However, when the functional F[A] is gauge invariant, for example a Wilson loop, they cancel each other [34].

[8] They might be associated with different values of the parameter s

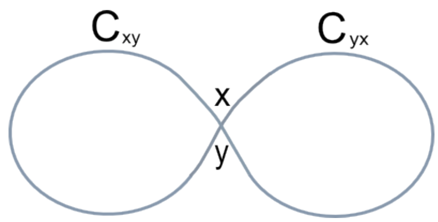

Figure 4.2: *Contours in equation* (4.19).

on, we will keep in mind that computing any observable, in the limit of a large number of colours, is equivalent to evaluating a Wilson loop along a specific contour. In addition, with the loop equation in hand, we are able to control loop kinematics.

4.4 Note on the cusp anomalous dimension

The last thing we promised to mention is the origin of the cusp anomalous dimension. To see where it enters our game, let us consider a Wilson loop along the contour with a single cusp:

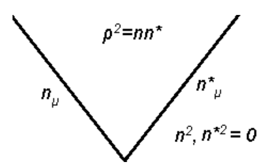

Figure 4.3: *Contour with a cusp.*

In the past, people have studied such objects extensively [44], [27]. They realised that the formulation of non-abelian gauge theories in terms of Wilson loops enables us to study their behaviour under the renormalisation group (RG) [39]. When they applied the standard RG equation to a Wilson loop in a particular theory, they could see how it changes with a coupling constant. The cusp anomalous dimension $f(\lambda)$ appeared when the equation was applied to the cusped Wilson loop like 4.4. In such a case, $f(\lambda)$ determines the leading divergence

$$\left(\rho\frac{\partial}{\partial\rho}+\beta\frac{\partial}{\partial g}\right)\ln W_{C_{cusp}}=-2f(\lambda)\ln\rho^2+O(\rho^0), \tag{4.20}$$

where $O(\rho^0)$ are the constant terms and higher powers of ρ. The parameter ρ is defined in Fig. 4.4. This was the last part of the gauge theory background. In the next chapters, we will introduce some string theory methods which will help us to understand the gluon scattering amplitudes at strong coupling.

Part II

String Theory

5
Bosonic string theory

In this chapter, we are going to briefly recall a few basic concepts from bosonic string theory. Starting from the simplest bosonic model and its connection to minimal surfaces, we will motivate a geometrical equivalence between the Nambu-Goto and the Polyakov actions. This is naturally a basic fact, but we will show its practical meaning, crucial in computing the area of minimal surfaces. Next, we will present string actions in general backgrounds. This will be followed by a short review of T-duality where we will show how the dual system emerges from the non-linear sigma model perspective. Finally, the result of Gross and Mende about scattering amplitudes in flat space-time will be presented.

5.1 Minimal surfaces and classical bosonic strings

Let us first start with a simple and illustrative example, a minimal surface in Euclidean space (like a soap bubble). To compute the area of the surface we can use the coefficients of second-fundamental form. For the parametrisation $X^\mu = X^\mu(u,v)$ they are defined as

$$A_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \partial_u X \cdot \partial_u X & \partial_u X \cdot \partial_v X \\ \partial_v X \cdot \partial_u X & \partial_v X \cdot \partial_v X \end{pmatrix}.$$

The element of area is simply the square root of the determinant of $A_{\mu\nu}$

$$dA = \sqrt{\det(A_{\alpha\beta})} = \sqrt{EG - F^2}. \tag{5.1}$$

Hence, the area of a surface is

$$A = \iint dudv\sqrt{EG - F^2}. \tag{5.2}$$

This is a general formula for any surface embedded in the Euclidean $\mathbb{R}^n$. On the other hand, if we want to find an unknown parametrisation X^μ of a minimal surface with given boundary conditions (Plateau's problem), we have to vary the area functional

$$\delta A = 0. \tag{5.3}$$

Already here, we see that a special choice of coordinates parametrising a surface, such that

$$E = G \qquad F = 0, \tag{5.4}$$

makes the functional very simple

$$A = \iint dudvE = \frac{1}{2}\iint dudv(E + G). \tag{5.5}$$

This fact proves to be very useful in the theory of minimal surfaces, since instead of trying to find a general solution for the Euler-Lagrange equation corresponding to (5.3), we can just solve

$$(\partial_u^2 + \partial_v^2)X = 0, \tag{5.6}$$

and make sure that our solution satisfies two extra constraints

$$\partial_u X \cdot \partial_v X = 0 \qquad (\partial_u X)^2 = (\partial_v X)^2. \tag{5.7}$$

This special set of coordinates is called the "isothermal coordinates". Moreover, it was proved by L. Bers that one can always find a parametrisation such that equations (5.4) are satisfied.

With this knowledge we can easily construct actions for bosonic strings. First of all, string theory as a description of particles emerged by generalisation of point particles. As we know, from classical mechanics, the action for a point particle is proportional to the length of the world-line which the particle marks propagating in space-time. By analogy one dimensional

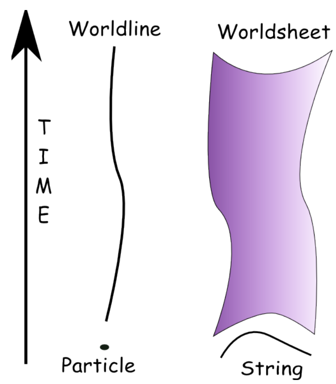

Figure 5.1: *Strings world-sheet is a generalisation of a particles world-line.*

string traces a world-sheet (surface in space-time) (see Fig. 5.1), so their action is proportional to the area of the world-sheet

$$S_{NG} = -T \int dA, \tag{5.8}$$

where T is the string tension and dA is the area element. Like every action, it must be dimensionless, so $[T] = [length]^{-2} = [mass]^2$.
A string world-sheet can be parametrised by (τ, σ), and the action becomes

$$S = -T \iint d\tau d\sigma \sqrt{\partial_\sigma X^2 \partial_\tau X^2 - (\partial_\sigma X^\mu \partial_\tau X^\nu \eta_{\mu\nu})^2}, \tag{5.9}$$

where $\eta_{\mu\nu}$ is the usual Minkowski metric $\eta_{\mu\nu} = (-1, 1, 1 \ldots)$ in n-dimensions (26 for bosonic string). This is the well known Nambu-Goto action for classical strings.
We can use Bers' theorem to rewrite it in a form that is more convenient to work with. However, we should remember that the world-sheet has Minkowski signature, so the condition for isothermal coordinates will slightly change to

$$-(\partial_\tau X)^2 = (\partial_\sigma X)^2 \qquad \partial_\tau X \cdot \partial_\sigma X = 0. \tag{5.10}$$

In such coordinates the string action is

$$S_P = -\frac{T}{2} \iint d\tau d\sigma \left(-\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X\right). \tag{5.11}$$

This is known as the Polyakov action. In addition, the isothermal constraints (5.10) can be put into a more compact form

$$(\partial_\tau X \pm \partial_\sigma X)^2 = 0. \tag{5.12}$$

In string theory this condition is known as the Virasoro constraint.
It is sometimes convenient to introduce light-cone coordinates

$$\sigma^{\pm} = \tau \pm \sigma, \tag{5.13}$$

and corresponding light-cone derivatives

$$\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma). \tag{5.14}$$

In light-cone language, the Polyakov action takes a nice and compact form[1]

$$S_P = 2T \int d^2\sigma \partial_+ X^\mu \partial_- X^\nu \eta_{\mu\nu}. \tag{5.15}$$

Strings can be easily embedded in general spaces with the metric $G_{\mu\nu}(X)$. Since coefficients of the second-fundamental form are simply equal to those of the induced metric on the world-sheet, the N-G action is

$$S_{NG} = -T \int d^2\sigma \sqrt{-\det \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X)}, \tag{5.16}$$

whereas the Polyakov action reads

$$S_P = 2T \int d^2\xi \partial_+ X^\mu \partial_- X^\nu G_{\mu\nu}. \tag{5.17}$$

Physically, with such actions, we would like to describe strings propagating in curved backgrounds. It turns out that a complete description of these kind of systems requires including the antisymmetric two-form gauge field $B_{\mu\nu}(X)$ and the dilaton field $\Phi(X)$ (see [26]). Then, the action for a bosonic string in a curved background is

$$S = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \partial_\alpha X^\mu \partial_\beta X^\nu \left(\sqrt{g} g^{\alpha\beta} G_{\mu\nu}(X) + \epsilon^{\alpha\beta} B_{\mu\nu}(X)\right) + \frac{1}{4\pi} \int d^2 R^{(2)} \Phi(X), \tag{5.18}$$

where $T = 1/2\pi\alpha'$ and $R^{(2)}$ is the scalar curvature of the world-sheet metric $g^{\alpha\beta}$.

[1] In the literature it is usually called the Polyakov action in the conformal gauge, where the world-sheet metric $g_{\alpha\beta} = \eta_{\alpha\beta}$

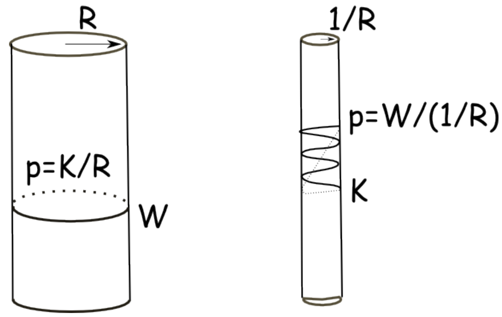

Figure 5.2: *T duality of closed strings.*

5.2 T-duality on the world-sheet

The idea of T-duality arose from studies of particles on a circle. The Hamiltonian for this system is

$$H = \frac{1}{2}\left(W^2 \frac{1}{R^2} + K^2 R^2\right), \tag{5.19}$$

with winding number W and momentum excitation number K. It is not too difficult to notice that H does not change upon the following manipulation

$$R \to \frac{1}{R}, \qquad W \leftrightarrow K. \tag{5.20}$$

This way, two systems

- particle on a circle of a radius R, winding number W and momentum number K
- particle on a circle of a radius $1/R$, winding number K and momentum W

are completely equivalent. In string theory, we can study a similar model. The bosonic strings in 26 dimensions can have some of their coordinates periodic (then we say that strings are compactified on a circle). In practise, they can wrap circles. Then, T-duality manifests that we cannot distinguish between strings with momentum K/R wound W times around the cylinder of radius R, and strings with momentum $W/(1/R)$ wound K times around the cylinder of radius $1/R$ (see Fig. 5.2). More formally, we can analyse what happens with the world-sheet when we switch from one system to its dual. In the parametrisation $X^\mu = X^\mu_R + X^\mu_L$, for closed strings, T-duality flips the sign of the right-movers

$$X_R \to -X_R \qquad X_L \to X_L. \tag{5.21}$$

For the open strings T-duality interchanges boundary conditions. There are two of them: Neumann, when the string can freely move at the end-points, and Dirichlet, when the end-points of the open string are fixed. The Dirichlet condition breaks Poincare invariance. Therefore, they are equivalent to introducing a new object in the theory, namely a hyperplane on which open strings end. These hyperplanes are called D-branes, and are fully dynamical objects. This means that we can study how they deform space-time around them. We will see a bit more about D-branes in the next chapter about the AdS/CFT duality.
T-duality can be studied in a general and formal way [9]. No matter whether strings are open or closed, we can write the Polyakov action, which is an example of a sigma model. For a string on the cylinder with a radius R:

$$S = \frac{R^2}{4\pi}\int d^2\sigma \partial^\alpha X \cdot \partial_\alpha X, \tag{5.22}$$

where $\partial^\alpha = g^{\alpha\beta}\partial_\beta$. To read out the properties of the dual system with world-sheet Y, we can first "gauge" the action (make it invariant under local translations). For that, we will have to introduce the gauge field A_α, where $\alpha \in (\tau, \sigma)$. To make sure that the new action is equivalent to the original, we can add Y which will play the role of the Lagrange multiplier. Hence, integrating out Y from the action (substituting it by the solution of its equation of motion) we will get the old Polyakov integral. On the other hand, integrating out the A_α, we will arrive at the dual sigma model. Let us see how this works explicitly. Naturally, the action (5.31) is invariant under global translations $X(\sigma,\tau) \to X(\sigma,\tau) + a$. When translations are local we have to introduce the gauge field A_α which transforms as

$$A_\alpha \to A_\alpha - \partial_\alpha a, \tag{5.23}$$

and the corresponding field strength F

$$F = \partial_\tau A_\sigma - \partial_\sigma A_\tau \equiv \epsilon^{\alpha\beta}\partial_\alpha A_\beta. \tag{5.24}$$

After we incorporate this into the action, it becomes

$$S = \frac{1}{2\pi}\int d^2\sigma \frac{R^2}{2}\left(\partial^\alpha X + A^\alpha\right)\cdot\left(\partial_\alpha X + A_\alpha\right) - Y \cdot F, \tag{5.25}$$

which is invariant under the local shift transformation. Now, we can perform a gauge transformation to absorb $\partial_\alpha X$ into A_α

$$A_\alpha \to A_\alpha - \partial_\alpha X. \tag{5.26}$$

Finally, the action becomes

$$S = \frac{1}{2\pi} \int d^s\sigma \frac{R^2}{2} A^\alpha \cdot A_\alpha - Y \cdot F. \tag{5.27}$$

The equation of motion for Y

$$F = \partial_\tau A_\sigma - \partial_\sigma A_\tau = 0 \Rightarrow A_\alpha = \partial_\alpha \tilde{X}, \tag{5.28}$$

when plugged into (5.27), shows the equivalence to the original sigma model

$$S = \frac{R^2}{4\pi} \int d^2\sigma \partial^\alpha \tilde{X} \cdot \partial_\alpha \tilde{X}. \tag{5.29}$$

On the other hand, the equation of motion for the gauge field A_α reads

$$A_\alpha = -\frac{1}{2R^2} \epsilon_{\alpha\delta} \partial^\delta Y. \tag{5.30}$$

Inserting this solution into (5.27) we obtain the dual sigma model

$$S = \frac{1}{4\pi} \frac{1}{R^2} \int d^2\sigma \partial^\alpha Y \cdot \partial_\alpha Y. \tag{5.31}$$

In this way we see that T-dual systems have inverted radii

$$R \Leftrightarrow \frac{1}{R}. \tag{5.32}$$

We will apply this formal T-duality to string amplitudes in order to simplify boundary conditions for minimal surfaces in AdS.

5.3 Remark on string amplitudes

Scattering amplitudes in string theory can be naturally studied in the language of two-dimensional conformal field theory (CFT) (see e.g.[18]). For example, four closed or open string scattering realised as an appropriate deformation of the string world-sheet, in CFT corresponds to the sphere or the disk respectively, with four vertex-operator insertions. The amplitude is then evaluated as a path integral over all possible world-sheets with these

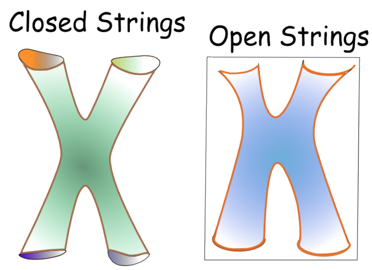

Figure 5.3: *String scattering process.*

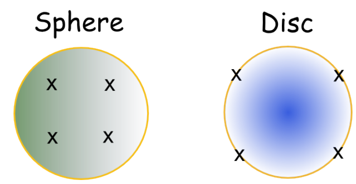

Figure 5.4: *Corresponding amplitudes in CFT.*

operators inserted at the boundary. According to the operator-state correspondence, we can construct vertex operators for all kinds of string states. For example, for the massless string-tachyon with momentum k, we define the vertex operator at σ_i as

$$V(\sigma_i) = \int d^2\sigma e^{ik\cdot X(\sigma_i)}, \tag{5.33}$$

and the corresponding 4-tachyon amplitude is

$$A_4 = \int \mathcal{D}g_{\alpha\beta}\mathcal{D}X^\mu e^{-\frac{1}{2\pi}\int d^2\sigma\sqrt{g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu}\prod_{i=1}^{4}V(\sigma_i). \tag{5.34}$$

Depending on whether we consider an open or a closed string, we have to use different correlators for X^μ (see [18]).
In [11] D. Gross and P. Mende studied four-point open string amplitudes at fixed angles and very high momenta. They showed that when the momentum transferred in the process is very high, the amplitude is dominated by a classical world-sheet area in flat space-time (see Fig. 5.5). They evaluated a disc with four insertions of vertex operators in points $\sigma_1 = 0$, $\sigma_3 = 1$, $\sigma_4 = \infty$ and the remaning one parametrised in terms of the Mandelstam variables

$$\sigma_2 = \frac{s}{s+t}, \qquad s = -(k_1+k_3)^2, \qquad t = -(k_1+k_4)^2. \tag{5.35}$$

The four massless string amplitude became

$$A_4 \sim e^{S(s,t)}, \tag{5.36}$$

where the on-shell action S is

$$S(s,t) = s\ln(-s) + t\ln(-t) - (s+t)\ln(-s-t). \tag{5.37}$$

5.3 Remark on string amplitudes

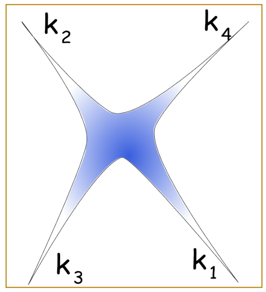

Figure 5.5: *Open string amplitudes at very high momentum transfer s, are dominated by the saddle point of the action (classical, minimal surface).*

Moreover, lightlike open strings ending on a D-brane can be identified with gluons in a gauge-theory. This way, we are able to reconstruct the gauge theory amplitudes as the low energy limit of the string ones. This is a very remarkable result and we should keep it in mind, especially when we will study scattering amplitudes in AdS. As we will see, the result of Alday and Maldacena is a sort of generalisation of the Gorss and Mende computation to AdS space.

6
The AdS/CFT correspondence

In this chapter we will briefly review the idea of the correspondence between string theory in Anti-de Sitter space and a conformal field theory, the non-abelian $\mathcal{N}$=4 SYM theory. Starting from brane solutions in supergravity, we will see the original system, a stack of N D3 branes, which motivated Maldacena to formulate his famous proposal. Then, we will briefly present the general proposal and explain the limits which allowed for explicit computations supporting that bold conjecture. The last section explains how Wilson loops can be interpreted in terms of open strings stretching into the interior of AdS. This result will be crucial in the computation of the gluon scattering amplitudes in planar $\mathcal{N}$=4 SYM at strong coupling. We mentioned before that in theories with a large number of colours, amplitudes are equivalent to Wilson loops along specific contours. That is why understanding how they behave in different regimes is so important.

6.1 D3 branes

We have learned that for open strings in d dimensions, one has to specify what happens at their end-points. If we impose Dirichlet conditions (string is fixed) in p directions, we breake Poincare symmetry in these directions. In other words, we introduce a new object, a D(d-p-1) brane[1]. In this way,

[1]As we remember a Dp brane is labeled by the number of its spatial directions p

the open strings attached to a D(d-p-1)-brane can freely move in $d-p$ directions.
Let us now consider a single open string attached to a D3 brane (Fig.6.1) in 10 dimensional Minkowski space-time. The string can have an arbitrarily short length, so it must be massless. If the end-points carry extra labels

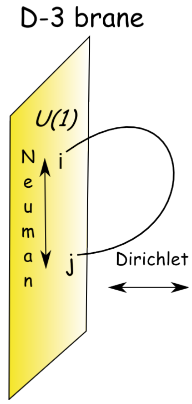

Figure 6.1: *Massless open string on a D3 brane.*

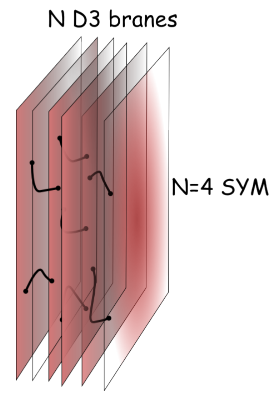

Figure 6.2: *Stack of N D3 branes.*

$i, j = 1 \dots N$, this string excitation mode induces a massless $U(1)$ gauge theory on the D3 brane. If we look at the brane as a 3+1 dimensional Minkowski world-brane, our brane+string system defines a non-abelian gauge theory in this space. The system can be enlarged by adding more D3 branes. If we put N of them close to each other (Fig. 6.1), we can not only have strings that starts and ends at the same brane, but also there are $N^2 - N$ open strings which start at one brane and end on the neighboring. In the limit where branes coincide, all strings are massless and the gauge symmetry induced on them becomes $U(N)$. After precise and extensive studies of the stack of N D3 branes, people realised that the low energy limit of this system is equivalent to the $\mathcal{N}$=4 SYM theory [19].
Alternatively, N D3 branes can be described in terms of closed strings propagating in their presence. In order to see how D3 branes modify space-time one needs to solve type IIB supergravity equations. Analysing symmetries of D3 branes and their SUSY properties, the metric which solves them for N coincident branes is

$$ds^2 = H^{-\frac{1}{2}} \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + H^{\frac{1}{2}} \left(dr^2 + r^2 d\Omega_5^2\right), \qquad (6.1)$$

where

$$H \equiv 1 + \frac{R^4}{r^4}, \qquad R^4 \equiv 4\pi g_s(\alpha')^2 N. \tag{6.2}$$

The first bracket in the metric describes the coordinates "on" D3 branes (parallel to the branes). Whereas the second bracket contains the coordinates perpendicular to the branes. $d\Omega_5^2$ is the metric of the five dimensional sphere. This will be the supergravity metric for space-time where open strings propagate. In addition, type IIB theory contains the Ramond-Ramond (RR) four-form A_4. The flux coming from integrating the field strength $F_5 = dA_4$ will also contribute to the open string background [19]. Factors $H(r)$ have the following physical interpretation: if we measure the energy of the D3 brane system E_r from some constant distance r, it would be related to the energy observed at infinity E_∞ by

$$E_\infty = H^{-\frac{1}{4}} E_r. \tag{6.3}$$

Therefore, the low energy limit, according to the observer at ∞, is equivalent to focusing on the branes ($r \to 0$). In this limit, the fraction in H dominates and the metric becomes (after change of variables $r \to R^2/r$)

$$ds^2 = R^2 \left(\frac{\eta_{\mu\nu} dx^\mu dx^\nu}{r^2} + \frac{dr^2}{r^2}, \right) + R^2 d\Omega_5^2. \tag{6.4}$$

which is precisely the metric on the product manifold $AdS_5 \times S^5$ with the same radius R [2]. On the other hand, if we "zoom out" far from the branes, the space-time geometry is flat (see Fig.6.3)

6.2 The AdS/CFT duality

Intuitively, we would expect that approaching branes at low energies requires the $\mathcal{N}$=4 SYM theory (without gravity) to be the correct description of the physics we see. On the contrary, the low energy limit of type IIB supergravity solution tells us to use string theory in $AdS_5 \times S^5$ with R-R flux (with gravity). In 1998, this argument encouraged J. Maldacena to conjecture the exact equivalence between string theory in $AdS_5 \times S^5$ and conformal $\mathcal{N}$=4 SYM (hence, the abbreviation: AdS/CFT duality)[3].

[2] AdS metric is in Poincare coordinates. For more details about the AdS/CFT see App.A.

[3] String theory in $AdS_5 \times S^5$ is compactified on the S^5, so that the gravity theory in $5d$ is equivalent to the gauge theory in $4d$. This makes the duality holographic.

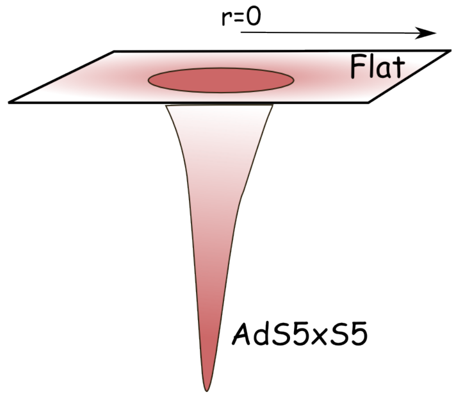

Figure 6.3: *Space-time in the presence of D3 branes. Flat geometry, far from D3 branes turns into the "throat"* $AdS_5 \times S^5$ *when* $r \to 0$.

Naturally, Maldacena did not finish pointing the potential equivalence between these gauge and gravity systems. First of all, in order to compare observables on both sides, and check whether they really "match", he had to explain how to relate the different parameters from dual theories. Observables in $\mathcal{N}$=4 SYM depend on usual kinematical parameters (Mandelstam variables, helicity etc.), the Yang-Mills coupling constant g_{YM} and the number of colours N of the non-abelian gauge group $SU(N)$. Simultaneously, the type IIB solution in $AdS_5 \times S^5$ also possess two parameters; radius R and the string coupling g_s. Maldacena's idea was to identifay them in a following way

$$g_s^2 \equiv g_{YM}^2 \qquad \frac{R^4}{(\alpha')^2} \equiv 4\pi g_{YM}^2 N \equiv 4\pi\lambda, \tag{6.5}$$

where λ is the 't Hooft coupling and $\alpha' = l_s^2$ is the Regge slope, equal to the square of the string length. Already from these relations we can deduce when the AdS/CFT duality will be a useful tool for physics. When the coupling λ is small

$$g_{YM}^2 N = \frac{R^4}{4\pi l_s^4} \ll 1, \tag{6.6}$$

we can trust the perturbative expansion in the gauge theory, whereas on the gravity side we have the radius of AdS comparable with the string length l_s (difficult quantum gravity). On the other hand, when λ becomes strong and we loose control of the gauge theory, classical supergravity for large R in l_s

units is a good description. Therefore, we see that these two disconnected regimes make AdS/CFT very difficult to prove, since it would require a non-perturbative solution of either the $\mathcal{N}$=4 SYM or the string theory in $AdS_5 \times S^5$ background with R-R flux. Nevertheless, we can look at the other side of this coin. If the duality is true, we have a very powerful tool at hand. More specifically, we are able to study the non-perturbative regime of gauge theories by classical supergravity in AdS space, and a quantum gravity in AdS by perturbative $\mathcal{N}$=4 SYM.
Let us then present, in some detail, this limit where we can test and use AdS/CFT. We begin with a general string theory in $AdS_5 \times S^5$ background and a general $\mathcal{N}$=4 SYM theory. Then, we perform the following steps

1. **The 't Hooft Limit**
   On the gauge side we fix the coupling $\lambda \equiv g_{YM}^2 N$, while $N \to \infty$. As we saw in the part about the gauge theory, in the 't Hooft limit only planar diagrams contribute to the perturbative series. On the gravity side, since $g_s = \lambda/N$, we end up with non-interacting strings $g_s \to 0$ in curved space with constant radius R. This way we weaken the AdS/CFT to a duality between large N, planar $\mathcal{N}$=4 SYM and classical strings (no quantum corrections) on $AdS_5 \times S^5$.

2. **The Large λ Limit**
   Taking the 't Hooft coupling $\lambda \to \infty$, physically, makes the string tension $T \sim \lambda$ very large, such that all the massive modes become extremely heavy. They decouple from low energies. An effective theory is approximated by type IIB supergravity on $AdS_5 \times S^5$. More intuitively, we can just say that the large λ translates simply to the large R so we can use the classical, low energy type IIB supergravity. The gauge theory after this limit becomes strongly coupled.

Our two limits can summarised in table 6.1
There has been a lot of progress in work done in these testable limits of AdS/CFT. People realised that symmetries on both sides match and this way they could precisely relate certain operators on the gauge side with supergravity observables. This allowed for a fruitful confirmation of the AdS/CFT in these regimes. Among many observables mapped from the gauge to the gravity, the most established are Wilson loops. In the next section we will briefly describe what is the gravity counterpart of Wilson loops in $\mathcal{N}$=4 SYM at strong coupling.

$\mathcal{N} = 4SYM$	String in $AdS_5 \times S^5$
The Parameters	
g_{YM} colours N	g_s radius R
Relation between the parameters	
$g_s = g_{YM}^2, \qquad \frac{R^4}{(\alpha')^2} \equiv 4\pi g_{YM}^2 N$	
The 't Hooft limit	
$\mathrm{N} \to \infty, \qquad \lambda = g_{YM}^2$ fixed planar limit	$g_s \to 0, \qquad$ fixed R free strings
The Strong λ limit	
$\mathrm{N} \to \infty, \qquad \lambda \gg 1$ strongly coupled $\mathcal{N}$=4 SYM	$g_s \to 0, \qquad R \gg 1$ classical type IIB supergravity

Table 6.1: Limits of the AdS/CFT correspondence.

6.3 Wilson loops in AdS/CFT

Wilson loops at strong coupling can be evaluated using the AdS/CFT correspondence. The way it is done is actually quite surprising and geometrically beautiful. The standard loop in $\mathcal{N}$=4 SYM along a contour C is identified with the world-sheet surface of the open string in AdS_5, with the end points attached to the contour C (Fig. 6.3). This is like a soap bubble fastening to a 2-dimensional surface on the edges while the entire part stretches into the third dimension. Let us give to this nice picture a slightly more pre-

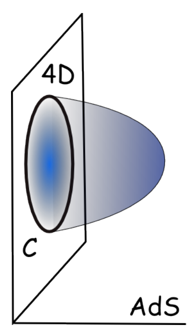

Figure 6.4: *Wilson loop interpretation in the AdS/CFT.*

Figure 6.5: *Soap bubbles.*

cise description. We have learned from [12] that the Wilson loop in $\mathcal{N}$=4 SYM required the W's from the symmetry braking. We can perform a similar construction using the D3 branes. We start with the stack of N D3 branes in flat 10 dimensional Monkowski space. Next, we seperate one D3 brane from the rest. This way $SU(N)$ is broken to $SU(N-1) \times U(1)$. The open strings will stretch between the stuck and the separated brane. Their ground state will correspond to W bosons and their superpartners of broken $SU(N)$. The string trajectories should give the same effect as a very heavy particle in the fundamental representation [12]. This way, we identify the expectation value of a Wilson loop in $\mathcal{N}$=4 SYM over a contour C with the string partition function in $AdS_5 \times S^5$ space

$$\langle W[C, \theta]\rangle = \int \mathcal{D}X\mathcal{D}\theta\mathcal{D}h_{ab} e^{-\frac{\sqrt{\lambda}}{4\pi}\int_B d^2\sigma\sqrt{h}h^{ab}G_{MN}\partial_a X^M \partial_b X^N + F}. \tag{6.7}$$

"F" stands for fermions, λ is the 't Hooft coupling and h_{ab} is the world-sheet metric. The action is the Polyakov string sigma model with the metric G_{MN}

$$ds^2 = R^2\left(\frac{\eta_{\mu\nu}dx^\mu dx^\nu}{r^2} + \frac{dr^2}{r^2}\right) + R^2 d\Omega_5^2. \tag{6.8}$$

This is the ideal, "Platonic" AdS/CFT proposal for Wilson loops, valid at any regime of λ. However, as we pointed out before, it is difficult to analytically solve the non-linear sigma model with RR fluxes. Hence, we usually study the proposal after taking the 't Hooft and large λ limits. The partition function is then dominated by the classical, saddle point. This is naturally the area of the minimal surface in AdS_5.
The negative curvature of the AdS space forces the string world-sheet to stretch into the "bulk". Therefore, the minimal surface is attached to C in four dimensions and the entire part is in the AdS. This constrains the boundary conditions for the world-sheet parametrisation at $r = 0$

$$X^\mu|_{r=0} = C(x) = x^\mu(s), \qquad \Theta^i|_{r=0} = C(\theta) = \theta^i(s) \tag{6.9}$$

where Θ parametrises the position of the world-sheet in S^5. Summarising, computing the Wilson loops at strong coupling is equivalent to solving Plateau's problem in Anti-de Sitter space[4]. Of course, there will be a quantum corrections to this picture: the perturbations around the classical solution. Though this is a very interesting problem, we will not discuss it in this thesis.
We should conclude this part witha last remark about the divergence of the area in AdS. Form the term $\frac{1}{r^2}$ in the metric, we can easily guess that the

[4] Actually finding the area of the minimal surface for given boundary conditions.

divergences will occur when want to evaluate the area close to $r = 0$. There are two ways to remedy this problem. The first way is to introduce a cut-off $r = r_\epsilon$, then compute the integrals and at the end take $r_\epsilon \to 0$. The second scheme is the dimensional regularisation on the gravity side. We will discuss both of them later in the Alday-Maldacena computation. Let us now use all the knowledge we have from the previous chapters to compute the gluon scattering amplitude at strong coupling.

Part III

Gluon scattering amplitudes. From weak to strong coupling and back.

7

The Alday-Maldacena computation

We will start this final part by presenting the computation of the 4-gluon scattering amplitude in $\mathcal{N}$=4 SYM at strong coupling. This was first obtained by Alday and Maldacena (A-M)[1] in 2007. We will follow their derivation and stress the most important conceptual steps. Hopefully, the reader, having passed through the two previous parts, will be able to fully appreciate this interesting result.

In the discussion below, we will see, that the amplitude at strong coupling will turn out to be dual to the Wilson loop along the contour parametrised by the gluons' momenta. In chapter 8, we will show that this duality, surprisingly, survives in the weak regime of λ. As is suggested by the title of this part, "Gluon scattering amplitudes. From weak to strong coupling and back", we will start from the BDS ansatz, then move to the Alday-Maldacena computation and end up at the Wilson loops at weak coupling again. In chapter 9 we will sketch the mechanism allowing for this strange passage. Finally, we will present the results for 6-gluon amplitudes at weak and strong coupling, and conclude with problems and open questions which need to be answered.

7.1 Motivation

The motivation for computing gluon scattering amplitudes at strong coupling started from the BDS ansatz. Just to remind the reader, according to the BDS, the IR regularised gluon amplitude can be expressed in terms of the usual kinematical variables and two universal functions: the cusp anomalous dimension $f(\lambda)$ and the collinear anomalous dimension $g(\lambda)$. Therefore, since we know the form of these functions at strong coupling (see chapter 3), we are able to write the ansatz for large λ. Having done this, Alday and Maldacena decided to check whether the strong coupling observable in the $\mathcal{N}$=4 theory SYM matched the dual result conjectured by the AdS/CFT correspondence. There was only one small problem: no gravity dual description of the gluon scattering amplitudes existed.

Naturally, the authors had to come up with a construction of the gluon amplitudes using available string theory tools. The main difficulty that they faced was related to the divergent structure of the amplitudes. Namely, we cannot define them for a definite number of gluons. As we saw before, they have to be regularised (dimensional regularisation in $D = 4 - 2\epsilon$). This way we "dress" the gluon propagator and study jets with an indefinite number of gauge particles. In string theory this picture is quite awkward to realise (we do not have a notion of a jet in terms of string modes). Nevertheless, A-M managed to give a more geometrical and elegant prescription using AdS/CFT. They started by putting a single D3 brane at some value r_{IR} in the radial direction of the AdS space. The aim of this step was to use it as the IR regulator (at the end take its position to infinity). The asymptotic states on the brane were massless open strings[1] and A-M defined gluons at their end points. Hence, the n-point gluon amplitude was identified with the correlation function of these massless string states. The states had the proper momentum $k_{pr} = kr_{IR}/R$, where k is the momentum conjugate to string position $X^\mu(\sigma, \tau)$, and R is the radius of the AdS. Therefore, when the D3 brane is moved to $r_{IR} \to \infty$, kinematically the process becomes identical to the one studied by Gross and Mende [11] (see chapter 5). Similar to the flat space example, they conjectured that the n-point amplitude has to be dominated by the classical saddle point of the action. This was naturally the minimal surface, but this time, due to the presence of the D3 branes background, the surface had to be computed in the AdS_5 space (see Fig. 7.1).

As we clearly see, the A-M computation does not refer to any particular he-

[1] With both end-points on the D3 brane.

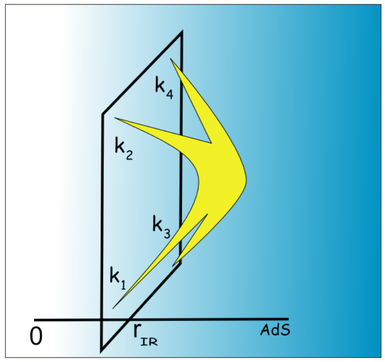

Figure 7.1: *Gluon amplitude in the AdS/CFT.*

licity configuration of scattered gluons. Therefore, though the BDS ansatz was proposed for MHV amplitudes, the string theory computation is more general. Let us examine this proposal in more detail.

7.2 Set-up

Our goal is to compute the scattering amplitude for four massless, open strings at strong coupling λ. The amplitude in this case has the topology of a disc with four vertex operators[2] inserted at the boundary

$$A_4 \sim \langle V_{k_1}(\sigma_1)V_{k_2}(\sigma_2)V_{k_3}(\sigma_3)V_{k_4}(\sigma_4)\rangle. \tag{7.1}$$

To describe the kinematics, we use the standard Mandelstam variables. For the process $1+2 \longmapsto 3+4$, they are

$$\begin{aligned} s = -(k_1+k_2)^2 = -2k_1\cdot k_2 = -4k^2 sin^2\frac{\phi}{2} \\ t = -(k_1+k_4)^2 = -2k_1\cdot k_4 = -4k^2 cos^2\frac{\phi}{2} \\ u = -(s+t). \end{aligned} \tag{7.2}$$

We fix the scattering angle ϕ and increase the momenta of the incoming particles (by shifting the IR brane to ∞). In this regime, the amplitude is dominated by the saddle point of the action[11]

$$A_4 \sim e^{\delta S=0}, \tag{7.3}$$

[2]We are not going to specify the form of vertex operators because it will not be important for computing the minimal surface.

where S is either the Polyakov or Nambu-Goto action for the open string embedded in the AdS_5 space. The classical saddle point of S is proportional to the minimal surface with particular boundary conditions. In our case, the minimal surface on the boundary should be attached to the D3 brane at $r = r_{IR}$. In addition, close to the vertex operators insertions, it should be fixed by the gluons' momenta [1]. From the geometrical point of view, this does not say too much. Fortunately, A-M showed that there exists a trick that changes the boundary conditions into a -very convenient to work with- form; especially when we want to find the minimal surface. We will analyse it very carefully in the next section.

7.3 T-duality and boundary conditions

The trick which involves simplifying the boundary conditions is based on performing the T-duality along the coordinates of the D3 brane. Precisely, if the minimal surface is parametrised by $X = (x^0, x^1, x^2, x^3, r)$, the T-duality can be formally applied to $x^\mu, \mu \in (0,1,2,3)$, by gauging the shift symmetry in these directions (see chapter 5). From the quantum point of view this step seems to be forbidden since x^0 is non-compact (no winding modes). However, we are only interested in the classical minimal surface (without handles etc.). In this case, the T-duality is just a map between two world-sheets with different boundary conditions, but the same areas.
To see explicitly how this procedure acts upon the world-sheets' boundaries, let us consider the n-point, massless, open string amplitude, and T-dualise it. The n-point correlation function is

$$A_n = \langle \prod_{i=1}^{n} V(k_i, \sigma_i) \rangle \sim \int_D \mathcal{D}X e^{iS} e^{i\sum_{i=1}^{n} k_i \cdot x(\sigma_i)}. \tag{7.4}$$

Vertex operators V are inserted at the boundary of the disc D (that is why they only depend on x^μ)[3]. The product of n operators will give the exponent of the sum

$$i \sum_{j=1}^{n} k_j \cdot x(\sigma_j). \tag{7.5}$$

[3]We use tachyon vertex operators just to illustrate the action of the T-duality. They should not be confused with the proper vertex operators representing the scattering of the open super-strings in the AdS.

First, we will "massage" it a bit, and add as the boundary term to the exponent of the action in (7.4)

$$e^{iS} = e^{\frac{1}{4\pi}\int d\sigma d\tau \partial_\alpha x^\mu \partial^\alpha x^\nu G_{\mu\nu} + \frac{R^2 dr^2}{\partial_\alpha r \partial^\alpha r}}, \tag{7.6}$$

where $G_{\mu\nu}$ is the "Minkowski" part of the AdS metric

$$ds^2 = R^2 \left[\frac{dx_{3+1}^2 + dr^2}{r^2}\right] \equiv G_{\mu\nu} dx^\mu dx^\nu + \frac{R^2 dr^2}{r^2}. \tag{7.7}$$

Then, we can perform the T-duality and read out the new boundary conditions[4] and the new, "dual" metric.
The sum (7.5) can be written as

$$i\sum_{j=1}^{n} k_j \cdot X(\sigma_j) = i\sum_{j=1}^{n} k_j \cdot \int d\sigma X(\sigma)\delta(\sigma - \sigma_j). \tag{7.8}$$

In our scattering process, the momenta of massless gluons are conserved. Hence, we can replace one of them by the sum of the others (with a minus sign). For instance, let us pick the momentum k_n

$$k_n = -\sum_{j=1}^{n-1} k_j, \tag{7.9}$$

substituting into (7.8) we obtain

$$i\sum_{j=1}^{n-1} k_j \cdot \int d\sigma X(\sigma)(\delta(\sigma - \sigma_j) - \delta(\sigma - \sigma_n)). \tag{7.10}$$

Using the equality of the distributions

$$\partial_\sigma \theta(\sigma; \sigma_j, \sigma_n) = \delta(\sigma - \sigma_j) - \delta(\sigma - \sigma_n), \tag{7.11}$$

where the periodic Heaviside's step function θ is defined as

$$\theta(\sigma; \sigma_i, \sigma_j) = \begin{cases} 1 & \sigma_i < \sigma < \sigma_j \\ ,0 & otherwise \end{cases} \tag{7.12}$$

[4]In further computations we will omit the term with derivatives of r since the T-duality will be applied only to x^μ.

our product becomes

$$-\sum_{j=1}^{n-1} k_j \cdot \int d\sigma X(\sigma)\partial_\sigma \theta(\sigma;\sigma_j,\sigma_n). \tag{7.13}$$

In order to gauge this boundary piece, we would like to have it in the form of $\partial x \cdot (\ldots)$. Let us then integrate by parts the second factor in (7.13)

$$\int d\sigma X(\sigma)\partial_\sigma \theta(\sigma;\sigma_j,\sigma_n) \;=\; -\int d\sigma (\partial_\sigma X(\sigma))\theta(\sigma;\sigma_j,\sigma_n) + \int d\sigma \partial_\sigma (X(\sigma)\theta(\sigma;\sigma_j,\sigma_n)). \tag{7.14}$$

The second integral gives the value of $X(\sigma)$ between points σ_n and σ_j. Because of this, it can be written as

$$\int d\sigma \partial_\sigma (X(\sigma)\theta(\sigma;\sigma_j,\sigma_n)) = -c\int d\sigma \partial_\sigma X(\sigma), \tag{7.15}$$

where c is some constant that, multiplied by four-momentum vector, becomes a constant 4-vector $\mathbf{c}$. In the first integral, the theta function "cuts" the region of integration to $\int_{\sigma_j}^{\sigma_n}$. Finally, putting all this together, the contribution from the vertex operators to the action is

$$i\sum_{j=1}^{n} k_j \cdot X(\sigma) = -i\sum_{j=1}^{n-1}\int_{\sigma_j}^{\sigma_{j+1}} d\sigma \partial_\sigma X(\sigma)\cdot \left(\sum_{i\le j} k_i + \mathbf{c}\right). \tag{7.16}$$

We are ready to perform the T-duality. The procedure was already studied in chapter 5. Here, we just generalise it to the AdS metric. First, we write the -local shift symmetry invariant- action(5.25) (only the Minkowski part of the AdS metric), together with the gauged boundary piece

$$\frac{1}{4\pi}\int_D d\sigma d\tau \left(\frac{R^2(\partial_\alpha x - A_\alpha)^2}{r^2} - iy\cdot F\right) + i\sum_{j=1}^{n-1}\int_{\sigma_j}^{\sigma_{j+1}} d\sigma [\partial_\sigma x - A_\sigma]\cdot \left(\sum_{i\le j} k_i + \mathbf{c}\right). \tag{7.17}$$

New coordinates y play the role of a Lagrange multiplier. This means that the equation of motion for y leads to the equivalence between (7.17) and the original sigma model in the AdS. By gauge transforming $A_\alpha \to A_\alpha + \partial_\alpha x$ we can absorb $\partial_\alpha x$ into A_α. Hence, (7.17) becomes

$$\frac{1}{4\pi}\int_D d\sigma d\tau \left(\frac{R^2 A_\alpha \cdot A^\alpha}{z^2} - iy\cdot F\right) + i\sum_{j=1}^{n-1}\int_{\sigma_j}^{\sigma_{j+1}} d\sigma A_\sigma \cdot \left(\sum_{i\le j} k_i + \mathbf{c}\right). \tag{7.18}$$

As we remember, the final step in the T-duality was to integrate out the gauge field A_α (substituting A_α by the solution of its equation of motion). Before doing that, we integrate by parts the term with y in (7.18)

$$\frac{-i}{4\pi}\int_D d\sigma d\tau y\cdot(\partial_\tau A_\sigma - \partial_\sigma A_\tau) \;=\; \frac{-i}{4\pi}\int_D d\sigma d\tau(\partial_\sigma y\cdot A_\tau - \partial_\tau y\cdot A_\sigma) - \frac{i}{4\pi}\left(\int d\sigma y\cdot A_\sigma - \int d\tau y\cdot A_\tau\right). \tag{7.19}$$

After incorporating the second term into c, we end up with a neat form of the action and the boundary term

$$\frac{1}{4\pi}\int_D \left(R^2\frac{A_\alpha\cdot A^\alpha}{r^2} - i(A_\tau\cdot\partial_\sigma y - A_\sigma\partial_\tau y)\right) -i\sum_{j=1}^{n-1}\int_{\sigma_j}^{\sigma_{j+1}} d\sigma A_\sigma\cdot(\sum_{i\leq j}k_i + \mathbf{c} + \frac{R^2}{4\pi}y). \tag{7.20}$$

There are two equations of motion for A_α. One emerging from A_σ at the boundary of the disc

$$\sum_{i\leq j}k_i + \mathbf{c} + \frac{R^2}{4\pi}y(\sigma_j \leq \sigma \leq \sigma_{j+1}) = 0. \tag{7.21}$$

The other, coming from A_α in the "bulk" of the disc

$$A_\alpha = \frac{r^2}{R^2}i\epsilon_{\alpha\beta}\partial_\beta y. \tag{7.22}$$

After substituting (7.22) into the "bulk" part of (7.20), we obtain the dual sigma model

$$\frac{1}{4\pi}\int d\sigma d\tau \frac{r^2}{R^2}\partial_\alpha y \partial^\alpha y. \tag{7.23}$$

This means that after the T-duality, the minimal surface is embedded in the space with metric

$$d\tilde{s}^2 = \frac{r^2}{R^2}dy_{3+1}^2 + \frac{R^2}{r^2}dr^2. \tag{7.24}$$

If in addition we invert the radial direction $r = \frac{R^2}{r}$, the dual space again becomes Anti-de Sitter with y

$$ds^2 = R^2\left[\frac{dy_{3+1}^2 + dr^2}{r^2}\right]. \tag{7.25}$$

Let us summarise what has actually happened. In the original coordinates, we had momenta k^μ conjugated to x^μ. In the new coordinates (at the boundary), the difference between two neighboring y^μ (close to the insertion points σ_i and σ_{i+1}), is proportional to the gluon momenta[5]

$$y^\mu(\sigma_i) - y^\mu(\sigma_{i+1}) = \frac{R^2}{4\pi} k_i^\mu. \tag{7.26}$$

This basically means that the entire boundary contour C consists of the light-like segments l_i, so that

$$C = \bigcup_{i=1}^{4} l_i,$$

and l_i is defined as

$$l_i \equiv y_i^\mu - \tau_i k_i^\mu| \quad \tau_i \in (0,1), \tag{7.27}$$

where y_i^μ denotes the position of the i-th cusp in four dimensions. This can be viewed in Fig. 7.2 presenting the new boundary, where cusps are placed at the vertices of the cube. This way, by the T-duality, we mapped the

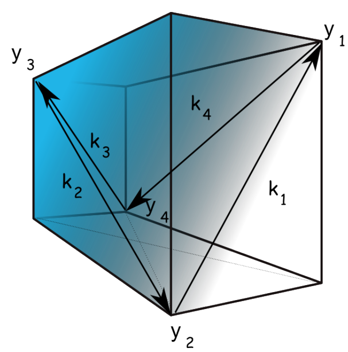

Figure 7.2: *After T-duality we end up with the coordinates y^μ. The boundary contour is then fully determined by gluons' momenta.*

problem of computing the 4-gluon scattering amplitude at strong coupling to finding the minimal surface which ends on the four light-like momenta. There is only one thing which remains to be clarified: the position of the single D3 brane. Originally, we placed it at some value r_{IR}. Then, in order to map the IR regularisation it was moved to $r = \infty$. In the new coordinates

[5] The proportionality constant can be removed by rescaling of all the AdS coordinates.

$r = \frac{R^2}{r^2}$ this corresponds to $r = 0$, whereas the new boundary of the AdS (previously at r=0) moved to $r = \infty$.
Putting this all together, we have to find the minimal surface in dual-Anti-de Sitter space which fastens to the light-like contour C on the D3 brane at $r = 0$ (Fig. 7.3) Before attacking this task, we will play with a simpler

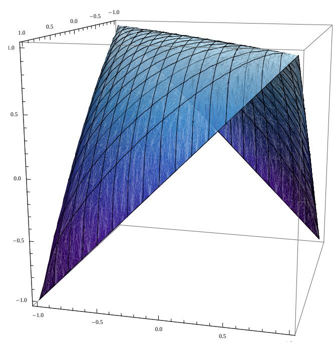

Figure 7.3: *World-sheet bounded by light-like momenta and stretching into AdS.*

example, the minimal surface on the light-like cusp. This will give us some intuition for how to work with minimal surfaces embedded in the AdS_5 space. Later we will see that the 4-cusps solution we are looking for can be constructed from the one-cusp toy model.

7.4 Minimal surface on the light-like cusp

Let us consider a light-like cusp $y^1 = \pm y^0$ (see Fig. 7.4) on the boundary of the AdS_5. Intuitively, the surface with such a boundary is an object from the AdS_3 embedded in the AdS_5. This is analogous to the shortest path in space-time (x, y, z). A line on the $x - y$ plane is minimal when its parametrisation has $y = z = 0$. The AdS_3 is endowed with scaling and boost symmetries and we will use them to construct the minimal surface in quite an elegant way [38]. The symmetries are transparent when we parametrise the AdS_3 metric

$$ds^2 = R^2 \left[\frac{-dy_0^2 + dy_1^2 + dr^2}{r^2} \right], \tag{7.28}$$

with $(\tau, \sigma, \omega(\tau))$

$$y_0 = e^\tau cosh\sigma, \qquad y_1 = e^\tau sinh\sigma, \qquad r = e^\tau \omega$$

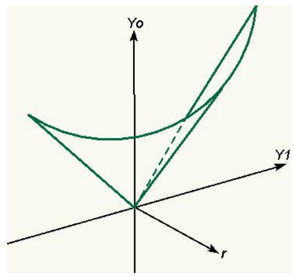

Figure 7.4: *The minimal surface on the light-like cusp is an object from AdS_3 embedded in AdS_5.*

In these parameters, scalings correspond to shifts in τ and boosts correspond to shifts in σ. Already from a simple analysis of the relations between τ, σ and $\omega(\tau)$, we notice that

$$r^2 = \omega(\tau)\left(y_0^2 - y_1^2\right) \tag{7.29}$$

This is precisely the surface on the two light-like cusps in Fig. 7.5. The extra parameter ω governs the amplitude of the semi-circles far from the vertex. We will only consider the "upper" cusp ($y^0 > 0$). In order to find

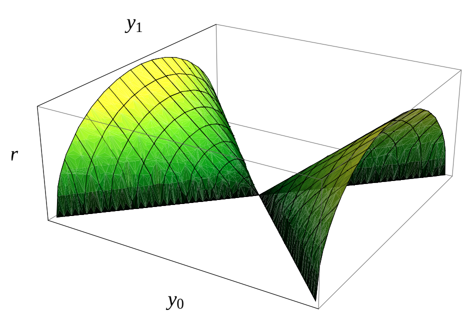

Figure 7.5: 2 cusps for an arbitrary ω

out which value of the ω minimises the surface, we will solve the equation of motion for the Nambu-Goto action. The action with the AdS_3 metric 7.28 reads

$$S = \frac{1}{2\pi}\int d\tau d\sigma \sqrt{-det(G_{\mu\nu}\partial_\alpha y^\mu \partial_\beta y^\nu)} \tag{7.30}$$

Where $\mu \in (0, 1, 2, 3)$ and $y^3 = r$. According the discussion before, we set $y^2 = 0$. Hence, the determinant in (7.30) becomes

$$det \begin{pmatrix} G_{\mu\nu}\partial_\tau y^\mu \partial_\tau y^\nu & G_{\mu\nu}\partial_\tau y^\mu \partial_\sigma y^\nu \\ G_{\mu\nu}\partial_\sigma y^\mu \partial_\tau y^\nu & G_{\mu\nu}\partial_\sigma y^\mu \partial_\sigma y^\nu \end{pmatrix} = \frac{1}{\omega^2} det \begin{pmatrix} -1 + (\omega(\tau) + \omega'(\tau))^2 & 0 \\ 0 & 1 \end{pmatrix}$$

This gives the Nambu-Goto action in terms of ω

$$S = \frac{R^2}{2\pi} \int d\sigma \int d\tau \frac{\sqrt{1 - (\omega(\tau) + \omega'(\tau))^2}}{\omega^2(\tau)} \tag{7.31}$$

The corresponding equation of motion is the second order differential equation

$$\partial_\tau \left(\frac{\omega + \omega'}{\omega^2 \sqrt{1 - (\omega + \omega')^2}} \right) = \frac{2 - (\omega + \omega')(\omega + 2\omega')}{\omega^3 \sqrt{1 - (\omega + \omega')^2}} \tag{7.32}$$

A general solution seems to be quite difficult, but we are interested in a particular stationary one. This allows us to take the ansatz $\omega = const$. From this, it easily follows that

$$0 = 2 - \omega^2 \rightarrow \omega = \sqrt{2} \tag{7.33}$$

This way we find the minimal surface on the single cusp

$$\boxed{r = \sqrt{2}\sqrt{y_0^2 - y_1^2}} \tag{7.34}$$

It is very instructive to study this solution in the embedding coordinates of the AdS space (see appendix A). Unlike the Poincare coordinates which parametrise only half of the AdS cylinder (see Fig. 7.6), the global ones cover all the space. Let us see what kind of interesting surprises can be found after a simple change of coordinates. The AdS_5 can be seen as a surface embedded in six-dimensional space. Its quadric equation is

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = 0 \tag{7.35}$$

The coordinates Y are usually known as the embedding coordinates and are related to the Poincare y by

$$Y^\mu = \frac{y^\mu}{r}, \quad \mu = 0, \dots, 3$$
$$Y_{-1} + Y_4 = \frac{1}{r}, \qquad Y_{-1} - Y_4 = \frac{r^2 + y_\mu y^\mu}{r} \tag{7.36}$$

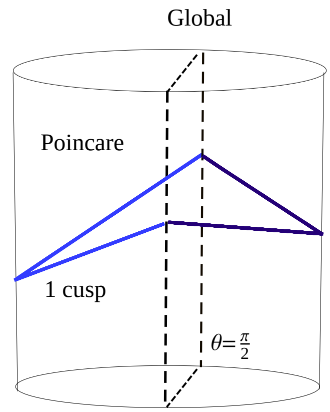

Figure 7.6: *Single cusp solution in the Poincare coordinates, in the embedding, corresponds to four cusps.*

Using these relations, we can easily check that the single cusp solution in the embedding coordinates reads

$$Y_0^2 - Y_1^2 = Y_{-1}^2 - Y_4^2, \qquad Y_2 = Y_3 = 0 \tag{7.37}$$

This solution represents the minimal surface on four cusps 7.6. We will see this explicitly after finding the minimal surface for the four light-like cusps. They will be related by the $SO(2,2|4)$ isometry of AdS_5. This interesting result will give us the motivation to construct the minimal surfaces for 6-gluons from the 4-point ones. With some confidence in the environment of minimal surfaces in AdS, we are ready for the final, Alday-Maldacena computation.

7.5 The minimal surface for 4 cusps

In this section are going to find the minimal surface which, at $r = 0$, ends on the contour depicted in Fig. 7.2. It is easier to start with the Poincare coordinates, so that the embedded surface we are looking for is $y^\mu = (y_0, y_1, y_2, y_3, r)$. Analogous to the single cusp, we fix $y_3 = 0$. From the theory of minimal surfaces (see chapter 5.1), we know that a general surface embedded in $\mathbb{R}^n$ is the image of a map $\mathbb{R}^2 \to \mathbb{R}^n$. This way, any two coordinates of $\mathbb{R}^n$ can serve as the parameters. Following the same logic, the most natural way to choose a parametrisation in our case seems to be by expressing y_0 and r in terms of y_1 and y_2. Hence, we have

$$y^\mu = (y_0(y_1, y_2), y_1, y_2, 0, r(y_1, y_2)) \tag{7.38}$$

In addition, using the rescaling invariance of AdS, we put $y_1, y_2 \in (-1, 1)$. After these important steps, we can apply all the machinery we have learned in the previous parts. First, we construct the Nambu-Goto action for (7.38) embedded in

$$ds^2 = R^2 \left[\frac{-dy_0^2 + dy_1^2 + dy_2^2 + dr^2}{r^2} \right] \tag{7.39}$$

The determinant $det(G_{\mu\nu}\partial_\alpha y^\mu \partial_\beta y^\nu)$ is then

$$det(\ldots) = \frac{1}{r^2} det \begin{pmatrix} -(\partial_1 y_0)^2 + (\partial_1 r)^2 + 1 & -\partial_1 y_0 \partial_2 y_0 + \partial_1 r \partial_2 r \\ -\partial_2 y_0 \partial_1 y_0 + \partial_2 r \partial_1 r & -(\partial_2 y_0)^2 + (\partial_2 r)^2 + 1 \end{pmatrix}$$

where the derivatives ∂_α are naturally with respect to y_1 and y_2. The emerging Nambu-Goto action stays

$$S = \frac{R^2}{2\pi} \int dy_1 dy_2 \frac{\sqrt{1 - (\partial_i y_0)^2 + (\partial_i r)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}}{r^2} \tag{7.40}$$

The associated equation of motion is relatively complicated (see appendix E). Nevertheless, by studying the behaviour of y_0 and r at the boundary, we might come up with an ansatz for the solution. From the figure 7.2 we can read off the boundary for y_0

$$y_0(\pm 1, y_2) = \pm y_2, \qquad y_0(y_1, \pm 1) = \pm y_1 \tag{7.41}$$

Simultaneously, on all the light-like intervals r has to vanish

$$r(\pm 1, y_2) = r(y_1, \pm 1) = 0 \tag{7.42}$$

At this point, the ambitious reader might use his geometrical intuition or other more metaphysical methods to guess the form of r and y_0. Here, we just follow Alday and Maldacena, and check that

$$\boxed{y_0(y_1, y_2) = y_1 y_2}, \qquad \boxed{r(y_1, y_2) = \sqrt{(1 - y_1^2)(1 - y_2^2)}} \tag{7.43}$$

satisfy not only the boundary conditions (7.41) and (7.42), but also the equations of motion for (7.40). This beautiful solution is depicted in Fig. 7.7 and Fig. 7.8.

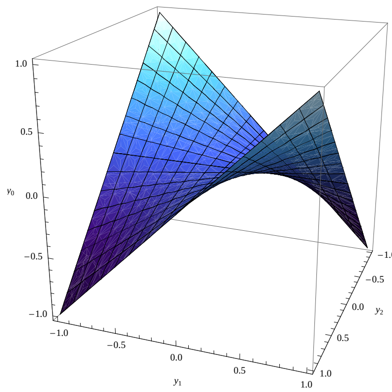

Figure 7.7: $y_0(y_1, y_2)$

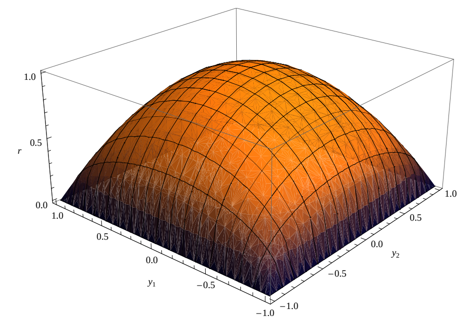

Figure 7.8: $r(y_1, y_2)$

So far, so good, we have found the minimal surface which ends on four light-like momenta. The last thing to do is to find its area, exponentiate it, and compare it with the BDS ansatz for four gluons ...almost. The first problem that stops us from doing this is the kinematical configuration of the momenta we used. In figure 7.2 we placed the momenta on the diagonals of the unit squares. This obviously means that we have made them all equal. Therefore

$$s = -(k_1 + k_3)^2 = -(k_1 + k_2)^2 = t \tag{7.44}$$

On the other hand, a very important part of the BDS ansatz consists of the finite term of the form $\ln^2 s/t$. To avoid this inconvenience we have to find a similar minimal surface but with $s \neq t$. Fortunately, A-M solved this puzzle in a very clever way. Their trick requires switching to the embedding coordinates.

7.6 The solution with non-trivial kinematics

In order to generate the solution $s \neq t$, we have to write the solution (7.43) in global coordinates and perform a boost in the Y_4 direction. Why boost? First of all, the equations of motion from the string actions are boost invariant. Hence we are allowed to do it freely. But the main motivation goes as follows: The kinematics of our scattering is basically determined by the size of the bottom square in Fig. 7.2. Namely, its diagonals are equal to $\sqrt{s}$ and $\sqrt{t}$. In the embedding coordinates the square lies in the $Y_1 - Y_2$

plane. Therefore, if we want to change the kinematics, we have to deform the square into a rhombus. A boost in the perpendicular direction Y_4 does this modification in a very elegant way. Since $ds^2 = G^{\mu\nu} dY_\mu dY_\nu$ is by default the invariant distance, after a boost in Y_4 the length of intervals in the $Y_1 - Y_2$ plane must change. To see how this idea works in practice, let us write (7.43) in the embedding coordinates. The surface is an object in AdS_4, so we can set two of the coordinates to zero: $Y_4 = Y_3 = 0$. As a consequence (using (7.36)) we get $Y_{-1} = \frac{1}{r}$. With Y_{-1} in this form we can write the following equation

$$Y_0 Y_{-1} = \frac{y_0}{r}\frac{1}{r} = \frac{y_1 y_2}{r^2} = Y_1 Y_2, \tag{7.45}$$

where in the second equality we plugged in the solution 7.43 for y_0. Simultaneously, we can rewrite r

$$r^2 = (1 - y_1^2)(1 - y_2^2) = Y_{-1}^2(1 - Y_{-1}^2 + Y_1^2 + Y_2^2) = Y_1^2 Y_2^2. \tag{7.46}$$

In the last equality we made use of the quadric equation for AdS_5 (7.35). Finally, the solution (7.43) in the embedding coordinates is simply:

$$Y_{-1} Y_0 = Y_1 Y_2, \qquad Y_4 = Y_3 = 0. \tag{7.47}$$

Before boosting, we will make one more side remark. We note that the single-cusp toy model, in the embedding coordinates, is equivalent to a four-cusp solution. Now the reader can easily check that indeed the solution (7.47) can be generated from (7.37) by the conformal transformations

$$\begin{aligned} Y_2 \to Y_4, \qquad Y_0 \to \frac{1}{\sqrt{2}(Y_0 + Y_{-1})}, \qquad Y_{-1} \to \frac{1}{\sqrt{2}}(Y_0 - Y_{-1}) \\ Y_1 \to \frac{1}{\sqrt{2}}(Y_1 + Y_2), \qquad Y_4 \to \frac{1}{\sqrt{2}}(Y_1 - Y_2). \end{aligned} \tag{7.48}$$

Naively, one might expect that since we found the four-cusp solution in Poincare coordinates, in the embedding ones it will describe a minimal surface on eight light-like intervals. Unfortunately, this is not the case[6].
To perform a boost in Y_4 we will take the Y_0 direction as playing the role of "time". Then, the new coordinates are

$$\begin{pmatrix} Y_0' \\ Y_4' \end{pmatrix} = \begin{pmatrix} \gamma & v\gamma \\ v\gamma & \gamma \end{pmatrix} \begin{pmatrix} Y_0 \\ Y_4 \end{pmatrix} = \begin{pmatrix} \gamma(Y_0 + vY_4) \\ \gamma(Y_4 + vY_0) \end{pmatrix}.$$

[6] see Goedel theorem.

In the original reference frame we had $Y_4 = 0$, whereas now, $Y_4' = vY_0$. Plugging it into the inverted relation for Y_0 gives

$$Y_0 = \gamma(Y_0' - v^2 Y_0') = \gamma^{-1} Y_0'. \tag{7.49}$$

So that the solution for $s \neq t$ in boosted global coordinates is

$$Y_4' - vY_0' = 0, \qquad \frac{1}{\gamma^{-1}} Y_0' Y_{-1} = Y_1 Y_2. \tag{7.50}$$

To see how the boosted surface looks from the original frame in the Poincare coordinates, we just substitute the new solution into

$$Y_{-1}' + Y_4' = \frac{1}{r'} = Y_{-1} + vY_0', \tag{7.51}$$

which gives

$$\frac{1}{r} + v\gamma\frac{y_0}{r} = \frac{1}{r'} \Rightarrow r' = \frac{1}{1 + v\gamma y_0} r = \frac{1}{1 + \frac{v}{\sqrt{1-v^2}} y_0} r. \tag{7.52}$$

Redefining $b = v\gamma$ (in our scattering $b < 1$ so $v < \frac{c}{\sqrt{2}}$, [1])

$$b^2 = \frac{v^2}{1 - v^2} \rightarrow v^2 = \frac{b^2}{1 + b^2} \; \gamma = \frac{1}{\sqrt{1 - v^2}} = \sqrt{1 + b^2},$$

leads to r' as a function of the old coordinates

$$r' = \frac{r}{1 + by_0} \tag{7.53}$$

Simultaneously for y_0'

$$Y_0' = \frac{y_0'}{r'} = \gamma \frac{y_0}{r} \tag{7.54}$$

hence

$$y_0' = \frac{\gamma}{1 + v\gamma y_0} y_0 = \frac{\sqrt{1 + b^2} y_0}{1 + by_0} \tag{7.55}$$

Both of these are depicted in Fig. 7.9 and Fig. 7.10. Naturally, after boosting, the parameters y_1 and y_2 change as well. They are given by

$$y_i' = \frac{y_i}{1 + by_0} \tag{7.56}$$

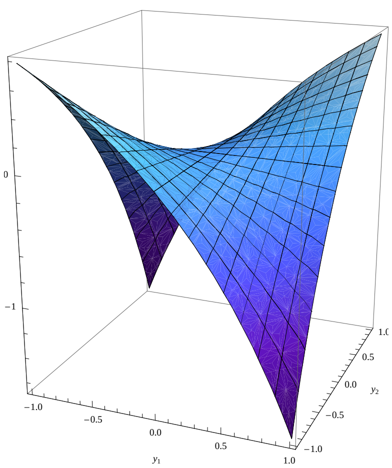

Figure 7.9: *Boosted* y_0

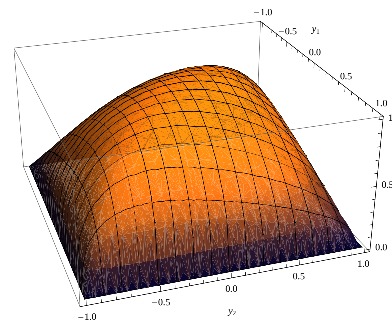

Figure 7.10: *Boosted* r

The results for a general boost in Y_4 can be presented in a more systematic way. For this purpose we will introduce special coordinates u_i, $i = 1, 2$:

$$y_i = \tanh u_i. \tag{7.57}$$

The $s = t$ solution takes the form

$$r = (\cosh u_1 \cosh u_2)^{-1}, \qquad y_0 = \tanh u_1 \tanh u_2 \tag{7.58}$$

Similarly the boosted surface (7.53) and (7.55)

$$\begin{aligned} r &= \frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, & y_0 &= \frac{a\sqrt{1+b^2} \sin u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \\ y_1 &= \frac{a \sinh u_1 \cosh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2}, & y_2 &= \frac{a \cosh u_1 \sinh u_2}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \end{aligned} \tag{7.59}$$

Parameter a is the size of the base square's side[7] in Fig. 7.2, and it sets the scale of the overall momentum (the bigger the square the higher the momentum). We would like to see how the contour $y_0(y_1, y_2)$ depends on a and the boost parameter b at the boundary $r = 0$. In the new coordinates the boundary corresponds to $u_{1,2} \to \pm\infty$. Therefore we can check the asymptotic behaviour of (7.59) and projecting y_0 into the $y_1 - y_2$ plane, we obtain a generic rhombus. The table 7.1 shows all four asymptotic solutions.

The corresponding rhombus is shown in Fig. 7.11. Lines (1,2,3,4) from

u_1	u_2	y_1	y_2	y_0	line
$+\infty$	u_2	$\frac{a}{1+b\tanh u_2}$	$\frac{a\tanh u_2}{1+b\tanh u_2}$	$\frac{a\sqrt{1+b^2}\tanh u_2}{1+b\tanh u_2}$	$(1) y_1+by_2=a$
$-\infty$	u_2	$\frac{-a}{1-b\tanh u_2}$	$\frac{a\tanh u_2}{1-b\tanh u_2}$	$\frac{-a\sqrt{1+b^2}\tanh u_2}{1-b\tanh u_2}$	$(2) y_1+by_2=-a$
u_1	$+\infty$	$\frac{a\tanh u_1}{1+b\tanh u_1}$	$\frac{a}{1+b\tanh u_1}$	$\frac{a\sqrt{1+b^2}\tanh u_1}{1+b\tanh u_1}$	$(3) by_1+y_2=a$
u_1	$-\infty$	$\frac{a\tanh u_1}{1-b\tanh u_1}$	$\frac{-a}{1-b\tanh u_1}$	$\frac{-a\sqrt{1+b^2}\tanh u_1}{1-b\tanh u_1}$	$(4) by_1+y_2=-a$

Table 7.1: Possible boundary (r=0) behaviour of the boosted solution

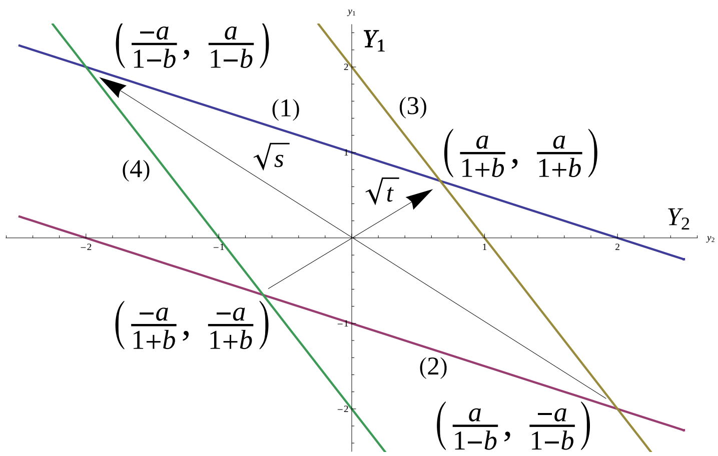

Figure 7.11: *The projection of the boundary solution y_0 gives a contour determined by parameters b and a*

the last column of the table are labelled on the plot in the same way. As we can see, the position of the cusp is a function of a and b. From very simple geometrical manipulations we can read out the relations between the physical gluon momenta and the two parameters

$$-s(2\pi)^2 = \frac{8a^2}{(1-b)^2}, \quad -t(2\pi)^2 = \frac{8s^2}{(1+b)^2}, \quad \frac{s}{t} = \frac{(1+b)^2}{(1-b)^2} \tag{7.60}$$

Hence, we have a neat control of the kinematics which can be quickly adjusted to an appropriate rhombus shape in Fig. 7.11.
Summarising, starting from the simplest kinematical case, $s = t$, we obtained a minimal surface in the dual Poincare coordinates. However, this was not the most useful solution to compare with the BDS ansatz. It turned

[7]previously we set it to 1

out that, in quite a clever way, we were able to generate the solution with $s \neq t$. Now, with our minimal surface in hand, we can proceed towards the string theory prediction for the gluon scattering at strong coupling.

7.7 Area of the minimal surface

We found the minimal surface for a specific light-like contour and it solves the equation of motion. If we plug it back to the action and integrate, we get an expression proportional to the area of our surface. Though this procedure seems to be straightforward, there are two pieces of news waiting for us, one bad and one terrible. The bad news is that the integral in the N-G action looks very complicated [8]. Fortunately, we learned from the minimal surfaces in chapter 5 that it is always possible to parametrise a world-sheet in terms of the isothermal coordinates. Since we use them, the induced metric on the surface becomes Euclidean (or Minkowskian) and we can use the Polyakov action (usually more treatable than N-G). At this point, we have a nice surprise for the reader. We already know what the isothermal coordinates for our solutions are. It is easy to check that the induced metric[9], when parametrised by $y_i = \tanh u_i (i = 1, 2)$ becomes Euclidean

$$ds^2 = \frac{dy_1^2}{(1-y_1^2)^2} + \frac{dy_2^2}{(1-y_2^2)^2}. \tag{7.61}$$

This means that we can compute the area using the Polyakov action

$$iS = -\frac{R^2}{2\pi}\int du_1 du_2 \frac{1}{2}\frac{\partial_1 r\partial_1 r + \partial_2 r\partial_2 r + \partial_i y^\mu \partial_i y^\nu \eta_{\mu\nu}}{r^2}. \tag{7.62}$$

Despite this nice surprise, we have terrible news as well. Both the N-G action and the Polyakov action, due to the presence of r^2 in the denominators, give an infinite area[10]. This means that if we want to get a reasonable result we must regularise it. In the next section we will see how Alday and Maldacena solved this problem.

[8] The reader passionate about "nasty" integrals can automatically skip this irrelevant argument.

[9] On the solution for $s = t$

[10] The careful reader remembers that we run into the same problem studying Wilson loops in AdS/CFT. The A-M computation is nothing but the evaluation of a specific Wilson loop at strong coupling.

7.8 Regularisation

In the context of the AdS/CFT correspondence one aspect of regularisation is very subtle. Using string theory in AdS, we want to solve the corresponding physical problem in the gauge theory. Therefore we would like to have a consistent regularization scheme on both sides of the duality. As we remember, in $\mathcal{N}$=4 SYM we use the dimensional regularization in $D = 4-2\epsilon$. More precisely, we start with the supersymmetric Yang-Mills theory in eleven dimensions (with 16 supercharges) and dimensionally reduce to $D = 4 - 2\epsilon$ (keeping the number of supercharges fixed) [50]. Introducing the $AdS_5 \times S^5$ geometry from D3 branes, we stressed that at low energies the branes effectively describe $\mathcal{N}$=4 SYM in 4 dimensions. The general rule is that an effective action for the integer dimensions corresponds to the supergravity solutions for D_p brane theories with $p = D - 1$ (hence D3 branes for D=4 dim). In the string frame, the metric for these configurations of D branes in D dimensions is (in the near horizon (NH) limit)

$$ds^2 = H^{-1/2} dy_D^2 + H^{1/2}[dr^2 + r^2 d\Omega_{9-D}^2], \qquad H_{NH} = \frac{c_D \lambda_D}{r^{8-D}}, \tag{7.63}$$

where $c_D \equiv 2^{4\epsilon}\pi^{3\epsilon\Gamma(2+\epsilon)}$, $\lambda_D = g_D^2 N$ is the t' Hooft coupling and g_D^2 the Yang-Mills coupling, both in D dimensions, N is the number of colours and Γ is the Euler Gamma function (see appendix C). Now the idea of Alday and Maldacena for dimensional regularisation in the dual gravitational theory, is to compute the minimal surface in the space with the metric (dropping the S^5 part):

$$ds^2 = f^{1/2}\left(dy_D^2 + dr^2\right) = \frac{\sqrt{c_D \lambda_D}}{r^\epsilon}\left(\frac{dy_D^2 + dr^2}{r^2}\right) = \frac{\sqrt{c_D \lambda_D}}{r^\epsilon} \cdot (AdS). \tag{7.64}$$

With this metric we end up with a modified action

$$S = \frac{\sqrt{\lambda_D c_D}}{2\pi} \int \frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon}, \tag{7.65}$$

where $\mathcal{L}$ is either the N-G or the Polyakov Lagrangian density evaluated in AdS_5.

Finally, to make "numerical" connection with the regularisation scheme used in the BDS proposal, λ_D is parametrised by the infrared cut-off μ

$$\lambda_D = \frac{\lambda \mu^{2\epsilon}}{(4\pi e^{-\gamma})}, \qquad \gamma = -\Gamma'(1). \tag{7.66}$$

Let us see how this procedure works on the single-cusp toy-model.

7.8.1 Regularised cusp

The algorithm for finding the surface on a single cusp remains exactly the same, we just use a different action functional. Plugging (7.4) into the modified action (7.65) leads to

$$S = \frac{\sqrt{c_D \lambda_D}}{2\pi} \int d\sigma \int d\tau \frac{e^{-\epsilon\tau}}{\omega^\epsilon} \frac{\sqrt{1-(\omega(\tau)+\omega'(\tau))^2}}{\omega^2(\tau)}. \tag{7.67}$$

Then, we just have to solve the corresponding equations of motion with an ansatz $\omega = const$:

$$\partial_\tau \left(-e^{-\epsilon\tau} \frac{\omega^2}{\omega^{3+\epsilon}\sqrt{1-\omega^2}} \right) = e^{-\epsilon\tau} \left(\frac{-2-\epsilon+\omega^2(1+\epsilon)}{\omega^{3+\epsilon}\sqrt{1-\omega^2}} \right). \tag{7.68}$$

Taking the time derivative of the left hand side gives

$$0 = e^{-\epsilon\tau} \left(\frac{-\epsilon\omega^2}{\omega^{3+\epsilon}\sqrt{1-\omega^2}} + \frac{-2-\epsilon+\omega^2(1+\epsilon)}{\omega^{3+\epsilon}\sqrt{1-\omega^2}} \right) = e^{-\epsilon\tau} \frac{\omega^2-(2+\epsilon)}{\omega^{3+\epsilon}\sqrt{1-\omega^2}}. \tag{7.69}$$

Hence the regularised solution for ω is

$$\omega = \sqrt{2}\sqrt{1+\frac{\epsilon}{2}}. \tag{7.70}$$

This naturally corresponds to r

$$r = \sqrt{2+\epsilon}\sqrt{y_0^2-y_1^2} = \sqrt{2+\epsilon}\sqrt{y_+ y_-}. \tag{7.71}$$

Computing the area integral simplifies when we use light-cone coordinates. That is why we substituted $y_\pm = y_0 \pm y_1$. Inserting into the action (7.67) and changing the integration variables

$$dy_+ \wedge dy_- = 2dy_0 \wedge dy_1 = 2e^{2\tau}(\cosh^2\sigma - \sinh^2\sigma) d\tau \wedge d\sigma \tag{7.72}$$

reduces our computation to

$$S = \frac{\sqrt{c_D\lambda_D}}{2\pi} 2 \int dy_+ dy_- \frac{\sqrt{1-\omega^2(\tau)}}{(e^\tau \omega(\tau))^{2+\epsilon}} = \frac{\sqrt{c_D\lambda_D}\sqrt{1+\epsilon}}{2\pi(1+\frac{\epsilon}{2})^{1+\frac{\epsilon}{2}}} 2 \int \frac{i dy_+ dy_-}{(2y_+y_-)^{1+\frac{\epsilon}{2}}} \tag{7.73}$$

We should notice the factor "i" in the integral above. Because of it, the action is imaginary and the amplitude[11] will be exponentially suppressed. Finally, the dimensionally regularised area of the cusp is

$$-iS = \frac{4A_\epsilon}{\epsilon^2} \frac{1}{(2y_+y_-)^\epsilon} \tag{7.74}$$

[11] Eventual amplitude on the cusp.

where

$$A_\epsilon = \frac{\sqrt{c_D \lambda_D}\sqrt{1+\epsilon}}{8\pi(1+\frac{\epsilon}{2})^{1+\frac{\epsilon}{2}}}. \tag{7.75}$$

With some confidence we can pursue regularising the surface for four gluons.

7.8.2 Regularised four-cusps surface

The regular surface we are looking for is a solution of the equation of motion for a general ϵ (E.5), (E.6). One could naively expect that such a solution would be just a factor-function of ϵ multiplying boosted solutions (7.59). However, this does not seems to be the case[12]. Therefore, it is very difficult to solve the equation directly or even come up with some ansatz for r or y^μ. Fortunately, there is a way to evaluate S with the accuracy we need. Namely, we are interested in all terms with divergences of order $\frac{1}{\epsilon^2}$, $\frac{1}{\epsilon}$, finite terms and finally the constant ones. It turns out that for this purpose one can consider the contribution from the surface close to the cusps [1]. Hence, we can assume that these near-cusp solutions are regularised like a single cusp

$$r_\epsilon \sim \sqrt{1+\frac{\epsilon}{2}}\, r_{\epsilon=0}, \qquad y^\mu_\epsilon \sim y^\mu_{\epsilon=0} \tag{7.76}$$

Where $r_{\epsilon=0}$ and y^μ_ϵ are the boosted solutions (7.59). Then, inserting them into (7.67) and expanding in ϵ gives

$$-iS = \frac{\sqrt{\lambda_D c_D}}{2\pi a^\epsilon}\int_{-\infty}^{\infty} du_1 du_2 (\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2)^\epsilon (1+\epsilon I_1 + \epsilon^2 I_2 + \ldots) \tag{7.77}$$

where the terms to integrate are

$$\begin{aligned} I_1 &= \frac{(b^2-1)(\cosh 2u_1 + \cosh 2u_2) - 2(1+b^2)}{8(\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2)^2} \\ I_2 &= \frac{1+b^2-(1+b^2)\cosh 2u_1 \cosh 2u_2 - 2b \sinh 2u_1 \sinh 2u_2}{16(\cosh u_1 \cosh u_2 + b\sinh u_1 \sinh u_2)^2} \end{aligned} \tag{7.78}$$

After integrating (see App.C) we get the regularised action which should be expanded in ϵ up to finite terms

$$iS = -\frac{\sqrt{\lambda_D c_D}}{2\pi a^\epsilon}\left(\frac{\pi\Gamma\left[\frac{-\epsilon}{2}\right]^2}{\Gamma\left[\frac{1-\epsilon}{2}\right]^2}\, {}_2\mathrm{F}_1\left(\frac{1}{2}, -\frac{\epsilon}{2}, \frac{1-\epsilon}{2}; b^2\right) + 1\right) \tag{7.79}$$

[12] A proof of this statement has not been achieved but many checks point to such a scenario.

7.8.3 Cut-off

An alternative method of regularisation is simply introducing a radial cut-off $r = r_C$. This basically means that we have to evaluate the part of our minimal area enclosed by

$$r_C = \frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2} \tag{7.80}$$

Though this method seems to be more brutal and straightforward than the aforementioned dimensional regularisation, they are equivalent [1]. After this long tour through the A-M computation we are ready to see the final answer.

7.9 The final result

Putting all the ingredients together and expanding in ϵ, we finish with the following string theory prediction for the 4 gluon scattering amplitude at strong coupling

$$\boxed{\mathcal{A}_4 = \exp\left[i(2S_{div}(s) + 2S_{div}(t)) + \frac{\sqrt{\lambda}}{8\pi}\left(\ln\frac{s}{t}\right)^2 + C\right]} \tag{7.81}$$

where the divergent pieces contains of sum of a function evaluated at s and t

$$\boxed{S_{div}(t) = -\frac{1}{\epsilon^2}\frac{1}{2\pi}\sqrt{\frac{\lambda\mu^{2\epsilon}}{(t)^\epsilon}} - \frac{1}{\epsilon}\frac{1}{4\pi}(1 - \ln 2)\sqrt{\frac{\lambda\mu^{2\epsilon}}{(t)^\epsilon}}} \tag{7.82}$$

and the constant term is

$$\boxed{C = -\frac{\sqrt{\lambda}}{4\pi}(1 - 2\ln 2 + (\ln 2)^2 + \frac{\pi^2}{3})} \tag{7.83}$$

Comparing these results to the BDS, we clearly see that the finite piece matches perfectly. Moreover, to match the IR divergences, according to A-M the cusp anomalous dimension and the collinear dimension at strong coupling should satisfy

$$\begin{aligned} \left(\lambda\frac{d}{d\lambda}\right)^2 f^{-2}(\lambda) &= f(\lambda) \\ \lambda\frac{d}{d\lambda} g^{-1}(\lambda) &= g(\lambda) \end{aligned} \tag{7.84}$$

and at strong coupling should be equal to

$$\boxed{f(\lambda) = \frac{\sqrt{\lambda}}{4\pi}, \qquad g = \frac{\sqrt{\lambda}}{2\pi}(1 - \ln 2)} \tag{7.85}$$

Remarkably, the strong coupling result for $f(\lambda)$ is precisely equal to the one obtained by Kruczenski [38]. The constant term will not play an important role, as we will see later.

Summarising, Alday and Maldacena developed the method which allows for computing the gluon scattering amplitudes in planar $\mathcal{N}$=4 SYM at strong coupling. The main part of the computation is mapped by T-duality into finding the minimal surface on a specific light-like contour. After finding the analytical formula for the surface, the action in AdS is divergent and has to be regularised. The authors explained how to gravitationally mimic the dimensional regularisation in $D = 4 - 2\epsilon$ from the gauge theory. Performing the regularisation the string theory prediction for four gluons perfectly matches the BDS ansatz.

After such a success one would naturally want to perform more tests of this interesting string theory method. Before any further steps, we will first focus on a very surprising result related to the four cusps, light-like contour.

8

The Wilson loop at weak coupling

In this chapter we will follow a simple and very interesting computation which emerged after the proposal of Alday and Maldacena. As we saw in the previous chapter, at strong coupling, the computation of the four-gluon amplitude is dual to the Wilson loop along the light-like contour. Two groups: Brandhuber, Heslop, Travaglini, [42] and Drummond, Korchemsky, Sokatchev [22] decided to check whether this duality is still valid in the weak regime of the coupling. Their idea was to take the same contour parametrised by gluon momenta, and compute perturbatively a Wilson loop in $\mathcal{N}$=4 SYM along it. In other words, evaluate the propagators between the light-like segments in four dimensions (see Fig. 8.1). Let us analyse all the one loop diagrams that contribute to the 4-segments Wilson loop, and see whether the duality holds at weak coupling as well.

8.1 Light-like Wilson loop at one-loop

Naturally we already have all the necessary tools to smoothly compute the Wilson loop in $\mathcal{N}$=4 SYM. It was introduced in chapter 4, and has the following form (in the dual coordinates y^μ)

$$W_C = \frac{1}{N} Tr\mathbf{P}exp\left(ig\oint_C ds\left(\dot{y}^\mu A_\mu - i\Phi_i\theta^i|\dot{y}|\right)\right), \tag{8.1}$$

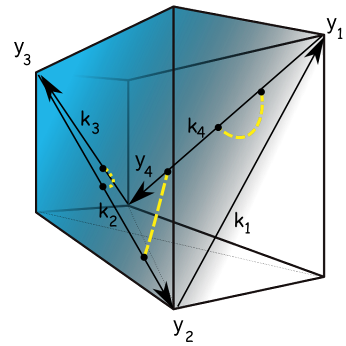

Figure 8.1: *Wilson loop at weak coupling along the light-like contour at the boundary of AdS_5. Yellow dashed lines denote three examples of the gluon propagator.*

where the contour C is parametrised by the gluon momenta

$$C = \bigcup_{i=1}^{4} l_i, \quad l_i \equiv \{y^\mu(\tau_i) = y_i^\mu - \tau_i k_i^\mu \quad \tau_i \in (0,1)\}\,. \tag{8.2}$$

With this form of C, the derivative of $y^\mu(\tau_i)$ is equal to the momentum of the external gluon k_i^μ to which the end-point of the propagator is attached. Since the propagator for the six scalars is proportional to the absolute value of the parametrisation $|\dot{y}(\tau_i)| = \sqrt{k_i^2} = 0$, the expectation value of the Wilson loop simplifies to

$$\langle W_C \rangle = 1 + \frac{(ig_{YM})^2 C_F}{2} \int_0^1 d\tau_i \int_0^{\tau_i} d\tau_j k_i^\mu k_j^\nu G_{\mu\nu}[y(\tau_i) - y(\tau_j)], \tag{8.3}$$

where $G_{\mu\nu}$ is the dimensionally regularised, in $D = 4 - 2\epsilon_{uv}$, gluon propagator (4.7).
To simplify the figures representing each diagram, we project the contour C onto a two dimensional plane. There are ten possible, path ordered, diagrams contributing to (8.3). Furthermore, they can be divided into three families. The first one consists of the graphs with both ends of the propagator attached to the same light-like interval (Fig. 8.2). The second contains the diagrams whith the propagator between two neighboring edges which meet at a cusp (Fig.8.3). Finally, to the third family contributes two diagrams with propagator end points on the opposite segments (Fig. 8.4).

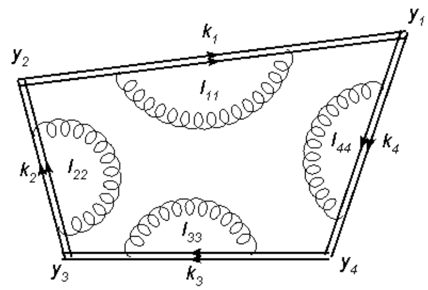

Figure 8.2: *The diagrams proportional to k_i^2.*

Analytically, all ten terms can be written in a compact form (we take the logarithm to avoid the unit element in the expansion)

$$\ln \langle W[C_4] \rangle = \frac{(ig_{YM})^2 C_F}{4\pi^2} \sum_{1 \leq i \leq j \leq 4} I_{ij} + O(g^4) \tag{8.4}$$

where I_{ij} is the integral over the gluon propagator which begins at the i-th segment and ends at the j-th segment (see Figures: 8.4, 8.3, 8.2). If we express the propagator (4.7) in terms of the parametrisation (8.2) , I_{ij} reads

$$I_{ij} = -\int_0^1 d\tau_i \int_0^1 d\tau_j \frac{k_i \cdot k_j \Gamma(1-\epsilon_{uv})(\pi\tilde{\mu}^2)^{\epsilon_{uv}}}{\left[-(y_i - y_j - \tau_i k_i + \tau_j k_j)^2 + i0\right]^{1-\epsilon_{uv}}} \tag{8.5}$$

Let us now analyse each of the three families separately. The easiest, depicted in figure 8.2, consists of I_{11}, I_{22}, I_{33}, and I_{44} proportional to k_1^2, k_2^2, k_3^2 and k_4^2 respectively. Therefore none of these diagrams contribute.
As a representative of the second group (Fig. 8.3), we can evaluate I_{12}. The integral is simply

$$I_{12} = -\Gamma(1-\epsilon_{uv})(\pi\tilde{\mu}^2)^{\epsilon_{uv}} \int_0^1 d\tau_1 \int_0^1 d\tau_2 \frac{k_1 \cdot k_2}{\left[-(y_1 - y_2 - \tau_1 k_1 + \tau_2 k_2)^2\right]^{1-\epsilon_{uv}}} = (*) \tag{8.6}$$

After substituting $y_1 - y_2 = k_2$ in the denominator we obtain

$$(*) = -\Gamma(1-\epsilon_{uv})(\pi\tilde{\mu}^2)^{\epsilon_{uv}} \int_0^1 d\tau_1 \int_0^1 d\tau_2 \frac{k_1 \cdot k_2}{\left[-2(k_1 \cdot k_2)(1-\tau_1)\tau_2\right]^{1-\epsilon_{uv}}} \tag{8.7}$$

To evaluate this integral it is convenient to re-define the parameter $\tilde{\tau}_1 = 1-\tau_1$ and use the Mandelstam variable $s = 2k_1 \cdot k_2$. Then it becomes

$$I_{12} = \frac{\Gamma(1-\epsilon_{uv})[(-s)\pi\tilde{\mu}^2]^{\epsilon_{uv}}}{2} \int_0^1 \frac{d\tilde{\tau}_1 d\tau_2}{\left[\tilde{\tau}_1 \tau_2\right]^{1-\epsilon_{uv}}} \tag{8.8}$$

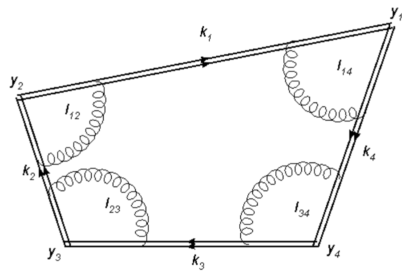

Figure 8.3: *The diagrams contributing to the divergent term.*

By en elementary integration we get the double pole in ϵ_{uv}

$$\int_0^1 \frac{d\tau_1 d\tau_2}{[\tau_1\tau_2]^{1-\epsilon_{uv}}} = \frac{1}{\epsilon_{uv}^2} \tag{8.9}$$

Hence the value of I_{12} is

$$I_{12} = \frac{\Gamma(1-\epsilon_{uv})[(-s)\pi\tilde{\mu}^2]^{\epsilon_{uv}}}{2\epsilon_{uv}^2} \tag{8.10}$$

Evaluation of I_{12} immediately gives us the answers for the remaining I_{ij}s in this group. From the conservation of the momentum, $(k_1{\cdot}k_2)^2 = (k_3{\cdot}k_4)^2$, we have the equality $I_{12} = I_{34}$. Similarly, from the form of (8.5) we notice that the integrals $I_{23} = I_{14}$ are simply generated from I_{12} by interchange of the Mandelstam variables $s = (p_1+p_2)^2 \leftrightarrow t = (p_1+p_4)^2$. This gives the second family in Fig. 8.3. At this point we should notice one very important issue. The integrals I_{ij} should be dimensionless numbers. But, since I_{12} contains the product of s and $\tilde{\mu}^2$, our UV scale is forced to have a dimension of $[\tilde{\mu}^2] = [mass]^{-2}$ (instead of $[mass]^2$ which is the usual energy scale). This is caused by the fact that "dual" parametrisation of the contour C gives the dimension of momentum to the light-like segments (they are precisely the momenta), whereas they should have the unit of the distance. On the other hand, we evaluate the integrals in $D = 4-2\epsilon$ dimensions, where the Yang-Mills coupling "runs" according to (3.1). Fortunately, we can actually kill two birds with one stone and introduce the new scale

$$\mu_{uv}^2 = (\tilde{\mu}\pi e^{\gamma})^{-1} \tag{8.11}$$

The last two integrals, $I_{13} = I_{24}$ (Fig.8.4), do not contain the UV divergences, hence they give a finite answer when $\epsilon_{uv} = 0$. This allows us to

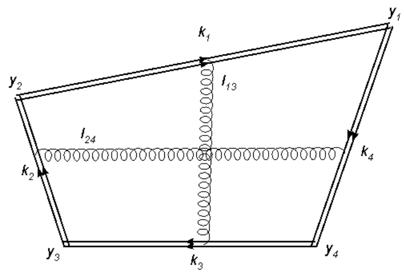

Figure 8.4: *The diagrams that gives rise to a finite term.*

compute them in $D = 4$ dimensions. Let us take I_{24}

$$I_{24} = \int_0^1 d\tau_2 d\tau_4 \frac{-k_2 \cdot k_4}{[k_2(1-\tau_2) + k_3 + \tau_4 k_4]^2}. \tag{8.12}$$

Again shifting $\tilde{\tau}_2 = 1 - \tau_2$, and manipulating the momenta we obtain

$$I_{24} = -\frac{1}{2}\int_0^1 d\tau_2 d\tau_4 \frac{s+t}{t\tau_2 + s\tau_4 - (s+t)\tau_2\tau_4} = -\frac{\ln^2(s/t) + \pi^2}{4} \tag{8.13}$$

Now we just have to add all the contributing terms together and separate the Wilson loop W_C into the divergent, finite and constant pieces. This procedure leads us to the finial answer

$$\boxed{\ln\langle W[C_4]\rangle = \frac{-\lambda}{8\pi^2\epsilon_{uv}^2}\left[\left(\frac{\mu_{uv}^2}{s}\right)^{-\epsilon_{uv}} + \left(\frac{\mu_{uv}^2}{t}\right)^{-\epsilon_{uv}}\right] + \frac{\lambda}{8\cdot 2\pi^2}\ln^2(s/t) + 2\frac{\lambda}{4\pi^2}\zeta_2} \tag{8.14}$$

where $\lambda = g_{YM}^2 N$ is the 't Hooft coupling in four dimensions. This result agrees with the BDS ansatz if we impose two conditions:

- The UV scale for the Wilson loop is equal to the IR scale for the scattering amplitudes

$$\boxed{\mu_{uv}^2 = \mu_{IR}^2}$$

- The value of the UV regulator ϵ_{uv} is equal to the value of the IR regulator ϵ_{IR} (opposite signs)

$$\boxed{\epsilon_{IR}^2 = -\epsilon_{uv}^2}$$

Similarly, the 4-gluon Wilson loop along C, at weak coupling, is dual to the minimal surface along the same contour at strong coupling. Duality here is understood as an equivalence between the methods. So that the final answer from any of the approaches (direct computation at weak coupling, Wilson loop and minimal surface in AdS) will depend on the functions $f(\lambda)$ and $g(\lambda)$. And in order to evaluate the four-gluon amplitude at strong or weak coupling λ, we have to insert an appropriate form for both of them (3.2), (3.4). Figure 8.5 summarises the results for 4 gluons. We should stress that this relation holds for four gluon amplitudes (we will show that for 5 as well).

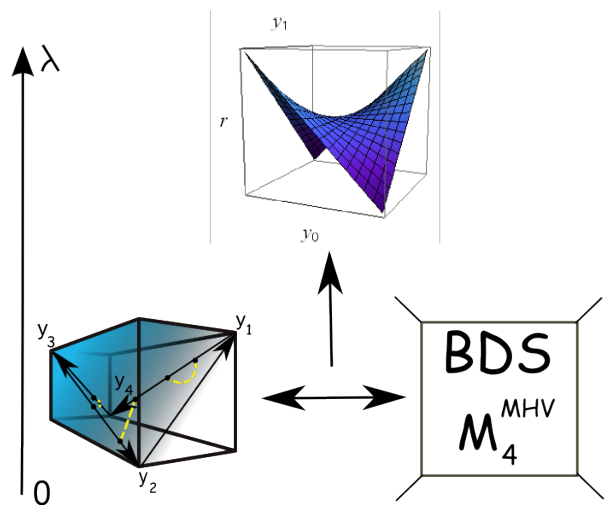

Figure 8.5: *Four-gluon amplitude in planar $\mathcal{N}=4$ SYM can be obtained from direct computation and light-like Wilson loop at weak coupling, and the minimal surface in AdS_5 at strong λ*

In the next chapter we will investigate what fixes the form of the four-gluon MHV amplitude, the four cusps light-like Wilson loop and finally the minimal surface found by Alday and Maldacena. We will analyse the range of this mechanism (number of scattered gluons) and see what happens beyond it.

9 Conformal symmetry

This chapter will involve the discussion of 4-gluon amplitudes at weak and strong coupling. First, we will show that integrals appearing in the direct computations at weak coupling, after changing the integration variables into "dual", exhibit the conformal symmetry. Then we will see that the $SO(2,4)$ conformal symmetry of the minimal surfaces in AdS_5 leaves the light-like contour C parametrised by the gluons' momenta invariant as well. Finally, we will argue that due to the requirement of regularisation of the Wilson loop (as we remember the gluons' propagator is divergent along the loop), the path integral from the Wilson loop is not $SO(2,4)$ invariant and anomalous Ward identities can be derived. Moreover the identities completely fix the form of 4 and 5 gluon amplitudes.

9.1 Dual conformal symmetry and MHVs

The main difficulty of the Alday-Maldacena computation was to find the minimal surface on the light-like momenta. As we recall, the surface in AdS_5 was invariant under boosts and scalings so we could use that to deduce the analytical form for the surface. Let us pursue a similar analysis and check whether the conformal transformation leaves the explicit gluon amplitudes invariant. Below we will follow the discussion from [22] and [13]. The $SO(2,4)$ group has 15 generators: 6-rotations $\mathbb{M}^{\mu\nu}$ (3 rotations and 3 boosts), 1-dilatation $\mathbb{B}$, 4-translations $\mathbb{P}^\mu$ and 4-special conformal transformations $\mathbb{K}^\mu$

$$\begin{aligned}\mathbb{M}^{\mu\nu} &= (x^\mu\partial^\nu - x^\nu\partial^\mu) + m^{\mu\nu}\\ \mathbb{D} &= x\cdot\partial + d\\ \mathbb{P}^\mu &= \partial^\mu\\ \mathbb{K}^\mu &= (2x^\mu x\cdot\partial - x^2\partial^\mu) + 2x^\mu d + 2x_\nu m^{\mu\nu},\end{aligned} \tag{9.1}$$

where d is the conformal weight of the fields upon which the generators act, whereas $m^{\mu\nu}$ denotes the generator of spin rotations. These are basically the usual Poincare transformations plus two extra operations: scaling

$$x^\mu \to \alpha x^\mu, \tag{9.2}$$

and special conformal transformation which is a sequence of inversion-translation-inversion. The conformal inversion acts as

$$x^\mu \to \frac{x^\mu}{x^2}, \qquad dx\cdot dx \to \frac{dx\cdot dx}{(x^2)^2}. \tag{9.3}$$

To check whether gluon amplitudes are invariant under the above operation we consider the simplest (but sufficient for us) example: the box integral. It was shown in [21] that the one loop, planar MHV amplitude for four gluons (see for e.g. (3.6)) is proportional to an integral $I^{(1)}$

$$M_4^{(1)} = 1 - \frac{\lambda st}{16\pi^2} I^{(1)} + O(\lambda^2), \tag{9.4}$$

where

$$I^{(1)} = \int \frac{d^D k}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2}. \tag{9.5}$$

The name for $I^{(1)}$ comes naturally from the corresponding Feynmann diagram which is displayed in Fig. 9.1. Now we will change the integration variables into the one that has the property of the dual parametrisation used by Alday and Maldacena. Namely, the difference between the two neighboring edges is equal to gluon's momenta. Denoting the difference between dual coordinates (see Fig. 9.1) as

$$y_{ij} \equiv y_i - y_j, \quad , \tag{9.6}$$

the momenta in (9.5) become

$$p_1 = y_{12}, \quad p_2 = y_{23}, \quad p_3 = y_{34}, \quad p_4 = y_{41}, \quad k = y_{15}. \tag{9.7}$$

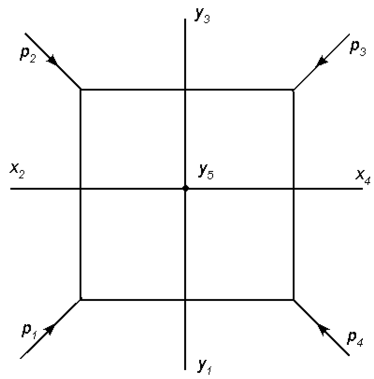

Figure 9.1: *Box diagram corresponding to I^1.*

With this substitution the box integral becomes

$$I^{(1)} = \int \frac{d^D y_5}{y_{12}^2 y_{25}^2 y_{35}^2 y_{45}^2}. \tag{9.8}$$

Now, the conformal properties are clearly exposed. Translations, boosts and scalings leave it invariant because of the dependence on $y_i - y_j$ only, and equal powers of y's in the numerator and the denominator. The inversion transformation acts upon the inverse square of y_{ij} and the measure $d^D x_5$ as

$$\frac{1}{y_{ij}^2} \to \frac{y_i^2 y_j^2}{y_{ij}^2}, \qquad d^D y_5 \to \frac{d^D y_5}{(y_5^2)}, \tag{9.9}$$

hence appropriate terms in the integral cancel and this proves its conformal invariance in dual variables (hence the dual conformal invariance). Of course one would have to repeat this procedure for higher loops. This has been done up to five loops and seems to hold for higher orders as well. One obvious question arising here is whether this dual conformal symmetry for explicitly computed amplitudes is not just the conformal symmetry of $\mathcal{N}$=4 SYM. The paradigm in most of the literature regarding this issue seems to be that it is not, at least not in a direct way (for further references see [21], [22]).
As a summary of the above result we can give an intuitive reasoning about the form of MHV for four (and five) gluons. Since the amplitude is invariant under $SO(2,4)$, the 15 generators enable one to constrain 15 parameters in the theory. In a scattering of gluons we need only three numbers for each of them (kinematical 3-momentum). Therefore the four and five gluon

amplitude is endowed with 12 and 15 parameters which can be fully fixed by the group generators.

9.2 Conformal symmetry and Wilson loops

Let us try to perform a similar reasoning to the light-like Wilson loops. The first thing we can easily see is that the contour C (8.2) preserves the dual conformal symmetry as well. If we take a segment:

$$y_i^\mu = y_i^\mu - \tau_i k_i^\mu = y_i^\mu - \tau_i \left(y_i^\mu - y_{i+1}^\mu \right); \tag{9.10}$$

and apply the conformal inversion to it, we end up with another light-like segment parametrised by:

$$\tau_i' = \frac{\tau_i}{\tau_i + (1-\tau_i)(y_i')^2/(y_{i+1}')^2}. \tag{9.11}$$

Naturally, parametrisation should not matter in a physical reasoning so a full contour remains invariant. Despite this promising sign of conformal invariance, the perturbative, light-like Wilson loop causes a more subtle problem. As we remember, due to the presence of cusps in the contour C, propagators are UV divergent and require regularisation $(D = 4 - 2\epsilon)$[1]. Therefore, though the W_C is $SO(2,4)$ invariant, the path integral:

$$\langle W[C_n] \rangle = \int \mathcal{D}A\mathcal{D}\chi\mathcal{D}\phi e^{iS_\epsilon} tr \mathbf{P} \exp\left(i \oint_{C_n} dy^\mu A_\mu(y) \right), \tag{9.12}$$

contains an action with the integral in $D = 4-2\epsilon$ dimensions over a covariant Lagrangian (with the canonical weight (see. [39]) d=4)

$$S_\epsilon = \frac{1}{g^2 \tilde{\mu}^{2\epsilon}} \int d^{D=4-2\epsilon} x \mathcal{L}(x). \tag{9.13}$$

This way the dimension does not match and when two of the conformal transformations (dilatations and special conformal transformation) change the fields in $\mathcal{L}$, we get anomalous terms. This can be summarised in the corresponding anomalous Ward identities [22] :

$$\begin{aligned} \mathbb{D} \ln\langle W[C_n] \rangle &= -\frac{2i\epsilon}{g^2 \tilde{\mu}^{2\epsilon}} \int d^D x \frac{\langle L(x) W[C_n] \rangle}{\langle W[C_n] \rangle} \\ \mathbb{K}^\mu \ln\langle W[C_n] \rangle &= \frac{4i\epsilon}{g^2 \tilde{\mu}^{2\epsilon}} \int d^D x x^\nu \frac{\langle L(x) W[C_n] \rangle}{\langle W[C_n] \rangle}. \end{aligned} \tag{9.14}$$

[1] This is also the case without cusps.

They can be analysed order by order in perturbation theory and the anomaly terms at one loop can be found in [22].
The anomalous Ward identities can be used further to constraint the form of Wilson loops. If we split the logarithm into a divergent and a finite part

$$\ln W[C_n] = \ln Z_n + \ln F_n, \tag{9.15}$$

the first identity in (9.14) boils down to

$$\sum_i (x_i \cdot \partial_{x_i}) F_n = 0, \tag{9.16}$$

and requires F_n to be a dimensionless scalar function of cross ratios

$$u_i \equiv \frac{(y_i - y_j)^2}{(y_k - y_l)^2}. \tag{9.17}$$

For example in the case of four gluons there is only one independent cross ratio and the finite part is:

$$F_4 \sim \ln^2 \left(\frac{y_{13}^2}{y_{24}^2} \right) = \ln^2(s/t), \tag{9.18}$$

where the second equality comes from the dual parametrisation. Incorporating this requirement to a full Wilson loop we obtain the dilatation Ward identity

$$\mathbb{D} \ln\langle W_n \rangle = -\frac{1}{2} \sum_{l \geq 1} \left(\frac{\lambda}{8\pi^2} \right)^l \sum_{i=1}^{n} (-y_{i-1,i+1}^2 \mu^2)^{l\epsilon} \left(\frac{f(\lambda)^{(l)}}{l\epsilon} + g^{(l)}(\lambda) \right). \tag{9.19}$$

The special conformal identity requires a bit more effort but after some approximations (see [22]) becomes

$$\sum_{i=1}^{n} \left(2y_i^\nu y_i \cdot \partial_i - y_i^2 \partial_i^\nu \right) \ln F_n = \frac{1}{2} \Gamma_{cusp}(a) \sum_{i=1}^{n} \ln \frac{y_{i,i+2}^2}{y_{i-1,i+1}^2} y_{i,i+1}^\nu. \tag{9.20}$$

The two identities, (9.20) and (9.19), are also very special for 4 and 5 gluons. In these cases they have a unique solution (up to an additive constant) and fix the form of finite pieces in the amplitudes:

$$\begin{aligned} \ln F_4 &= \frac{f(\lambda)}{4} \ln^2 \left(\frac{y_{13}^2}{y_{24}^2} \right) + const \\ \ln F_5 &= -\frac{f(\lambda)}{8} \sum_{i=1}^{5} \ln \left(\frac{y_{i,i+2}^2}{y_{i,i+3}^2} \right) \ln \left(\frac{y_{i+1,i+3}^2}{y_{i+2,i+4}^2} \right) + const, \end{aligned} \tag{9.21}$$

where

$$y_{i,j}^2 := (p_i + \ldots + p_{i+j-1})^2. \tag{9.22}$$

Summarising, though perturbative Wilson loops are not conformally invariant, the corresponding anomalous Ward identities governs the form of F_n. Combining this result with the one from the previous section we can conclude that the dualitites between Wilson loops at weak coupling and MHV amplitudes were caused by the dual conformal symmetry, which at the level of 4 and 5 point amplitudes does not leave any freedom for the structure of gluon amplitudes. This basically means that if there exist any non-trivial dualitites between the aforementioned objects, we should be able to spot them starting from a six gluon amplitude. We will see in the next chapter that not only can we spot potential dualities but we can also obtain some interesting conclusions about the BDS ansatz.

10
6-gluon amplitude

In this last chapter we will briefly summarise results for six gluon amplitudes. We will start by presenting an argument of Alday and Maldacena about potential problems of BDS at large gluon number. This will be supported by recent computations at weak coupling for six gluons. On top of that, since the dual conformal symmetry does not fix the amplitudes completely, we will obtain a clearer picture of the relations between Wilson loops and MHV amplitudes.

10.1 BDS failure

In the second article about gluon scattering amplitudes [2], Alday and Maldacena noticed that, at strong coupling, the BDS ansatz should fail for a large number of gluons. Their argument goes as follows: in the case of four gluons the minimal surface ends on four light-like segments. If we increase the number of particles (very large), a contour will approximate a rectangle (C_{rec}) with sides T and L (see Fig. 10.1)[1]. Luckily, a minimal surface in AdS_5 for such a contour was found in the original paper of Maldacena [23] where he postulated the interpretation for Wilson loops at strong coupling.

[1]At weak coupling a Wilson loop along this contour is considered in appendix D.

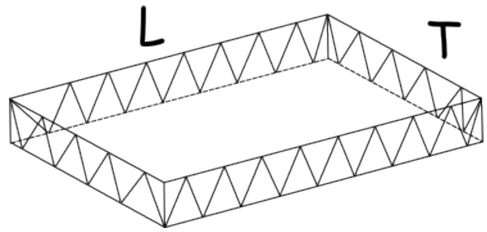

Figure 10.1: *Light-like contour for a large number of gluons.*

Its form is

$$\ln\langle W[C_{rec}]\rangle = \frac{\sqrt{\lambda}4\pi^2}{\Gamma(1/4)^4}\frac{T}{L}, \tag{10.1}$$

whereas BDS with for large n and strong coupling predicts

$$\ln\langle W_{rec}^{BDS}\rangle = \frac{\sqrt{\lambda}}{4}\frac{T}{L}, \tag{10.2}$$

hence according to A-M the BDS ansatz needed to be revised for at least a large number of gluons. This point about BDS failure appeared when the exact computations of six gluon amplitudes were not known. At the same time, motivated by duality between Wilson loops and MHV amplitudes for 4 gluons, Korchemsky and his collaborators postulated that this equality holds at weak coupling for any number of gluons. This gave even more motivation to evaluate the 6-cusp loop, M_6^{MHV} and the minimal surface for six segments. The first two tasks where accomplished perturbatively up to two loops and we will discuss these results in the next section.

10.2 6 MHV = 6 Wilson loop

It took some time and a lot of effort to compute the six-point amplitude explicitly. Nevertheless, the 6-point MHV amplitude and a corresponding Wilson loop were obtained almost at the same time. This allowed to be compared immediately and check what kind of new information they carry. To appreciate these results let us go through this systematically. We saw that anomalous Ward identities govern a form of Wilson loops at weak coupling. Moreover, the finite part of MHV scattering amplitudes also satisfies the identities [22]. These two facts were used to constrain a finite form for six gluons in both approaches (Wilson loops and MHVs). From the dilatation Ward identity we know that 6-point functions have to depend on so-called invariant cross ratios. They are defined as invariant (under $SO(2,4)$) combinations of y_i^μ. For six gluons (or six cusps) there are three of them:

$$
\begin{aligned}
u_1 &= \frac{y_{13}^2 y_{46}^2}{y_{14}^2 y_{36}^2} \\
u_2 &= \frac{y_{24}^2 y_{51}^2}{y_{25}^2 y_{41}^2} \\
u_3 &= \frac{y_{35}^2 y_{62}^2}{y_{36}^2 y_{52}^2}
\end{aligned}
\tag{10.3}
$$

The approach of two teams which were trying to compute a Wilson loop and MHV was to achieve the following structures:

$$F_6^{(MHV)} = F_6^{(BDS)} + R^{(MHV)}(u_1, u_2, u_3), \tag{10.4}$$

for a six gluon MHV, and

$$F_6^{(Wil)} = F_6^{(BDS)} + R_6^{(Wil)}(u_1, u_2, u_3), \tag{10.5}$$

for a six-cusp Wilson loop. Functions R denote "finite remainders", pieces that differ from BDS and do not vanish when $\epsilon \to 0$. If the BDS ansatz was right (for 6 gluons), both remainders should vanish.
The "MHV team" [52] computed an even contribution[2] to the two-loop planar MHV amplitude using unitarity methods for dual conformal integrals [3]. Here, we are not able to present their computation in detail but we can mention that there were 26 contributing box integrals which at the end had to be evaluated numerically with some assumptions for gluon momenta. In table 10.1 we present their numerical results for the remainder function evaluated at two loops for five configurations of momenta (so-called kinematical points, see appendix F). As we see the function is nonvanishing and BDS indeed fails already at six gluons. As we can expect, the 6-cusp Wilson loop did not bring good news for BDS neither. The group of Korchemsky [14] evaluated a two-loop contribution to

$$\ln\langle W[C_6]\rangle = \frac{g^2}{4\pi^2} C_F \omega^{(1)} + \left(\frac{g^2}{4\pi^2}\right)^2 C_F N \omega^{(2)} + O(g^2) \tag{10.6}$$

[2] The six-point amplitude consists of two kinds of terms. The first group contains odd powers of the Levi-Civita tensor contracted with the external momenta. They flip sign under the parity transformation which reverses helicities of gluons. These terms are of order ϵ and do not contribute to $R^{(MHV)}$. The remaining pieces are classified as even.

kinematic point	(u_1, u_2, u_3)	$R_6^{(2)}$
$K^{(0)}$	(1/4,1/4,1/4)	1.0937±0.0057
$K^{(1)}$	(1/4,1/4,1/4)	1.076±0.022
$K^{(2)}$	(0.547253,0.203822,0.881270)	-1.659±0.014
$K^{(3)}$	(28/17,16/5,112/85)	-3.6508±0.0032
$K^{(4)}$	(1/9,1/9,1/9)	5.21±0.10
$K^{(5)}$	(4/81,4/81,4/81)	11.09±0.50

Table 10.1: The remainder function for 6 gluon MHV amplitude.

One loop part ω^1 was already studied in [42] and we saw it in chapter 8. Unfortunately the two loop contribution ω^2 turned out to be much more difficult. Finally they also evaluated $R_6^{(Wil)}$ numerically for the same kinematical points. Their results are presented in table 10.2 which compares them with MHV. We can clearly see not only a breakdown of BDS but also serious evidence that their conjecture about the duality between Wilson loops and MHVs at weak coupling might be true.

kinematic point	$R_6^{Wil} - R_6^{Wil}(K^{(0)})$	$R_6^{MHV} - R_6^{MHV}(K^{(0)})$
$K^{(1)}$	$\leq 10^{-5}$	-0.018±0.23
$K^{(2)}$	-2.75533	2.753±0.015
$K^{(3)}$	-4.74460	-4.7445±0.0075
$K^{(4)}$	4.09138	4.12±0.10
$K^{(5)}$	9.72553	10.00±0.50

Table 10.2: Comparison of the remainder function for 6-point MHV and 6-cusp Wilson loop. At every kinematical point a value of $R(K^{(0)})$ is subtracted.

10.3 Minimal surface for six gluons

The ideal end of the six gluon story would be to find a minimal surface attached to light-like segments in Fig. 10.2. Unfortunately this did not appear to be an easy task. In our research project we tried to construct a surface with such a boundary using many possible ansatzes and guesses but none of them have (yet) turned out to be the right one. It is not worth presenting all of our attempts here, however, if after reading this thesis one feels inspired enough to try to solve the remaining puzzle, we will give some

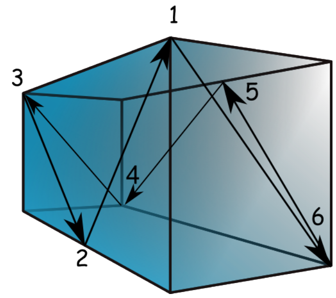

Figure 10.2: *Time-like contour for six gluon amplitude*

general hints and show where not to go. First of all there is no proof of the Plateau problem in AdS_5 [3]. This should not bother us too much since we have already found one for 4 cusps. On the other hand, the existence of the minimal surface we are looking for would not change the fact that we need its explicit and analytical form. Assuming that it exists we would like to be sure that the surface we will eventually find is unique. Naturally it has to satisfy the Euler-Lagrange equation for either the Nambu-Goto or Polyakov action. This is not all. It also has to be smooth everywhere except at the cusps ("bulk" and interior). As an example, we found a surface (it was also found in [4]) that satisfies the equation of motion but is not smooth where it should be. It can be simply constructed by gluing two four-cusp solutions together (see Fig. 10.3 and Fig. 10.4).

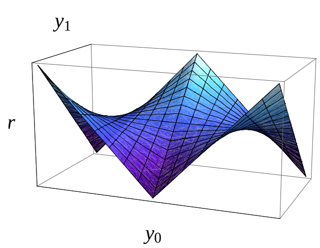

Figure 10.3: *Minimal surface on the light-like six contour from "gluing" two four-cusp solutions.*

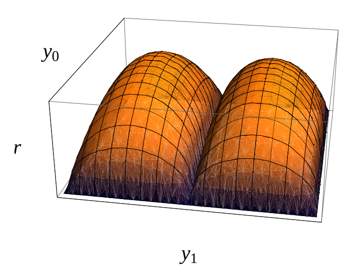

Figure 10.4: *Glued solutions for 4 cusps and the six-segment contour.*

Though it is smooth in the "bulk" there is an additional gluing line which

[3]The existence of the minimal surface for given boundary.

spoils the solution. Now if we wanted to evaluate its area, it would be simply a sum of two 4-cusp solutions, whereas we would like to have a function of the three invariant cross-ratios (to compare with BDS, Wilson loops etc.). There are many similar interesting examples but we will leave them to the interested reader for exploring on his/her own.
Summarising, the weak coupling results shed more light on relations between Wilson loops and MHV amplitudes. It is very likely that there exists a duality between these two objects (at least in planar theory with a large number of colours). Simultaneously, the BDS ansatz turned out to be wrong already for six gluons and we should reveal its construction again.
That was the last part of the review and though there is so much more to do and to write about, we will just try to conclude and sketch what else might be done and how.

11

Conclusions and open questions

We tried to lead the reader through the shortest possible path to one of the frontiers of research in the AdS/CFT correspondence: computation of gluon scattering amplitudes at strong coupling. Whether this has been achieved in a clear and understandable way or not can only be decided by the reader. Nevertheless, we can conclude and briefly summarise wha we have done and what still needs to be done.

Staring almost from scratch, we presented necessary background information to study gluon scattering amplitudes in planar $\mathcal{N}$=4 SYM from weak to strong coupling. Most of the results presented in this thesis (especially those at strong coupling) are not older than one year and they are still in they evolutionary stage. Therefore it was important to clarify the ideas and motivations which led to them. We tried to be as explicit as possible but on the other hand we did not want to make the reader drown in the cumbersome details. According to us this approach will later allow the reader to easily embark on research in strong or weak coupling gluon scattering amplitudes. In addition, some basic and easy to follow computations were presented. We hope that studying them will give insights into the logic being used in more advanced calculations in the literature.

To conclude about the physics content of this thesis, we started from the BDS proposal for colour-ordered, planar MHV amplitudes in the $\mathcal{N}$=4 SYM theory. Its construction allowed us to write it at strong coupling as well. This linked gluon scattering amplitudes to string theory through the AdS/CFT

correspondence. The computation of Alday and Maldacena based on evaluation of a Wilson loop at strong coupling, confirmed not only the BDS ansatz for four gluons but also predictions for strong coupling behaviour of the anomalous dimensions. Then the Wilson loop along the four-cusp light-like contour was evaluated at weak coupling, and surprisingly turned out to be equal to corresponding MHV amplitude. This led to a proposal for a duality between Wilson loops and MHV amplitudes at weak coupling. After more precise studies people realised that the dual conformal symmetry completely fixes forms of 4 and 5 point amplitudes and Wilson loops. The results for more are urgently needed. In the meantime, Alday and Maldacena pointed out that according to their approach of computing minimal surfaces in AdS, BDS must fail for a large number of gluons. This turned out to be the case already at $n = 6$. The results of the two groups working on 6-cusp Wilson loops and 6-gluon MHVs confirmed failure of the BDS ansatz. Nevertheless the duality between the two weak coupling approaches survived.
Finally, there are many open questions which can become interesting paths of research. We will list them and comment on possible solutions.

11.1 Open questions

- The minimal surface.
  The problem of finding an explicit solution for the minimal surface in AdS for six light-like segments remains unsolved . The only possilbe way to solve it: just find the solution.

- The BDS ansatz and loop equations.
  Since we agreed that BDS is wrong, what is the way to correct it? There is no too much progress in the literature however; since the duality between MHVs and Wilson loops seems to be there, the loop equations might give some interesting relations similar to the iterative one. We tried to pursue this topic as well but the main difficulty was related with an application of the Makeenko-Migdal equation into the light-like Wilson loop. Clarifying these issues will definitely shed more light on the subject.

- T-duality.
  One could repeat the T-duality applied to a string amplitude for superstrings at weak coupling. If there exists the duality between dual Wilson loops and scattering amplitudes, we should be able to see it

from string theory as well.

- Strings and MHV.
  Finally, the computation of Alday and Maldacena was "blind" for gluons' helicities (so important for MHV). It would be good to have a strong coupling computation which explicitly contains these structures.

Part IV

Appendices

A
The Anti-de Sitter space

Anti-de Sitter (acronym AdS) space is a solution of the Einstein's equations in a vacuum cosmological constant Λ

$$R_{\mu\nu} = \left(\frac{1}{2}R + \Lambda\right) g_{\mu\nu} \tag{A.1}$$

for maximally symmetric space-time (highest possible number of Killing vectors) and attractive (negative) cosmological constant, the solution describes a hyperbolic space. This can be analysed in the context of homogeneous spaces defined by quadrics in a vector space. Well known example is (n)-sphere embedded in Euclidean $\mathbb{R}^{n+1}$:

$$X_0^2 + \cdots + X_n^2 = R^2 \tag{A.2}$$

positive curvature of a sphere is proportional to the inverse of R^2. In general quadric equations, R sets the natural length scale in corresponding spaces. In this way, we can construct n-dimensional **Anti-de Sitter** space, as quadric with a negative curvature

$$X_0^2 + X_{n+1}^2 - \sum_{i=1}^{n} X_i^2 = -R^2 \tag{A.3}$$

Embedded in flat manifold with metric:

$$ds^2 = -dX_0^2 - dX_{n+1}^2 + \sum_{i=1}^{n} X_i^2 \tag{A.4}$$

By the construction, AdS is homogeneous and isotropic space with $SO(2, n-1)$ isometry (Poincare transformations in embedding space).

A.1 Coordinates

As any geometrical object AdS can be studied in any convenient coordinates. However, few of them are of special interest for physical purposes.

A.1.1 Global coordinates

We can easily check that quadric condition (A.3) is satisfied when we substitute

$$X_0 = R\ cosh\rho\ cos\tau \qquad X_{n+1} = R\ cosh\rho\ sin\tau \qquad X_i = R\ sinh\rho\ \Omega_i \quad \text{(A.5)}$$

where Ω_i ale the standard coordinates on S^{n-1} and $\sum_{i=1}^{n} \Omega_i^2 = 1$.
Coordinates ρ, τ and Ω_i give rise to induced metric on AdS

$$ds^2 = R^2 \left(-\ cosh^2\rho\ d\tau^2 + d\rho^2 + sinh^2\rho\ d\Omega_{n-1}^2 \right) \quad \text{(A.6)}$$

For the limit $\rho \to 0$ we have approximately

$$ds^2 = R^2(-d\tau^2 + d\rho^2 + \rho^2 d{\Omega_{n-1}}^2) \quad \text{(A.7)}$$

what makes transparent that $S^1 \times \mathbb{R}^{n-1}$ is the topology of AdS. One more

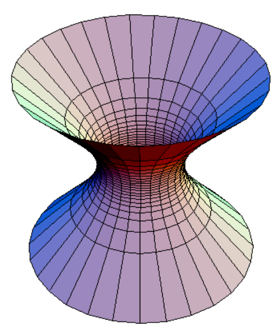

Figure A.1: AdS_n has the topology of $S^1 \times \mathbb{R}^{n-1}$. Picture shows AdS_{1+1} embedded in 2+1 dimensions

important point is, that (A.6) contain closed time-like curves around the "waist". Naturally, we want to avoid this fact in physics and in most of the literature, take the universal cover of anti-de Sitter space to cure this problem. In the language of the parameters this means ranging τ from $-\infty$ to $+\infty$.

A.1.2 Poincare coordinates

Poincare coordinates z, $\overrightarrow{x}$, t visualize AdS as a half plane. They ale related to global via:

$$\begin{aligned} X_0 &= \frac{1}{2z}\left(z^2 + R^2 + \overrightarrow{x}^2 - t^2\right) \\ X_i &= \frac{Rx^i}{z} \\ X_n &= -\frac{1}{2z}\left(z^2 - R^2 + \overrightarrow{x}^2 - t^2\right) \\ X_{n+1} &= \frac{Rt}{z} \end{aligned} \tag{A.8}$$

$\overrightarrow{x}$ contains $n-1$ components and $z \in \langle 0, \infty)$ hence the metric induced on half plane has a boundary at $z = 0$:

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 - dt^2 + d\overrightarrow{x}^2\right) \tag{A.9}$$

We can clearly see not only the Poincare symmetry ($x'^{\mu} \to \lambda^{\mu}_{\nu}x^{\nu} + a^{\mu}$) but also SO(1,1) scale invariance under $(z, t, \overrightarrow{x}) \to (\lambda z, \lambda t, \lambda\overrightarrow{x})$. Therefore, many papers about physics in AdS refers to modified Poincare coordinates where $z \to \frac{R^2}{r}$ which brings the metric to the form:

$$ds^2 = \frac{R^2 dr^2}{r^2} + \frac{R^2}{r^2}\left(-dt^2 + d\overrightarrow{x}^2\right) \tag{A.10}$$

The second line emerged after rescaling the parameters by R.
After T-duality in the text we did use the relation of dual Poincare coordinates to the dual embedding ones

$$Y^i = \frac{y^i}{r} \qquad , Y_{-1} + Y_4 = \frac{1}{r}, \qquad Y_{-1} - Y_4 = \frac{r^2 - y_0^2 + y_i y^i}{r} \tag{A.11}$$

B

Explicit BDS ansatz for 6 and 8 gluons

In order to see how the BDS formula works, we can write it explicitly for 6 and 8 gluon amplitudes. If will be important to notice, what are the variables in this expansion. The area of the proper (unique) minimal surface also has be the function of all of them.

B.1 BDS for 6 gluons

$$F_6^{BDS} = \frac{1}{2}\Gamma_{cusp}(a)\sum_{i=1}^{6}\left[-\ln\left(\frac{x^2_{i,i+2}}{x^2_{i,i+3}}\right)\ln\left(\frac{x^2_{i+1,i+3}}{x^2_{i,i+3}}\right)+\right.$$
$$\left.+\frac{1}{4}\ln^2\left(\frac{x^2_{i,i+3}}{x^2_{i+1,i+4}}\right)-\frac{3}{2}Li_2\left(1-\frac{x^2_{i,i+2}x^2_{i-1,i+3}}{x^2_{i,i+3}x^2_{i-1,i+2}}\right)\right] \tag{B.1}$$

Explicitly:

$$F_6^{BDS} = \frac{1}{2}\Gamma_{cusp}(a)\,(1+2+3) \tag{B.2}$$

$$1 = \sum_{i=1}^{6} \left[-\ln\left(\frac{x_{i,i+2}^2}{x_{i,i+3}^2}\right) \ln\left(\frac{x_{i+1,i+3}^2}{x_{i,i+3}^2}\right) \right] = -\left[\ln\left(\frac{x_{1,3}^2}{x_{1,4}^2}\right) \ln\left(\frac{x_{2,4}^2}{x_{1,4}^2}\right) + \ln\left(\frac{x_{2,4}^2}{x_{2,5}^2}\right) \ln\left(\frac{x_{3,5}^2}{x_{2,5}^2}\right) + \ln\left(\frac{x_{3,4}^2}{x_{3,6}^2}\right) \ln\left(\frac{x_{4,6}^2}{x_{3,6}^2}\right) + \ln\left(\frac{x_{4,6}^2}{x_{4,1}^2}\right) \ln\left(\frac{x_{5,1}^2}{x_{4,1}^2}\right) + \ln\left(\frac{x_{5,1}^2}{x_{5,2}^2}\right) \ln\left(\frac{x_{6,2}^2}{x_{5,2}^2}\right) + \ln\left(\frac{x_{6,2}^2}{x_{6,3}^2}\right) \ln\left(\frac{x_{1,3}^2}{x_{6,3}^2}\right) \right] \tag{B.3}$$

$$2 = \sum_{i=1}^{6} \frac{1}{4} \ln^2\left(\frac{x_{i,i+3}^2}{x_{i+1,i+4}^2}\right) = \frac{1}{4} \left[\ln^2\left(\frac{x_{1,4}^2}{x_{2,5}^2}\right) + \ln^2\left(\frac{x_{2,5}^2}{x_{2,6}^2}\right) + \ln^2\left(\frac{x_{3,6}^2}{x_{4,1}^2}\right) + \ln^2\left(\frac{x_{4,1}^2}{x_{5,3}^2}\right) + \ln^2\left(\frac{x_{5,2}^2}{x_{6,3}^2}\right) + \ln^2\left(\frac{x_{6,3}^2}{x_{1,4}^2}\right) \right] \tag{B.4}$$

$$3 = -\frac{3}{2} \sum_{i=1}^{6} Li_2\left(1 - \frac{x_{i,i+2}^2 x_{i-1,i+3}^2}{x_{i,i+3}^2 x_{i-1,i+2}^2}\right) = -\frac{3}{2} \left[Li_2\left(1 - \frac{x_{1,3}^2 x_{6,4}^2}{x_{1,4}^2 x_{6,3}^2}\right) + Li_2\left(1 - \frac{x_{2,4}^2 x_{1,5}^2}{x_{2,5}^2 x_{1,4}^2}\right) + Li_2\left(1 - \frac{x_{3,5}^2 x_{2,6}^2}{x_{3,6}^2 x_{2,5}^2}\right) + Li_2\left(1 - \frac{x_{4,6}^2 x_{3,1}^2}{x_{4,1}^2 x_{3,6}^2}\right) + Li_2\left(1 - \frac{x_{5,1}^2 x_{4,2}^2}{x_{5,2}^2 x_{4,1}^2}\right) + Li_2\left(1 - \frac{x_{6,2}^2 x_{5,3}^2}{x_{6,3}^2 x_{5,2}^2}\right) \right] \tag{B.5}$$

B.2 BDS for 8 gluons

$$F_8 = \frac{1}{2} \sum_{i=1}^{8} g_{8,i} = (1 + 2 + 3) \tag{B.6}$$

$$g_{8,i} \equiv -\sum_{r=2}^{3} \ln\left(\frac{x_{i,i+r}^2}{x_{i,i+r+1}^2}\right) \ln\left(\frac{x_{i+1,i+r+1}^2}{x_{i,i+r+1}^2}\right) + D_{8,i} + L_{8,i} + \frac{3}{2}\zeta_2 \tag{B.7}$$

$$1 = -\sum_{i=1}^{8} \left[\ln\left(\frac{x_{i,i+2}^2}{x_{i,i+3}^2}\right) \ln\left(\frac{x_{i+1,i+3}^2}{x_{i,i+3}^2}\right) + \ln\left(\frac{x_{i,i+3}^2}{x_{i,i+4}^2}\right) \ln\left(\frac{x_{i+1,i+4}^2}{x_{i,i+4}^2}\right) \right] \tag{B.8}$$

$$2 = \sum_{i=1}^{8} \frac{1}{4} \ln^2\left(\frac{x_{i,i+3}^2}{x_{i+1,i+4}^2}\right) \tag{B.9}$$

B.2 BDS for 8 gluons

$$3 = -\frac{3}{2}\sum_{i=1}^{8} Li_2\left(1 - \frac{x_{i,i+2}^2 x_{i-1,i+3}^2}{x_{i,i+3}^2 x_{i-1,i+2}^2}\right) \tag{B.10}$$

C

Polylog's and all that..

Euler's identity

$$Li_2(z) = -Li_2(1-z) - \log(z)\log(1-z) + \frac{\pi^2}{6} \tag{C.1}$$

Handy relations

$$Li_2(0) = 0, \qquad Li_2(1) = \zeta_2 = \frac{\pi^2}{6}, \qquad Li_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \tag{C.2}$$

$$Li_2(1+z) + Li_2(1+x^{-1}) = -\frac{1}{2}\ln^2(-z) \tag{C.3}$$

Euler's Γ function

$$\Gamma(n+1) = \int_0^{\infty} t^n e^{-t} dt \tag{C.4}$$

it satisfies $\Gamma(n+1) = n\Gamma(n) = n!$ and $\Gamma(1/2) = \sqrt{\pi}$.
Euler's β function

$$\beta(\mu,\nu) = \int_0^1 x^{\mu-1}(1-x)^{\nu-1} dx = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \tag{C.5}$$

Euler's hypergeometric function can be represented as an integral for $\Re c > \Re b > 0$

$$ {}_2\mathrm{F}_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt \qquad \text{(C.6)} $$

in the paper [1] it was expanded for small $\epsilon < 0$ as

$$ {}_2\mathrm{F}_1(\frac{1}{2}, -\frac{\epsilon}{2}; \frac{1-\epsilon}{2}; b^2) = 1 + \frac{1}{2}\ln(1-b^2)\epsilon + \frac{1}{2}\ln(1-b)\ln(1+b)\epsilon^2 + o(\epsilon^3) \quad \text{(C.7)} $$

D

Wilson loops and interaction potentials

To understand how Wilson loops work we can try to find a simple Coulomb's potential between two charges excited along the loop C. Taking a rectangular C and computing only the depicted propagator[1] we can recover for $T >> R$

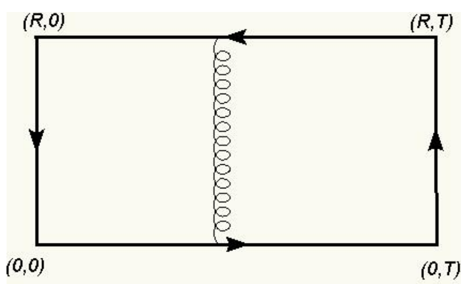

Figure D.1: *From a rectangular loop we can obtain information about potential between two charges*

$$< W[C] >= e^{-V(R)T} \qquad \ln < W[C] >= -V(R)T \tag{D.1}$$

[1]other gives the divergent terms

where $V(R)$ is the potential. Taking the standard, free Yang-Mills theory with a photon propagator

$$G_{\mu\nu}(x-y) = \frac{1}{4\pi^2}\frac{\delta_{\mu\nu}}{(x-y)^2} \tag{D.2}$$

We use the formula (4.4) with coupling constant e and parametrising upper bar by $z(s) = (R, Ts)$ and the lower $z(t) = (0, Ts)$, where $0 \geq s, t \leq 1$. What remains to compute, is the integral

$$\begin{aligned} &\frac{e^2}{2}\oint_C dx^\mu \oint_C dy^\nu G_{\mu\nu}(x-y) \\ = \frac{e^2}{4\pi^2}\int_0^1 dsdt \frac{\dot{z}(s)\cdot\dot{z}(t)}{[z(s)-z(t)]^2} &= \frac{e^2}{4\pi^2}\int_0^1 dsdt \frac{1}{(s-t)^2+(\frac{R}{T})^2} \end{aligned} \tag{D.3}$$

substitution $u = \frac{1}{2}(s+t)$ and $v = s+t$ which gives $dsdt = dudv$ and reduce it to

$$\frac{2e^2}{4\pi^2}\int_0^1 du \int_0^1 \frac{1}{v^2+(\frac{R}{T})^2} = \frac{2e^2}{4\pi^2}\frac{T}{R}\arctan(\frac{T}{R}) \tag{D.4}$$

The assumption $T >> R$ means we need a value of $x\arctan(x)$ for very large x. This can be easily checked that the correct form is $x\arctan(x) \approx x\frac{\pi}{2}$, hence we obtained the Coulomb's potential between two charges

$$V(R)T = \frac{e^2}{4\pi R}T \tag{D.5}$$

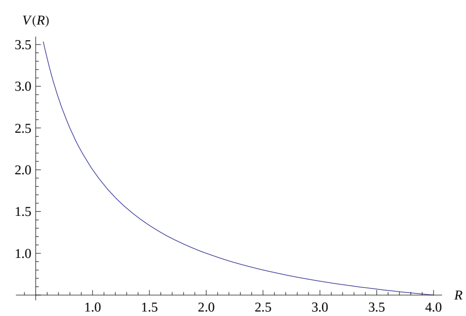

Figure D.2: Coulomb's potential from rectangular Wilson loop

E
Equations of motion in AdS

The equations of motion for the Nambu-Goto action

$$\partial_1 \left(\frac{\partial_1 r(1-(\partial_2 y_0)^2) + \partial_2 r \partial_1 y_0 \partial_2 y_0}{r^2 \sqrt{()}} \right)$$
$$+\partial_2 \left(\frac{\partial_2 r(1-(\partial_1 y_0)^2) + \partial_1 r \partial_1 y_0 \partial_2 y_0}{r^2 \sqrt{()}} \right) = -\frac{2\sqrt{()}}{r^3} \tag{E.1}$$

$$\partial_1 \left(\frac{-\partial_1 y_0(1+(\partial_2 r)^2) + \partial_2 r \partial_1 r \partial_2 y_0}{r^2 \sqrt{()}} \right)$$
$$+\partial_2 \left(\frac{-\partial_2 y_0(1+(\partial_1 r)^2) + \partial_2 r \partial_1 r \partial_1 y_0}{r^2 \sqrt{()}} \right) = 0 \tag{E.2}$$

where

$$\sqrt{()} \equiv \sqrt{1+(\partial_1 r)^2+(\partial_2 r)^2-(\partial_1 y_0)^2-(\partial_1 y_0)^2-(\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2} \tag{E.3}$$

For the dimensional regularisation, the equations has to be solved for the action

$$S \sim \int \frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon} \tag{E.4}$$

this modify the previous case to

$$\partial_1\left(\frac{\partial_1 r(1-(\partial_2 y_0)^2)+\partial_2 r\partial_1 y_0\partial_2 y_0}{r^{2+\epsilon}\sqrt{()}}\right)$$
$$+\partial_2\left(\frac{\partial_2 r(1-(\partial_1 y_0)^2)+\partial_1 r\partial_1 y_0\partial_2 y_0}{r^{2+\epsilon}\sqrt{()}}\right)=-\frac{(2+\epsilon)\sqrt{()}}{r^{3+\epsilon}} \quad \text{(E.5)}$$

$$\partial_1\left(\frac{-\partial_1 y_0(1+(\partial_2 r)^2)+\partial_2 r\partial_1 r\partial_2 y_0}{r^{2+\epsilon}\sqrt{()}}\right)$$
$$+\partial_2\left(\frac{-\partial_2 y_0(1+(\partial_1 r)^2)+\partial_2 r\partial_1 r\partial_1 y_0}{r^{2+\epsilon}\sqrt{()}}\right)=0 \quad \text{(E.6)}$$

F Kinematical points

In chapter 10 we used special configurations of momenta for which the reminder function in [52] was evaluated. They are called kinematical points and are defined below.
For six momenta we have nine dual distances between vertices on the hexagonal Wilson loop (or nine points in dual parametrisation of box integrals):

$$K = \{y_{13}^2, y_{14}^2, y_{15}^2, y_{24}^2, y_{25}^2, y_{35}^2, y_{26}^2, y_{36}^2, y_{46}^2\} \tag{F.1}$$

From the conservation of momenta there are only 8 independent left. Naturally, in four dimensions, there are at most four linearly independent momenta. Therefore, the Gram determinant of any five out of six momenta vanish

$$det(k_i \cdot k_j) = 0, \qquad i, j = 1, \ldots, 5 \tag{F.2}$$

These are the constraints for momenta in a scattering process of six particles (this can be naturally generalised to any number). Now the kinematical point is a configuration of momenta which is either compatible with Gram constraint or not. In chapter 10 we showed only six points where four satisfied Gram constraint and two $K^{(4)}$ and $K^{(5)}$ not[1]. They are:

$$
\begin{aligned}
K^{(0)} &: \quad s_{i,i+1} = -1, \quad s_{i,i+1,i+2} = -2 \\
K^{(3)} &: \quad s_{i,i+1} = -1, \quad s_{123} = -1/2, \quad s_{234} = -5/8, \quad s_{345} = -17/14 \\
K^{(4)} &: \quad s_{i,i+1} = -1, \quad s_{i,i+1,i+2} = -3 \\
K^{(5)} &: \quad s_{i,i+1} = -1, \quad s_{i,i+1,i+2} = -9/2,
\end{aligned}
\tag{F.3}
$$

where generalised Mandelstam variables are

$$
s_{i\dots j} = (k_i, \dots, k_j)^2, \qquad ij = mod(n). \tag{F.4}
$$

[1]notation like in the article

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