AN INTRODUCTION TO

DEFORMATION QUANTIZATION

AFTER KONTSEVICH

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Summary

Deformation quantization is a way to quantize a classical mechanical system described by a Poisson manifolds and characterized by its Poisson structure. In contrast to most methods of quantization, where a Hilbert space of states with operators acting on it is somehow constructed, deformation quantization allows for the definition of a non-commutative star product directly on the classical phase space. This product then allows for the description of quantum behaviour in algebraic terms.

As an introduction to the subject, the simplest star product is first considered. It is called the Moyal product and was already known of in the 1940s as it naturally emerged in the Weyl-Wigner correspondence of phase space distributions. When quantizing a classical theory, there is a certain freedom of ordering of non-commutative operators. Two such orderings are discussed. Moreover, the Moyal product is seen to correspond to the star product induced by a symmetric ordering.

After this, some symplectic theory is introduced as a natural framework for classical mechanics. In particular, Hamilton's equations of motion are cast in symplectic language. Anticipating a more general theorem, every symplectic manifold is shown to carry a natural Poisson structure.

The concept of differential graded Lie algebra's is defined and two examples are considered. The Formality theorem, due to Maxim Kontsevich, is mathematical results formulated in terms of these objects. It is interpreted, and leads to a one-one correspondence between formal deformations of null Poisson structures and equivalence classes of star products. As for its application in physics, it shows that every classical mechanical system can essentially be quantized uniquely.

Finally, an explicit formula for the construction of a star product is given. It is in terms of a certain class of graphs, reminiscent of the calculation of transition amplitudes in field theory using Feynman diagrams. This formula is applied to the constant Poisson structure. The corresponding star product is found to be the Moyal product.

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Information

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1 Introduction

In the beginning of the twentieth century, Max Planck theorized that electromagnetic energy could only be emitted in quantized form. For this, he introduced the action quantum h now known as Planck's constant. This deep assumption allowed Planck to explain the ultraviolet catastrophe, one of the big open problems of that time. A few years later, Albert Einstein used Planck's postulate to explain the photo-electric effect. The following twenty years led to the birth of quantum mechanics through the works of most notably Niels Bohr, Werner Heisenberg and Erwin Schrödinger. Quantum mechanics gave a remarkably accurate theoretical description of the spectrum of the hydrogen atom [1].

Since it is at the basis of these three major breakthroughs[1], it is clear that Planck's postulate was not some sort of trick, but an expression of a more fundamental theory. The question what quantization actually entails, or how it should be done is therefore important. One could wonder why quantization is necessary and if there is some sort of deeper principle underlying it. Also, one could imagine different methods of quantization leading to different theories.

In this thesis, the first two questions will nót be pursued. For the necessity of quantization, the reader will have to content himself with the very accurate empirical proof supporting the quantum mechanical explanation of the above three major problems (again, see [1]); as for the deeper principle, nature apparently *is* quantized. Instead, one of the methods of quantization is studied in detail: quantization by deformation.

Quantization

A quantization is a recipe for associating a quantum system to any given classical system in a consistent and reasonable manner. To make this recipe more clear, recall that quantum mechanics is best described by self-adjoint operators acting on states in a Hilbert space [2]. However, a fundamental quality of quantum mechanics that sets it apart from its classical counterpart is the non-commutativity of certain of these operators. The most obvious example of this statement is the position operator and momentum operator. Position and momentum cannot be simultaneously measured, as is confirmed by experiments. This is a very odd thing in comparison to the classical analogs of position and momentum, which are commutative functions on classical phase space [2]. This behaviour should be captured by any quantization.

However, it appears to be difficult to simply construct a Hilbert space suitable to describe a given (quantum) system. The usual way to do quantum mechanics (in flat space) is by starting with a classical system, and *quantizing* it: promoting smooth functions on the classical system to operators and imposing the so-called *canonical commutation relations* (see for instance [3]). This method is called *canonical quantization* and it is the one found in text books. Although this method of quantization perfectly allowes one to do quantum mechanics in the sense that predicted results agree well with experiments, it does not say anything about the foundational aspects of quantization.

Now, normally one demands of a quantum theory to become classical when the quantum constant $\hbar$ tends to zero, that is: when the parameters of the studied phenomenon are such that quantum effects can be ignored. This requirement seems reasonable. It is called the *correspondence principle*, and we want a quantization to satisfy this principle. Moreover, it leads to an important remark: the quantum description of a physical phenomenon contains more information than the classical one. This means that there are certain effects that only appear at the quantum level and in the quantum description, nót in the classical theory [3]. An example is the phase transition

[1] For more information on the history and development of quantum mechanics, see further volumes of the series [1].

known as Bose-Einstein condensation, which is due to the quantum statistics of bosons; classical theories are not capable of describing this effect (see §7.6 of [4]).

Since a quantum description of a phenomenon is more accurate than the classical one, there could be different quantizations satisfying the correspondence principle: different quantum systems reducing to the same classical system when $\hbar$ tends to zero. Hence, there could be different methods of quantization possible. This means that a choice has to be made, and it has to be motivated. In the ideal case, however, one method of quantization is clearly superior to others and yields a unique quantization.

To summarize: quantization is a recipe for associating a quantum system to every classical system, such that non-commutative behaviour is captured, the canonical commutation relations hold[2] and the correspondence principle is satisfied.

Methods of quantization

There is a long history of studying methods of quantization, the roots of which can be traced back to the work of Dirac and Weyl in the early twenties of the past century. Dirac noted a resemblance between the classical Poisson bracket and the quantum commutator. He proposed to introduce a product on classical phase space that captures this resemblance; this can be considered the foundation of so-called *star products* that form the cornerstone of quantization by deformation [6]. Weyl, on the other hand, was in search of an alternative formulation of quantum mechanics. He proposed a correspondence between classical phase space distribution functions and quantum mechanical operators. Also in this context, a star product is seen to emerge [7].

Another notable effort is functorial quantization. In this method, the idea is to construct a covariant functor that assigns to each classical system (symplectic manifold) a quantum system (Hilbert space). It turns out, however, that there does not exist a functor that yields a full quantization as defined above. This is meant in the sense that in almost all imaginable classical systems, not all classical observables can be given operator equivalents in a physically satisfying manner. Equivalently: not all principles mentioned in the above definition of quantization can be consistently incorporated in such a functor. See §4 of [8] for more details.

In the influential paper [9], published in 1978, it was proposed and motivated to look for a quantization by means of deforming an already present structure. Since quantum mechanics can be completely described in algebraic terms, in the article it was suggested to combine this algebraic description with Dirac's earlier mentioned intuition. This yielded the idea of deforming the commutative algebra of smooth functions on a classical system (its phase space) into a non-commutative one, thus effectively introducing a star product. This deformed algebra is then capable of capturing the odd quantum behaviour of non-commuting operators.

For an overview of the birth and development of this method of quantization, see [10].

Why deformation?

Quantization by deformation, or simply *deformation quantization*, is the method of quantization studied in this thesis. That this study is relevant can be seen by the following two arguments:

The first argument is Kontsevich's proof of his Formality conjecture in 1997 [11]. A corollary of this conjecture states that every classical system (described by a Poisson manifold) can be uniquely[3] quantized by deformation. Moreover, his method satisfies the correspondence principle

[2] As an added benefit of the canonical commutation relations, the Stone-von Neumann theorem - a certain statement about uniqueness - is seen to hold (see [5]).

[3] Uniquely up to a certain equivalence.

ánd Dirac's mentioned intuition. In a sense, Kontsevich has shown that deformation is a viable approach to the problem of quantization: he has shown existence and uniqueness.

A second argument is that the correspondence principle motivates a deformation theoretical approach. It is something that is also observed in other domains of physics. This rough analogy is meant in the following sense. When studying physics, one notices that all processes are governed by a small set of fundamental parameters: the constants of nature. Examples are the speed of light c, Planck's constant h and the gravitational constant G. The domain of physics that is used to solve a particular problem is dependent on the relation between the problem's parameters and these constants of nature. In the case of relativistic particles, such as photons, one works in the domain of Einstein's theory of relativity. When considering objects that move much slower than the speed of light, one works in the domain of Newton's classical mechanics. Letting v denote the object's speed, this suggests that the value of the ratio v/c plays an important role. When this ratio is negligeble, relativity is seen to reduce to classical mechanics. Although Einstein's theory is not built on a deformation theoretical footing, it can be considered as a deformation of classical mechanics in this sense, just as the correspondence principle indicates for quantum mechanics: in the first case space itself is deformed (it is curved), in the latter an algebraic structure is deformed.

These two arguments motivate the relevance and the study of deformation quantization.

Structure of the text

It is the goal of this thesis to develop the machinery to state Kontsevich's Formality theorem, and to give his explicit formula for a star product on an arbitrary Poisson manifold. As for every mathematical theory, there is a simplest example. In this case, it is the Moyal product: the star product that emerges from Weyl's correspondece between classical phase space distributions and quantum operators. Application of Kontsevich's formula to again find this example is our final goal.

In the first section of this thesis, the Moyal product will be discussed. This is done by defining and studying the Weyl-Wigner correspondence. A star product naturaly emerges, which is later seen to correspond to the constant Poisson structure. The Weyl-Wigner correspondence links a commutative to a non-commutative theory. It is therefore a priori not clear that it is well-defined. For this purpose, the question of ordering of quantum operators is considered. When no inconsistencies can be found, two arguments - a physical, and a mathematical one - are given to support the correspondence. It is proven up to equivalence, but a conclusive proof is outside the scope of this text.

In the second section, the theory of symplectic geometry is developed. Definitions and examples of symplectic manifolds are given, where the most notable one is the cotangent bundle $M = T^*Q$ of an arbitrary smooth manifold Q. When the latter one is interpreted as configuration space of a system, the first one is its phase space. Then it is shown that symplectic manifolds are a natural framework to describe classical dynamics. In particular, Hamilton's equations of motion are reformulated in a symplectic language. Since Kontsevich's statement is in terms of Poisson manifolds, it is finally shown that every symplectic manifolds carries a natural Poisson structure.

In the third section, the definition of a star product on an algebra is given; this defines deformation quantization. On a general Poisson manifold, it is natural to define the product on the manifold's Poisson algebra $\mathcal{C}^\infty(M)$ of smooth functions $(M, \{\cdot,\cdot\})$. For reference and intuition, the Moyal product should be kept in mind. A group action is then introduced on the set of possible star products, yielding so-called gauge equivalence classes. It is shown that each equivalence class contains a star product in which the Poisson bracket $\{\cdot,\cdot\}$ plays a prominent role. This is done using Hochschild cohomology, which is explained in greater detail in the appendix. Then the notion of differential graded Lie algebra is introduced. It is shown that the Hochschild cocomplex is a first and important example. Finaly, Kontsevich's Formality theorem and its corollary relevant to deformation quantization are stated and explained.

The last section concludes this theis by describing Kontsevich's explicit formula for the star product on an open subset of $\mathbb{R}^d$, hence locally on an arbitrary Poisson manifold. It is applied to the Poisson structure with constant coefficients that corresponds to flat space. The associated star product is again seen to be the Moyal product.

2 The Moyal product

Although the Moyal product was already known of in the 1940s, it was not defined to be a star product in some sort of deformation theory. It rather emerged in the work of Groenewold and Moyal as an alternative formulation of Schrödinger's quantum theory: the so-called phase space picture, where position and momentum are treated on equal footing; notice the resemblance with Hamilton's description of classical mechanics. In this picture, quantum states are described by phase space distributions. These are functions that are quasiprobability distributions, which means that locally they need not be positive semi-definite. For more information, see [7], [12], [13].

Decades later, however, the Moyal product turned out to be the first and most elementary example of a star product. It is the quantization by deformation of a Poisson manifold with a constant Poisson structure. This represents the deformation of a classical commutative theory (the algebra of smooth functions on the Poisson manifold) in the sense that it turns its commutative product into a non-commutative one. This deformed algebra can then be interpreted and used as the space of quantum operators, since it is capable of capturing non-commutative behaviour.

To anticipate on what is to come, let us first give the definition of a deformation in the most general terms of category theory. As its most elementary example of a quantization by deformation, the Moyal product should turn out to satisfy this definition.

Definition 2.1 (Deformation)**.** *Say X is an object in a certain category $\mathcal{C}$. A deformation of X is a family of objects $X_\epsilon \in Obj(\mathcal{C})$ depending on a parameter ϵ such that $X_{\epsilon_0} = X$ for a certain ϵ_0.*

In the case of the Moyal product, the objects will be associative unital algebras and the role of parameter will be fulfilled by $\hbar$. [4]

The general definition of a deformation quantization will be given in section 4. It is this setting in which Kontsevich has solved the problem of existence and uniqueness of a deformation quantization of an arbitrary Poisson manifold.[11] However intricate his solution may be, its principle rests on deforming a product on an algebra, just as in the Moyal case. Similarities can therefore be found. It will be useful to keep the Moyal product in mind when considering the general case. Due to this introductory and relatively simple nature, definitions and basic properties of the Moyal product will first be discussed.

Preliminaries

In this section only the classical configuration space $Q = \mathbb{R}$ is considered. Symplectic geometry (see section 3) tells us that the associated classical phase space is given by the cotangent bundle of Q, which will be denoted by $M = T^*Q$ and can be identified with $\mathbb{R}^2$ in this case. The (classical) state of the system is fully determined by the pair $(q, p) \in M$ of position and momentum. Observable quantities like position, angular momentum and the hamiltonian can now be seen as smooth functions of (q, p). Therefore, observables f are elements of the commutative algebra of smooth function on phase space $\mathcal{C}^\infty(M) = \{f : M \longrightarrow \mathbb{R} \,|\, f \text{ is smooth}\}$; they are interpreted as phase space distributions (see [13]).

2.1 Weyl-Wigner correspondence

In this setting, a way to quantize the classical theory was first introduced by German physicist Hermann Weyl [7], and was later called *Weyl quantization* in his honour. Actually, it shall be shown that Weyl's procedure leads to a complete and alternative formulation of quantum mechanics.

[4] Actually, this is just a simplification of notation: in physics, the parameter is $i\hbar/2$.

The idea is to associate a quantum operator $\hat{W}[f]$ to every phase space distribution function $f \in \mathcal{C}^\infty(M)$, and to do this by a one-one correspondence. The procedure relies heavily on the invertible character of the Fourier transform on a certain class of 'nice' functions.

Definition 2.2. *The Weyl-Wigner correspondence consist of the following steps*

1. *Let $a, b \in \mathbb{R}$, define the Fourier transform of $f \in L^2(\mathbb{R}, \mu)$ as*

$$\tilde{f}(a,b) = \iint dq dp \exp\left[-i(ap + bq)\right] f(q,p). \tag{1}$$

2. *Perform a formal substitution $p \to \hat{p}$, $q \to \hat{q}$.*
3. *Then define*

$$\hat{W}[f](\hat{q},\hat{p}) = \iint \frac{da}{2\pi}\frac{db}{2\pi} \exp\left[i(a\hat{p} + b\hat{q})\right]\tilde{f}(a,b)\,, \tag{2}$$

which is known as the associated Weyl-operator.

The correspondence between the classical phase space distribution function f and the quantum operator $\hat{W}[f]$ is the before mentioned Weyl-Wigner correspondence. There are a few remarks to be made about this definition.

1. For mathematical simplicity, the procedure is only defined for distribution functions $f \in L^2(\mathbb{R})$. Since this space is a Hilbert space, integration theory tells us that the Fourier transform ánd its inverse are well-defined for square-integrable functions.[5] The fact that this definition is not yet satisfactory is seen by the fact that the harmonic oscillator's hamiltonian $H(q,p) = c(q^2 + p^2)$ is not even included; nevertheless, the procedure can be extended to all physical relevant functions[13].
2. It is know that the Fourier transform is a linear automorphism on the so-called Schwartz space, defined as $\mathcal{S}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C}\,|\, f \text{ is smooth}, sup_x\, |x^k f^{(l)}(x)| < \infty\, \forall\, k, l \in \mathbb{N},\, \forall x \in \mathbb{R}\}$ [14].
3. However, although there is no problem with Fourier transforms on $L^2(\mathbb{R}^2)$, it is a priori not clear that the total *procedure* of assigning an operator to a phase space distribution function is well-defined. It links the commutative theory of phase space distribution functions with the non-commutative one of quantum operators. In the latter case, different choices of ordering yields different expressions due to non-commutativity. The point of ordering will be adressed in section 1.4.

It shall presently be made plausible that this procedure also yields a (formal) one-one mapping for less 'nice' functions, such as polynomials in q, p. However, for the moment there are two points that first need addressing. First of all, the structure of the operator space is not yet clear. Expressing the product of two associated operators in terms of the operator of some function of the two phase space distributions will give information about the space's product. This product will be the Moyal product.[6]. Secondly, the aforementioned problem of ordering is dealt with: two natural choices are presented and discussed. Moreover, these orderings lead to two possible star products.

[5] To be precise, there is the following inclusion of dense spaces: the space $\mathcal{C}_c(\mathbb{R})$ of compactly supported continuous functions (even the space of step functions) is dense in $L^2(\mathbb{R})$ (see [15], §4.3). The space of smooth bump functions can be used to uniformly approximate functions in $\mathcal{C}_c^\infty(\mathbb{R})$, the space of compactly supported smooth functions. The smooth bump functions are dense in $\mathcal{C}_c(\mathbb{R})$ (see [16], §13); hence, the space of compactly supported *smooth* functions is dense in $L^2(\mathbb{R})$. Moreover, it is contained in the space of Schwartz-functions $\mathcal{S}(\mathbb{R})$. So, Fourier transform and inverse are well-defined.

[6] This function will turn out to be the star product of the two distributions.

2.2 Introducing the star product

The development in [17] is followed. Let $f, g \in \mathcal{C}^\infty(M)$. The question is to find a function $h \in \mathcal{C}^\infty(M)$ such that $\hat{W}[f]\hat{W}[g] = \hat{W}[h]$. Furthermore, it is desirable to express h in terms of f and g. By definition, the left-hand side is

$$\hat{W}[f]\hat{W}[g] = \int \frac{da}{2\pi}\frac{db}{2\pi} \exp\left[i(a\hat{p} + b\hat{q})\right]\tilde{f}(a,b) \int \frac{da'}{2\pi}\frac{db'}{2\pi} \exp\left[i(a'\hat{p} + b'\hat{q})\right]\tilde{g}(a',b') \,.$$

Mixing the integrals, the exponents can be but together by using a descendent of the Baker-Campbell-Hausdorff formula [18]. It states that for two linear operators A, B that both commute with their commutator $[A, B]$, the following holds

$$\exp A \, \exp B = \exp\left(A + B\right) \, \exp\left[A, B\right]/2 \,. \tag{3}$$

Implementing this equation yields

$$\exp\left[i(a\hat{p} + b\hat{q})\right] \exp\left[i(a'\hat{p} + b'\hat{q})\right] = \exp\left[i((a + a')\hat{p} + (b + b')\hat{q})\right] \exp\left[i\hbar(ab' - ba')/2\right] .$$

Performing the shifts $a \mapsto a - a'$, $b \mapsto b - b'$ yields the requested expression

$$\hat{W}[f]\hat{W}[g] = \int \frac{da}{2\pi}\frac{db}{2\pi} \exp\left[i(a\hat{p} + b\hat{q})\right] \left(\widetilde{f \star_\hbar g}\right)(a,b) \equiv \hat{W}[f \star_\hbar g] \,, \tag{4}$$

where the pseudo star-product $\widetilde{f \star_\hbar g}$ is defined as

$$\left(\widetilde{f \star_\hbar g}\right)(a,b) := \int \frac{da'}{2\pi}\frac{db'}{2\pi} \tilde{f}(a - a', b - b') \exp\left[i\hbar((a - a')b' - (b - b')a')/2\right] \tilde{g}(a',b') \,. \tag{5}$$

The Moyal product or star-product is then finally defined as the ordinary Fourier inverse of (5):

$$\left(f \star_\hbar g\right)(q,p) := \mathcal{F}^{-1}[\widetilde{f \star_\hbar g}] = \int \frac{da}{2\pi}\frac{db}{2\pi} \exp\left[i(ap + bq)\right] \left(\widetilde{f \star_\hbar g}\right)(a,b) \,.$$

Note that the star product depends on the *classical* variables (q, p), and that it defines a smooth function on classical phase space; it is therefore a product on the algebra, so $h \in \mathcal{C}^\infty(M)$, which is easily verified to be non-commutative. To obtain the product's form as found in the papers by Moyal and Groenewold ([12], [13]), it suffices to perform the substitutions $a, a' \mapsto -i\partial/\partial p$, $b, b' \mapsto -i\partial/\partial q$ under the inverse Fourier transform, and to remark that derivatives of smooth functions commute; then

$$\left(f \star_\hbar g\right)(q,p) = f(q,p) \exp\left[\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)\right] g(q,p) = \left(f \cdot g\right)(q,p) + \sum_{n=1}^{\infty} \hbar^n \mathcal{C}_n[f,g](q,p) \tag{6}$$

where $\cdot$ will denote the commutative classical product and where

$$\mathcal{C}_n[f,g](q,p) := \frac{1}{n!}\left(\frac{i}{2}\right)^n f(q,p) \left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)^n g(q,p) \,. \tag{7}$$

There are a few remarks that can be made about this final form of the Moyal product:

1. Clearly $f \star_\hbar g = f \cdot g + \mathcal{O}(\hbar)$.

2. With respect to the Moyal product, $\hat{W}$ is a homomorphism of algebras by (4):

$$\hat{W} : (\mathcal{C}^\infty(M), \star_\hbar) \longrightarrow (\mathcal{C}^\infty(M), \cdot), \quad \text{where } \hat{W}[f \star_\hbar g] = \hat{W}[f] \cdot \hat{W}[g] \,. \tag{8}$$

3. The Moyal commutator is defined, then found to be

$$[f \star_\hbar g] := f \star_\hbar g - g \star_\hbar f = i\hbar\{f, g\} + \mathcal{O}(\hbar^2) \,, \tag{9}$$

   where $\{\cdot, \cdot\}$ denotes the standard Poisson bracket on the Poisson algebra $\mathcal{C}^\infty(M)$. It is interesting to note that apart from the $\mathcal{O}(\hbar^2)$, this is the precise form of Dirac's intuition about quantization mentioned in the introduction.[6]

4. The Moyal bracket is defined to be $\{f, g\}_\hbar := [f \star_\hbar g]/i\hbar = \{f, g\} + \mathcal{O}(\hbar)$. It can be interpreted as Lie bracket on the space of Weyl-operators, although this fact is not proven here.

5. Since $\hat{W}(\cdot)$ is a linear operator, it follows that $\left[\hat{W}(f), \hat{W}(g)\right] = \hat{W}\left([f \star_\hbar g]\right)$. [7]

These remarks allow for a few conclusions regarding the Moyal product.

First of all, since $f \star_\hbar g \to f \cdot g$ and $\{f, g\}_\hbar \to \{f, g\}$ when $\hbar \to 0$, it is seen by the first three remarks that the Moyal product indeed defines a deformation of the Poisson algebra $\mathcal{C}^\infty(\mathbb{R}^2)$ of smooth functions on phase space. Note that the deformed object X is precisely this associative algebra of smooth functions $\mathcal{C}^\infty(\mathbb{R}^2)$, the family of objects is the family of deformed algebras and the formal parameter is indeed $\hbar$; hence, definition 2.1 is satisfied.

Secondly, for the space $L^2(\mathbb{R})$ of square-integrable functions the fourth remark shows that there is a one-one correspondence between the classical product of phase space distribution functions and the (quantum) Moyal product of associated operators. Furthermore, since the notion of commutator is conserved, *we have constructed a complete, alternative formulation of quantum mechanics*, one in terms of distribution functions on phase space. [17] This is because a quantum mechanical system is fully determined by its C^*-algebra of operators. The Stone-von Neumann theorem then gives the unique unitary representation of this algebra that respects canonical quantization: the representation space is the Hilbert space in which we are familiar to do quantum mechanics [5].

Furthermore, let us deem the upcoming arguments in section 2.4 about the Weyl-Wigner correspondence sufficiently plausible, and let us assume that it is also a (formal) bijection for less 'nice' functions, such as polynomials. Then the fourth remark permits us to do ordinary quantum physics as we are used to whilst using this alternative formulation of Weyl. This is again because commutators are preserved: the non-commutative quantum mechanical behaviour is exactly transferred to the classical phase space.

2.3 Ordering

Quantum mechanics is a non-commutative theory in the sense that some quantum operators do not commute [2]. On the other hand, operators in classical mechanics always commute. Since the Weyl-Wigner correspondence links the two, it is not a priori clear that this procedure is well-defined. This is due to the ordering of operators in quantum mechanics. As an example, consider $f \in \mathcal{C}^\infty(M)$ given by $f(q, p) = q \cdot p = p \cdot q$. Its quantum analog, however, is not immediately well-defined: $f(\hat{q}, /hp) = \hat{q}\hat{p} = \hat{p}\hat{q} + i\hbar \neq \hat{p}\hat{q} = f(\hat{q}, \hat{p})$.

[7] This equality actually implies a homomorphism of Lie algebras: the Lie bracket on the left (for the space of quantum operators) is the commutator bracket, on the right the Lie algebra for the phase space distribution functions is $(L^2(\mathbb{R}), \{\cdot, \cdot\}_\hbar)$; for definitions, see section 2.2.

Following [19], two natural choices of ordering will be discussed: the standard (or naive) ordering and the symmetric or Weyl-Moyal ordering. It will be shown that the two can be linked by a bijective linear map. Furthermore, the symmetric ordering will be shown to yield exactly the same product as the Weyl-Wigner correspondence (hence the ordering's name): the Moyal product. This proves that the correspondence is a well-defined quantization.

2.3.1 Standard ordering

The *naive* quantization procedure would be the following:

Definition 2.3 (Naive quantization)**.** *The linear operator $Q_N : \mathbb{C}[q,p] \longrightarrow \mathit{Diffop}(\mathbb{R})$ is defined by*

$$\begin{aligned} 1 &\mapsto Q_N(1) := \mathbb{1} & q &\mapsto Q_N(q) := \hat{q} \\ p &\mapsto Q_N(p) := \hat{p} & q^n \cdot p^m &\mapsto Q_N(q^n \cdot p^m) := \hat{q}^n \hat{p}^m \,, \end{aligned}$$

where $\mathbb{C}[q,p]$ is the ring[8] *of complex polynomials of two variables and $\mathit{Diffop}(\mathbb{R})$ denotes the space of differential operators with polynomial coefficients in the space $\mathcal{C}^\infty(\mathbb{R},\mathbb{C})$, i.e. an element $D \in \mathit{Diffop}(\mathbb{R})$ takes the form $D = \sum_{k=0}^N f_k \partial^k/\partial q^k$ where $f_0, \cdots, f_N \in \mathbb{C}[q]$.*

Firstly, note that the map Q_N is well-defined since it is defined on a basis of the ring $\mathbb{C}[q,p]$; it is extended to the entire ring by $\mathbb{C}$-bilinearity. Furthermore, note that Q_N is bijective since its inverse is evidently well-defined.

Secondly, note that applying this procedure to the aforementioned example $f(q,p) = q \cdot p = p \cdot q$ would yield $Q_N(p \cdot q) = Q_N(q \cdot p) \equiv \hat{q}\hat{p} = \hat{p}\hat{q} + i\hbar \neq Q_N(p)Q_N(q)$. This means that Q_N is not a homomorphism of algebras. Note however that classically, so when $\hbar \to 0$, this map ís a homomorphism.

Using naive quantization, it is possible to construct an associative non-commutative product on $\mathbb{C}[q,p]$ that almost satisfies the correspondence principle. The idea is to pullback the non-commutative associative multiplication int the space of differential operators $\mathit{Diffop}(\mathbb{R})$ to the ring $\mathbb{C}[q,p]$ using the bijective map Q_N [19].

First of all let $f \in \mathbb{C}[q,p]$, $\phi \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C})$ and recall the actions of $\hat{q}$ and $\hat{p}$: $\hat{q}\phi(q) = q \cdot \phi(q)$ and $\hat{p}\phi(q) = (\hbar/i)(\partial\phi/\partial q)$. Write $f = \sum_{n,m=0}^\infty a_{n,m} q^n \cdot p^m$ where $a_{n,m} \in \mathbb{C}$. Since $Q_N(f)$ yields a differential operator, we can apply it to ϕ:

$$\begin{aligned} Q_N(f)(\phi) &= \sum_{n,m=0}^\infty a_{n,m} Q_N(q^n \cdot p^m)(\phi) = \sum_{n,m=0}^\infty a_{n,m} \hat{q}^n \hat{p}^m(\phi) \\ &= \sum_{n,m=0}^\infty \left(\frac{\hbar}{i}\right)^m a_{n,m}\, \hat{q}^n \,\frac{\partial^m \phi}{\partial q^m} = \sum_{m=0}^\infty \left(\frac{\hbar}{i}\right)^m \left(\sum_{n=0}^\infty a_{n,m}\hat{q}^n\right) \frac{\partial^m\phi}{\partial q^m} \,. \end{aligned}$$

The first equality holds, for Q_N is a linear operator. Now note the following identity:

$$\frac{1}{r!}\frac{\partial^r f}{\partial p^r}\bigg|_{p=0} = \frac{1}{r!}\sum_{n=0}^\infty a_{n,r}\,(r!)\,q^n = \sum_{n=0}^\infty a_{n,r} q^n \,.$$

Using the definition of the action of the operator $\hat{q}$, we see that the previous two equations yield

$$Q_N(f)(\phi) = \sum_{m=0}^\infty \frac{(\hbar/i)^m}{m!} \frac{\partial^m f}{\partial p^m}\bigg|_{p=0} \frac{\partial^m \phi}{\partial q^m} \tag{10}$$

This expression is at the basis of the following proposition.

[8] For an introduction into groups and rings, see for example [25].

Proposition 1. *Let $f, g \in \mathbb{C}[q,p]$ and $\phi \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C})$. Then*

$$f \star_N g := Q_N^{-1}\left(Q_N(f)Q_N(g)\right) = \sum_{m=0}^{\infty} \frac{(\hbar/i)^m}{m!} \frac{\partial^m f}{\partial p^m} \frac{\partial^m g}{\partial q^m} = \exp\left((\hbar/i)\,\partial_p \otimes \partial_q\right)(f,g) \tag{11}$$

defines an associative non-commutative product on the ring $\mathbb{C}[q,p]$.

Proof. The proof of the formula is a lengthy calculation using the above expression; it is omitted. Since Q_N is a bijective linear mapping, it is clear that the associativity of the product in *Diffop*$(\mathbb{R})$ directly carries over to $\star_N$; in a sense, these two products are the same. The first product is in general non-commutative, so the same holds for $\star_N$. □

In the classical limit when $\hbar$ tends to zero, only keeping terms up to and including $\mathcal{O}(\hbar^1)$ yields

$$f \star_N g = f \cdot g - i\hbar \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} + \mathcal{O}(\hbar^2)\,,$$

which is almost the aforementioned intuition of Dirac [6]: the derivatives in the term $i\hbar$ should equal the classical Poisson bracket, which is

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\,.$$

Finally, note that Q_N ís a homomorphism of algebras with respect to the star product $\star_N$: for all $f, g \in \mathbb{C}[q,p]$, it follows immediatelly from 11 that $Q_N(f \star_N g) = Q_N(f)Q_N(g)$. This means that although the correspondence principle is not exactly satisfied, Q_N does define a complete quantization by deformation.

2.3.2 Weyl-Moyal ordering

To construct a deformation quantization that also satisfies the correspondence principle, a more symmetric operator might be able to yields the Poisson bracket in the classical limit. An operator symmetric under the exchange $\hat{q} \leftrightarrow \hat{p}$ could be used; let's call it Q_S. The definition for this quantization procedure is then

Definition 2.4 (Symmetric quantization)**.** *The linear operator $Q_S : \mathbb{C}[q,p] \longrightarrow$ Diffop$(\mathbb{R})$ is defined on the basis of monomials of the ring $\mathbb{C}[q,p]$ by*

$$\begin{aligned} 1 &\mapsto Q_S(1) := \mathbb{1} & q &\mapsto Q_S(q) := \hat{q} \\ p &\mapsto Q_S(p) := \hat{p} & q^n \cdot p^m &\mapsto Q_S(q^n \cdot p^m) := S_{n,m}(\hat{q},\hat{p}) \end{aligned}$$

where by definition $S_{n,m}(\hat{q},\hat{p})$ is a polynomial expression in $\hat{q},\hat{p}$ symmetric under the exchange $(\hat{q},n) \leftrightarrow (\hat{p},m)$. The operator is extended to the whole ring by $\mathbb{C}$-bilinearity.

For the operator Q_S to be well-defined, an explicit expression of $S_{n,m}(\hat{q},\hat{p})$ is needed.

Since $S_{n,m}(\hat{q},\hat{p})$ has a symmetry property, the idea is to exploit the properties of Newton's binomium. Consider the operator-valued function $f(x,y) := (x\hat{q}+y\hat{p})^{n+m}$, which can be expanded as $f(x,y) = \cdots + W_{n,m}(\hat{q},\hat{p})x^n y^m + \cdots$ by the binomium, where $W_{n,m}$ is the sum of all $\binom{n+m}{n}$ different orders in which one can multiply n $\hat{q}$'s and m $\hat{p}$'s; in particular, $W_{n,m}$ is symmetric under $(\hat{q},n) \leftrightarrow (\hat{p},m)$.

Now, the crucial step here is to remark that by taking an appropriate derivative of f and by putting $x, y = 0$, only $W_{n,m}(\hat{q},\hat{p})$ is left. To be precise

$$\left.\frac{\partial^{n+m}}{\partial x^n \partial y^m}\right|_{x,y=0} f(x,y) = n!\,m! W_{n,m}(\hat{q},\hat{p}) =: (n+m)! S_{n,m}(\hat{q},\hat{p})\,, \tag{12}$$

where in the last line $S_{n,m}(\hat{q},\hat{p})$ is explicitly defined. Since Q_S is now fully given on a basis of the ring $\mathbb{C}[q,p]$, its definition again extends to the entire ring by $\mathbb{C}$-bilinearity. Rewriting and generalising $S_{n,m}(\hat{q},\hat{p})$ yields

$$
\begin{aligned}
S_{n,m}(\hat{q},\hat{p}) &= \left.\frac{\partial^{n+m}}{\partial x^n \partial y^m}\right|_{x,y=0} \frac{(x\hat{q}+y\hat{p})^{n+m}}{(n+m)!} \\
&= \left.\frac{\partial^{n+m}}{\partial x^n \partial y^m}\right|_{x,y=0} \sum_{k+l=0}^{\infty} \frac{(x\hat{q}+y\hat{p})^{k+l}}{(k+l)!} = \left.\frac{\partial^{n+m}}{\partial x^n \partial y^m}\right|_{x,y=0} \exp\left[x\hat{q}+y\hat{p}\right] ,
\end{aligned}
$$

where the second equality holds since all terms with powers of x strictly smaller than n and/or powers of y strictly smaller than m vanish due to the derivative, whereas all the strictly higher powers vanish due to the evaluation $x, y = 0$; so this only leaves $S_{n,m}(\hat{q},\hat{p})$.

Note that in the last equality a generating function is found for all the symmetrized forms $S_{n,m}(\hat{q},\hat{p})$ of the monomials $\hat{q}^n\hat{p}^m$. In particular, this generating function is itself symmetric under the exchange $(\hat{q},n) \leftrightarrow (\hat{p},m)$, so $Q_S(\exp[xq+yp]) = \exp[x\hat{q}+y\hat{p}]$.

In conclusion: Q_S is a well-defined, symmetric linear operator.

By again invoking the Baker-Campbell-Hausdorff formula (3) and by noting that $\hat{q}$ and $\hat{p}$ commute with their commutator, it is clear that

$$
\begin{aligned}
Q_S(e^{xq+yp}) &= e^{x\hat{q}+y\hat{p}} = \exp\left[\hbar xy/2i\right] e^{x\hat{q}} e^{y\hat{p}} = \exp\left[\hbar xy/2i\right] Q_N(e^{xq}e^{yp}) \\
&= \exp\left[\hbar xy/2i\right] Q_N(e^{xq+yp}) = Q_N(\exp\left[\hbar xy/2i\right](e^{xq+yp}))\,.
\end{aligned}
$$

So there is a map $N : \mathbb{C}[q,p] \to \mathbb{C}[q,p]$ defined by $f \mapsto Nf := \exp\left[(\hbar/2i)\,\partial^2/\partial q\partial p\right] f$ under the usual correspondence $x \mapsto \partial/\partial q, y \mapsto \partial/\partial p$; this map is linear. If $Nf = 0$ the power series representation of the exponential shows that $f = 0$, so N is injective; also, one can quickly convince oneself that this map is surjective, since $\mathbb{C}[q,p]$ is the ring of polynomials of *all orders* in q, p. This means N is the bijection linking the two orderings above by

$$
\boxed{Q_N(Nf) = Q_S(f)} \qquad \text{for all } f \in \mathbb{C}[q,p]\,.^9 \tag{13}
$$

Since a bijection between the naive and the symmetric ordering has been found, it is possible to give the analogous proposition to 1:

Proposition 2. *Let $f, g \in \mathbb{C}[q,p]$ and $\phi \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C})$. Then*

$$
f \star_S g := Q_S^{-1}\left(Q_S(f)Q_S(g)\right) = \sum_{m=0}^{\infty} \frac{(i\hbar/2)^m}{m!} \sum_{k=0}^{m} \binom{n}{k} (-1)^{m-k} \frac{\partial^m f}{\partial q^k p^{m-k}} \frac{\partial^m g}{\partial q^{m-k} p^k} \tag{14}
$$

defines an associative non-commutative product on the ring $\mathbb{C}[q,p]$. Moreover, this product is seen to be isomorphic to the product $\star_N$ of naive quantization via the linear bijective map N:

$$
\boxed{N(f \star_N g) = (Nf) \star_S (Ng)} \qquad \textit{for all } f, g \in \mathbb{C}[q,p]\,. \tag{15}
$$

Proof. The formula follows directly from (11) by substituting N and expliciting. Since it was shown that $Q_N(Nf) = Q_S(f)$ for all $f \in \mathbb{C}[q,p]$, the isomorphism of the two star products follows directly from proposition 1. The associativity of the product in *Diffop*$(\mathbb{R})$ again carries over to $\star_S$, since $Q_S = Q_N \circ N$ is a bijection. Hence, $\star_S$ is an in general non-commutative product on $\mathbb{C}[q,p]$. $\square$

[9] Note that N's inverse is given by $N^{-1} = \exp\left[-(\hbar/2i)\,\partial^2/\partial q\partial p\right]$, so $Q_N(f) = Q_S(N^{-1}f)$ for all $f \in \mathbb{C}[q,p]$.

Again note that equation (14) implies that Q_S is a homomorphism of algebras with respect to $\star_S$: for all $f, g \in \mathbb{C}[q,p]$ we have $Q_S(f \star_S g) = Q_S(f)Q_S(g)$. This time, however, the correspondence principle is satisfied:

$$f \star_S g = f \cdot g + \frac{i\hbar}{2}\{f, g\} + \mathcal{O}(\hbar^2)\,. \tag{16}$$

We can now conclude that Q_S defines a deformation quantization on the ring of polynomials $\mathbb{C}[q,p]$ in the sense that to every function in the ring, it associates a differential operator that acts on the space $\mathcal{C}^\infty(\mathbb{R},\mathbb{C})$. A star product $\star_S$ is found that captures the non-commutative behaviour of these differential operators, and transfers it back to the ring $\mathbb{C}[q,p]$. Moreover, this star product satisfies the correspondence principle.

2.4 Weyl-Wigner is one-one

Although it is beyond the scope of this thesis to prove rigorously that Weyl-Wigner is a one-one correspondence, it is however possible to give some arguments that make it plausible. What's more important is that proposition 2 allows us to link the Moyal product of the Weyl-Wigner correspondence to the the star product $\star_S$ of the symmetric quantization method in the previous section.

Two arguments will be given to support the statement that the Weyl-Wigner correspondence and the symmetric quantization yield the same operators. The first is an argument based on a physical reasoning. The second is making the above mentioned link between Moyal product and $\star_S$ explicit, which is a mathematical argument

It is important to note that the arguments given below should be read as a motivation for defining the Weyl-Wigner correspondence as it is. They motivate a formal manipulation of expressions that do not necessarily represent well defined mathematical objects, but dó represent physical quantities of interest.

2.4.1 A physicist's point of view

In quantum mechanics it is important to be able to calculate means and expectation values of position, momentum, energy and other functions of $\hat{q}, \hat{p}$. Once a smooth function representing a physical and measurable quantity is given, its expectation value is calculated by integrating over the total system with respect to a certain weight function: the absolute square of the wave function, that plays the role of probability distribution.

In general, measurable quantities encountered in physics are polynomials or analytic functions[10]; in particular they are smooth. The Fourier transform of such polynomials in $\mathbb{C}[q,p]$ such as position $\hat{q}$ or momentum $\hat{p}$ is in general not well-defined. This is due to the fact that integrating over the whole of $\mathbb{R}^2$ yields infinities. However, one need not integrate over the whole of $\mathbb{R}^2$ per se, but only over the *total system* to obtain expectation values. Since we live in a finite world, one can restrict these polynomials to a certain compact domain. For example $\hat{q} \longrightarrow \hat{q}\mathbb{1}_{[-n,n]^2}$ for a certain $n \in \mathbb{N}$. As was explained in the footnote that accompanied the definition of the Weyl-Wigner correspondence, the space of compactly supported smooth functions $\mathcal{C}_c^\infty(\mathbb{R}^2)$ is contained

[10] To be precise: analycity of a function $f : \mathbb{R}^2 \to \mathbb{C}$ implies $\forall (q,p) \in \mathbb{R}^2, \exists U \subset \mathbb{R}^2$ open around (q,p) : $f(q,p) = \sum_{n+k=0}^{\infty} a_{n,k} q^n p^k = \lim_{N\to\infty} \sum_{n+k=0}^{N} a_{n,k} q^n p^k$. Hence, one could be tempted to prove the bijectiveness for monomials of *finite order* (as was the author at first), and then take limits to obtain the result for all analytic functions. However, one actually needs to verify commutation of taking the limit (to obtain the sought after *infinite* sum of monomials) and applying the linear operator $\hat{W}(\cdot)$; this limit appears in the above definition of analycity. The bigger problem is that monomials are not integrable, and certainly not square-integrable as is required by definition 2.2. This approach does not work.

in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$. On this space, the Fourier transform is a linear automorphism so in particular, it is well defined.

From this point of view, one need not worry about infinities and convergence. Expectation values of physically measurable functions are well defined.

2.4.2 A mathematical argument

Recall that the Weyl-Wigner correspondence associates a quantum operator $\hat{W}[f]$ to a phase space distribution function $f \in L^2(\mathbb{R}^2)$. This means that f is a function of classical position q and momentum p. The restriction of the space of functions was for mathematical simplicity, in the general settng $f \in \mathcal{C}^\infty(M)$ the algebra of smooth functions on phase space $M = \mathbb{R}^2$. In section 2.2, the Moyal product $\star_\hbar$ was found to be a non-commutative product on the algebra $\mathcal{C}^\infty(M)$. Moreover, with respect to this product, it was noted that $\hat{W}$ is a homomorphism of algebras:

$$\hat{W} : (\mathcal{C}^\infty(M), \star_\hbar) \longrightarrow (\mathcal{C}^\infty(M), \cdot), \quad \text{where } \hat{W}[f \star_\hbar g] = \hat{W}[f] \cdot \hat{W}[g] \tag{17}$$

by virtue of equation (4).

On the other hand, symmetric quantization also yielded a homomorphism of algebras: Q_S, which was defined in equation 2.4 as

$$Q_S : (\mathbb{C}[q,p], \star_S) \longrightarrow \mathrm{Diffop}(\mathbb{R}), \quad \text{where } Q_S(f \star_S g) = Q_S(f) Q_S(g)\,. \tag{18}$$

Proposition 2 gives an expression for $f \star_S g$. It will now be shown that this expression is *exactly the same* as the one found for the Moyal product of f and g, in equation (6). Let $f, g \in \mathbb{C}[q,p]$, consider the expression from proposition 2:

$$\begin{aligned}
(f \star_S g)(q,p) &= \sum_{m=0}^{\infty} \frac{(i\hbar/2)^m}{m!} \sum_{k=0}^{m} \binom{n}{k} (-1)^{m-k} \frac{\partial^m f(q,p)}{\partial q^k p^{m-k}} \frac{\partial^m g(q,p)}{\partial q^{m-k} p^k} \\
&= \sum_{m=0}^{\infty} \frac{(i\hbar/2)^m}{m!} \sum_{k=0}^{m} \binom{n}{k} f(q,p) \left(\frac{\overleftarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial p} \right)^k (-1)^{m-k} \left(\frac{\overleftarrow{\partial}}{\partial p} \frac{\overrightarrow{\partial}}{\partial q} \right)^{m-k} g(q,p) \\
&= \sum_{m=0}^{\infty} \frac{(i\hbar/2)^m}{m!} \sum_{k=0}^{m} \binom{n}{k} f(q,p) \left(\frac{\overleftarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p} \frac{\overrightarrow{\partial}}{\partial q} \right)^m g(q,p) \\
&= f(q,p) \exp\left[\frac{i\hbar}{2} \left(\frac{\overleftarrow{\partial}}{\partial q} \frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p} \frac{\overrightarrow{\partial}}{\partial q} \right) \right] g(q,p) \\
&= (f \star_\hbar g)(q,p)\,.
\end{aligned}$$

In conclusion: for all $f, g \in \mathbb{C}[q,p]$, we have $\boxed{f \star_S g = f \star_\hbar g}$. This means that although the Moyal product is only defined on $L^2(\mathbb{R}^2)$, apparently its definition also work for functions in the ring $\mathbb{C}[q,p]$. Since symmetric quantization is well-defined, we see now that the Weyl transform of polynomial functions, such as the harmonic oscillator hamiltonian, are also well-defined. Moreover, symmetric quantization is in one-one correspondence with naive quantization which is a bijective quantization.

The found relations for both star products that turn the mappings in homomorphism are

$$\hat{W}[f \star_\hbar g] = \hat{W}[f]\hat{W}[g] \qquad \text{and} \qquad f \star_S g = Q_S^{-1}(Q_S(f) Q_S(g)) \quad \text{for all } f, g \in L^2(\mathbb{R}^2) \cap \mathbb{C}[q,p]\,.$$

This means that

$$(\hat{W} \circ Q_S^{-1})\,(Q_S(f) Q_S(g)) = \hat{W}[f]\hat{W}[g] \tag{19}$$

In total: the Weyl-Wigner correspondence yields a well-defined quantization by introducing a star product on the algebra $\mathbb{C}[q, p]$ of polynomial functions of phase space. Although the star products $\star_N$ and $\star_S$ are seen to be formaly equivalent by equation (15), the Moyal star product $\star_\hbar$ and symmetric star product $\star_S$ are seen to be *exactly the same* by the above boxed equation. Hence, also the Moyal product is a symmetric operation, and it is seen to be equivalent to $\star_N$ via $Q_S = Q_N \circ N$ which is bijective. This is a strong argument for $\hat{W}$'s invertibility is. However, to make this proof precise, the space $\hat{W}$ acts on needs to be defined exactly, as well as some more properties. As announced earlier, this lies outside the scope of this text.

3 Symplectic geometry: a framework for classical mechanics

The second part of this thesis is devoted to showing that symplectic geometry is a natural category to work in when it comes to classical mechanics. To this extent, the first section discusses the basics of the theory of symplectic geometry. In particular, it is shown that hamiltonian classical mechanics can be fully described by this theory. In the second section, Poisson manifolds are defined since Kontsevich's theorem is stated in these terms. It is shown that symplectic manifolds carry a natural Poisson structure. This proofs the claim that deformation quantization - which is expressed in terms of Poisson manifolds - has physical relevance, since almost all hamiltonian classical mechanical systems can be described by the symplectic formalism.

3.1 Classical mechanics as symplectic theory

Throughout this section, definitions and mathematical statements as found in chapters of the lecture notes [21] will be followed, unless specified otherwise; physical interpretation is added by the author. It is assumed that the reader is familiar with the basic theory of smooth manifolds as found in for example [16]. Nevertheless, some aspects of the theory will be recalled when needed.

3.1.1 Some symplectic theory

Let M be a smooth manifold without boundary. A 2-form $\omega \in \Gamma^\infty(\Lambda^2 T^*M)$ on M is a smooth section of the second exterior product of the cotangent bundle T^*M. This means that for all $p \in M$, the map $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is bilinear and skew-symmetric. In symplectic theory, a particular type of 2-form is associated to a smooth manifold, turning it into a so-called *symplectic* manifold; this particular type of 2-form is also called symplectic. It is defined in the following way:

Consider a real vector space V of dimension m. Let $\Omega : V \times V \to \mathbb{R}$ be a bilinear, skew-symmetric[11] map. There is the following theorem:

Theorem 1. *There exists a basis $\{u_1, \dots, u_k, e_1, \dots, e_n, f_1, \dots, f_n\}$ of V, such that $\Omega(u_i, v) = 0$, $\Omega(e_i, e_j) = 0 = \Omega(f_i, f_j)$ and $\Omega(e_i, f_j) = \delta_{i,j}$ for all i, j and for all $v \in V$.*

Note that pictorially, this means that for all $u, v \in V$

$$\Omega(u,v) = \begin{bmatrix} & u & \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbb{1} \\ 0 & -\mathbb{1} & 0 \end{bmatrix} \begin{bmatrix} \\ v \\ \\ \end{bmatrix} \tag{20}$$

Definition 3.1 (Nondegeneracy)**.** *If the linear map $\tilde{\Omega} : V \to V^*$, defined by $v \mapsto \Omega(v, \cdot)$ is bijective, then Ω is called symplectic or nondegeneratie; also, (V, Ω) is called a symplectic vector space.*

A few remarks can be made about this definition:

1. When $\tilde{\Omega}$ is bijective, in particular $\ker(\tilde{\Omega}) = \{0\}$. Denote $U = \mathrm{span}\{u_1, \dots, u_k\}$. Since for all $i \in \{1, \dots, k\}$, $v \in V$, the above theorem states that $\Omega(u_i, v) = 0$, a trivial kernel means that $\dim(U) = k = 0$. Hence $\dim(V) = 2n$ and the manifold's dimension is seen to be even.

[11] Skew-symmetry means that for all $u, v \in V$, $\Omega(u, v) = -\Omega(v, u)$.

2. The symplectic vector space (V, Ω) has $\mathcal{B} = \{e_1, \dots, e_n, f_1, \dots, f_n\}$ as skew-orthonormal basis. Since $\Omega(e_i, e_j) = 0 = \Omega(f_i, f_j)$ and $\Omega(e_i, f_j) = \delta_{i,j}$ for all $i, j \in \{1, \dots, n\}$, it is called a symplectic basis. Pictorially, this means that for all $u, v \in \Omega$
$$\Omega(u, v) = u^t \begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix} v \tag{21}$$
3. It can be show that the above definition of Ω being nondegenerate is equivalent to the following one found in [20]:
$$\Omega(v, w) = 0 \quad \text{for all} \quad w \in V \quad \text{implies} \quad v = 0\,. \tag{22}$$

Definition 3.2 (Symplectic Manifold). *The pair (M, ω) of smooth manifold M and 2-form $\omega \in \Gamma^\infty(\Lambda^2 T^*M)$ is called a symplectic manifold if $d\omega = 0$ and if ω_p is symplectic as defined above for all $p \in M$; the form is said to be closed and nondegenerate respectively.*

Examples

1. The first example of a symplectic manifold is the trivial one: consider $M = \mathbb{R}^{2n}$ with chart $(\mathbb{R}^{2n}, x^1, \dots, x^n, y^1, \dots, y^n)$ and symplectic form $\omega_0 = \sum_{j=1}^n dx^j \wedge dy^j$. Since $d^2 = 0$, it is clear that ω_0 is closed. To see that it is also nondegenerate, fix a $p \in M$ and consider the mapping
$$\tilde{\omega}_{0,p} : T_pM \longrightarrow T_p^*M,\; a_p \mapsto \Big(\tilde{\omega}_{0,p}(a_p) := \omega_{0,p}(a_p, \cdot\,) : T_pM \longrightarrow \mathbb{R},\; b_p \mapsto \omega_{0,p}(a_p, b_p)\Big)\,.$$
Note that the set $\mathcal{B}_p = \{\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p, \partial/\partial y^1|_p, \dots, \partial/\partial y^n|_p\}$ of tangent vectors forms a basis of T_pM for all $p \in M$. Denote $\partial/\partial x^i|_p = \partial_i^x|_p$, $\partial/\partial y^i|_p = \partial_i^y|_p$ and observe that
$$\tilde{\omega}_{0,p}(\partial_i^x|_p) = \omega_{0,p}(\partial_i^x|_p, \cdot\,) = dy^i|_p \qquad \text{and} \quad \tilde{\omega}_{0,p}(\partial_i^y|_p) = -dx^i|_p\,. \tag{23}$$
By bilinearity of ω_0, this yield for a general $a_p = \sum_{i=1}^n \alpha_i \partial_i^x|_p + \beta_i \partial_i^y|_p \in T_pM$
$$\tilde{\omega}_{0,p}(a_p) = \sum_{i=1}^n \beta_i dy^i|_p - \alpha_i dx^i|_p\,. \tag{24}$$
Using the equivalent definition of nondegeneracy given in remark 3, assume that $\tilde{\omega}_{0,p}(a_p)(v) = \omega_p(a_p, v) = 0$ for all $v \in T_pM$. Evaluating $\tilde{\omega}_{0,p}(a_p)$ on the basis $\mathcal{B}_p$ of T_pM shows that this implies $\alpha_i = 0 = \beta_i$ for all $i \in \{1, \dots, n\}$ whence $a_p = 0$. Thus, ω_0 is nondegenerate and we conclude that (M, ω_0) is a symplectic manifold. Note that $\mathcal{B}_p$ is a symplectic basis for T_pM for all $p \in M$.

2. Let Q be a smooth manifold, for instance the configuration space of a certain system. Its cotangent bundle $M = T^*Q \equiv \cup_{p \in Q}\{p\} \times T_p^*Q$ carries a natural structure of symplectic manifold as follows. Let $\{(\mathcal{U}_\alpha, \phi_\alpha)_{\alpha \in A}\}$ be an atlas of Q and $(\mathcal{U}, x^1, \dots, x^n)$ a chart centered at $p \in Q$ with coordinate functions $x^i : \mathcal{U} \to \mathbb{R}$. Since $\{dx^1|_p, \dots, dx^n|_p\}$ is a basis for T_p^*Q, for every $\xi \in T_p^*Q$ there exist $\{\xi^1, \dots, \xi^n\} \in \mathbb{R}$ such that $\xi = \sum_{j=1}^n \xi^j dx^j|_p$.

   This induces a smooth map $F : T^*\mathcal{U} \to \mathbb{R}^{2n}$ defined by $(x, \xi) \mapsto (x^1, \dots, x^n, \xi^1, \dots \xi^n)$ which is a chart for $T^*\mathcal{U}$ centered at (x, ξ). It allows for the construction of an atlas for T^*Q and equips it with a smooth manifold structure of dimension twice the dimension of Q.

Smoothness comes from the fact that transition mappings are smooth: let $(\mathcal{U}, x^1, \dots, x^n)$, $(\mathcal{V}, y^1, \dots, y^n)$ be two charts and $p \in \mathcal{U} \cap \mathcal{V}$, then if $\xi \in T_p^*Q$

$$\xi = \sum_{j=1}^{n} \xi^j dx^j|_p = \sum_{i,j} \xi_i \left(\frac{\partial x^j}{\partial y^i}\right) dy^i|_p =: \sum_{i=1}^{n} \tilde{\xi}^i dy^i|_p \quad \text{where} \quad \tilde{\xi}^i := \sum_{j=1}^{n} \xi^j \left(\frac{\partial x^j}{\partial y^i}\right) \text{ is smooth} . \tag{25}$$

One can define a symplectic 2-form on the cotangent bundle both globally and locally; in both ways it can be shown to be intrinsic, i.e. independent of choice of coordinates. Due to this property it is called the *canonical* symplectic form. We'll follow the local definition according to [21].

On the aforementioned chart $(T^*\mathcal{U}, x^1, \dots, x^n, \xi^1, \dots \xi^n)$, define $\alpha := \sum_{j=1}^n \xi^j dx^j|_p$. Consider a point $p \in T^*Q$ in an overlapping of charts as before. By using the transition functions as defined in 25, one sees that

$$\alpha_{\mathcal{U}} \equiv \sum_{j=1}^{n} \xi^j dx^j|_p = \sum_{i,j} \xi_i \left(\frac{\partial x^j}{\partial y^i}\right) dy^i|_p = \sum_{i=1}^{n} \tilde{\xi}^i dy^i|_p \equiv \alpha_{\mathcal{V}} .$$

Hence, α is independent of choice of coordinates and for this reason, it is called the tautological 1-form. Now, $\omega := -d\alpha$ defines the canonical symplectic 2-form. In local coordinates (on $\mathcal{U}$), it looks like $\omega = \sum_{j=1}^n dx^j \wedge d\xi^j$, which is the expression of the 2-form on the trivial symplectic manifold in the first example. This specific expression of ω will later on prove to be important. Clearly, ω is closed, nondegenerate and skew-symmetric, turning $(M = T^*Q, \omega)$ into a symplectic manifold.

Anticipating on what is to come, recall that the dynamics of a classical mechanical system are determined by the hamiltonian function, which will be $H \in \mathcal{C}^\infty(M, \mathbb{R}) =: \mathcal{C}^\infty(M)$ where $M = T^*Q$ is the cotangent bundle of the configuration space Q; now H is in fact the infinitesimal generator of time translation. It will be shown that there is a unique vector field X_H associated to the hamiltonian, whose flow controls the dynamics of the system. Since a symplectic manifold (phase space) is defined as a *pair* of manifold and 2-form, it is important that this evolution leaves the 2-form invariant[12]. The following definition says when two symplectic manifolds are essentially the same.

Definition 3.3 (Symplectomorphism)**.** *Let (M_1, ω_1), (M_2, ω_2) be two symplectic manifolds of dimension $2n$. If there is a smooth map $\varphi : M_1 \to M_2$, it is called a symplectomorphism if $\varphi^*\omega_2 = \omega_1$ [13]. Also, M_1 and M_2 are the same as symplectic manifolds when φ is a symplectic diffeomorphism, i.e. both φ and φ^{-1} are symplectomorphisms and bijections.*

A hamiltonian system is a triple (M, ω, H), where $H \in \mathcal{C}^\infty(M)$ is a smooth function called the hamiltonian function. Before accepting such a triple as a physical system, one needs to verify the invariance of ω under the flow induced by H, i.e. the invariance of the symplectic structure on the manifold under the dynamical evolution. For this, the concepts of isotopy and Lie derivative are needed.

Definition 3.4 (Isotopy)**.** *Let M be a smooth manifold, $\rho : M \times \mathbb{R} \to M$ a smooth map and set $\rho_t(p) := \rho(p, t)$. This map is called an isotopy if*

[12] As was noted by J. de Boer, the phase space of a system subject to time-dependent constraints is in general also dependent on time. In this case, the symplectic structure need not be conserved. However, this type of problems falls outside the scope of this thesis.

[13] Note that in the (physical) literature, such a mapping is also known as a *canonical transformation*, see [20].

1. *For* $t = 0$, $\rho_0 = \mathbb{1}_M$,

2. $\forall t \in \mathbb{R}$, $\rho_t : M \to M$ *is a diffeomorphism.*

Given an isotopy ρ, one can build a time-dependent vector field or equivalently a family of vector fields $\{X_t, t \in \mathbb{R}\}$ in the following manner. Define for all $p \in M$

$$X_t(p) = \frac{d}{ds}\rho_s(q)\big|_{s=t} \quad \text{where} \quad q = \rho_t^{-1}(p) \in M \tag{26}$$

which is equivalent to defining $d\rho_t/dt = X_t \circ \rho_t$ by definition of an isotopy. Note that his corresponds with the flow $c(t)$ of a vector field Y defined by $\dot{c}(t) = Y_{c(t)}$. Conversely, given a family of vector fields on a (compact) manifold M, one can construct an isotopy (see [21], p.35)

Definition 3.5 (Exponential map)**.** *When* $X_t = X$, $\forall t \in \mathbb{R}$ *the associated isotopy is called the exponential map or the flow of* X*. It is denoted by* $\alpha_X(t) = \exp tX$*.*

As a consequence, $\{\exp tX : M \to M \,|\, t \in \mathbb{R}\}$ is the unique smooth family of diffeomorphisms satisfying

1. $\exp(tX)\big|_{t=0} = \mathbb{1}_M$,

2. $\frac{d}{dt}(\exp tX)(p) = X\Big((\exp tX)(p)\Big) \quad \forall p \in M, t \in \mathbb{R}.$

Using the concept of exponential map, the Lie derivative can now be defined. Recall that on a smooth manifold M of dimension m, k-forms are sections of the $\mathrm{k^{th}}$ exterior power of the cotangent bundle $\Lambda^k(T^*M) = \coprod_{p\in M}\{p\} \times \Lambda^k(T_p^*M)$. The triple $(\Lambda^k(T^*M), \pi, M)$ has a natural vector bundle structure, where $\pi : \Lambda^k(T^*M) \to M$, $\alpha \mapsto \pi(\alpha) = p$ when $\alpha \in \Lambda^k(T_p^*M)$ is the projection map [16].

Since we are interested in smooth manifolds, only smooth sections of this bundle matter for the moment; denote these by $\Omega^k(M) := \Gamma^\infty(\Lambda^k T^*M)$ and note that $\Omega^0(M) = \mathcal{C}^\infty(M)$ the smooth functions on M. Since (smooth) k-forms can be added and multiplied by constants and smooth functions, this space has a natural structure of vector space. Adding the wedge-product $\wedge$ as operation, the space $\Omega^*(M) := \bigoplus_{k=0}^n \Omega^k(M)$ is seen to be a graded algebra with respect to the gradation of forms [16].

On every smooth manifold M there exists a unique and intrinsically defined concept of antiderivation (of degree 1) that is compatible with the gradation of $\Omega^*(M)$ (see [16], §19): it is called the *exterior derivative* and it is defined as follows:

Definition 3.6 (Exterior derivative)**.** *The exterior derivative on a smooth manifold* M *is an* $\mathbb{R}$*-linear map* $d : \Omega^*(M) \to \Omega^*(M)$ *such that*

1. $\forall \omega \in \Omega^k(M), \tau \in \Omega^l(M)$, $d(\omega \cdot \tau) = (d\omega)\cdot\tau + (-1)^k\omega\cdot(d\tau)$,

2. $d \circ d = 0$,

3. $\forall f \in \mathcal{C}^\infty(M), X \in \mathcal{X}(M)$ *then* $(df)(X) = X \circ f$.

Note that d being an antiderivation of degree 1 means that if $\omega \in \Omega^k(M)$ then $d\omega \in \Omega^{k+1}(M)$; exterior derivation is seen to increase the degree of a k-form by 1.

Both the wedge product and the exterior derivative have the property of commutation with the pullback by a smooth map. Furtermore, the pullback of a smooth k-form by a smooth map is again a smooth k-form [16].

The Lie derivative is defined for smooth vector fields, so for smooth sections of the tangent bundle over M; denote this vector space by $\Gamma^\infty(TM) := \mathcal{X}(M)$. The Lie derivative of $Y \in \mathcal{X}(M)$ (or ω) with respect to $X \in \mathcal{X}(M)$ is a measure for the extent in which the vector field Y (or the k-form ω) flows like X. This is a bit vague. Here are two definitions, one in terms of vector fields and one in terms of smooth k-forms.

Definition 3.7 (Lie derivative)**.**

1. *Let $X_t = X, \forall t \in \mathbb{R}$ a family of smooth vector fields. The Lie derivative of $\omega \in \Omega^k(M)$ with respect to the vector field X is the operator $\mathcal{L}_X : \Omega^k(M) \to \Omega^k(M)$, $\omega \mapsto \mathcal{L}_X\omega := \frac{d}{dt}(\exp tX)^*\omega\big|_{t=0}$.*
2. *Let $X, Y \in \mathcal{X}(M)$. The Lie derivative of the vector field Y with respect to the vector field X is defined as their Lie bracket, so $\mathcal{L}_X : \mathcal{X}(M) \to \mathcal{X}(M), Y \mapsto \mathcal{L}_X Y := [X, Y] = XY - YX$.*

These definitions only hold for the Lie derivative with respect to time-independent vector fields. In the time-dependent case, it is not clear wether or not the above definitions work correctly since the relation between the vector field's time parameter and the flow's time parameter is not clear. In any case, by Picard's theorem, a time-dependent vector field's flow exists locally [21]. Hence in a neighbourhood of any point $p \in M$, for sufficiently small t, there is an associated (local) isotopy ρ_t consisting of diffeomorphisms satisfying the properties in definition 3.4. It is therefore possible to define the Lie derivative of a k-form ω with respect to a time-dependent vector field X_t as

$$\mathcal{L}_{X_t} := \frac{d}{dt}\left(\rho_t\right)^* \omega\big|_{t=0} . \tag{27}$$

Note that this reduces to the original definition for a time-independent family $\{X_t = X\,, \forall t \in \mathbb{R}\}$ by definition of the exponential map.

Definition 3.8 (Inner product)**.** *The inner product of a k-form $\omega \in \Omega^k(M)$ and a smooth vector field $X \in \mathcal{X}(M)$ is defined as the contraction of the two. To be more precise:*

$$i_X : \Omega^k(M) \to \Omega^{k-1}(M),\ \omega(\underbrace{\cdot, \ldots, \cdot}_{k\ times}) \mapsto \omega(X, \underbrace{\cdot, \ldots, \cdot}_{k-1\ times}) .$$

It is now possible to state two important formula's [21]

Proposition 3 (Cartan's magic formula)**.**

1. *For $X \in \mathcal{X}(M)$, $\omega \in \Omega^k(M)$ we have $\mathcal{L}_X\omega = i_X d\omega + d i_X \omega$*[14]*.*
2. *Let $\{X_t, t \in \mathbb{R}\}$ a family of smooth vector fields, ρ_t the associated (local) isotopy. Then*

$$\frac{d}{dt}\left(\rho_t\right)^*\omega = \rho_t^* \mathcal{L}_{X_t}\omega . \tag{28}$$

Some remarks about this proposition and the preceding definitions:

1. Note that for $f \in \Omega^0(M) = \mathcal{C}^\infty(M)$, Cartan's formula yields $\mathcal{L}_X f = df(X) \equiv i_X df$.
2. It follows from the definition (or from Cartan) that $\mathcal{L}_X d\omega = d\mathcal{L}_X\omega$.
3. The Lie derivative is a derivation, i.e. an $\mathbb{R}$-linear map such that for $\omega \in \Omega^k(M)$, $\tau \in \Omega^l(M)$ we have that $\mathcal{L}_X(\omega \wedge \tau) = (\mathcal{L}_X\omega) \wedge \tau + \omega \wedge (\mathcal{L}_X\tau)$.
4. The inner product is an anti-derivation, i.e. an $\mathbb{R}$-linear map such that for $\omega \in \Omega^k(M)$, $\tau \in \Omega^l(M)$ we have that $i_X(\omega \wedge \tau) = (i_X\omega) \wedge \tau + (-1)^k\omega \wedge (i_X\tau)$.

[14]For a proof of this remarkable formula, see [22] pages 476, 477.

3.1.2 Hamiltonian vector fields

After this intermezzo on symplectic theory, it is now possible to construct Hamiltonian vector fields and to find Hamilton's equations, showing symplectic theory to be a framework for classical mechanics.

Let (M, ω) be a smooth symplectic manifold and $H \in \mathcal{C}^\infty(M)$ a smooth function; then $dH \in \Omega^1(M)$ is a smooth 1-form. Since the 2-form ω is nondegenerate, there is exactly one smooth vector field $X_H \in \Gamma^\infty(TM) = \mathcal{X}(M)$ such that $i_{X_H}\omega = \omega(X_H, \cdot) = dH$. A vector field that satisfies this property is called a hamiltonian vector field associated to the hamiltonian function H.

To be more precise, recall that for a smooth vector field $Y \in \mathcal{X}(M)$ one can construct a real $\mathbb{R}$-linear function[16]

$$dH(Y) : M \to \mathbb{R} \quad \text{by pointwise defining} \quad (dH)(Y)_p := (dH)_p(Y_p)\,,$$

where $(dH)_p : T_pM \to \mathbb{R}$ is linear; $\mathbb{R}$-linearity means that for $f \in \mathcal{C}^\infty(M)$, $(dH)(fY)_p = (f \cdot dH(Y))_p$. [15] Furthermore, recall that ω is bilinear and that it is nondegenerate. These two properties allow for the construction of the vector field X_H by assigning the one tangent vector in T_pM to $(X_H)_p$ such that $\omega_p((X_H)_p, Y_p) = (dH)_p(Y_p)$ holds for all $Y_p \in T_pM$; since dH is smooth, X_H is smooth as well.

Or: since $\{T_pM\}_{p\in M}$ is a collection of vector spaces, the above process should be carried out on a frame $\{(\partial/\partial x^1), \ldots, (\partial/\partial x^n)\}$ of the tangent bundle TM; this completely determines X_H on TM by bilinearity and non-degeneracy of ω.

There are two observations one can make about this construction:

1. $X \in \mathcal{X}(M)$ is hamiltonian $\iff i_X\omega$ is exact.

2. The Lie derivative of a hamiltonian function H with respect to its hamiltonian vector field X_H is zero. Explicitly:

$$\mathcal{L}_{X_H}H = i_{X_H}dH + di_{X_H}\omega = i_{X_H}i_{X_H}\omega + 0 = 0\,, \tag{29}$$

   where the first equality holds by Cartan's formula, and the second by skew-symmetry of ω. Note that $\mathcal{L}_{X_H}H = 0$ means that each integral curve $\{\rho_t(x)\,|\,t \in \mathbb{R}\}$ of X_H, where $x \in M$, is contained in a level set of H. Denote $H(x)$ for the level set of H that contains x. The above means that $H(x) = \rho_t^*H(x) = H(\rho_t(x))$ for all $t \in \mathbb{R}$ [21].

   Physically, one can state this as the well known result that the dynamical evolution of a classical mechanical system makes particules travel around the level curves of the (time-independent) Hamiltonian governing the system.

Now assuming M to be compact, the remark after definition 3.4 of isotopy states that there exists a family of diffeomorphisms - an isotopy of M - that is generated by X_H, say $\{\rho_t : M \to M,\ t \in \mathbb{R}\}$, such that

1. $\rho_0 = \mathbb{1}_M$,

2. For all $t \in \mathbb{R}$, $\frac{d\rho_t}{dt} \circ \rho_t^{-1} = X_H$ or equivalently $\frac{d\rho_t}{dt} = X_H \circ \rho_t$.

Proposition 4. *This family of maps is in fact a family of symplectic diffeomorphisms. Equivalenty, every ρ_t preserves ω, so $\rho_t^*\omega = \omega$ for all $t \in \mathbb{R}$.*

[15] Explicitly: $(dH)(fY)_p = (dH)_p(f(p)Y_p) = f(p)(dH)_p(Y_p) = (f \cdot dH(Y))_p$ by pointwise definition.

Proof. By the above, $\rho_0 = \mathbb{1}_M$ so by functoriality of the pullback $\rho_0^* \omega = \omega$. Furthermore, first using the second result of proposition 3 and then the Cartan magic formula, one sees that

$$\begin{aligned}\frac{d}{dt}\rho_t^* \omega &= \rho_t^* \mathcal{L}_{X_H}\omega = \rho_t^* \left(i_{X_H} d\omega + d i_{X_H}\omega\right) \quad \text{(Cartan)} \\ &= \rho_t^* \left(i_{X_H} 0 + ddH\right) = \rho_t^* \left(0+0\right) = 0\,,\end{aligned}$$

where the third equality follows by closedness of ω and by the defining property of X_H. It follows that, indeed, for all $t \in \mathbb{R}$ there holds $\rho_t^* \omega = \omega$. □

It is important to note that the closedness of the 2-form ω is needed to proof the above proposition. Equivalently: due to the closedness of ω, the family of diffeomorphisms generated by the hamiltonian vector field X_H leaves the symplectic structure on (M, ω) invariant. The mathematical requirement that $d\omega = 0$ is seen to gain physical relevance: without this requirement, the phase space of the system under consideration need not be time-independent.

By the above proposition, every smooth function $f \in \mathcal{C}^\infty(M)$ on (M, ω) induces a family of symplectic diffeomorphisms [21]. This means that the evolution of a symplectic system governed by the hamiltonian function H leaves the system itself invariant. Note in particular that the evolution is given by diffeomorphisms, so that it is seen to be *reversible*. This coincides with the notion of invertible evolution known from classical mechanical system where the Hamiltonian is time-independent.

3.1.3 Hamilton's equations

Following the brief description in [21], consider a Hamiltonian system (M, ω_0, H) consisting of

- the symplectic manifold $M = (\mathbb{R}^{2n}, q_1, \ldots, q_n, p_1, \ldots, p_n)$,
- the symplectic 2-form $\omega_0 = \sum_{j=1}^n dq^j \wedge dp^j$, and
- the hamiltonian function $H \in \mathcal{C}^\infty(M)$.

The triple (M, ω_0, H) represents the phase space of a classical mechanical system consisting of n particules with positions q_i and momenta p_i[16]. Denote $\rho_t = (q(t), p(t))$ where $q(t) = (q_1(t), \ldots, q_n(t))$ and $p(t) = (p_1(t), \ldots, p_n(t))$. This curve is an integral curve of the associated hamiltonian vector field X_H if for all $i \in \{1, \ldots, n\}$ and for all $t \in \mathbb{R}$

$$\frac{dq_i(t)}{dt} = X_H(q_i(t)) \quad \text{and} \quad \frac{dp_i(t)}{dt} = X_H(p_i(t))\,.$$

Note that the tangent vectors $\mathcal{B}_x = \{\partial/\partial q_1|_x, \ldots, \partial/\partial q_n|_x, \partial/\partial p_1|_x, \ldots, \partial/\partial p_n|_x\}$ form a basis of T_xM for all $x \in M$. So in particular, $\mathcal{B}_x^* = \{dq^1|_x, \ldots, dq^n|_x, dp^1|_x, \ldots, dp^n|_x\}$ forms a basis of T_x^*M, $\forall x \in M$. This basis is called the dual basis, since $(dz^i)(\partial/\partial y_j) = \delta_{i,j}\delta_{z,y}$ where $z, y \in \{q, p\}$.

By construction of X_H, the inner product of ω_0 with X_H is equal to dH, so $i_{X_H}\omega_0 = dH$. Now

$$\begin{aligned}i_{X_H}\omega_0 &= \sum_{j=1}^n i_{X_H}(dq^j \wedge dp^j) = \sum_{j=1}^n \left[(i_{X_H} dq^j) \wedge dp^j - dq^j \wedge (i_{X_H} dp^j)\right] \\ dH &= \sum_{j=1}^n \left(\frac{\partial H}{\partial q_j} dq^j + \frac{\partial H}{\partial p_j} dp^j\right)\,.\end{aligned}$$

[16] Actually, one could take $\mathbb{R}^{6n}$ to represent n particules in $\mathbb{R}^3$ with positions $\vec{q}_i = (q_i^x, q_i^y, q_i^z)$ and momenta $\vec{p}_i = (p_i^x, p_i^y, p_i^z)$. However, this only complicates notation whilst not adding much to the idea.

By evaluating both $i_{X_H}\omega_0$ and dH on the aforementioned basis of vector fields $\mathcal{B}$, one sees that

$$\text{For all } j \in \{1, \ldots, n\} : \quad i_{X_H}\, dq^j = \frac{\partial H}{\partial p_j} \quad \text{and} \quad i_{X_H}\, dp^j = -\frac{\partial H}{\partial q_j}\,.$$

This yields

$$X_H = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial}{\partial p_j} \right)^{17}.$$

Since ρ_t is an integral curve, Hamilton's equations are found:

$$\boxed{\frac{dq_j(t)}{dt} = X_H \circ q_j(t) = \frac{\partial H}{\partial p_j}} \quad \text{and} \quad \boxed{\frac{dp_j(t)}{dt} = X_H \circ p_j(t) = -\frac{\partial H}{\partial q_j}}\,. \tag{30}$$

In conclusion, the symplectic formalism developped in the previous paragraphs leads to the same equations for a classical mechanical system as Hamilton's principle of least action. Note that for now, this is only shown for the trivial symplectic manifold $M = \mathbb{R}^{2n}$. However, an important theorem due to Darboux (see [21], §8) states that locally every symplectic manifold looks like the trivial one. More formally:

Theorem 2 (Darboux's theorem)**.** *Let (M, ω) be a symplectic manifold. Then for every $p \in M$ we can find a coordinate system $\{\mathcal{U}, x_1, \ldots, x_n, y_1, \ldots, y_n\}$ centered at p such that on $\mathcal{U}$ we have $\omega_0 = \sum_{j=1}^{n} dx^j \wedge dy^j$. This collection of coordinate systems forms an atlas of (M, ω).*

The most important consequence of this theorem is that if a certain property holds locally, i.e. on $(\mathcal{U} \subset \mathbb{R}^{2n}, \sum dx^j \wedge dy^j)$, it also holds globally. This result shows in particular that Hamilton's equations are valid on every symplectic manifold (M, ω). This observations allows for the treatment of all classical systems described by symplectic manifolds using symplectic theory; hence its importance.

3.2 Poisson manifolds

Having set up a formalism for symplectic manifolds that is shown to be applicable to classical mechanics, it is important to note that Kontsevich's theorem is stated in terms of Poisson manifolds [11]. In this section, they are defined and it is shown that every symplectic manifold carries a natural Poisson structure, i.e. symplectic manifolds are a particular case of elements of the greater category of Poisson manifolds.

First of all, a few definitions are needed (following [23]), in particular the concept of a Lie algebra.

Definition 3.9 (Lie & Poisson algebra)**.**

1. *A (real) Lie algebra is a real vector space V equipped with a skew-symmetric bilinear operation $[\cdot, \cdot] : V \times V \to V$ called the Lie bracket, that sends $(f, g) \mapsto [f, g]$ and satisfies the Jacobi identity*
$$[[f, g], h] + [[g, h], f] + [[h, f], g] = 0 \quad \forall f, g, h \in V\,. \tag{31}$$
2. *A (real) Poisson algebra is a real vector space P equipped with two products: a commutative product*
$$P \times P \to P,\ (f, g) \mapsto fg$$

[17] Note that one recognizes the classical Poisson bracket: $X_H(\cdot) = \{H, \cdot\}$.

that makes P into a commutative algebra; and the Poisson bracket

$$\{\cdot,\cdot\} : P \times P \to P,\ (f,g) \mapsto \{f,g\}$$

that makes P into a real Lie algebra. These two product are related by a compatibility rule

$$\{fg,h\} = f\{g,h\} + \{f,h\}g \quad \forall f,g,h \in P\,, \tag{32}$$

called the Leibniz rule. Note that this rule is a generalization of the ordinary product rule for differentiation.

Examples

1. Let M be a smooth manifold. The space $V = (\Gamma^\infty(TM), [\cdot,\cdot])$ of smooth vector fields on M, equipped with the commutator bracket as Lie bracket, is a first example of a Lie algebra. Skew-symmetry and bilinearity of $[\cdot,\cdot]$ are clear and the Jacobi identity follows from a straightforward calculation.

2. Let M again be a smooth manifold. Then $P = (\mathcal{C}^\infty(M), \{\cdot,\cdot\} \equiv 0)$ is a first albeit trivial example of a Poisson algebra, where the commutative product is just the multiplication of two smooth function (which is again smooth).

3. Let (M,ω) be a symplectic manifold. Define $\{f,g\} := \omega(X_f, X_g)$ for $f,g \in \mathcal{C}^\infty(M)$. Here, X_f and X_g are the associated hamiltonian vector fields of f and g respectively, as defined in subsection 3.1.2. The rest of this section is devoted to showing that with these definitions $P = (\mathcal{C}^\infty(M), \{\cdot,\cdot\})$ carries the structure of Poisson algebra.

Definition 3.10 (Poisson manifold[23])**.** *Let M a smooth manifold, $A = \mathcal{C}^\infty(M)$ its algebra of smooth functions. Then M is a Poisson manifold if A is a Poisson algebra with $(f,g) \mapsto fg$ the commutative product, and with a Poisson bracket $\{\cdot,\cdot\} : A \otimes A \to A$ defined as follows. Since $\{\cdot,\cdot\}$ must satisfy the Leibniz rule, it comes from a skew-symmetric bivectorfield. So $\exists\, \alpha \in \Gamma^\infty(\Lambda^2 TM)$ such that $\{f,g\} = \alpha(df,dg)$ for all $f,g \in A$. Furthermore the Jacobi identity must hold.*

Let's check explicitly that this bracket satisfies the Leibniz rule:

Leibniz Let $f,g,h \in A$, then $\{fg,h\} = \alpha(d(fg),dh) = \alpha(df \cdot g + (-1)^0 f \cdot dg, dh)$ since d is an anti-derivation of degree 1 and f a 0-form. By bilinearity of α, this yields $\{fg,h\} = \alpha(df,dh)g + f\alpha(dg,dh) = \{f,h\}g + f\{g,h\}$ which establishes the Leibniz rule.

The condition for this bracket to satisfy the Jacobi identity can be reformulated as follows:

Jacobi Let again $f,g,h \in A$, then $\{\{f,g\},h\} = \alpha\big(d\alpha(df,dg),dh\big)$. Now, in local coordinates $\{x^i\}$ the bracket of two functions f,g reads $\{f,g\} = \alpha^{ij}\partial_i f \partial_j g$ where $\partial_i f := \partial f/\partial x^i$. Keeping in mind that the bivectorfield's coefficients need not be constant, we find

$$\begin{aligned}
\{\{f,g\},h\} &= \alpha^{ij}\partial_i \left(\alpha^{kl}\partial_k f \partial_l g\right)\partial_j h = \alpha^{ij}\alpha^{kl}\left(\partial_{ik} f \partial_l g + \partial_k f \partial_{il} g\right)\partial_j h + \alpha^{ij}(\partial_i \alpha^{kl})\,\partial_k f\,\partial_l g\,\partial_j h\\
\{\{g,h\},f\} &= \alpha^{ij}\partial_i \left(\alpha^{kl}\partial_k g \partial_l h\right)\partial_j f = \alpha^{ij}\alpha^{kl}\left(\partial_{ik} g \partial_l h + \partial_k g \partial_{il} h\right)\partial_j f + \alpha^{ij}(\partial_i \alpha^{kl})\,\partial_k g\,\partial_l h\,\partial_j f\\
\{\{h,f\},g\} &= \alpha^{ij}\partial_i \left(\alpha^{kl}\partial_k h \partial_l f\right)\partial_j g = \alpha^{ij}\alpha^{kl}\left(\partial_{ik} h \partial_l f + \partial_k h \partial_{il} f\right)\partial_j g + \alpha^{ij}(\partial_i \alpha^{kl})\,\partial_k h\,\partial_l f\,\partial_j g\,.
\end{aligned}$$

Observing that $\alpha^{ij} = -\alpha^{ji}$ and that the expressions without derivatives of α^{ij} are invariant under switching $\{ij\} \leftrightarrow \{kl\}$, one concludes that adding the above three terms on the left yields zero. For the Jacobi identity to hold, the remaining three terms on the right need to sum up to zero as well. This translates into the condition

$$\alpha^{ij}(\partial_i\alpha^{kl})\,\partial_k f\,\partial_l g\,\partial_j h + \alpha^{ij}(\partial_i\alpha^{kl})\,\partial_k g\,\partial_l h\,\partial_j f + \alpha^{ij}(\partial_i\alpha^{kl})\,\partial_k h\,\partial_l f\,\partial_j g = 0$$

which is equivalent to the vanishing of the so-called Schouten-Nijenhuis bracket[18]:

$$\boxed{\alpha^{ij}(\partial_i\alpha^{kl})\,\partial_j \wedge \partial_k \wedge \partial_l =: [\alpha,\alpha]_{SN} = 0}\,. \tag{33}$$

This bracket plays an important role in Kontsevich's Formality theorem in the next section.

When $[\alpha,\alpha]_{SN} = 0$, so when the Jacobi identity holds, $\{\cdot,\cdot\}$ is indeed seen to define a Poisson bracket on A since α is by assumption bilinear and skew-symmetric.

It is now possible to show that the algebra of smooth functions $\mathcal{C}^\infty(M)$ of a symplectic manifold (M,ω) can be equipped with a Poisson bracktet, turning it into a Poisson manifold. For this, recall that by nondegeneracy of the symplectic 2-form ω, there is a unique vector field $X_f \in \mathcal{X}(M)$ assigned to each $f \in \mathcal{C}^\infty(M)$ such that $i_{X_f}\omega = \omega(X_f,\cdot) = df$. Using this fact, a bracket is defined as follows:

$$\{\cdot,\cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M),\ (f,g) \mapsto \{f,g\} := \omega(X_f, X_g)\,. \tag{34}$$

Proposition 5. *The bracket defined above is in fact a Poisson bracket: it is a skew-symmetric bilinear mapping (i) that satisfies both the Leibniz rule (ii) and the Jacobi identity (iii).*

Proof.

(i) Since ω is a skew-symmetric bilinear mapping, the bracket is as well.

(ii) Let $f,g,h \in \mathcal{C}^\infty(M)$, hence by the commutative product also $fg \in \mathcal{C}^\infty(M)$. Now, let X_{fg} be the hamiltonian vector field corresponding to fg, so $\omega(X_{fg},\cdot) = d(fg) = df\cdot g + (-1)^0 f\cdot dg$. By bilinearity and skew-symmetry

$$\begin{aligned}\{fg,h\} &\equiv \omega(X_{fg},X_h) = d(fg)(X_h)\\ &= df(X_h)\cdot g + f\cdot dg(X_h) = \omega(X_f,X_h)\cdot g + f\cdot\omega(X_g,X_h)\\ &\equiv \{f,h\}g + f\{g,h\}\,.\end{aligned}$$

Hence, the bracket satisfies the Leibniz rule. □

Now, proving the Jacobi identity is always a somewhat laborious exercise. The proof found in [24] is actually quite elegant, so this is the one that will be followed. For the elegance to become apparant, however, three auxiliary results are needed. All of them can be proven in a quite straightforward manner from their definitions, so these proofs are only indicated or omitted.

Proposition 6 (Auxiliary results)**.** *Let (M,ω) a symplectic manifold and let $f,g \in \mathcal{C}^\infty(M)$. Then*

1. $X_{\{f,g\}} = -[X_f,X_g]$[19];
2. $[\mathcal{L}_{X_f},\mathcal{L}_{X_g}] = \mathcal{L}_{[X_f,X_g]}$ *or equivalently, the Lie derivative is a Lie algebra homomorphism;*
3. $\{f,g\} = i_{X_g}df = \mathcal{L}_{X_g}f$ *since by Cartan* $\mathcal{L}_{X_f}g = i_{X_f}dg$.

[18] See §3.2 of [30] for more information.
[19] The proof of this uses $i_{[X,Y]}\omega = d\omega(Y,X)$ for all $X,Y \in \mathcal{X}(M)$ (see [21]).

Now for the proof of the Jacobi identity:

Proof.

(iii) Let $f, g, h \in \mathcal{C}^\infty(M)$ then consider

$$\begin{aligned}\{\{f,g\},h\} + \{\{h,f\},g\} &\overset{(3)}{=} \mathcal{L}_{X_h}\{f,g\} + \mathcal{L}_{X_g}\{h,f\} \overset{(3)}{=} \mathcal{L}_{X_h}\mathcal{L}_{X_g}f - \mathcal{L}_{X_g}\mathcal{L}_{X_h}f \\ &= [\mathcal{L}_{X_h}, \mathcal{L}_{X_g}]f \overset{(2)}{=} \mathcal{L}_{-[X_g,X_h]}f \\ &\overset{(1)}{=} \mathcal{L}_{X_{\{g,h\}}}f \overset{(3)}{=} \{f,\{g,h\}\}\,.\end{aligned}$$

Hence $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$ for all $f, g, h \in \mathcal{C}^\infty(M)$, so this establishes Jacobi. $\square$

Finally, the work done in this section is summarized in the following theorem:

Theorem 3. *Every symplectic manifold (M, ω) carries a natural structure of Poisson manifold. The Poisson algebra is $P = (\mathcal{C}^\infty(M), \{\cdot,\cdot\})$, where the bracket is defined as $\{\cdot,\cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$, $(f,g) \mapsto \{f,g\} := \omega(X_f, X_g)$.*

In conclusion: it is now proven that symplectic manifolds are a special case of Poisson manifolds, on which the general result of Kontsevich as found in [11] applies. Furthermore, it was shown that for a hamiltonian system, specified by a triple (M, ω, H), Hamilton's equations of classical mechanics hold. Therefore, classical mechanics can indeed be described in terms of Poisson geometry. This concludes the development of symplectic and Poisson theory for the purposes of this thesis.

4 General statement of deformation quantization

In this third part, the way will be cleared to state Kontsevich's formality theorem. For this purpose, the problem of quantizing a classical theory in general is discussed in the preliminaries, following a classic article by [8] about functorial geometric quantization. Afterwards, given a Poisson manifold $(P, \{\cdot, \cdot\})$ associated to a certain classical mechanical system, its quantization by deformation is defined and discussed.

The object being deformed is the product of the algebra $A = \mathcal{C}^\infty(P)$ of smooth functions on the manifold. We are only interested in products up to equivalence under a certain group action. Hochschild (co)homology measures to great extent the deformation of an algebra and captures this equivalence. The second section discusses definitions, small proofs and statement of results about the group action, the Hochschild complex and its cohomology.

Finally, in the third section, Kontsevich's theorem is formulated in terms of the differential graded Lie algebra (DGLA) structure of the Hochschild complex and its cohomology. The explicit formula for finding a star-product on a given manifold is stated. It is then shown in the last section that by applying this formalism to the trivial Poisson structure, the Moyal product is found once more.

4.1 Deformation quantization

4.1.1 Preliminaries: quantization in general

Following [8], let's take a step back for a moment and consider a classical mechanical system described by the symplectic manifold (M, ω), which is the total phase space. In section 3.1.1, symplectomorphisms were defined as maps leaving the symplectic structure of a manifold invariant. The symplectic category $\mathcal{S} = \{symplectic\, manifolds,\, symplectomorphisms\}$ is formed by these objects and maps; it can be used to describe classical mechanics.

Quantum mechanical systems however, are described by a Hilbert space of states. The corresponding class of maps that leave the Hilbert structure invariant are unitary transformations, since they leave the inner product invariant. Together, these objects and maps define the unitary category $\mathcal{U} = \{hilbert\, spaces,\, unitary\, transformations\}$. A reasonable definition of a quantization of a classical theory would therefore be a covariant functor $\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{U}$ satisfying:

1. To every symplectic manifold (M, ω) is associated a Hilbert space $\mathcal{F}(M, \omega) = (\mathcal{F}[M], \mathcal{F}[\omega])$. In this space, $\mathcal{F}[\omega] : \mathcal{F}[M] \times \mathcal{F}[M] \longrightarrow \mathbb{C}$ is the inner product.

2. To every symplectomorphism $\varphi : (M, \omega_M) \to (N, \omega_N)$ is associated a unitary transformation $\mathcal{F}[\varphi] : \mathcal{F}(M, \omega_M) \to \mathcal{F}(N, \omega_N)$.

The question is, however, if it is possible to find a functor consistent with Schrödinger's theory of quantum mechanics. In 1946, it was shown by Groenewold in [13] that it is not possible to find a *full* quantization, meaning that not every smooth function in $\mathcal{C}^\infty(M)$ can be quantized consistently: he showed that it is not possible to send the classical Poisson bracket of any two functions onto their quantum commutator (their Lie bracket) [30]. This results is known as Groenewold's no-go theorem.[20] It is partially for this reason that it was proposed by the authors of the influential paper [9] to instead look for an approach by deformation theoretical means. Their idea is to realize quantum mechanics in an autonomous manner, directly on classic phase

[20]Still another approach is the one taken by Abraham and Marsden (see [20]), to obtain a 'reasonable' quantization using representation theoretical means. For this, an irreducibility requirement on representations is necessary so as to not violate Heisenberg's uncertainty principle. Including this irreducibility criterion also yielded inconsistencies. For a discussion on this approach, see §4 of [8].

space, by deforming its product to be non-commutative. As a result, since the mapping of brackets need only be satisfied up to $\mathcal{O}(\hbar^2)$ so as to find classical mechanics in the limit where $\hbar$ tends to zero, the inconsistencies are resolved. Moreover, by the Stone-von Neumann theorem, a unitary algebra representation satisfying the canonical commutations yields a unique Hilbert space (the representation space), up to unitary equivalence. This Hilbert space contains the regular quantum states and expectation values can be calculated by virtue of its inner product[5].

The physical inspiration for deformation theory is that different domains of physics are linked by deforming their algebraic structure encoded by certain parameters. As an example, consider relativistic mechanics where the important parameter is c, the speed of light. When it tends to infinity, or equivalenty, when $v/c \ll 1$, Einstein's theory reduces to Newton's classical mechanics.

The authors of the aforementioned paper imagined a similar construct in quantum mechanics: when the characteristic paramater $\hbar$ goes to 0, quantum mechanics reduces to classical mechanics. This can equivalently be stated as $S/\hbar \gg 1$ where S denotes the classical action of the system under consideration. So in this case, $\hbar$ is to be considered as a deformation parameter in the sense of definition 2.1; note that this was already mentioned in relation to the Moyal product in the introduction of section 2.

4.1.2 Definition of deformation quantization

Let us now pursue the above sketched idea; this will be done by following the structure of [23].

Let $(P, \{\cdot,\cdot\})$ be a Poisson manifold, denote $A = \mathcal{C}^\infty(P)$ its algebra of smooth functions. This is the algebraic structure that is to be deformed. In section 3.2 it was shown that $(A, \{\cdot,\cdot\})$ is indeed a Poisson algebra. Moreover, all classical mechanical systems that can be described in the symplectic formalism carry a Poisson structure. The discussion about deformation in the previous section suggests that for two functions $f, g \in A$ we would want something like

$$f \star g = f \cdot g + \frac{i\hbar}{2}\{f, g\} + \mathcal{O}(\hbar^2)\,, \tag{35}$$

where $\star$ denotes the star product that results from deforming the algebra A. [21]

This can be seen from the fact that this proto-star product can be related to canonical quantization, in which smooth functions are simply promoted to operators and where the brackets correspond via $i\hbar\{\cdot,\cdot\} \longleftrightarrow [\cdot,\cdot]$. Up to $\mathcal{O}(\hbar^2)$, this structure is already present in the above formule, which is clear from the so-called star commutator

$$[f, g]_\star := f \star g - g \star f = \frac{i\hbar}{2}\left(\{f, g\} - \{g, f\}\right) + \mathcal{O}(\hbar^2) = i\hbar\{f, g\} + \mathcal{O}(\hbar^2)\,.$$

Expanding on the above intuition, the sought after deformation of the algebra $A = \mathcal{C}^\infty(P)$ now lies in the following definition of a star product on it [23]:

Definition 4.1 (Star product)**.** *A deformed product or star product is an associative, $\hbar$−adic continuous, $\mathbb{C}[[\hbar]]$-bilinear product*

$$\star : A[[\hbar]] \times A[[\hbar]] \longrightarrow A[[\hbar]]\,,$$

that takes values in the algebra of formal[22] *power series in $\hbar$. It is defined explicitly on A as*

$$f \star g := \sum_{n=0}^{\infty} B_n(f, g)\,\hbar^n \qquad \textit{for all}\, f, g \in A \tag{36}$$

[21] According to [23], this intuition can already be found in early work of Dirac (see [6]), although he did not state the fact that this is up to $\mathcal{O}(\hbar^2)$; presumebly, though, he had it in mind.

[22] Formal means that there is not necessarily a notion of convergence in $A[[\hbar]]$, nor reason to worry about it. The powers of $\hbar$ are used to index contributions of the bidifferential operators per order of magnitude.

and extended to $A[[\hbar]]$ by $\mathbb{C}[[\hbar]]$-bilinearity. Furthermore, in zeroth order the star product reduces to the commutative product, so $f \star g\big|_{\hbar=0} = f \cdot g$. The $B_n : A \times A \longrightarrow A$ are bidifferential operators on A, so they are differential operators in both arguments.

There are some remarks to be made about this definition:

1. First of all, the star product is defined on A. The extension to formal power series by $\mathbb{C}[[\hbar]]$-bilinearity is given by
$$\left(\sum_{k=0}^{\infty} f_k \hbar^k\right) \star \left(\sum_{l=0}^{\infty} g_l \hbar^l\right) = \sum_{n=0}^{\infty} \left(\sum_{k+l+m=n} B_m(f_k, g_l)\right) \hbar^n \,, \quad \text{where } f_k, g_l \in A\,.$$
This extension is done so as to be able to relax the requirement on the mapping between classical Poisson brackets and quantum commutators: instead of a direct mapping, only the correspondence principle is to be satisfied.

2. Again by $\mathbb{C}[[\hbar]]$-bilinearity, the associativity on A extends to the algebra of formal power series. Explicitly, this means for $f, g, h \in A$ that
$$(f{\star}g){\star}h = f{\star}(g{\star}h) \quad \text{or} \quad \sum_{k+l=n} B_k(B_l(f,g),h) = \sum_{k+l=n} B_k(f, B_l(g,h)) \quad \forall n \in \mathbb{N}{\cup}\{0\}\,. \tag{37}$$

3. The fact that the star product should reduce to the ordinary commutative product in zeroth order means $B_0(f,g) = f \cdot g$. It allows for the notation
$$f \star g := f \cdot g + \sum_{n=1}^{\infty} B_n(f,g)\, \hbar^n$$
which shows the deformative nature of the product more clearly: $\lim_{\hbar \to 0} f \star g = f \cdot g$.

4. The B_n could just be bilinear operators. However, bi*differential* operators are 'local' in the sense that derivation in a point only depends on a (arbitrarily) small neighbourhood about that point (cf. the concept of *germ*, see [16]). Hence, the differential aspect of these operators encodes locality from a physical point of view [30].

5. Although this remark will not be discussed in more detail in this thesis, the claim that the star product be $\hbar$-adic continuous is still relevant. It entails the following: to talk about continuity, a topology is needed. Take an element $f = \sum_{n=0}^{\infty} f_n \,\hbar^n \in A[[\hbar]] = \mathcal{C}^{\infty}(M)[[\hbar]]$. The order $ord(f)$ is defined as the smallest $n \in \mathbb{N} \cup \{0\}$ for which $f_n \neq 0$; if $f \equiv 0$ then $ord(f) := \infty$. Using this function, a metric can be defined on $A[[\hbar]]$ by
$$d : A[[\hbar]] \times A[[\hbar]] \longrightarrow [0,1],\, (f,g) \mapsto d(f,g) = \begin{cases} 2^{-ord(f-g)} & \text{if} f \neq g \\ 0 & \text{if} f = g \end{cases}$$
It can be shown that the collection of open balls $\mathcal{T} = \{B(f \equiv 0, r) \,|\, r \in [0,1]\}$ induces a topology that is Hausdorff. It is called the $\hbar$-adic topology on $A[[\hbar]]$. The assertion in definition 4.1 is, thus, that applying the star product is a continuous operation with respect to this $\hbar$-adic topology.[19] This essentially means that the product is continuous in every order of $\hbar$, an assumption usually made in physics.

6. By the universal property of tensor products (see [25], 1 chapter XVI), $\mathbb{C}[[\hbar]]$-bilinearity of the star product $\star$ implies that the following diagram commutes

$$\begin{tikzcd}
A[[\hbar]] \times A[[\hbar]] \arrow[r, "\star"] \arrow[d, "\iota"'] & A[[\hbar]] \\
A[[\hbar]] \otimes_{\mathbb{C}[[\hbar]]} A[[\hbar]] \arrow[ur, dashed, "\bar{\star}"'] &
\end{tikzcd}$$

which reads as $\star = \bar{\star} \circ \iota$. By definition of the tensor product, we have $\iota(a, b) := a \otimes_{\mathbb{C}[[\hbar]]} b$ for $a, b \in A[[\hbar]]$. So equivalently, the notation using the tensor product space and the star product $\bar{\star}$ could be used; this emphasizes the fact that $\star$ is a $\mathbb{C}[[\hbar]]$-bilinear product.

As was mentioned in the introduction, quantization idealy associates a unique quantum system (star product) to a classical system (Poisson bracket). It is therefore important to keep in mind that our goal essentially is to relate a Poisson structure to a star product in a one-one manner. This goal will be clarified step-by-step as we proceed.

Now, using both the associativity on $A[[\hbar]]$ and the second remark, a relation between these two can be shown: the antisymmetric part of B_1 is seen to carry the structure of Poisson bracket [23]. This means that even when the manifold M carries no Poisson structure, its algebra $\mathcal{C}^\infty(M)$ can be given one by defining B_1^- as bracket; hence, it is turned into a Poisson manifold.[23] Conversely, when M already carries such a structure, it would be natural to ask for $B_1^- \equiv \{\ ,\ \}$. However, note that the symmetric part B_1^+ need not be zero. The relation between B_1^+ and Poisson brackets is discussed in more detail in the next section.

The above leads naturally to the following definition of quantization of a Poisson manifold:

Definition 4.2 (Quantization by deformation)**.** *A quantization by deformation of a Poisson manifold* $(P, \{\cdot, \cdot\})$ *is a star product as in definition 4.1 on its algebra of smooth functions* $\mathcal{C}^\infty(P)$*, such that the equality* $B_1^-(\cdot, \cdot) = \{\cdot, \cdot\}$ *between differential operators holds.*

To make all this precise, let's prove the assertion that B_1^- carries the structure of Poisson bracket:

Proposition 7. *The antisymmetric part of the bidifferential operator* B_1 *is a Poisson bracket on* $\mathcal{C}^\infty(P)$*.*

Proof. Let us first split B_1 in its symmetric and antisymmetric parts as follows

$$\begin{aligned}
\forall f, g \in A: \quad B_1(f, g) &\equiv B_1^-(f, g) + B_1^+(f, g) \\
&\equiv \frac{1}{2}\left(B_1(f, g) - B_1(g, f)\right) + \frac{1}{2}\left(B_1(f, g) + B_1(g, f)\right).
\end{aligned}$$

So by definition, B_1^- is antisymmetric. Using the star product's associativity as made explicit in the second remark after definition 4.1, both the Leibniz rule and the Jacobi identity can be shown. Consider the associativity condition (37) for $n = 1, 2$, this reads

$$\begin{gathered}
B_1(fg, h) + B_1(f, g)h = B_1(f, gh) + fB_1(g, h) \\
B_2(fg, h) + B_1(B_1(f, g), h) + B_2(f, g)h = B_2(f, gh) + B_1(f, B_1(g, h)) + fB_2(g, h).
\end{gathered}$$

Including the two cyclic permutations of the first equation, this yields after some manipulation

$$B_1^-(fg, h) = fB_1^-(g, h) + B_1^-(f, h)g$$

[23] One could also define $\{f, g\} = B_1(f, g) - B_1(g, f)$ as is done in [28] to turn $\mathcal{C}^\infty(M)$ into a Poisson manifold. Since this only differs from our definition by a factor 2, it is just a matter of taste which definition to use.

where the Leibniz rule has been found explicitly. Using this rule, and adding the six permutations of the second equation above yields the Jacobi identity for B_1^-. Explicitly:

$$B_1^-(B_1^-(f,g),h) + B_1^-(B_1^-(g,h),f) + B_1^-(B_1^-(h,f),g) = 0 \quad \forall f,g,h \in A\,.$$

Thus, B_1^- can be considered a Poisson bracket on $A = \mathcal{C}^\infty(P)$. □

4.2 Equivalence of products

In the previous section, it was shown that the antisymmetric part of the first order bidifferential operator of a star product on $\mathcal{C}^\infty(M)$ carries a Poisson structure. When deforming Poisson manifolds $(P, \{\cdot,\cdot\})$, it would therefore be natural to ask for $B_1^- \equiv \{\ ,\ \}$. Note that from now on, M denotes a generic smooth manifold whereas P denotes a *Poisson* manifold.

First of all, a group action will be introduced on $A[[\hbar]] = \mathcal{C}^\infty(P)[[\hbar]]$ to explore the relation between star products on Poisson manifolds and their Poisson structure. Then, using the first Hochschild cohomology group of the algebra $\mathcal{C}^\infty(P)$, it will be shown that each equivalence class of star products induced by this action contains deformations as in definition 4.2 for which $B_1^+ \equiv 0$. In other words: by introducing this equivalence we need only consider *natural* quantizations of Poisson manifolds, those for which $B_1(\cdot,\cdot) \equiv \{\cdot,\cdot\}$. Again, the brief description in [23] will be elaborated on.

4.2.1 A group action

To construct an action on $A[[\hbar]]$, recall the definition of a star product on $A = \mathcal{C}^\infty(P)$: let $f, g \in A$, the product was defined as

$$f \star g = f \cdot g + \sum_{n=1}^{\infty} B_n(f,g)\,\hbar^n$$

where $B_n : A \times A \longrightarrow A$ are linear bidifferential operators, and the product was to be associative. Now, choosing an other family $\{\tilde{B}_n\}_{n\in\mathbb{N}}$ of bidifferential operators will result in a different star product. Since composing linear differential operators yields again a linear differential operator, let's consider the following mapping

$$D : A \longrightarrow A[[\hbar]],\ f \mapsto D(f) := f + \sum_{n=1}^{\infty} D_n(f)\,\hbar^n \tag{38}$$

where the $D_n : A \longrightarrow A$ are again linear differential operators; in particular, D is itself a linear mapping. This map can be uniquely extended to the whole of $A[[\hbar]]$ by asking for $\mathbb{C}[[\hbar]]$-linearity. The idea is to construct new star products on $A[[\hbar]]$ by first applying such a mapping on both functions, then the original star product. Explicitly:

$$\forall f,g \in A \quad \text{define} \quad f \star_D g := D(f) \star D(g)\,. \tag{39}$$

The above defines a star product since $\star$ is a star product and since D is a linear differential operator. Note that the requirement that $f \star_D g\big|_{\hbar=0} = f \cdot g$ of definition 4.1 is met: setting $D_0 = \mathbb{1}$ makes the product reduce to the commutative product in the classical limit.

Definition 4.3 (Gauge action)**.** *A mapping $D : A[[\hbar]] \longrightarrow A[[\hbar]]$ is called a gauge action if it is a $\mathbb{C}[[\hbar]]$-linear mapping of the form* (38) *where explicitly $D_0 = \mathbb{1}$. The collection of gauge actions is denoted by G. In the physics literature, such a gauge action is often called a field redefinition.*

Notice that D can be inverted if and only if D_0 can be inverted, since all the information about $f \in A$ is already present in

$$\lim_{\hbar \to 0} D(f) = D_0(f) = f\,.$$

By definition $D_0 = \mathbb{1}$, so all gauge actions are invertible. Their collection G is now seen to have nice properties under composition: those of a group.

Proposition 8. *The collection of gauge actions G forms a group under composition of maps, with identity element the identity mapping $\mathbb{1} : A[[\hbar]] \longrightarrow A[[\hbar]]$, $f \mapsto \mathbb{1}(f) := f$. This group is denoted by $\mathcal{G} = (G, \circ, \mathbb{1})$.*

Proof.

Composition Denote $\mu : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ the composition. Let $D_1, D_2 \in \mathcal{G}$, then for $f \in A[[\hbar]]$

$$\begin{aligned}\mu(D_1, D_2)(f) \equiv (D_1 \circ D_2)(f) &= D_1 \circ \big(\sum_{n=0}^{\infty} D_{2,n}(f)\,\hbar^n\big)\\ &= \sum_{m=0}^{\infty} D_{1,m} \circ \left(\sum_{n=0}^{\infty} D_{2,n}(f)\,\hbar^n\right)\hbar^m = \sum_{k=0}^{\infty}\left(\sum_{m+n=k} (D_{1,m} \circ D_{2,n})(f)\,\hbar^{m+n}\right)\\ &= \sum_{k=0}^{\infty}\left(\sum_{m+n=k} (D_{1,m} \circ D_{2,n})(f)\right)\hbar^k =: \sum_{k=0}^{\infty} D_k(f)\,\hbar^k\,,\end{aligned}$$

where the $\{D_k\}_{k=0}^{\infty}$ form a new family of linear differential operators. Note that since all the power series are formal and by $\mathbb{C}[[\hbar]]$-linearity, it is permitted to take operators inside infinite sums. Furthermore, $\sum_{m+n=0} D_{1,m} \circ D_{2,n} = D_{1,0} \circ D_{2,0} = \mathbb{1}$, so indeed $D_0 = \mathbb{1}$. Hence $\mu(D_1, D_2) \equiv D := \mathbb{1} + \sum_{k=1}^{\infty} D_k\,\hbar^k$ is again an element of $\mathcal{G}$.

Neutral element The identity mapping $\mathbb{1} : f \mapsto f$, $A[[\hbar]] \longrightarrow A[[\hbar]]$ is the neutral element under composition: for all $D \in \mathcal{G}$, $D \circ \mathbb{1} = D = \mathbb{1} \circ D$.

Inverse It was already noted that the requirement of D_0 being invertible suffices for the entire operator to be invertible; denote it's inverse by E. An explicit formula is

$$E_0 = \mathbb{1}\,, \quad E_n = -\sum_{m=0}^{n-1} E_m\,D_{n-m} \quad \forall n > 0\,, \tag{40}$$

which is easily verified by an explicit calculation.

Associativity Finally, associativity follows from the fact that composition of maps is associative.

We conclude that $\mathcal{G} = (G, \circ, \mathbb{1})$ is a group. □

In the above equation (39) a new star product $\star_D$ is defined by choosing some $D \in \mathcal{G}$. Although this does define a star product, it does not tell us much about the one we had. It is therefore interesting to use the group structure of $\mathcal{G} = (G, \circ, \mathbb{1})$ to again invert the operator D after taking the product $\star_D$: the total operation will be called a gauge transformation.

For this purpose, let $\mathcal{G} \times A \longrightarrow A[[\hbar]]$ be the action of $\mathcal{G}$ on A defined as

$$(D, f) \mapsto D \cdot f := D(f) = f + \sum_{n=1}^{\infty} D_n(f)\,\hbar^n$$

and extended by $\mathbb{C}[[\hbar]]$-linearity to $A[[\hbar]]$; note that this is the same construction as in equation (38), in particular, D is a linear mapping. By the above, it is now clear that this indeed defines an action on $A[[\hbar]]$, whence the name gauge action is justifed. This leads to

Definition 4.4 (Gauge transformation)**.** *Fix a $D \in \mathcal{G}$, let E be its inverse. A new star product $\tilde{\star}$ is defined on $A[[\hbar]]$ as follows: for $f, g \in A[[\hbar]]$, set $f \tilde{\star} g = D \circ (E(f) \star E(g))$.*

This product is associative and $\mathbb{C}[[\hbar]]$-bilinear since D, E and $\star$ are. Furthermore, $B_0 = D_0 = E_0 = \mathbb{1}$ which entails $f \tilde{\star} g|_{\hbar=0} = f \cdot g$, so it is a star product as in definition 4.1 indeed. Explicitly:

$$f \tilde{\star} g = f \cdot g + \sum_{n=1}^{\infty} C_n(f,g)\, \hbar^n \quad \text{where} \quad C_n(f,g) = \sum_{m+k+l+j=n} D_m \circ B_k \left(E_l(f), E_j(g)\right) . \tag{41}$$

Note that $\tilde{\star}$ is the one operator that makes the following diagram commute:

$$\begin{tikzcd}
A[[\hbar]] \times A[[\hbar]] \arrow[r, "\star"] & A[[\hbar]] \arrow[d, "D"] \\
A[[\hbar]] \times A[[\hbar]] \arrow[u, "E \times E"] \arrow[r, dashed, "\tilde{\star}"] & A[[\hbar]]
\end{tikzcd}$$

To quote [23], all this can be interpreted as follows:

> Given a star product $\star$, which is determined by a family of linear bidifferential operators $\{B_n\}_{n \geq 0}$, and a gauge action $D \in \mathcal{G}$ we can think of D as a formal change of coordinates, and call $\tilde{\star} = D(\star)$ the star product in the new coordinates.

Star products that can be linked by a gauge transformation are called *formally equivalent* (see §5 of [19]). Note that the naive star product $\star_N$ and symmetric star product $\star_S$ encountered in section 2.4.2 are formally equivalent in this sense. As for the Moyal product $\star_\hbar$, it was seen to be exactly the same as the symmetric star product $\star_S$ (so formally equivalent under the identity mapping).

The action of gauge transformations induces an equivalence relation on star products. The corresponding equivalence classes of star products play a prominent role in examing the uniqueness of deformation quantization. They appear in the corollary of Kontsevich's Formality theorem that is discussed in section 4.3.3.

4.2.2 Hochschild complex & cohomology

In the previous section, an action of gauge equivalence on the set of star products was constructed. To show that in each equivalence class of this action are star products for which $B_1^+ \equiv 0$, Hochschild (co)homology is a great tool. It is a (co)homology theory for associative algebra's and was introduced in 1945 by mathematician Gerhard Hochschild in the classic paper [26]. For some background information, proofs and generalities, especially regarding the interpretation of this particular homology theory, see the appendix.

Definition 4.5 (Hochschild cocomplex)**.** *Let $A = \mathcal{C}^\infty(M)$ be the associative $\mathbb{R}$-algebra of smooth real functions on a smooth manifold M. For $n \geq 0$, define the Hochschild chains with coefficient bimodule A itself (see appendix) as*

$$C^n(A,A) = \{\phi : A^n \longrightarrow A \,|\, \phi \textit{ multilinear, differential operator}\}$$

where by definition $A^0 \equiv \mathbb{R}$, so $C^0(A,A) = Hom(\mathbb{R}, A)$. The chains $\{C^n(A,A)\}_{n \geq 0}$ are linked by the differentials $d_n : C^n(A) \longrightarrow C^{n+1}(A)$ that are defined as

$$d_n(\phi)(a_1, \ldots, a_{n+1}) := \begin{cases} a\,\phi - \phi\,a & \textit{for } n = 0 \\ a_1\,\phi(a_2, \ldots, a_{n+1}) + (-1)^{n+1}\phi(a_1, \ldots, a_n)\,a_{n+1} & \\ + \sum_{j=1}^{n} (-1)^j \phi(a_1, \ldots, (a_j\, a_{j+1}), \ldots, a_n) & \textit{for } n > 0 \end{cases}$$

for $a, a_1, \ldots, a_{n+1} \in A$. Then, $(C^*(A,A), d_*)$ is the Hochschild complex of the algebra A.

In the appendix, it is shown that $d_{n+1} \circ d_n = 0$ holds for any arbitrary k-algebra A, where k is a commutative ring, on the condition that the $\phi \in C^n(A,A)$ are only asked to be multilinear. In this case however, only results on the Hochschild cohomology groups of the associative algebra $A = \mathcal{C}^\infty(M)$ are needed where M is a smooth manifold. It turns out that $d_{n+1} \circ d_n = 0$ still holds in that category when the functions are taken to be differential operators.

The Hochschild cohomology groups[24] are now defined in the normal way as

$$H^n_{Hoch}(A) := H^n_{Hoch}(C^*(A,A), d_*) = \frac{\ker\left(d_n : C^n(A,A) \longrightarrow C^{n+1}(A,A)\right)}{\mathrm{im}\left(d_{n-1} : C^{n-1}(A,A) \longrightarrow C^n(A,A)\right)} \,. \tag{42}$$

Furthermore, one can prove that for a smooth manifold M where its algebra of smooth functions is denoted by $A = \mathcal{C}^\infty(M)$ as before, its Hochschild cohomology is given by $H^n_{Hoch}(A) = \Gamma^\infty(\Lambda^n TM)$ (see for instance [27]). This is the algebra of smooth multivector fields on M. When relating Kontsevich's Formality theorem to deformation quantization, this result will be of great importance, since a Poisson structure (classical system) is defined by a smooth bivectorfield $\alpha \in \Gamma^\infty(\Lambda^2 TM)$ (see definition 3.10 in section 3.2). This importance will be explained in section 4.3.3.

It is now time to link the first order bidifferential operator $B_1 : A \otimes A \longrightarrow A$ of a star product to the second Hochschild cohomology group of $\mathcal{C}^\infty(M)$. By the above, this group is isomorphic to the space of smooth bivectorfields $H^2_{Hoch}(\mathcal{C}^\infty(M)) = \Gamma^\infty(\Lambda^2 TM)$. Note that $B_1 \in C^2(A,A)$. Recall that the associativity condition of star products imposes relations per order of $\hbar$. In first order

$$\forall f,g,k \in A : \; B_1(fg,k) + B_1(f,g)k = B_1(f,gk) + fB_1(g,k) \,. \tag{43}$$

Comparing this to definition 4.5 of the Hochschild chains, yields

$$\forall f,g,k \in A : \; d_2(B_1)\,(f,g,k) = fB_1(g,k) - B_1(fg,k) + B_1(f,gk) - B_1(f,g)k = 0 \,,$$

so B_1 is closed in the Hochschild complex: the closedness of a bidifferential operator in Hochschild cohomology is equivalent to its associativity condition. Furthermore, applying a gauge transformation to $\star$, equation (41) for $n = 1$ yields the transformed first order differential operator

$$C_1(f,g) = B_1(f,g) - \Big[fD_1(g) - D_1(fg) + D_1(f)g \Big] \qquad \text{where } f,g \in A \,,$$

since $E_1 = -E_0 \circ D_1 = -D_1$ by equation (40). Therefore, we have

$$\boxed{(B_1 - C_1)(f,g) = d_1(D_1)\,(f,g)} \qquad \text{for all } f,g \in A \,.$$

This expression is seen to have the structure of a Hochschild 1-coboundary, hence $d_2(B_1 - C_1) = 0$. This is important, since it means that B_1 and all first order operators C_1 of formally equivalent star products are the same element (i.e. the same cohomology class) in the second Hochschild cohomology group $H^2_{Hoch}(A)$. Consequently, up to gauge equivalence, the first order operator of a star product is a Hochschild cohomology class.

Moreover, $B_1 - C_1$ is symmetric in f, g : $(B_1 - C_1)(f,g) = (B_1 - C_1)(g,f)$. Hence, any gauge transformation only affects the symmetric part of B_1. Splitting B_1 and C_1 in their symmetric and antisymmetric parts as in the proof of proposition 7, it becomes clear that

$$C_1(f,g) = C_1^-(f,g) + C_1^+(f,g) = \underbrace{B_1^-(f,g)}_{antisymmetric} + \underbrace{\left[B_1^+(f,g) - \left(fD_1(g) - D_1(fg) + D_1(f)g\right)\right]}_{symmetric} \,.$$

[24] Actually, they have the structure of $\mathbb{R}$-module; this is shown in the appendix.

Since B_1 is a linear bidifferential operator, so is B_1^+. The assertion that there always is a linear differential operator D_1 (i.e. a formal sum of linear differential operators $D \in \mathcal{G}$) such that the symmetric part in the above equation vanishes is non-trivial. It amounts to stating that B_1^+ is a Hochschild 1-coboundary, which means that there is an element $K \in C^1(A, A)$ such that $B_1^+ = d_1(K)$. The Hochschild-Kostant-Rosenberg theorem (see §4 of [27]) says that this statement is true on $\mathbb{R}^d$, hence locally on a manifold. Then, by choosing a partition of unity, we may finally apply D_1 to any smooth function on the entire manifold M [30].

Assembling all the information, this shows the following equivalent statements:

Theorem 4.

1. *Any gauge equivalence class of star products contains at least one representative for which the first order operator B_1 is antisymmetric;*
2. *A Poisson manifold $(P, \{\cdot, \cdot\})$ can be quantized with $B_1^- = \{\ ,\ \}$ as in 4.2 if and only if it can be quantized with $B_1 = \{\ ,\ \}$[25];*
3. *The antisymmetric part of B_1 is invariant under gauge transformations;*

The above result entitles us to only consider *natural* quantizations of a Poisson manifold, those for which $B_1 = \{\ ,\ \}$. In the rest of this thesis, the attention will be restricted to this class of star products.

Remark

Note however that this only determines the representative of an equivalence class of star products up to $\mathcal{O}(\hbar^2)$; gauge transformations can still change higher order contributions of this representative. This will play an important role in the statement about uniqueness following from Kontsevich's Formality theorem in section 4.3.3.

4.3 Formulation of Kontsevich's theorem

The Formality conjecture that Kontsevich proved in 1997 in [11], is an assertation about differential graded Lie algebra's (DGLA). He showed that two objects are quasi-isomorphic as DGLAs. The concept of quasi-isomorphism is very intricate, so for details the reader is referred to the original [11] or the excellent [30]. In this thesis, only an approximate definition is given. The two objects that are quasi-isomorphic are the Hochschild cocomplex on the one hand, and its cohomology complex on the other hand. From this theorem, Kontsevich deduced a result on star products that we are interested in: one about classification.

First of all, the definition of a DGLA is given. It is shown that when such a structure is present on a complex $\mathfrak{g}$, it naturally induces a DGLA structure on its cohomology complex. Then, after introducing a so-called degree shift, it is proven that the Hochschild cocomplex is a first example of such a DGLA, whence its cohomology complex is one as well. After this, Kontsevich's Formality theorem is stated as in [11], including the corollary relevant for deformation quantization. These results are discussed and interpreted.

[25] It is interesting to note that a Poisson bracket can also be defined on $\mathcal{C}^\infty(M)$ by $\{f, g\} := B_1(f, g) - B_1(g, f)$, as is done in for example [28]. By definition, it is immediately skew-symmetric for B_1's symmetric part falls out. Since gauge transformations only affect the symmetric part of B_1, this construction yields the same equivalence classes under their action. Therefore, it is just a matter of taste (and a factor) which definition to follow.

4.3.1 Differential graded Lie algebra

Following [29], first of all the definition of this structure:

Definition 4.6 (DGLA)**.** *A differential graded Lie algebra over a field $\mathcal{K}$ of characteristic zero is a complex*

$$\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} g^k\,, \qquad d : g^k \longrightarrow g^{k+1}\,, \qquad d^2 = 0$$

that is also a graded Lie algebra with respect to the same gradation, with bracket

$$[\,,\,] : g^k \times g^l \longrightarrow g^{k+l}$$

that satisfies the requirements

1. $\forall x_i \in g^{s_i}$: $[x_1,[x_2,x_3]] + (-1)^{s_1(s_2+s_3)}[x_2,[x_3,x_1]] + (-1)^{s_3(s_1+s_2)}[x_3,[x_1,x_2]] = 0$
2. $\forall x_i \in g^{s_i}$: $[x_1,x_2] = (-1)^{1+s_1 s_2}[x_2,x_1]$.

Finally, there is a compatibility axiom relating the differential and the bracket:

$$\forall x_i \in g^{s_i} : \, d[x_1,x_2] = [dx_1,x_2] + (-1)^{s_1}[x_1,dx_2]\,.$$

It will presently be shown that the Hochschild cocomplex indeed carries all the structure required for a differential graded Lie algebra. This is a labourious exercise, so let us first of all proof the claim that a DGLA structure on a given complex naturally induces a DGLA structure on its cohomology complex [30].

Theorem 5 (Induced DLGA)**.** *Let $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} g^k$ be a differential graded Lie algebra with differential $d : g^k \longrightarrow g^{k+1}$ and Lie bracket $[\,,\,]_{\mathfrak{g}} : g^k \times g^l \longrightarrow g^{k+l}$. When equipped with the naturally induced Lie bracket $[\,,\,]_{\mathcal{H}}$ and zero differential $\delta \equiv 0$, its cohomology complex $\mathcal{H} := \bigoplus_{i\in\mathbb{Z}} \mathcal{H}^i(\mathfrak{g})$ has a natural structure of DGLA.*

Proof. Since $\mathfrak{g}$ is a complex, its cohomology is naturally defined as

$$\mathcal{H}^i(\mathfrak{g}) := \mathrm{Ker}(d : g^i \longrightarrow g^{i+1}) \Big/ \mathrm{Im}(d : g^{i-1} \longrightarrow g^i) \tag{44}$$

which equips $\mathcal{H}$ with the structure of graded vector space. Furthermore, a Lie bracket $[\,,\,]_{\mathcal{H}}$ can be defined on the cohomology classes as follows: for $|a| \in \mathcal{H}^k(\mathfrak{g})$ and $|b| \in \mathcal{H}^l(\mathfrak{g})$ set

$$\Big[|a|,|b|\Big]_{\mathcal{H}} := \Big|[a,b]_{\mathfrak{g}}\Big|\,. \tag{45}$$

We must verify that this is a well-defined graded Lie bracket. Let $\tilde{a} \in |a|$, $\tilde{b} \in |b|$ be other representatives of the respective cohomology classes. This means that $\tilde{a} = a + d\alpha$ and $\tilde{b} = b + d\beta$ for certain $\alpha \in g^{k-1}$ and $\beta \in g^{l-1}$. Their bracket yields

$$\Big[|\tilde{a}|,|\tilde{b}|\Big]_{\mathcal{H}} \equiv \Big|[\tilde{a},\tilde{b}]_{\mathfrak{g}}\Big| = \Big|[a+d\alpha, b+d\beta]_{\mathfrak{g}}\Big| = \Big|[a,b]_{\mathfrak{g}}\Big| + \Big|[d\alpha,b]_{\mathfrak{g}}\Big| + \Big|[a,d\beta]_{\mathfrak{g}}\Big| + \Big|[d\alpha,d\beta]_{\mathfrak{g}}\Big|\,.$$

The compatibility axiom of the DGLA reduces in cohomology to $|d[a,b]_{\mathfrak{g}}| = |0|$. This yields:

$$\Big|[d\alpha,b]_{\mathfrak{g}}\Big| = -(-1)^{k-1}\Big|[\alpha,db]_{\mathfrak{g}}\Big| = |0|$$

since $|b| \in H^l(\mathfrak{g})$ which means $db = 0$, so $[\alpha,db]_{\mathfrak{g}} = |0|$. The exact same reasoning yields $|[a,d\beta]_{\mathfrak{g}}| = |0|$. Furthermore, $|[d\alpha,d\beta]_{\mathfrak{g}}| = |d[\alpha,d\beta]_{\mathfrak{g}}| = |0|$, so we obtain

$$\Big[|\tilde{a}|,|\tilde{b}|\Big]_{\mathcal{H}} \equiv \Big|[\tilde{a},\tilde{b}]_{\mathfrak{g}}\Big| = \Big|[a,b]_{\mathfrak{g}}\Big| \equiv \Big[|a|,|b|\Big]_{\mathcal{H}} \tag{46}$$

whence the bracket is independent of representative and therefore well-defined. It inherits the graded Lie bracket structure from $[\,,\,]_{\mathfrak{g}}$. Furthermore, since the differential δ of the cohomology complex is defined to be identically zero, $\mathcal{H}$ is a DGLA indeed. $\square$

4.3.2 Hochschild cocomplex as DGLA: a degree shift

Let us proof that the Hochschild cocomplex is a first example of DGLA. Following definition 4.6, a graded Lie bracket is needed. This means that the bracket must satisfy graded antisymmetry and the graded Jacobi identity. Furthermore, the present differential structure on the cocomplex should be compatible with this Lie bracket.

In this section, such a bracket will first of all be defined. It is then shown that for the cocomplex to be a *graded* Lie algebra, it needs to be shifted by one degree; this will presently be made precise. After this, graded antisymmetry and the graded Jacobi identity are shown. Finally, a lemma suggested by [29] is proven to verify that the compatibility axiom holds.

We now equip the Hochschild cocomplex with a bracket, defined for $\phi_i \in C^{k_i}(A, A)$ as

$$[\phi_1, \phi_2] = \phi_1 \circ \phi_2 - (-1)^{k_1 k_2} \phi_2 \circ \phi_1 \,, \tag{47}$$

which is called the Gerstenhaber bracket [29]. The circle operation $\circ$ is known as the Gerstenhaber product. It builds a $k_1 + k_2 - 1$-cochain out of a k_1-cochain and a k_2-cochain: for $x_i \in A$

$$(\phi_1 \circ \phi_2)(x_1, \dots, x_{k_1+k_2-1}) := \sum_{j=1}^{k_1} (-1)^{(j-1)k_1} \phi_1(x_1, \dots, x_{j-1}, \phi_2(x_j, \dots, x_{j+k_2-1}), x_{j+k_2}, \dots, x_{k_1+k_2-1})$$

It is important to note that the Gerstenhaber bracket eats two elements - one of degree k_1, the other of degree k_2 - and produces an element of degree $k_1 + k_2 - 1$. Comparing this to the definition of a DGLA above shows that this does not define a *graded* Lie algebra, for we want $\deg(f) + \deg(g) = \deg([f, g])$. To remedie this, a degree shift on the cocomplex is needed. However, there are two natural gradings:

1. the homological degree (deg): the grading induced by the chain complex d;
2. and the dimension of the function (dim): the number of slots a function has.

It is the homological degree that is to be shifted. This is defined as follows:

Definition 4.7. *The k^{th} degree shift of a complex $C^\bullet$, denoted by $C^\bullet[k]$, is defined as $(C^\bullet[k])^n := C^{n+k}$. Then, the dimension of an element $f \in C^n_{Hoch}[k](A, A)$ is $\dim(f) = n + k$ and its degree is $\deg(f) = n$. The index of the differential operators is shifted accordingly, so $d_n : C^n[k](A, A) \longrightarrow C^{n+1}[k](A, A)$.*

As will be shown presently, shifting the degree of the Hochschild cocomplex by 1 will adjust the Gerstenhaber bracket in such a way as to turn the shifted cocomplex into a graded Lie algebra. As an added benefit, the compatibility axiom for differential operator and bracket can then be seen to hold, giving the complex the additional structure of a *differential* graded Lie algebra.

To show the impact of such a degree shift, we'll define a so-called circle-i operation in terms of which the Gerstenhaber product can be written[26]. In turn, the Lie bracket is defined as a sum of two Gerstenhaber products in equation 47. This links the circle-i operations and the Lie bracket [29].

To show the effect of such a shift and to clarify what is at stake here, we'll juxtapose the circle-i operations for the unshifted and shifted case to show the incongruency in gradation:

By definition, the circle-i operations take $f \in C^n(A, A)$, $g \in C^m(A, A)$ and build a $f \circ_i g \in C^{n+m-1}(A, A)$ for $1 \le i \le n$.

[26] This product is in fact a signed sum of circle-i operations. Note that this operation looks a lot like the decomposition of the Hochschild differential in terms of partial operations, see the appendix.

Unshifted Consider two elements of the unshifted Hochschild cocomplex, e.g. $f \in C^n(A, A)$, $g \in C^m(A, A)$; this means that for both functions, their dimension and homological degree are the same, so for example $\dim(f) = n = \deg(f)$. The circle-i operation now builds a function of *dimension* $\dim(f) + \dim(g) - 1 = n + m - 1$ as follows:

$$f \circ_i g(a_1, \ldots, a_{n+m-1}) = f(a_1 \otimes \ldots \otimes a_{i-1} \otimes g(a_i \otimes \ldots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \ldots \otimes a_{n+m-1}), \quad (48)$$

where $a_1, \ldots, a_{n+m-1} \in A$ and $i \in \{1, \ldots, n\}$. Hence, $f \circ_i g \in C^{n+m-1}(A, A)$. The problem, now, is that $\deg(f \circ_i g) = n + m - 1 \neq n + m = \deg(f) + \deg(g)$, so the homological degree is seen to mismatch: the unshifted Hochschild cocomplex is not a graded Lie algebra.

Shifted After shifting the Hochschild cocomplex with 1 degree in total, take $f \in C^n[1](A, A)$, $g \in C^m[1](A, A)$. Recall that this means that $\dim(f) = n + 1$ but its homological degree still is $\deg(f) = n$. The circle-i operation in this case constructs a functions of *dimension* $\dim(f) + \dim(g) - 1 = (n + 1) + (m + 1) - 1 = n + m + 1$:

$$f \circ_i g(a_1, \ldots, a_{n+m+1}) = f(a_1 \otimes \ldots \otimes a_{i-1} \otimes g(a_i \otimes \ldots \otimes a_{i+m}) \otimes a_{i+m+1} \otimes \ldots \otimes a_{n+m+1}), \quad (49)$$

where $a_1, \ldots, a_{n+m+1} \in A$ and $i \in \{1, \ldots, n + 1\}$. But due to the shift, although the *dimension* of $f \circ_i g$ is $n + m + 1$, its *degree* is $n + m$. This means $f \circ_i g \in C^{n+m}[1](A, A)$ so indeed $\deg(f \circ_i g) = \deg(f) + \deg(g)$.

In conclusion: by shifting the degree of the Hochschild cocomplex by 1, the gradations are seen to match since we now have $\deg(f) + \deg(g) = \deg([f, g])$ for all homogeneous elements f, g. It is now possible to state

Theorem 6 (Hochschild cocomplex)**.** *The 1^{st} degree shifted Hochschild cocomplex $(C^*[1](A, A), d_*)$ endowed with the Gerstenhaber bracket $[\,,\,]$ is a differential graded Lie algebra $(C^*[1](A, A), d_*, [\,,\,])$. For $f \in C^n[1](A, A)$, $g \in C^m[1](A, A)$, the bracket is defined in* (47) *in terms of the Gerstenhaber product. Explicitly, these definitions are*

$$[f, g] = f \circ g - (-1)^{nm} g \circ f \qquad \text{and} \qquad f \circ g = \sum_{i=1}^{n+1} (-1)^{(i-1)m} f \circ_i g \quad (50)$$

where $\circ_i$ denotes the shifted circle-i operation defined above.

Proof. In definition 4.5, the Hochschild cochains $C^n[1](A, A) = C^{n+1}(A)$ are only defined for $n \geq -1$. To fit definition 4.6 of DGLA, the cocomplex needs to be extended to the negative integers. This is most easily done by setting $C^n[1](A, A) := 0$ $(n < -1)$ for the cochains and by putting $d_n \equiv 0$ $(n < -1)$ for the differentials linking them; we so obtain the complex $(C^*[1](A, A), d_*) = (\bigoplus_{k \in \mathbb{Z}} C^k[1](A, A), d_k)$. It is shown in the appendix that this indeed defines a complex (i.e. that $d^2 = 0$), hence the first part of the definition of DGLA is satisfied.

Let us now turn to the aspect of Lie algebra. First we'll consider the graded antisymmetry: let $\phi_i \in C^{k_i}[1](A, A)$, then for the Gerstenhaber bracket

$$\begin{aligned} [\phi_1, \phi_2] &\overset{(47)}{=} \phi_1 \circ \phi_2 - (-1)^{k_1 k_2} \phi_2 \circ \phi_1 \\ &= (-1)^{1+k_1 k_2} \phi_2 \circ \phi_1 - (-1)^{1+2k_1 k_2} \phi_1 \circ \phi_2 \\ &= (-1)^{1+k_1 k_2} \big(\phi_2 \circ \phi_1 - (-1)^{k_1 k_2} \phi_1 \circ \phi_2\big) \overset{(47)}{=} (-1)^{1+k_1 k_2} [\phi_2, \phi_1] \end{aligned}$$

Note that the graded antisymmetry follows directly from the bracket's definition. In particular, the definition of the Gerstenhaber *product* as in the statement of the theorem is irrelevant for this part of the proof; this will also hold for the Jacobi identity.

For the proof of the graded Jacobi identity, note that the Gerstenhaber bracket is associative[27]:

$$\begin{aligned}[\phi_1,[\phi_2,\phi_3]] &= \phi_1\circ[\phi_2,\phi_3]-(-1)^{k_1(k_2+k_3)}[\phi_2,\phi_3]\circ\phi_1\\ &= \phi_1\circ\big(\phi_2\circ\phi_3-(-1)^{k_2k_3}\phi_3\circ\phi_2\big)-(-1)^{k_1(k_2+k_3)}\big(\phi_2\circ\phi_3+(-1)^{k_1(k_2+k_3)}\phi_3\circ\phi_2\big)\circ\phi_1\\ &= \phi_1\circ\phi_2\circ\phi_3-(-1)^{k_2k_3}\phi_1\circ\phi_3\circ\phi_2-(-1)^{k_1(k_2+k_3)}\phi_2\circ\phi_3\circ\phi_1\\ &\quad+(-1)^{k_1(k_2+k_3)}\phi_3\circ\phi_2\circ\phi_1\end{aligned}$$

The other two requested double commutators can be obtained by permuting $(1\to 2,\ 2\to 3,\ 3\to 1)$ and $(1\to 3,\ 2\to 1,\ 3\to 2)$ respectively[28]. Adding the weighted signs as in the definition of the graded Jacobi identity yields

$$\begin{aligned}&[\phi_1,[\phi_2,\phi_3]]+(-1)^{k_1(k_2+k_3)}[\phi_2,[\phi_3,\phi_1]]+(-1)^{k_3(k_1+k_2)}[\phi_3,[\phi_1,\phi_2]]\\ &=\phi_1\circ\phi_2\circ\phi_3-(-1)^{k_2k_3}\phi_1\circ\phi_3\circ\phi_2-(-1)^{k_1(k_2+k_3)}\phi_2\circ\phi_3\circ\phi_1+(-1)^{k_1(k_2+k_3)+k_2k_3}\phi_3\circ\phi_2\circ\phi_1\\ &+(-1)^{k_1(k_2+k_3)}\big(\phi_2\circ\phi_3\circ\phi_1-(-1)^{k_3k_1}\phi_2\circ\phi_1\circ\phi_3-(-1)^{k_2(k_3+k_1)}\phi_3\circ\phi_1\circ\phi_2\\ &\quad+(-1)^{k_2(k_3+k_1)+k_3k_1}\phi_1\circ\phi_3\circ\phi_2\big)\\ &+(-1)^{k_3(k_1+k_2)}\big(\phi_3\circ\phi_1\circ\phi_2-(-1)^{k_1k_2}\phi_3\circ\phi_2\circ\phi_1-(-1)^{k_3(k_1+k_2)}\phi_1\circ\phi_2\circ\phi_3\\ &\quad+(-1)^{k_3(k_1+k_2)+k_1k_2}\phi_2\circ\phi_1\circ\phi_3\big)\,.\end{aligned}$$

Expliciting all the factors of -1 yields expressions like $(-1)^{2k_1k_3+k_1k_2}=(-1)^{k_1k_2}$. After making these simplifications in the second and third row of the above equation, it is finally apparent that

$$\begin{aligned}&[\phi_1,[\phi_2,\phi_3]]+(-1)^{k_1(k_2+k_3)}[\phi_2,[\phi_3,\phi_1]]+(-1)^{k_3(k_1+k_2)}[\phi_3,[\phi_1,\phi_2]]\\ &=\phi_1\circ\phi_2\circ\phi_3-(-1)^{k_2k_3}\phi_1\circ\phi_3\circ\phi_2-(-1)^{k_1(k_2+k_3)}\phi_2\circ\phi_3\circ\phi_1+(-1)^{k_1(k_2+k_3)+k_2k_3}\phi_3\circ\phi_2\circ\phi_1\\ &+(-1)^{k_1(k_2+k_3)}\phi_2\circ\phi_3\circ\phi_1-(-1)^{k_1k_2}\phi_2\circ\phi_1\circ\phi_3-(-1)^{(k_1+k_2)k_3}\phi_3\circ\phi_1\circ\phi_2+(-1)^{k_2k_3}\phi_1\circ\phi_3\circ\phi_2\\ &+(-1)^{k_3(k_1+k_2)}\phi_3\circ\phi_1\circ\phi_2-(-1)^{k_1k_2+k_3(k_1+k_2)}\phi_3\circ\phi_2\circ\phi_1-\phi_1\circ\phi_2\circ\phi_3+(-1)^{k_1k_2}\phi_2\circ\phi_1\circ\phi_3=0\,.\end{aligned}$$

Although, admittedly, it can take some time to see the pairwise cancellation that yields Jacobi.

It was already shown above that the gradation of homogeneous elements is respected by the Lie bracket due to the degree shift. We therefore conclude that de Hochschild complex is a graded Lie algebra. Moreover, the compatibility axiom now *is* valid. For its proof, the aforementioned lemma found in [29] must first be shown:

Lemma 4.1 (Niesser)**.** *Let $\mu_A : A\otimes A\longrightarrow A$ be the associative algebra multiplication in A; note that $\mu_A\in C^1[1](A,A)$. Then for $f\in C^n[1](A,A)$ we have $d_n(f)=[\mu_A,f]$.*

Proof. This follows from a straightforward computation. Note that $\dim(\mu_A)=2$, $\deg(\mu_A)=1$ and that $\dim(f)=n+1$, $\deg(f)=n$. Thanks to equation (47), symbol calculus suffices for the bracket:

$$\begin{aligned}[\mu_A,f]&=\mu_A\circ f-(-1)^{1\cdot n}f\circ\mu_A\overset{(50)}{=}\sum_{i=1}^{2}(-1)^{(i-1)n}\mu_A\circ_i f-(-1)^n\sum_{j=1}^{n+1}(-1)^{(j-1)1}f\circ_j\mu_A\\ &=(-1)^n\left[\mu_A\circ_2 f+\sum_{j=1}^{n+1}(-1)^j f\circ_j\mu_A+(-1)^{n+2}\mu_A\circ_1 f\right].\end{aligned}$$

[27] See for example §1.2 of [29].

[28] Or we can apply the cycles (123) and (132) respectively.

Comparing this expression to the differential $d_n(f)$ and applying it to $a_1, \ldots, a_{n+2} \in A$ yields

$$(d_n f)(a_1, \ldots, a_{n+2}) = a_1\, f(a_2, \ldots, a_{n+2}) - (-1)^{n+1} f(a_1, \ldots, a_{n+1})\, a_{n+2} \\ + \sum_{j=1}^{n+1} (-1)^j f(a_1, \ldots, a_{j-1}, \underbrace{a_j\, a_{j+1}}, a_{j+2}, \ldots, a_{n+2})\,,$$

where $a_j\, a_{j+1} = \mu_A(a_j, a_{j+1})$ by definition of μ_A. Furthermore, note that for the same reason also $a_1\, f(a_2, \ldots, a_{n+2}) = \mu_A(a_1, f(a_2, \ldots, a_{n+2})) = (\mu_A \circ_2 f)(a_1, \ldots, a_{n+2})$. Rewriting the remaining two terms in the above expression, using μ_A in the same manner, shows that indeed $[\mu_A, f] = d_n(f)$ for all $f \in C^n[1](A, A)$. □

Now for the remainder of the proof of theorem 6:
Consider $\phi_i \in C^{k_i}[1](A, A)$. Using the previous lemma (i), the graded Jacobi identity (ii) and graded antisymmetry (iii), it is then clear that

$$[d\phi_1, \phi_2] + (-1)^{k_1}[\phi_1, d\phi_2] \overset{(i)}{=} [[\mu_A, \phi_1], \phi_2] + (-1)^{k_1}[\phi_1, [\mu_A, \phi_2]]\,. \tag{51}$$

Furthermore, the last term can be expanded using (ii) as

$$(-1)^{k_1}[\phi_1, [\mu_A, \phi_2]] \overset{(ii)}{=} (-1)^{k_1}\Big(-(-1)^{k_1(1+k_2)}[\mu_A, [\phi_2, \phi_1]] - (-1)^{k_2(k_1+1)}[\phi_2, [\phi_1, \mu_A]] \Big) \\ \overset{(iii)}{=} (-1)^{k_1 k_2}[\mu_A, [\phi_2, \phi_1]] - [[\mu_A, \phi_1], \phi_2]$$

since $[\phi_2, [\phi_1, \mu_A]] = (-1)^{1+k_1}[\phi_2, [\mu_A, \phi_1]] = (-1)^{1+k_1}(-1)^{1+(k_1+1)k_2}[[\mu_A, \phi_1], \phi_2]$ by the graded antisymmetry; this reduces to the last line of the previous equation by carefully keeping track of all the minus signs. Finally, combining the last expression with equation (51) yields

$$[d\phi_1, \phi_2] + (-1)^{k_1}[\phi_1, d\phi_2] \overset{(51)}{=} [[\mu_A, \phi_1], \phi_2] + (-1)^{k_1 k_2}[\mu_A, [\phi_2, \phi_1]] - [[\mu_A, \phi_1], \phi_2] \\ \overset{(iii)}{=} (-1)^{2k_1 k_2}[\mu_A, [\phi_1, \phi_2]] = d([\phi_1, \phi_2])\,.$$

In total, this amounts to $[d\phi_1, \phi_2] + (-1)^{k_1}[\phi_1, d\phi_2] = d([\phi_1, \phi_2])$ whence the compatibility axiom is valid. We conlude that with the introduced degree shift, the Hochschild cocomplex $(C^*[1](A, A), d_*, [\,,\,])$ endowed with differential and Gerstenhaber bracket is indeed a DGLA. □

4.3.3 The Formality theorem

As was stated earlier in this section, it was shown in [27] that for a Poisson manifold P with Poisson algebra $A = \mathcal{C}^\infty(P)$ the Hochschild cohomology complex of A is isomorphic to the (graded space of) smooth n-multivectorfields on P, so $H^n_{Hoch}(A) \cong \Gamma^\infty(\Lambda^n TP)$. Again, to give this object the structure of DGLA, the chain needs to be extended to all $n \in \mathbb{Z}$. Since the Hochschild cocomplex was already extended by setting $C^n_{Hoch}(A, A) := 0$ for all $n < 0$, it follows that $H^n_{Hoch}(A) = 0$ for $n < 0$.

Theorem 5 now says that the Hochschild *cohomology* complex indeed carries the structure of DGLA. Recall that the differential in the induced case is identically zero $\delta \equiv 0$. The bracket $[\,,\,]_{\mathcal{H}}$ is the one induced by the Hochschild cocomplex's Lie bracket, which is the Gerstenhaber bracket $[\,,\,]_G$, but up to cohomology. So $[\,,\,]_{\mathcal{H}} = |[\,,\,]_G|$. For the space of smooth multivectorfields, it is the earlier mentioned Schouten-Nijenhuis bracket that plays the role of Lie bracket (see §3.2 of [30]). Hence, this bracket is transferred to the Hochschild cohomology complex by isomorphism, so

$$[\,,\,]_{\mathcal{H}} = |[\,,\,]_G| = [\,,\,]_{SN}\,. \tag{52}$$

For more detailed information about this bracket, see [30] or the original article [11].

Before stating the Formality theorem, one more definition is needed [30]:

Definition 4.8 (Formal (DLGA))**.** *A differential graded Lie algebra is called formal if it is quasi-isomorphic to its cohomology, regarded as a DGLA with zero differential and the induced bracket.*

The precise definition of a quasi-isomorphism is quite elaborate and is a statement about a more general category of objects called L_∞-algebras. To give an approximate definition: a quasi-isomorphism is a morphism (structure-preserving mapping) that induces isomorphism in cohomology. This means that the DGLA structure of complex and cohomology complex are preserved, and the induced cohomology groups are isomorphic. For a DGLA to be *formal* a lot of requirements need to be met.

It is now possible to state Kontsevich's result:

Theorem 7 (Formality Theorem (Kontsevich 1997))**.** *Denote by* $\mathcal{C} := (C^*[1](A,A), d_*, [\,,\,]_G)$ *the Hochschild cocomplex endowed with differential and Gerstenhaber bracket. It is quasi-isomorphic to its cohomology complex* $\mathcal{H} := (H^*_{Hoch}[1](A,A), \delta \equiv 0, [\,,\,]_{\mathcal{H}} = |[\,,\,]_G| = [\,,\,]_{SN})$ *as DGLAs. In other words: the Hochschild cocomplex* $\mathcal{C}$ *is formal.*

The original proof can be found in [11]. Also, [30] presents a very detailed digression of the proof including all mathematical details which could not be included in this thesis.

Now, the claim is that this theorem is related to the quantization problem by deformation. To see this relation, let us recall the two objects that need to be linked:

1. On the one hand, there is a classical mechanical system described by a Poisson manifold P. In most of the cases, this represents the classical phase space of the system. Recall that by definition 3.10, P is Poisson if and only if its algebra of smooth functions $A = \mathcal{C}^\infty(P)$ carries the structure of Poisson algebra, that is: there is a skew-symmetric, bilinear bracket $\{\,,\,\} : A \otimes A \longrightarrow A$ on A that satisfies both the Jacobi identity and the Leibniz rule. This last requirement means that such a bracket comes from a skew-symmetric bivectorfield $\alpha \in \Gamma^\infty(\Lambda^2 TM)$ [23]. This is the Poisson structure characterizing the manifold. The bracket is then defined as:

   $$\{\,,\,\} : A \otimes A \longrightarrow A,\ (f,g) \mapsto \{f,g\} := \alpha(df, dg)\,. \tag{53}$$

   It was shown in section 3.2, that the Jacobi identity is identical to the vanishing of the aforementioned Schouten-Nijenhuis bracket of α: $[\alpha,\alpha]_{SN} = 0 \in \Gamma^\infty(\Lambda^3 TM)$. Recall that for any *symplectic* structure this bracket always vanishes, as was shown in section 3.

   The Hochschild-Kostant-Rosenberg theorem states that the Hochschild cohomology groups are isomorphic to the space of smooth multivector fields $H^n_{Hoch}[1](A,A) \equiv H^{n+1}_{Hoch}(A,A) \cong \Gamma^\infty(\Lambda^{n+1}TP)$ [27]. This means that the Poisson structure α is contained in a cohomology class of the cochain $H^1_{Hoch}[1](A,A)$ of the cocomplex $\mathcal{H}$.

2. On the other hand, classical systems are quantized by the introduction of a star product as in definition 4.1. Recall that this is a $\mathbb{C}[[\hbar]]$-bilinear mapping[29]

   $$A[[\hbar]] \otimes A[[\hbar]] \longrightarrow A[[\hbar]],\ (f,g) \mapsto f \star g := f \cdot g \sum_{n=1}^{\infty} B_n(f,g)\,\hbar^n\,. \tag{54}$$

   This means that $\star \in C^1[1](A[[\hbar]], A[[\hbar]])$. Star products were classified up to equivalence under gauge transformations, that is, the action of formal sums $D = \mathbb{1} + \sum_{n=1}^{\infty} D_n \hbar^n$ of

[29] For the tensor product notation, see the third remark after the star product's definition.

linear differential operators: star products $\tilde{\star}$ and $\star$ are called formally equivalent $\tilde{\star} \sim \star$ when they are linked by $D(f\tilde{\star}g) = D(f) \star D(g)$. Recall that it was shown in section 4.2.2 that the first order bidifferential operator B_1 of a star product is contained in the same Hochschild cohomology class of $H^1_{Hoch}[1](A, A)$ as all first order operators C_1 of formally equivalent star products: gauge transformations respect the cohomology structure.

Kontsevich's Formality theorem links the Hochschild complex $\mathcal{C}$ and its cohomology complex $\mathcal{H}$. However, star products are formal deformations of the regular 'zeroth order' (i.e. commutative) product - they are formal power series of bidifferential operators in $\hbar$ - whereas α remains the regular 'zeroth order' Poisson structure. With a relation between the two in mind, this suggests also considering *formal deformations* of Poisson structures. They are defined as follows [23]:

Definition 4.9 (Poisson structures)**.**

1. *A (null) Poisson structure on a smooth manifold M is a skew-symmetric bivectorfield $\alpha \in \Gamma^\infty(\Lambda^2 TM)$ that satisfies the 'Jacobi identity' in the sense that $[\alpha, \alpha]_{SN} = 0$.*
2. *A formal deformation of the null Poisson structure is a formal bivectorfield*

$$\alpha(\hbar) = \sum_{j=1}^{\infty} \alpha_j \, \hbar^j \in \Gamma^\infty(\Lambda^2 TM)[[\hbar]] \tag{55}$$

such that $[\alpha(\hbar), \alpha(\hbar)]_{SN} = 0 \in \Gamma^\infty(\Lambda^3 TM)[[\hbar]]$. Note that formal refers to the fact that the power series is formal, i.e. one need not worry about convergence.

Just as the notion of formal equivalence between star products (i.e. formal deformations of the commutative structure), there is also a notion of formal equivalence between formal deformation of the null Poisson structure of a Poisson manifold P. The equivalence classes are to be taken with respect to the action by formal paths in the diffeomorphism group of P, starting at the identity $\mathbb{1}_P$. This is meant in the following way [30]:

The set of Poisson structures $\mathcal{S} = \{\alpha \in \Gamma^\infty(\Lambda^2 TP) \,|\, [\alpha, \alpha]_{SN} = 0\}$ is acted on by the diffeomorphism group $\mathrm{Diff}(P)$ of P by pushforward:

$$\mathrm{Diff}(P) \times \mathcal{S} \longrightarrow \mathcal{S}, \; (\phi, \alpha) \mapsto \alpha_\phi := \phi_* \, \alpha \,. \tag{56}$$

Since the pushforward is a covariant functor, this indeed defines an action. It can be extended to the collection of formal deformations of the null Poisson structure on P, which will be denoted by

$$\mathcal{F}_P = \left\{ \alpha(\hbar) = \sum_{j=1}^{\infty} \alpha_j \, \hbar^j \in \Gamma^\infty(\Lambda^2 TP)[[\hbar]] \;\middle|\; [\alpha(\hbar), \alpha(\hbar)]_{SN} = 0 \right\}$$

Note that in the following, *formal* will mean that formal power series are considered.

To extend the action in equation (56) from $\mathcal{S}$ to $\mathcal{F}_P$, consider 'paths' of formal diffeomorphisms of P that start at the identity $\mathbb{1}_P$ diffeomorphism. Thus, these paths are formal power series of diffeomorphisms. They are of the form

$$\phi_\hbar(X) = \exp(\hbar X) \qquad \text{where} \qquad X = \sum_{k=0}^{\infty} X_k \, \hbar^k \text{ a formal vectorfield, so } X_k \in \Gamma^\infty(\Lambda^k TP) \,.$$

Since a group structure is needed to define an action, consider the set of exponentials of formal vector fields on P, denoted by $\mathcal{V}_P$. The product of two such exponentials is defined by the Baker-Campbell-Hausdorff formula [18]

$$\exp(\hbar X) \circ \exp(\hbar Y) = \exp\left(\hbar X + \hbar Y + \frac{1}{2}\hbar[X, Y] + \frac{1}{12}([X, [X, Y]] + [[X, Y], Y]) + \cdots \right), \tag{57}$$

which yields an object that is again in $\mathcal{V}_P$; the Lie bracket is again the Schouten-Nijenhuis bracket. One can check that this indeed satisfies the axioms of a group. To extend the action in equation (56) to the whole of $\mathcal{F}_P$, define

$$\mathcal{V}_P \times \mathcal{F}_P \longrightarrow \mathcal{F}_P,\ (\phi_\hbar(X), \alpha(\hbar)) \mapsto \exp(\hbar X)_* \alpha(\hbar) := \sum_{m=0}^{\infty} \left(\sum_{i+j+k=m}^{\infty} \left(\mathcal{L}_{X_i}\right)^j \alpha_k \right) \hbar^m , \tag{58}$$

where $\mathcal{L}_{X_i}$ is the Lie derivative on bivector fields. Again, since the pushforward is a covariant functor, equation (58) indeed defines an action on $\mathcal{F}_P$. The equivalence classes induced by this action are the mentioned classes of formal deformations of the null Poisson structure.

Remark

Note that every null Poisson structure $\alpha_{(0)} \in \Gamma^\infty(\Lambda^2 TP)$ on P can be associated with a formal deformation by choosing the 'path' $\alpha(\hbar) = \alpha_{(0)} \cdot \hbar \in \Gamma^\infty(\Lambda^2 TM)[[\hbar]]$.

As a corollary to his Formality theorem, Kontsevich proved in [11] a statement about deformation quantization:

Theorem 8 (Classification of quantization)**.** *Let M be a smooth manifold, $A = \mathcal{C}^\infty(M)$ its algebra of smooth functions. There is a natural one-to-one correspondence between star products on M modulo gauge equivalence $[\star_\pi]$ and equivalence classes of deformations $[\pi(\hbar)]$ of the null Poisson structure on M.*

A few remarks and conclusions regarding this final theorem:

1. The Formality theorem links the Hochschild complex $\mathcal{C}$ of the smooth manifold M and the space of smooth multivector fields on M. However, for an element in $C^1[1](A[[\hbar]], A[[\hbar]])$ to be a star product, restrictions such as associativity must hold. Moreover, in zeroth order the product must reduce to the regular commutative product and the operators are to be bidifferential ones. This means that for deformation theoretical purposes, one must consider a subDGLA of $\mathcal{C}$ that contains only these elements. The same holds for the space of smooth multivector fields: formal deformations $\alpha(\hbar)$ of the null Poisson structure have vanishing Schouten-Nijenhuis bracket. Also in this case, a subDGLA of $\mathcal{H}$ is to be considered. For details about these structures, see §3 of [30].

   Then, theorem 8 was proven by Kontsevich as a consequence of the Formality theorem. One considers formal power series of the mentioned subDGLAs, takes the cochains in first order and modulo formal equivalence. This amounts to:

   $$\tilde{C}^1[1]\Big(\mathcal{C}^\infty(M)[[\hbar]], \mathcal{C}^\infty(M)[[\hbar]]\Big) \Big/ \mathcal{G} \quad \longleftrightarrow \quad \tilde{\Gamma}^\infty(\Lambda^2 TM)[[\hbar]] \Big/ \mathcal{V}_P \tag{59}$$

   where the quotients are to be read as 'module the action of', and the ˜ denotes that the mentioned subDGLAs are to be considered. It should be noted, however, that the isomorphism $H^n_{Hoch}(A) \cong \Gamma^\infty(\Lambda^n TM)$ due to Hochschild-Kostant-Rosenberg [27] is of key importance in this result.

2. As was noted in the remark prior to the theorem, every null Poisson structure $\alpha_{(0)}$ can be associated to a formal deformation by choosing the path $\alpha(\hbar) = \alpha_{(0)} \cdot \hbar$. By theorem 8, its equivalence class of formal deformations $[\alpha(\hbar)]$ is in a canonical one-to-one correspondence with a well-defined gauge or formal equivalence class of star products $[\star_\alpha]$. Hence, choosing a representative of the equivalence class, the Poisson structure α comes from a natural star

product, one for which $B_1 = \{\ ,\ \}$. This makes that Dirac's intuition (the correspondence principle) is satisfied:

$$\text{For all } f, g \in \mathcal{C}^\infty(M) \ :\ \lim_{\hbar \to 0} \frac{[f \star_\alpha g]_\star}{i\hbar} \equiv \frac{f \star_\alpha g - g \star_\alpha f}{i\hbar} = \alpha(df, dg) = \{f, g\}\,.$$

Conversely, given a certain class of star products $[\star_\pi]$, it corresponds to a class of deformations $[\pi(\hbar)]$. The Poisson bracket on A associated with $\star_\pi$ then equals coefficient π_1 of $\pi = \pi_1\,\hbar + \pi_2\,\hbar^2 + \ldots$: for $f, g \in \mathcal{C}^\infty(M)$ we put $\{f, g\} = \pi_1(df, dg)$. This indeed defines a Poisson bracket since $[\pi(\hbar), \pi(\hbar)]_{SN} = 0$ by definition 4.9 of formal deformations of the null Poisson structure.

In conclusion: theorem 8 states that, up to formal equivalence, to every classical system (Poisson manifold P with Poisson structure α) a unique quantum system (star product $\star_\alpha$) can be associated in a canonical way. Hence, deformation quantization is a well-defined and unique quantization procedure. The formal equivalence translates into a freedom in choosing the symmetric part of the first order bidifferential operator B_1^+ of the star product, as well as a freedom in higher order (in terms of $\hbar$) contributions to the star product ánd to the formal deformation of the null Poisson structure. Kontsevich's Formality theorem is at the base of this relation that gives a positive answer to the question if deformation quantization is a viable approach to the quantization problem. However, it would be interesting to further examine the remaining freedom in higher order terms of $\hbar$ and the symmetric part of the first order bidifferential operator.

5 Kontsevich's formula

5.1 The setting

As was seen in section 3.2, a Poisson structure $\alpha \in \Gamma^\infty(\Lambda^2 TM)$ on a smooth manifold M defines a Poisson bracket as in equation (53). In [11], Kontsevich gave an explicit universal formula that allows for the construction of a star product associated to this Poisson structure. This formula is defined in local coordinates and, hence, only works when M is an open in $\mathbb{R}^d$. A globalization procedure is given in [31] to extend this construction to arbitrary Poisson manifolds.

This section describes Kontsevich's construction of a local star product on $\mathcal{C}^\infty(M)$ where $M \subset \mathbb{R}^d$ open, for a certain Poisson structure α given in local coordinates. Then, the construction is applied to the Poisson structure corresponding with the regular Poisson bracket; this is the constant structure corresponding to flat space. Kontsevich's formula is seen to yield the Moyal product as associated star product.

Recall definition 36 of the star product of $f, g \in \mathcal{C}^\infty(M)$

$$f \star g := f \cdot g + \hbar \{f, g\} + \sum_{n=2}^{\infty} B_n(f, g)\, \hbar^n \,. \tag{60}$$

Anticipating on what is to come, Kontsevich's formula is the following

$$f \star_K g = f \cdot g + \sum_{n=1}^{\infty} \left(\sum_{\Gamma \in G_{n,2}} w_\Gamma \, B_{\Gamma,\alpha}(f, g) \right) \frac{\hbar^n}{n!} \,, \tag{61}$$

where $G_{n,2}$ is a suitable subset of the collection of graphs of $n+2$ vertices; this subset indexes the bidifferential operators $B_{\Gamma,\alpha}$ and their weights w_Γ. These are the objects that will be constructed in this section. The development as found in [23] and [30] is followed closely.

5.1.1 Admissable graphs

For all $n \in \mathbb{N}$, $G_{n,2}$ denotes the collection of admissable graphs that index the bidifferential operators that appear in the star product $\star_K$ in n^{th} order of $\hbar$; this is clear from equation (61) above. For a graph $\Gamma \in G_{n,2}$, we write $|\Gamma| = n$ its order.

Only oriented, finite graphs appear in the star product. Following a definition by Serre [32]:

Definition 5.1. *An oriented, finite graph Γ consists of two sets E_Γ, V_Γ (edges and vertices) and two maps $\phi : E \to E$, $\iota : E \to V$ with the following rules, nomenclature and interpretation:*

1. *An $e \in E$ is called a directed edge, $\phi(e) := \bar{e}$ is the reverse edge.*
2. *$\forall e \in E$ we have $\bar{\bar{e}} = e$, so reversing the reverse edge yields the initial one.*
3. *$\forall e \in E$, $\phi(e) := \bar{e} \neq e$: an edge can't be its reverse edge.*
4. *Also, $\iota(e)$ is the initial vertex and $\tau(e) := \iota(\bar{e})$ the terminal vertex: the direction of the edge e is from $\iota(e)$ to $\tau(e)$.*

The bidifferential operators act on two functions $f, g \in \mathcal{C}^\infty(M)$. Kontsevich's construction is local, so suppose that local coordinates are chosen. The interpretation of the admissable graphs is that they index with respect to which of these coordinates f and g are to be differentiated. The properties that a star product must satisfy by definition 4.1 impose conditions on the different ways in which this process of differentiation can be carried out; it is most notably the associativity condition of $\star_K$ that imposes conditions.

The set $G_{n,2}$ of admissable graphs of order n is defined as follows:

Definition 5.2 (Admissable graphs)**.** *An admissable graph of order n is an oriented, finite graph consisting of $n+2$ vertices $V_\Gamma = \{1, \ldots, n, L, R\}$, $2n$ edges $E_\Gamma = \{i_1, j_1, i_2, j_2, \ldots, i_n, j_n\}$ and two maps $\phi : E_\Gamma \to E_\Gamma$ (reversal), $\iota : E_\Gamma \to V_\Gamma$ (initial vertex) that satisfy for all $k \in \{1, \ldots, n\}$:*

1. *We have $\iota(i_k) = k = \iota(j_k)$: this means that i_k, j_k both start at vertex k.*
2. *Again denoting $\phi(a) := \bar{a}$ for an $a \in E_\Gamma$, we have $\iota(\bar{i_k}) \neq \iota(\bar{j_k})$: the two edges i_k, j_k starting at vertex k end at different vertices.*
3. *There are no loops: $\iota(\bar{i_k}) \neq k \neq \iota(\bar{j_k})$.*

The set of admissable graphs of order n is denoted by $G_{n,2}$, the total set of admissable graphs of finite order by $G = \cup_{n\in\mathbb{N}} G_{n,2}$. Note that by item 2, $\#G_{n,2} = (n(n+1))^n$.

The $\{B_{\Gamma,\alpha}\}$ being *bi*differential operators, keep in mind that the vertex L encodes the left entry of this operator, R the right entry. The direction of edges determines on which vertex is being acted. We consider two examples of such admissable graphs. These will illustrate the characteristics of this type of graph and allow for the description of the algorithm used to associate a bidifferential operator $B_{\Gamma,\alpha}$ to a graph $\Gamma \in G$. The left graph will be denoted by Γ_a, the other by Γ_b.

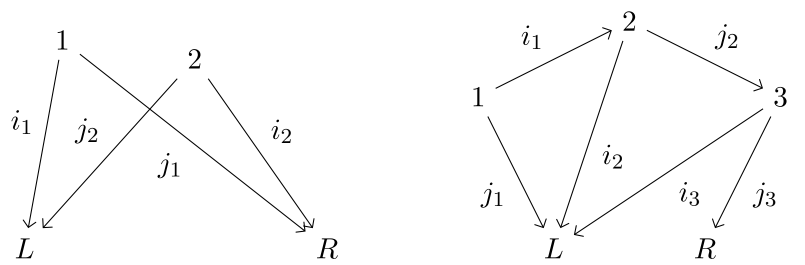

Figure 1: Admissable graphs Γ_a (left) and Γ_b.

Algorithm to assign $B_{\Gamma,\alpha}$ to $\Gamma \in G$

Let $f, g \in \mathcal{C}^\infty(M)$. The bidifferential operator will be of the form $B_{\Gamma,\alpha}(f,g) = \ldots f \otimes \ldots g$.

1. Write $\alpha^{i_k j_k}$ for vertex k; write f on the left side of the tensor product, g on the far right.
2. Place a ∂_{i_k} directly before the vertex where i_k ends; the same for j_k. If an edge ends at L or R, write the differential before f or g respectively.

Since differential operators commute when acting on smooth functions, their order is sufficiently characterized by the above procedure. Furthermore, since the graphs are finite, this algorithm is well-defined.
To clarify this procedure, we consider the above shown examples in which Einstein's convention is assumed: repeated indices are to be summed over.

(Graph Γ_a) First of all, $|\Gamma_a| = 2$ since $V_a = \{1, 2, L, R\}$. We have edges $i_1 = (1, L)$, $j_1 = (1, R)$, $i_2 = (2, R)$ and $j_2 = (2, L)$. Note that all edges end at L or R: although not required by definition 5.2, this is a special case that will turn up in the next section when examining the Moyal product[30].

[30] The product coming from a constant Poisson structure, i.e. one independent of the local coordinates.

By step one of the algorithm, replacing the vertices yields $\alpha^{i_1 j_1}\alpha^{i_2 j_2} f \otimes g$. Then, the differential operators are to be placed. For example, edge i_1 ends at L, so ∂_{i_1} is put directly before f. On the other hand, edge i_2 ends at R so ∂_{i_2} is placed before g at the right side of the tensor product. Finally:

$$B_{\Gamma_a}(f,g) = \left(\alpha^{i_1 j_1}\alpha^{i_2 j_2}\partial_{j_2}\partial_{i_1} \otimes \partial_{i_2}\partial_{j_1}\right)(f,g) = \alpha^{i_1 j_1}\alpha^{i_2 j_2}\left(\partial_{j_2}\partial_{i_1} f\right)\left(\partial_{i_2}\partial_{j_1} g\right).$$

(Graph Γ_b) Now $|\Gamma_b| = 3$ since $V_b = \{1,2,3,L,R\}$. We thus obtain three times the Poisson structure: $\alpha^{i_1 j_1}\alpha^{i_2 j_2}\alpha^{i_3 j_3} f \otimes g$. We have edges $i_1 = (1,2)$, $j_1 = (1,L)$, $i_2 = (2,L)$, $j_2 = (2,3)$, $i_3 = (3,L)$ and $j_3 = (3,R)$. This time, edge i_1 ends at 2, so it yields a differential operator ∂_{i_1} *in front* of Poisson structure $\alpha^{i_2 j_2}$. Note that this fact is non-trivial in case of a non-constant Poisson structure. In total:

$$\begin{aligned} B_{\Gamma_a}(f,g) &= \left(\alpha^{i_1 j_1}\partial_{i_1}\alpha^{i_2 j_2}\partial_{j_2}\alpha^{i_3 j_3}\partial_{i_3}\partial_{i_2}\partial_{j_1} \otimes \partial_{j_3}\right)(f,g) \\ &= \left(\alpha^{i_1 j_1}\partial_{i_1}\alpha^{i_2 j_2}\partial_{j_2}\alpha^{i_3 j_3}\partial_{i_3}\partial_{i_2}\partial_{j_1} f\right)\left(\partial_{j_3} g\right). \end{aligned}$$

For the general formula assigning to a graph $\Gamma \in G$ its bidifferential operator $B_{\Gamma,\alpha}$, see [11].

5.1.2 The weights w_Γ

The procedure that assigns a weight to every differential operator is somewhat more intricate to define. It seems that Kontsevich arrived at these weights by building his proof for the Formality theorem in its most general setting in §6 of [11]. His construction is too involved to fall within the scope of this thesis. However, the weights are needed to treat the constant Poisson structure as an example (and for the sake of completeness), which will yield the Moyal product.

Now, to describe deformation quantization, only corollary 8 is relevant. Therefore, only a recipe to calculate these weights in the setting of deformation quantization is given due to the advanced level of this construction. For questions regarding motivation, interpretation or a more detailed explanation, the reader is referred to §6 of the original article [11], the detailed and more readable §5 of [30], or the treatment of a particular case concerning Lie algebra's in [33].

Given $\Gamma \in G_{n,2}$ an admissable graph, the correspondening weight w_Γ is calculated by integrating a certain $2n$-form over the upper half-plane $\mathcal{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ endowed with the so-called Lobachevsky or Poincaré or hyperbolic metric. Following [23], the recipe is the following:

Recipe to calculate w_Γ

Denote $\mathcal{C}_n(\mathcal{H}) = \{(p_1,\ldots,p_n) \in \mathcal{H}^n \mid p_i \neq p_j \text{ iff } i \neq j\}$ for $n \geq 1$. For $p \neq q \in \mathcal{H}$, we want to define a function ϕ that gives an angle between 0 and 2π. This is done as follows:

1. Let $l(p,\infty)$ be the vertical line in $\mathcal{H}$ through p to ∞.
2. Let $l(p,q)$ be the unique geodesic joining[31] p and q.
3. Denote the angle from $l(p,\infty)$ to $l(p,q)$, measured counter clockwise, by $\phi(p,q)$. This defines a smooth map $\phi : \mathcal{C}_2(\mathcal{H}) \longrightarrow S^1$. It is made explicit by Kontsevich in [11] to simplify calculations using a 'trick with logarithms' as

$$\phi : \mathcal{C}_n(\mathcal{H}) \longrightarrow S^1,\ (p,q) \mapsto \phi(p,q) := \operatorname{Arg}\left(\frac{q-p}{q-\bar{p}}\right) = \frac{1}{2i}\log\frac{(q-p)(\bar{q}-p)}{(q-\bar{p})(\bar{q}-\bar{p})}. \tag{62}$$

[31] A geodesic is the shortest (with respect to some metric) line joining two points. In hyperbolic geometry as described on the upper half-plane $\mathcal{H}$, the unique geodesic between $p,q \in \mathcal{H}$ is either a vertical line (iff $\Re(p) = \Re(q)$) or a circular arc connecting both points. For proofs or a detailed introduction to hyperbolic geometry, see the excellent [34].

This map can be extended to $p \neq q \in \mathcal{H} \sqcup \mathbb{R} =: \overline{\mathcal{H}}$ by continuity, which will be necessary later on. See figures 2 and 3 below for an example and interpretation.

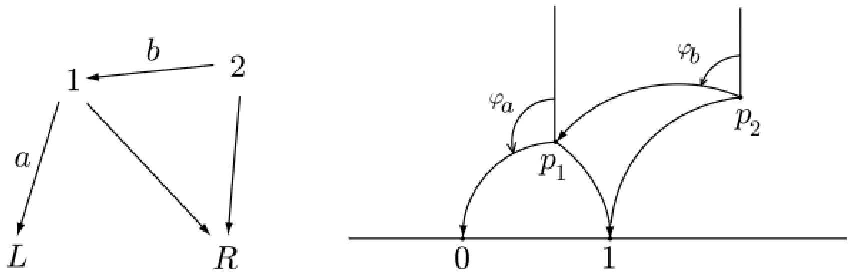

Figure 2: Associating geodesics and angles to a graph of order 2 [28].

Using this angle map, a $2n$-form β_Γ can be defined on $\mathcal{C}_n(\mathcal{H})$ using its differential:

$$\beta_\Gamma(u_1, \ldots, u_n) := \bigwedge_{l \in E_\Gamma} d\phi(u_{\iota(l)}, u_{\tau(l)}) \,, \tag{63}$$

where $u_1, \ldots, u_n \in \mathcal{C}_n(\mathcal{H})$ and the wedge product is taken in the order $i_1, j_1, i_2, j_2, \ldots, i_n, j_n$. Recall that $\iota(l)$ denotes the initial vertex of edge l, and $\tau(l)$ its terminal vertex. Furthermore, let $u_L = 0$ and $u_R = 1$. Finally, the weight w_Γ is defined as a multiple of the integral of this $2n$-form over the whole of $\mathcal{C}_n(\mathcal{H})$. To be precise:

$$w_\Gamma := \frac{1}{(2\pi)^{2n}} \int_{\mathcal{C}_n(\mathcal{H})} \beta_\Gamma \,, \tag{64}$$

where the orientation on $\mathcal{C}_n(\mathcal{H})$ is the one induced by $\mathcal{H}$.[32]

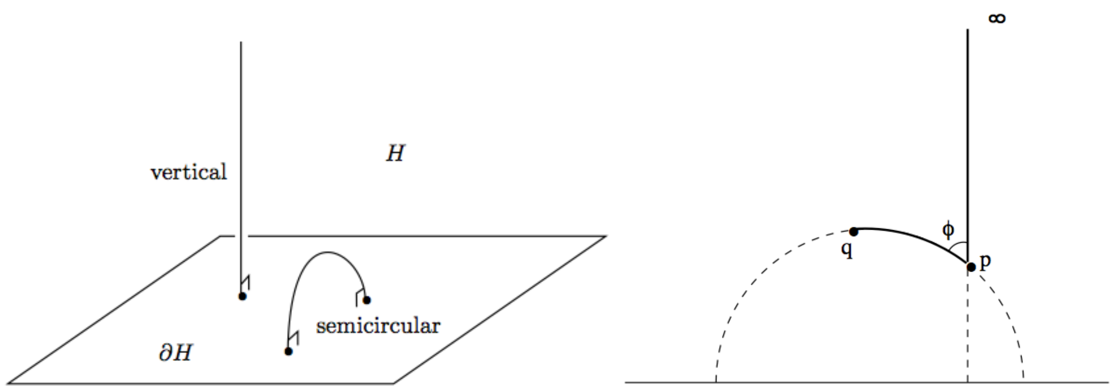

Figure 3: The two types of geodesics in $\mathcal{H}$ (left)[34]; the angle $\phi(p, q)$[33].

In [11], Kontsevich then proved the following:

Theorem 9.

1. *The integral in* (64) *is absolutely convergent for all* $\Gamma \in G$.

[32]For information about integrating forms over smooth manifolds, see [16] or [22].

2. Let α be a Poisson bivectorfield on a domain M in $\mathbb{R}^d$. For $f, g \in \mathcal{C}^\infty(M)$, the formula

$$f \star_K g = f \cdot g + \sum_{n=1}^{\infty} \left(\sum_{\Gamma \in G_{n,2}} w_\Gamma \, B_{\Gamma,\alpha}(f,g) \right) \frac{\hbar^n}{n!} \tag{65}$$

defines an associative star product on M. Under a change of coordinates, a formally equivalent star product is obtained: one related to $\star_K$ by a gauge transformation as in equation (41).

In particular: a change of coordinates respects the action of the group $\mathcal{G}$ of formal sums of linear differential operators on the set of star products as defined in section 4.2.1. This theorem is important since it not only establishes associativity, but also allows for changing coordinates to simplify calculating weights of operators.

5.2 Example: the Moyal product

Using the machinery from the previous section, it is now possible to construct a star product for the Poisson structure α^{ij} with constant coefficients on an open M in $\mathbb{R}^d$. This structure corresponds to the usual Poisson bracket known from classical mechanics. Kontsevich's construction should, up to formal equivalence, yield the Moyal product as described in equation (6) in section 2.

Before constructing the Moyal product, consider the graph Γ_b of third order for which the associated operator $B_{\Gamma,\alpha}$ was calculated. Recall that there were edges ending in vertices other than L or R. Now, a Poisson structure with *constant coefficients* is considered. It is clear that in this case, all contributions from such edges are zero, since they contain a term $\partial_i \alpha^{jk} = 0$. At order n, the only admissable graphs that contribute are the ones where every vertex k has one edge ending in L and one edge ending in R; this leaves 2^n graphs.

5.2.1 Orders $n = 0, 1$

There is only one graph of order 0, which contributes $f \cdot g$, as is expected by the correspondence principle. Let us now consider the admissable graphs of order 1. By the above, there are only two such graphs (denoting $i_1 \equiv i$, $j_1 \equiv j$):

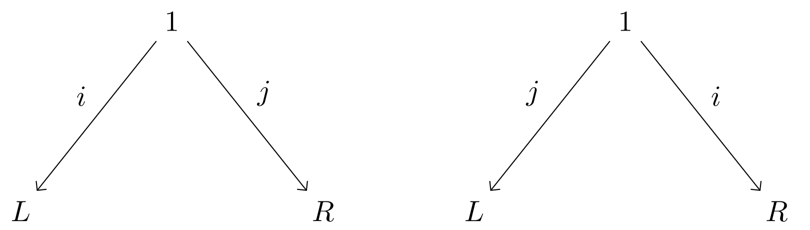

Figure 4: Graphs Γ_1 (left) and Γ_2

The graphs differ in switching the two edges. Recall that $u_L = 0$ and $u_R = 1$. Following the algorithm for assigning operators and for determining weights, we have for graph Γ_1

$$B_{\Gamma_1}(f,g) = \alpha^{ij} (\partial_i f) \, (\partial_j g) \qquad \text{and} \qquad w_{\Gamma_1} = \frac{1}{(2\pi)^2} \int_{\mathcal{H}} d\phi(u,0) \wedge d\phi(u,1)$$

where $f, g \in \mathcal{C}^\infty(M)$. Denote $\phi(u,0) = \phi_0(u)$ and $\phi(u,1) = \phi_1(u)$. First of all, by Kontsevich's trick with logarithms, ϕ can be calculated explicitly as in (62). We obtain for $\phi_0(u)$

$$\phi_0(u) = \mathrm{Arg}\left(\frac{0-u}{0-\bar{u}}\right) = \mathrm{Arg}\left(\frac{u^2}{|u|^2}\right) = \mathrm{Arg}(u^2) = 2\mathrm{Arg}(u)\,,$$

and for $\phi_1(u)$ the same equation yields

$$\phi_1(u) = \mathrm{Arg}\left(\frac{1-u}{1-\bar{u}}\right) = \mathrm{Arg}\left(\frac{1-u}{\overline{\overline{1-u}}}\right)\mathrm{Arg}\left(\frac{(1-u)^2}{|1-u|^2}\right) = \mathrm{Arg}((1-u)^2) = 2\mathrm{Arg}(1-u)\,.$$

Theorem 9 allows for a change of variables to simplify the integration. This is done by changing from cartesian coordinates $\{x = \Re(u), y = \Im(u)\}$ on the whole of the extended $\overline{\mathcal{H}} = \{z \in \mathbb{C} \mid \Im(z) \geq 0\}$ to $\{\phi_0(u) = \phi(u,0), \phi_1(u) = \phi(u,1)\}$. We claim that the integration domain is now given by $\mathcal{R} = \{0 \leq \phi_0(u) \leq \phi_1(u) \leq 2\pi\}$. This is seen as follows:

Fix $\phi_0(u) \equiv \alpha \in [0, 2\pi]$. By the above equation, this means that $\alpha = 2\mathrm{Arg}(u)$, so the set of complex numbers corresponding to the equation $\phi_0(u) = \alpha$ is given by

$$\mathcal{S}_\alpha = \{u \in \mathbb{C} \mid 2\mathrm{Arg}(u) = \alpha\} = \{u_r = r\exp(i\alpha/2) \mid r \in (0,\infty)\}\,.$$

Note that $\mathcal{S}_\alpha \subset \overline{\mathcal{H}}$ for all $\alpha \in [0, 2\pi]$. Therefore, $\mathcal{R}$ indeed yields the correct integration domain.

Denoting the new integration variables by $\phi_0(u) = \phi_0$, $\phi_1(u) = \phi_1$, the integral yields

$$\begin{aligned} w_{\Gamma_1} &= \frac{1}{(2\pi)^2}\int_{\mathcal{H}} d\phi_0(u) \wedge d\phi_1(u) = \frac{1}{(2\pi)^2}\int_{\mathcal{R}} d\phi_0\ d\phi_1 \\ &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\left(\int_{\phi_0}^{2\pi} d\phi_1\right) d\phi_0 = \frac{1}{(2\pi)^2} d\phi_0 \int_0^{2\pi} (2\pi - \phi_0) d\phi_0 = \frac{1}{2}\,. \end{aligned}$$

To calculate the weight associated to graph Γ_2, recall that it only differs from Γ_1 by switching the edges. By antisymmetry of the wedge product, its weight therefore only differs by a minus sign:

$$w_{\Gamma_2} = \frac{1}{(2\pi)^2}\int_{\mathcal{H}} d\phi_1(u) \wedge d\phi_0(u) = -\frac{1}{(2\pi)^2}\int_{\mathcal{H}} d\phi_0(u) \wedge d\phi_1(u) = -w_{\Gamma_1} = -\frac{1}{2}\,.$$

Now, the first order bidifferential operator B_1 contributing to the star product $\star_\alpha$, acting on $f, g \in \mathcal{C}^\infty(M)$ is found to be

$$\begin{aligned} B_1(f,g) &= \sum_{\Gamma \in G_{1,2}} w_\Gamma B_{\Gamma,\alpha}(f,g) = w_{\Gamma_1} B_{\Gamma_1,\alpha}(f,g) + w_{\Gamma_2} B_{\Gamma_2,\alpha}(f,g) \\ &= \frac{1}{2}\left(\alpha^{ij}(\partial_i f)\,(\partial_j g) - \alpha^{ji}(\partial_j f)\,(\partial_i g)\right) = \alpha^{ij}(\partial_i f)\,(\partial_j g) = \{f,g\}\,, \end{aligned}$$

where the fourth equation is valid due to antisymmetry $\alpha^{ij} = -\alpha^{ji}$.

5.2.2 Using prime graphs

The explicit form of Kontsevich's star product as given in equation (61) indicates that the product's expressions can be written as an exponential if all operators B_n could be written as B_1^n for some 'prime' operator B_1. This approach can be used to calculate the higher order terms of the star product $\star_\alpha$ where α^{ij} is a Poisson structure with constant coefficients. The principle of decomposing graphs in terms of such prime graphs can be used in a more general setting, see [33]. Following [23], the method is applied to α^{ij} to re-obtain the Moyal star product.

Consider a general admissable graph of order n, of which there are 2^n:
To decompose graphs in terms of prime graphs, a composition on the collection G of admissable graphs is needed. It is defined by [23] as follows:

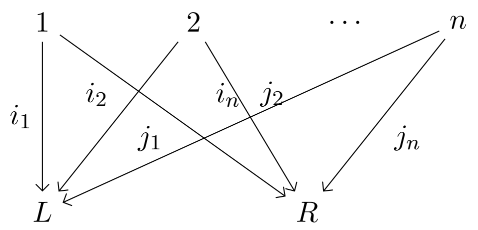

Figure 5: General graph Γ_n of order n.

Definition 5.3. *Let $\Gamma \in G_{m,2}$, $\Gamma' \in G_{k,2}$ be two admissable graphs. Their composition $\Gamma \cdot \Gamma' \in G_{m+k,2}$ is defined as the admissable graph obtained by putting together the two graphs, relabeling the vertices of Γ' from $1, 2, \ldots, k$ to $m+1, m+2, \ldots, m+k$ and identifying the two vertices labeled L and R. For $n \in \mathbb{N}$, $n \cdot \Gamma = \underbrace{\Gamma \cdot \ldots \cdot \Gamma}_{n \text{ times}}$.*

Equipped with this composition and with the unique graph Γ_0 of order 0 as identity element, the set of admissable graphs is seen to have the structure of a semigroup[33]; it is denoted by $\mathcal{A} = (G, \cdot, \Gamma_0)$. Note that this semigroup is in general not abelian.

As an example, consider the composition of the earlier mentioned admissable graphs Γ_1 and Γ_2:

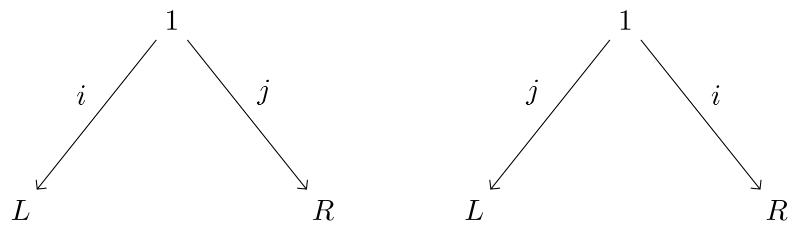

Figure 6: Graphs Γ_1 (left) and Γ_2

Their composition $\Gamma_1 \cdot \Gamma_2$ is seen to be the earlier considered admissable graph Γ_a:

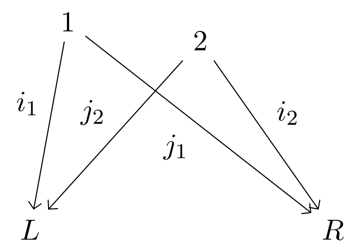

Figure 7: The composition $\Gamma_1 \cdot \Gamma_2 = \Gamma_a$

The key fact to note is that graph Γ_a is in fact one of the graphs of order 2 that dó contribute to the star product $\star_\alpha$ associated to the Poisson structure with constant coefficients α. Moreover, the set $\tilde{\mathcal{P}} = \{\Gamma_1, \Gamma_1'\}$ generates all the contributing graphs under the above defined composition; it is called the set of generating graphs.

[33] A semigroup is a triple $(S, \cdot, e_S)$ satisfying all group axioms but the inverse axiom.

However, as was noted before, both graphs are linked by switching edges. More formally: there is a natural S_2-action on the set $\tilde{\mathcal{P}}$ of generating graphs

$$S_2 \times \tilde{\mathcal{P}} \longrightarrow \tilde{\mathcal{P}}, (\sigma, \Gamma) \mapsto \sigma \cdot \Gamma \tag{66}$$

where $e \in S_2$ leaves the graphs intact and (12) switches edges, thus effectively switching graphs. In the case of a constant Poisson structure, $\mathcal{P} = \tilde{\mathcal{P}}/S_2 = \{[\Gamma_1]\}$: there is only one prime graph.

Let us now calculate the higher order terms of $f \star_\alpha g = f \cdot g + \hbar\{f, g\} + \mathcal{O}(\hbar^2)$. For all $n \in \mathbb{N}$, denote by Γ_n the unique graph of order n for which all edges i_k end in L, and all edges j_k end in R (it is in fact this graph that is depicted in figure 5). When calculating the weight of the two graphs of order 1, it was seen that switching edges yielded a minus sign due to antisymmetry of the wedge product. However, due to skew-symmetry of α^{ij}, this sign was again absorbed. In an analogous manner, when switching edges i_k and j_k with $k \in \{1, \dots, n\}$ in the graph Γ_n, a minus sign is picked up and again absorbed by α^{ij}. Hence:

$$\begin{aligned}\forall f, g \in \mathcal{C}^\infty(M) \, : \, B_n(f, g) &= \sum_{\Gamma \in G_{n,2}} w_\Gamma \, B_{\Gamma,\alpha}(f, g) = 2^n w_{\Gamma_n} B_{\Gamma_n,\alpha}(f, g) \\ &= \frac{2^n}{(2\pi)^{2n}} \left(\int_{\mathcal{C}_n(\mathcal{H})} \beta_{\Gamma_n} \right) (\alpha^{i_1 j_1} \dots \alpha^{i_n j_n}) (\partial_{i_1 \dots i_n} f) (\partial_{j_1 \dots j_n} g) \, ,\end{aligned}$$

where $\partial_{i_1 \dots i_n} = \partial_{i_1} \dots \partial_{i_n}$. Examing the defining equation (63) of β_{Γ_n}, it is possible to decompose the integral into n times the same integral, for all edges i_k end at L (where $u_L = 0$) and all edges j_k end at R (where $u_R = 1$); this yields

$$\begin{aligned}2^n w_{\Gamma_n} = \frac{2^n}{(2\pi)^{2n}} \int_{\mathcal{C}_n(\mathcal{H})} \beta_{\Gamma_n} &= \left(\frac{2}{(2\pi)^2} \right)^n \int_{\mathcal{C}_n(\mathcal{H})} \bigwedge_{k=1}^n (d\phi_0(u_k) \wedge d\phi_1(u_k)) \\ &= \prod_{k=1}^n \left(\frac{2}{(2\pi)^2} \int_{\mathcal{H}} d\phi_0(u_k) \wedge d\phi_1(u_k) \right) = \prod_{k=1}^n 2w_{\Gamma_1} = \left(\frac{2}{2} \right)^n = 1 \, .\end{aligned}$$

Inserting this result into the equation for bidifferential operator B_n and plugging that expression into the Kontsevich formula (61) yields

$$\begin{aligned}f \star_K g &= f \cdot g + \sum_{n=1}^\infty (\alpha^{i_1 j_1} \dots \alpha^{i_n j_n}) (\partial_{i_1 \dots i_n} f) (\partial_{j_1 \dots j_n} g) \, \frac{\hbar^n}{n!} \\ &= \exp(\hbar \alpha^{ij} \partial_i \otimes \partial_j)(f, g) = \exp(\hbar 2 w_{\Gamma_1} \beta_{\Gamma_1})(f, g) \, .\end{aligned}$$

Switching to the convention maintained by physicists by substituting $\hbar \mapsto i\hbar/2$ and by choosing the coefficients in α^{ij} so as to obtain the regular Poisson brackets in first order (so $|\alpha^{ij}| = 1$), the Moyal product as in equation (6) in section 2 is seen to emerge:

$$f \star_K g = \exp(\hbar \alpha^{ij} \partial_i \otimes \partial_j)(f, g) = f \exp\left[\frac{i\hbar}{2} \left(\overleftarrow{\partial_i} \overrightarrow{\partial_j} - \overleftarrow{\partial_j} \overrightarrow{\partial_i} \right) \right] g = f \star_\hbar g \, . \tag{67}$$

This result completes the Moyal product example, and thus the treatment of Kontsevich's explicit formula for associating a star product to a given Poisson structure.

A Hochschild (co)homology

In section 4.2.2, the first Hochschild cohomology group of $A = \mathcal{C}^\infty(M)$ was used to show that every manifold that admits a quantization as in definitions 4.2, also admits a *natural* quantization. Here A is the $\mathbb{R}$-linear algebra of smooth functions on the smooth manifold M. However, this is a particular case of Hochschild theory. In this appendix, it will be shown that Hochschild (co)homology can be defined for any associative unital k-algebra R, where k is a commutative ring. All results will be proven for homology. Definitions and results for cohomology are only stated, since they can be obtained in a similar manner. The article [35] will be followed, where the subject is studied in greater depth. For more information on groups and rings, aswell as algebras, see for example [25].

A.1 Algebras and bimodules

Every homology theory relies on some structured space to supply coefficients. In singular homology, for example, this space is a group. In the Hochschild case, it turns out that the appropriate coefficient spaces are bimodules over the k-algebra that is under consideration. First of all, some preliminary definitions and results.

Definition A.1. *Let k be a commutative ring. An associative unital k-algebra can be defined in two ways:*

1. *As a ring R equipped with a ring homomorphism $f : k \longrightarrow R$ such that $f(k) \subset Z(R)$. Here, $Z(R)$ denotes the center of R: the collection of elements that commute with all elements in R with respect to ring multiplication;*
2. *As a k-module R equipped with a multiplication $\mu : R \times R \longrightarrow R$ compatible with the module structure:*

$$\forall \lambda \in k,\, \forall a, b \in R:\ \lambda(a \cdot b) = (\lambda a) \cdot b = a \cdot (\lambda b) \qquad \textit{(for a left } k\textit{-module)}\,.$$

Note that these two definitions are equivalent. We'll call such an algebra simply a k-algebra.

Definition A.2. *Let R be a ring. An R-module is an abelian group $(M, +, 0)$ equipped with a unital (sending identity to identity) ring homomorphism*

$$\phi : R \longrightarrow \mathit{End}(M)\,,$$

which is equivalent to having a linear action $R \times M \longrightarrow M$. Note that this is in fact a left action defining a left R-module.

From now on, k denotes a commutative ring.

Definition A.3. *A bimodule M over a given k-algebra R is a k-module together with two commuting k-linear actions of R on M, one to the left and one to the right:*

$$l : R \times M \longrightarrow M \qquad \textit{and} \qquad r : M \times R \longrightarrow M\,.$$

Commutativity means that $\forall s, t \in R, m \in M$: $r(t)\big[l(s)(m)\big] = l(s)\big[r(t)(m)\big]$; or $(s \cdot m) \cdot t = s \cdot (m \cdot t)$.

An R-bimodule M of which the actions coincide is called *symmetric*: this reads $r \cdot m = m \cdot r$ for all $r \in R$, $m \in M$. Note that this implies $(r'r) \cdot m = (rr') \cdot m$ for all $r, r' \in R$, $m \in M$; this is a very restrictive condition on R, close to commutativity.

Before defining the Hochschild homology, let's first consider some examples of bimodules.

Examples

1. The easiest but surpisingly important example is taking the k-algebra R itself as bimodule. By definition (A.1), it is both a left and right k-module, where the action is just ring multiplication. Commutativity of the left and right actions is given by associativity of the multiplication. This choice of bimodule will be made for Hochschild homology in the main text.

2. Let M a right R-module, N a left R-module where R is a k-algebra. Denote $M \otimes_k N$ their tensor product. By definition, it comes equipped with a k-bilinear map

$$\iota : M \times N \longrightarrow M \otimes_k N,\ (m, n) \mapsto \iota(m, n) = m \otimes_k n$$

through which all k-bilinear maps from $M \times N$ to some other linear space factor; this is the universal property of tensors.

The claim is that $M \otimes_k N$ is an R-bimodule. [34]

Proof. First of all, $M \otimes_k N$ is both a left and right k-module. This follows from the fact that M is a right and N a left R-module (hence also a k-module) and furthermore their tensor product $M \otimes_k N$ is by definition k-linear.

Secondly, we need two commuting k-linear actions of R on $M \otimes_k N$. Since M and N come equipped with such structures, it is natural to use these:

$$\begin{aligned} l : R \times M \otimes_k N &\longrightarrow M \otimes_k N,\ (s, m \otimes_k n) \mapsto m \otimes_k s \cdot n \\ r : M \otimes_k N \times R &\longrightarrow M \otimes_k N,\ (m \otimes_k n, t) \mapsto m \cdot t \otimes_k n \end{aligned}$$

By virtue of the above and the tensor product's properties, it is clear that the above equations define k-linear actions. Let's verify explicitly that these two actions commute. For $s, t \in R$, $m \otimes_k n \in M \otimes_k N$

$$\begin{aligned} l\left(s, r\left(m \otimes_k n, t\right)\right) &= l\left(s, m \cdot t \otimes_k n\right) = m \cdot t \otimes_k s \cdot n \\ r\left(l\left(s, m \otimes_k n\right), t\right) &= r\left(m \otimes_k s \cdot n, t\right) = m \cdot t \otimes_k s \cdot n\,, \end{aligned}$$

which shows that $l \circ r = r \circ l$. The two k-linear actions commute and we conclude that $M \otimes_k N$ is an R-bimodule indeed. □

A.2 Hochschild homology

After having considered two examples of coefficient spaces, we'll go on to define the Hochschild homology complex and its homology. Let R be a k-algebra as before, and M an R-bimodule; note that all tensor products are to be taken over k, so $\otimes \equiv \otimes_k$.

Define $C_n(R, M) = M \otimes R^{\otimes n}$, denote by $b_n : C_n(R, M) \longrightarrow C_{n-1}(R, M)$ the n$^{\text{th}}$ differential operator; take $C_0(R, M) := M$, $C_{-1}(R, M) = \{0\}$ and $b_0 \equiv 0$. This yields the complex

$$\ldots \xrightarrow{b_4} M \otimes R^{\otimes 3} \xrightarrow{b_3} M \otimes R^{\otimes 2} \xrightarrow{b_2} M \otimes R \xrightarrow{b_1} M \xrightarrow{b_0} \{0\}\,,$$

[34] This is exercise 2 of [36].

where the boundary operator puts adjacent symbols together, summing them alternatingly. A tensor in $M \otimes R^{\otimes n}$ is written as $(m, a_1, \ldots, a_n)$ where $m \in M$, $a_i \in R$. As an example, the action of b_3 is

$$b_3(m, a_1, a_2, a_3) = (m \cdot a_1, a_2, a_3) - (m, a_1 a_2, a_3) + (m, a_1, a_2 a_3) - (a_3 \cdot m, a_1, a_2)\,.$$

Note that this is well-defined since M is an R-*bi*module.

Before defining Hochschild homology (in the usual way), the name *complex* needs justification.

Proposition 9. *The chain $(C_*(R, M), b_*)$ defines a complex, i.e. $b_{n-1} \circ b_n = 0$ for all $n > 0$.*

Proof. Define an operator $d_i^{(n)} : M \otimes R^{\otimes n} \longrightarrow M \otimes R^{\otimes (n-1)}$ for all $0 < i < n$ by

$$d_i^{(n)}(m, a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) = \begin{cases} (m \cdot a_1, a_2, \ldots, a_n) & \text{for } i = 0 \\ (m, a_1, \ldots, a_{i-1}, \underbrace{a_i a_{i+1}}_{\text{i}^{\text{th}} \text{ copy of } R}, a_{i+2}, \ldots, a_n) & \text{for } 0 < i < n \\ (a_n \cdot m, a_1, \ldots, a_{n-1}) & \text{for } i = n\,. \end{cases}$$

It is now clear that $b_n = \sum_{i=0}^{n} (-1)^i \, d_i^{(n)}$. From now on we omit the superscript (n), this number will be clear from the context. We now claim that

$$\boxed{d_i \circ d_j = d_{j-1} \circ d_i} \qquad \text{for all} \quad 0 \leq i < j \leq n\,. \tag{68}$$

This can be easily seen by writing out explicitly and is left to the reader.

For the sake of completeness, we'll now prove $b_{n-1} \circ b_n = 0$ by induction on the gradation of the boundary operator. The above result is key in this proof.

($k = 1$) By definition $b_0 \circ b_1 = d_0 \circ (d_0 - d_1) = d_0 \circ d_0 - d_0 \circ d_1 \overset{(68)}{=} d_0 \circ d_0 - d_0 \circ d_0 = 0$.

($k \leq n-1$) Assume that for all $k \leq n-1$ we have $b_{k-1} \circ b_k = 0$.

($k = n$) Note that the number of terms in $b_{n-1} \circ b_n$ is equal to $n\,(n+1)$ which is even; by using (68), pairwise cancellation will be obtained. First of all

$$\begin{aligned} b_{n-1} \circ b_n &= \big(d_0 - d_1 + \ldots + (-1)^{n-2} d_{n-2} + (-1)^{n-1} d_{n-1}\big) \circ \big(d_0 - d_1 + \ldots + (-1)^n d_n\big) \\ &= b_{n-2} \circ b_{n-1} + (-1)^{n-1} d_{n-1} \circ b_n + (-1)^n b_{n-2} \circ d_n \\ &= 0 + (-1)^{n-1} \big[d_{n-1} \circ b_n - b_{n-2} \circ d_n\big] \qquad \text{by the induction hypothesis}\,. \end{aligned}$$

Note that there are $(n+1) + (n-1) = 2n$ terms left, again an even number. Now

$$\begin{aligned} d_{n-1} \circ b_n - b_{n-2} \circ d_n = {} & \big[(d_{n-1}\, d_0 - d_0\, d_n) + (d_{n-1}\, d_1 - d_1\, d_n) + \ldots \\ & + (-1)^{n-2}(d_{n-1}\, d_{n-2} - d_{n-2}\, d_n)\big] + (-1)^{n-1} d_{n-1}\, d_{n-1} + (-1)^n d_{n-1}\, d_n \\ \overset{(68)}{=} {} & 0 + (-1)^{n-1}\big[d_{n-1}\, d_{n-1} - d_{n-1}\, d_n\big] \overset{(68)}{=} 0\,. \end{aligned}$$

So, indeed, $b_{n-1} \circ b_n = 0$ for all $n \geq 1$.

In conclusion: $(C_*(R, M), b_*)$ is indeed a complex. □

Letting $M \otimes R^{\otimes n}$ have homological degree n, we can now define the Hochschild homology of the k-algebra R with coefficients in the R-bimodule M. This is the usual definition for homology:

$$\text{For all } n \geq 0 \qquad H_n(R, M) := H_n(C_*(R, M)) = \frac{\ker\left(b_n : C_n(R, M) \longrightarrow C_{n-1}(R, M)\right)}{\operatorname{im}\left(b_{n+1} : C_{n+1}(R, M) \longrightarrow C_n(R, M)\right)} . \quad (69)$$

Now, about the structure of these objects. They are not only abelian groups, but also k-modules in general and even R-modules when R is commutative. This is a straighforward verification using definition A.2. Let's calculate the zeroth homology group to get some intuition about this specific homology theory. Since $b_0 \equiv 0$, $\ker(b_0) = M$. Furthermore, we see that for $m \in M$, $r \in R$ we have

$$b_1(m, r) = m \cdot r - r \cdot m \,,$$

so $\operatorname{im}(b_1) = \langle m \cdot r - r \cdot m \,|\, m \in M,\, r \in R \rangle =: [R, M]$ where $[R, M]$ is the k-module generated by the commutator of both actions. This yields in total

$$H_0(R, M) = M/[R, M] = M \,/\, \langle m \cdot r - r \cdot m \,|\, m \in M,\, r \in R \rangle \,.$$

This result has a nice interpretation: when R is commutative, $[R, M]$ is in fact an R-module. Then, the zeroth homology group can be seen as the symmetrized R-bimodule of M; in particular, finding $H_0(R, M) = M$ for a particular choice of R and M means that the bimodule M is symmetric, i.e. the left and right actions coincide. This means that the zeroth Hochschild homology group of a commutative k-algebra R measures to what extent its coefficient bimodule M acts on it in a symmetric way. In particular: the algebra $\mathcal{C}^\infty(P)$ considered in the main text - the $\mathbb{R}$-algebra of smooth functions on a smooth manifold P - is commutative.

Let's now consider a particular choice of bimodule that is usually made in Hochschild homology. As we have seen in the first section of this appendix, R is an R-bimodule in its own right.

Definition A.4. *Let R be a k-algebra, take R as R-bimodule with as k-linear action the left and right multiplication of the algebra. The n^{th} Hochschild homology group of R is defined as $H_{Hoch,n}(R) := H_n(C(R, R))$.*

This means that for each $n \geq 0$ we have a functor $H_{Hoch,n}$ from the category of pairs of k-algebra and bimodule over this algebra, to the category of k-modules. This is meant in the following sense:

Let (R, M) and (R', M') be such pairs, so R and R' are k-algebras, M is an R-bimodule and M' an R'-modules; the latter two are the coefficient spaces. These pairs define the objects of the category, now we need morphisms between these objects. Since these morphism need to preserve the object's structure, let $\alpha : R \longrightarrow R'$ be a map of k-algebras. This means that α is a k-linear ring homomorphism, which is the type of map that respects the structure of the k-algebras. Furthermore, let $\varphi : M \longrightarrow M'$ be a map of R-bimodules. It is important to note that although M' is an R'-bimodule, it can be *interpreted* as an R-bimodule via the map α in the following way: for $s \in R$, $m' \in M'$, $s \cdot m' := \alpha(s) \cdot m'$. On the other hand, for φ to preserve the R-bimodule structure, it should be a k-linear homomorphism of groups that respects both actions (equivariance)

$$\forall s, t \in R,\, m \in M : \, \varphi(s \cdot m) = \alpha(s) \cdot \varphi(m) \quad \text{and} \quad \varphi(m \cdot t) = \varphi(m) \cdot \alpha(t)$$

and the k-linear left and right actions should commute, so

$$\forall s, t \in R,\, m \in M : \, \varphi\big((s \cdot m) \cdot t\big) = \varphi\big(s \cdot (m \cdot t)\big) \quad \text{or} \quad \big(\alpha(s) \cdot \varphi(m)\big) \cdot \alpha(t) = \alpha(s) \cdot \big(\varphi(m) \cdot \alpha(t)\big) \,.$$

Such a pair of maps (α, φ) between pairs (R, M), (R', M') is a morphism in this category.

We'll show that associated to every pair of objects (R, M), (R', M') and morphism (α, φ) between them, there is an induced map in homology $H_n(\alpha, \varphi) : H_n(R, M) \longrightarrow H_n(R', M')$, turning H_n into a functor. Consider (α, φ) as above. It induces a natural map between complexes $\Gamma_* : C_*(R, M) \longrightarrow C_*(R', M')$, which means that for all $n \geq 0$ the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r, "b_{n+2}"] & C_{n+1}(R,M) \arrow[r, "b_{n+1}"] \arrow[d, dashed, "\Gamma_{n+1}"] & C_n(R,M) \arrow[r, "b_n"] \arrow[d, dashed, "\Gamma_n"] & C_{n-1}(R,M) \arrow[r, "b_{n-1}"] \arrow[d, dashed, "\Gamma_{n-1}"] & \cdots \\
\cdots \arrow[r, "b'_{n+2}"] & C_{n+1}(R',M') \arrow[r, "b'_{n+1}"] & C_n(R',M') \arrow[r, "b'_n"] & C_{n-1}(R',M') \arrow[r, "b'_{n-1}"] & \cdots
\end{tikzcd}$$

where $\Gamma_n := \varphi \otimes \alpha^{\otimes \mathrm{n}} : M \otimes R^{\otimes \mathrm{n}} \longrightarrow M' \otimes R'^{\otimes \mathrm{n}}$. To see that this diagram is commutative, let $m \in M, a_1, \ldots, a_n \in R$ and consider

$$\begin{aligned}
(\Gamma_{n-1} \circ b_n)(m, a_1, \ldots, a_n) = {} & [\varphi(m) \cdot \alpha(a_1), \alpha(a_2), \ldots, \alpha(a_n)] + [\alpha(a_n) \cdot \varphi(m), \alpha(a_1), \ldots, \alpha(a_n)] \\
& + \sum_{i=1}^{n-1} (-1)^i [\varphi(m), \alpha(a_1), \ldots, \alpha(a_{i-1}), \alpha(a_i)\, \alpha(a_{i+1}), \alpha(a_{i+2}), \ldots \alpha(a_n)] \\
& = (b'_n \circ \Gamma_n)(m, a_1, \ldots, a_n)\,.
\end{aligned}$$

So indeed, for all $n \geq 0$ we have $\Gamma_{n-1} \circ b_n = b'_n \circ \Gamma_n$ whence the diagram is commutative.

Now, the natural map of complexes Γ_* induces the sought after map in homology which turns H_n in general, and $H_{Hoch,n}$ in particular, into a covariant functor. There is a general construction to this extent, which can be found in chapter IV of [37], for example. This map can be denoted by

$$H_n(\alpha, \varphi) : H_n(R, M) \longrightarrow H_n(R', M')\,. \tag{70}$$

A.3 Hochschild cohomology

Although very similiar, the relevant definitions of Hochschild *co*homology are given in this section. This is not only for the sake of completeness, but also since it is this variant of Hochschild's theory that is used in the main text. Furthermore, the Hochschild *co*complex plays a prominent role in Kontsevich's Formality theorem, in section 4.3.

Definition A.5. *Let R be a k-algebra, M an R-bimodule. A cocomplex is defined as $C^0(R, M) := M = Hom_k(k,M)$ and for $n \geq 1$*

$$C^n(R, M) := Hom_k(R^{\otimes n}, M) = \{f : R^{\otimes n} \longrightarrow M \mid f \, \text{is } k\text{-multilinear}\}\,. \tag{71}$$

The codifferential operator β is defined analogously to the differential operator b: it puts adjacent symbols together, summing these contributions alternatingly. Note, however, that β raises the cohomological degree, so the cocomplex looks like

$$M \xrightarrow{\beta_0} Hom_k(R, M) \xrightarrow{\beta_1} Hom_k(R^{\otimes 2}, M) \xrightarrow{\beta_2} Hom_k(R^{\otimes 3}, M) \xrightarrow{\beta_3} \ldots$$

As an example, consider β_2: let $f \in \mathrm{Hom}_k(R^{\otimes 2}, M)$ and $a_1, a_2, a_3 \in R$. Then we have

$$\beta_2(f)(a_1, a_2, a_3) = a_1 f(a_2, a_3) - f(a_1 a_2, a_3) + f(a_1, a_2 a_3) - f(a_1, a_2)\, a_3\,.$$

It can be shown in the same manner as above that $\beta_{n+1} \circ \beta_n = 0$ so that the above indeed defines a (co)complex. The cohomology groups, that can again be interpreted as k-modules, are defined in the usual way and denoted by $H^n(R, M) = H^n(C^*(R, M))$. Again, one speaks of Hochschild cohomology groups when the special case of $M = R$ is considered. However, due to the fact that the codifferential operator raises the homological degree, this yields a functor H^n. It's construction is completely analogous to that of H_n, a part from the fact that for pairs $(R, M), (R', M')$ this time $\varphi : M' \longrightarrow M$ whilst still $\alpha : R \longrightarrow R'$. This leads to a *contra*variant functor, which means that the direction of the induced map in cohomology is opposite:

$$H_n(\alpha, \varphi) : H_n(R', M') \longrightarrow H_n(R, M) \,.$$

Having finished the general setting, let us now put [35] aside and do some interpretating.

Zeroth degree

Let $m \in M$, $r \in R$, then we have $H^0(R, M) = \{m \in M \,|\, r \cdot m = m \cdot r \,, \forall r \in R\}$. Just as in the homological case, the zeroth cohomology group is seen to be a measure for the symmetry of R's actions on the bimodule M. In particular, when $M = R$, it is seen to encode the commutativity of the algebras product since then

$$H^0(R, R) = \{z \in R \,|\, r \cdot z = z \cdot r \,, \forall r \in R\} \equiv Z(R) \,.$$

The last equality means that the zeroth cohomology group is equal to the center $Z(R)$ of R.

First degree

The first Hochschild cohomology group plays an important role in the gauge action on the collection of star products in section 4.2.2. By definition

$$H^1_{Hoch}(R) = \frac{\ker(\beta_1 : \mathrm{Hom}_k(R, M) \longrightarrow \mathrm{Hom}_k(R \otimes R, M))}{\mathrm{im}(\beta_0 : M \longrightarrow \mathrm{Hom}_k(R, M))}$$

where for $f \in \mathrm{Hom}_k(R, M)$ and $a_1, a_2 \in R$ we have

$$\beta_1(f)(a_1, a_2) = a_1 \cdot f(a_2) - f(a_1\, a_2) + f(a_1) \cdot a_2 \,,$$

which is in particular symmetric in a_1, a_2 if and only if the product on R is commutative.

Now, suppose $f \in \ker(\beta_1)$ so $\beta_1(f) = 0$. By the above, this means that for all $a_1, a_2 \in R$ we have

$$f(a_1\, a_2) = a_1 \cdot f(a_2) + f(a_1) \cdot a_2.$$

But this is just the Leibniz rule! Let us therefore return to the setting in the main text. Consider M a smooth (Poisson) manifold and set $R = \mathcal{C}^\infty(M)$ its $\mathbb{R}$-algebra of smooth functions. We were in the particular case of Hochschild cohomology, so the bimodule that figures as coefficient space is just $\mathcal{C}^\infty(M)$ itself. Let $B : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ an element of $C^1(R)$ (so it is an $\mathbb{R}$-linear map) and f, g smooth real functions on M. Then, the above equation reads

$$B(f, g) = fB(g) + B(f)g \,.$$

Again the Leibniz rule, but this time it can be given a nice interpretation: every element of $\mathrm{Hom}_k(\mathcal{C}^\infty(M), \mathcal{C}^\infty(M))$ that is closed in Hochschild cohomology is a derivation on the algebra $\mathcal{C}^\infty(M)$ of smooth functions.

Second degree

Lastly the second Hochschild cohomology group. Let $f \in \mathrm{Hom}_k(R \otimes R, M)$ and $a_1, a_2, a_3 \in R$. Applying the codifferential to f yields

$$\beta_2(f)(a_1, a_2, a_3) = a_1 \cdot f(a_2, a_3) - f(a_1\, a_2, a_3) + f(a_1, a_2\, a_3) - f(a_1, a_2) \cdot a_3\,.$$

Again taking an $f \in \ker(\beta_2)$, we obtain

$$f(a_1\, a_2, a_3) + f(a_1, a_2) \cdot a_3 = a_1 \cdot f(a_2, a_3) + f(a_1, a_2\, a_3)\,.$$

Let us return to the setting used in the main text, where $M = R$ and the M's actions are just algebra multiplication. Denoting this multiplication by $\mu_R : R \otimes R \longrightarrow R$, the above translates to

$$f(\mu_R(a_1, a_2), a_3)) + \mu_R(f(a_1, a_2), a_3) = \mu_R(a_1, f(a_2, a_3)) + f(a_1, \mu_R(a_2, a_3))\,.$$

Rewriting this expression in terms of the Gerstenhaber bracket $[\,,\,]$ introduced in section 4.3.1, one can recognize a particular case of lemma 4.1 used in the same section: here $f \in C^1[1](R, R)$, which is equal to $C^2(R, R) = \mathrm{Hom}_k(R \otimes R, R)$, and

$$\beta_2(f)(a_1, a_2, a_3) = [\mu_R, f](a_1, a_2, a_3)\,.$$

In turn, the Gerstenhaber bracket is related to the differential graded Lie algebra structure of the Hochschild cohomology complex, which plays a prominent role in Kontsevich's Formality theorem.

These three examples demonstrate that the Hochschild (co)homology groups encode a lot of information about associative algebras. Their cohomology complex is even seen to carry a differential graded Lie algebra structure. It should therefore be no surprise that Hochschild cohomology naturaly presents itself in the study of deformation theory in general, where DGLAs play an important role, and the study of deformation quantization in particular, in which associative algebras are deformed.

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Index

Populaire samenvatting

Kwantummechanica is ontwikkeld in het begin van de twintigste eeuw om sommige tekortkomingen van Newton's klassieke mechanica te ondervangen. Ook de speciale en de algemene relativiteitstheorie zijn ontwikkeld met dit doel. Waar deze laatste twee vooral bedoeld zijn om beweging met extreem hoge snelheden correct te beschrijven, is kwantummechanica juist bedoeld voor het beschrijven van hele kleine fenomenen, zoals de werking van atomen of zelfs subatomaire deeltjes.

Deze scriptie gaat over het verband tussen klassieke kwantummechanica. Aangezien deze laatste niet in het middelbaar onderwijs behandeld wordt, geef ik hier eerst een voorbeeld dat Newton's mechanica niet goed kan beschrijven. Kwantummechanica blijkt dit wel correct te kunnen. Daarna bekijken we een van de tegenintuïtieve voorspellingen die kwantummechanica doet. Ter afsluiting ligt ik nog iets gedetailleerder toe welk relatie tussen klassieke en kwantummechanica ik nu precies bestudeerd heb ik deze scriptie.

Laten we eerst een voorbeeld bekijken dat Newton's mechanica niet correct beschrijft: het waterstofatoom. Dit atoom bestaat uit een kern met één positief geladen proton, en uit één negatief geladen elektron dat om de kern cirkelt. Dit doet denken aan een miniatuurvariant van Aarde en maan: het proton fungeert als middelpunt waar het elektron als maan omheen cirkelt. Analoog aan de elliptische baan die de maan beschrijft om de Aarde, kan je je afvragen welke baan dit elektron beschrijft relatief tot de kern. Wanneer we hiervoor klassieke mechanica aanwenden, blijkt dit te voorspellen dat het atoom al na een fractie van een seconde instort! Immers, net als bij magneten trekken positieve en negatieve lading elkaar aan: het elektron cirkelt door deze aantrekking steeds dichter naar het positief geladen proton, totdat ze op elkaar belanden.

Daarentegen lees jij nog steeds deze samenvatting, en draait de wereld gewoon door. Oftewel, van deze *theoretische* voorspelling blijkt in de *werkelijkheid* niets te kloppen. We zijn aanbeland bij een tegenspraak. Dat wil zeggen: we hebben aangenomen dat de baan van het elektron in een waterstofatoom net als de baan van de Aarde correct te beschrijven is met behulp van klassieke mechanica, maar dit leidt tot instortende atomen: *non-sense*. Dientengevolge moeten we concluderen dat er een ander soort mechanica nodig is om atomen te beschrijven; de huidige voldoet niet in het domein van atomen.

Deze andere mechanica blijkt kwantummechanica te zijn. Het is een veralgemenisering van klassieke mechanica, in de zin dat het meer effecten en fenomenen correct beschrijft. Denk bij wijze van voorbeeld aan het hiervoor genoemde waterstofatoom: kwantummechanica voorspelt diens gedrag. En dan niet alleen de baan van het elektron om het proton, maar ook wat er gebeurt als men licht schijnt op het atoom, of als het extreem wordt afgekoeld. Er zijn vele experimenten gedaan waarbij waterstofatomen werden bestookt met licht, onder de meest extreme temperaturen, en de theoretische voorspellingen van de kwantummechanica blijken zeer nauwkeurig overeen te komen met de gedane metingen.

Kwantummechanica blijkt dus een 'correctere' theorie te zijn in zekere zin. Echter, het doet ook een aantal vreemde en tegenintuïtieve voorspellingen. Het boek *Alice in Quantumland*, geschreven door Brits natuurkundige Robert Gilmore, illustreert deze vreemde gedragingen aan de hand van een allegorie gebaseerd op Lewis Carroll's *Alice in Wonderland*. Laten we één van deze voorbeelden bekijken: het onzekerheidsprincipe van Heisenberg[35].

Stel je voor dat je een voetbal een flinke trap geeft. Klassieke mechanica beschrijft dan nauwgezet de baan van de bal. Om precies te zijn: wanneer een vriend op het moment dat jij de bal lanceert een stopwatch aanzet, vertelt Newton's mechanica ons op elk volgend tijdstip waar de bal in de lucht is (positie) en hoe hard deze gaat (snelheid, of liever: impuls). Dit kunnen we ook meten: de bal is duidelijk zichtbaar in de lucht, en met wat mooie apparatuur is zijn

[35] Werner Heisenberg (1901-1976), Duits natuurkundige, was een van de grondleggers van de kwantummechanica.

snelheid nauwkeurig te achterhalen. Oftewel, positie en impuls zijn gelijktijdig en nauwkeurig meetbaar.

In kwantummechanica blijkt dit niet zo te zijn. Dit is wegens het eerder genoemde onzekerheidsprincipe van Heisenberg. Beschouw wederom het rondcirkelende elektron in een waterstofatoom. Dit principe zegt dan dat diens positie en impuls *niet* tegelijk meetbaar zijn. In meer formele taal: de positie- en impulsoperator commuteren niet. Dit betekent dat het meten van de één invloed heeft op de ander. Klassiek gezien is dit heel vreemd. Dit zou in het voetbalvoorbeeld betekenen dat meten (lees: zien) waar de voetbal is, spontaan zijn snelheid zou veranderen. Of omgekeerd, dat het vastliggen van de snelheid van de voetbal ervoor zorgt dat het ding ineens vijf meter verplaatst is. Geen voetbalwedstrijd zou ooit hetzelfde zijn. Vandaar ook de naam: *onzekerheids*principe.

Op atomaire schaal blijkt dit principe echter wel te gelden. Het wordt geïllustreerd door onderstaande tekening[36], eveneens gebaseerd op het boek Alice in Quantumland. In het midden staat een verwarde Alice die het hele kwantumgebeuren waarneemt. Links zien we een chaotisch bewegend elektron dat versnelt, afremt, van richting verandert; er valt geen touw aan vast te knopen. Kortom, zijn snelheid (impuls) is niet goed te bepalen. Tegelijkertijd is zijn positie op elk moment van de tijd helder: hoewel het in de praktijk misschien lastig is het elektron te volgen, zien we duidelijk waar het zich bevindt. Aan de rechterkant van Alice zien we het omgekeerde geval. Een groot, uitgesmeerd, bijna stilstaand elektron neemt alle ruimte in. Aan de ene kant is zijn snelheid goed te bepalen - het staat immers stil. Aan de andere kant zijn zijn contouren zo onduidelijk en is het zo verspeid over de ruimte, dat spreken over de ‘positie’ van het elektron weinig zin meer heeft.

Figure 8: (v.l.n.r.) een chaotisch bewegend elektron, Alice, een stilstaand en uitgespreid elektron.

Ter herhaling: bij het beschrijven van kwantummechanische fenomenen die plaatshebben op atomaire schaal, zijn positie en impuls niet gelijktijdig meetbaar. De positie- en impulsoperator

[36]Gemaakt door Lou Ripoll. *Merci à toi, Lou.*

commuteren niet. Dit feit staat bekent als de *kanonieke commutatierelaties*, en het ligt aan de basis van bijna al het tegenintuïtieve in de kwantummechanica.

Nu we een kijkje hebben gehad in de vreemde kwantumkeuken, zijn we geïnteresseerd in een theorie om dit alles te beschrijven en, belangrijker, om al dit geks correct te voorspellen. Het blijkt echter nogal lastig te zijn om zomaar met een kwantumtheorie op de proppen te komen. Meestal begint men daarom met een klassieke theorie, om deze vervolgens te *kwantiseren.* Wat dit kwantiseren inhoudt, is ervoor zorgen dat kwantumfenomenen die bij experimenten waargenomen zijn, correct beschreven worden. Het zorgt er bijvoorbeeld voor dat de eerdergenoemde kanonieke commutatierelaties gelden. Dit betekent niets anders dan dat positie en impuls niet gelijktijdig meetbaar zijn, zoals we hiervoor al zagen. Aangezien kwantummechanica nauwkeuriger is dan Newton's mechanica is niet duidelijk op welke manier dit kwantiseren moet gebeuren.

In deze scriptie heb ik gekeken naar één van de manieren om een klassieke theorie te kwantiseren, zogeheten *kwantisatie door deformatie.* Hiervoor heb ik allereerst symplectische geometrie bestudeerd: een wiskunde formalisme dat klassieke mechanica in een algemeen jasje steekt. Het voordeel van dit jasje is dat allerlei lastige systemen goed te beschrijven zijn. Het wegschieten van een voetbal over een veld is redelijk eenvoudig, maar ook beweging op gekromde oppervlakken (zogeheten *variëteiten*) zoals een bol of een cilinder is zo te vangen. Deze gekromde structuur wordt vervolgens gedeformatiekwantiseerd door een zogeheten *sterproduct* $\star$ te introduceren. Dit is een ander soort product dan het reguliere, dat we gebruiken om bijvoorbeeld vijf maal zeven uit te rekenen. Merk op dat $5 \times 7 = 35 = 7 \times 5$, wat betekent dat 5 en 7 commuteren ten opzichte van regulier vermenigvuligen; in beide volgorden is hun product gelijk aan 35. Echter, ten opzichte van het sterproduct commuteren positie r en impuls p niet. Oftewel, $r \star p \neq p \star r$. Dit is precies het vreemde kwantumgedrag wat we hierboven zagen. Dankzij dit sterproduct zijn andere kwantumfenomenen ook te beschrijven.

De Russisch wiskundige Maxim Kontsevich heeft in 1997 een stelling bewezen waaruit volgt dat elk klassiek systeem (*symplectische variëteit* in jargon) te deformatiekwantiseren valt tot één kwantumsysteem door definitie van zo een sterproduct.[37] Ik heb deze stelling en zijn gevolgen onderzocht. Daarnaast geeft Kontsevich een formule om op elk oppervlak (variëteit) het bijbehorende sterproduct te vinden. Deze berekening gaat met behulp van grafieken en ik heb de formule op het makkelijkste voorbeeld, het platte vlak, toegepast. Dit levert het zogeheten Moyalproduct, dat in de jaren '40 al bekend was bij onder andere Nederlands natuurkundige Hip Groenewold. Ter introductie tot het onderwerp zijn de eigenschappen van dit product onderzocht en beschreven.

Kwantisatie door deformatie blijkt een goede manier te zijn om aan een klassiek systeem een kwantummechanisch systeem te associëren. Dankzij Kontsevich' bewijs is zo een associatie voor elk klassiek systeem mogelijk. Bovendien is de toekenning in zekere zin 'uniek'. Dit zijn twee sterke argumenten ter ondersteuning van deze kwantisatiemethode. Echter, deformatiekwantisatie blijft één van de methoden. Aangezien de natuurkunde een empirische wetenschap is, zal een experiment op een of andere manier uitsluitsel moeten geven; *that's just how the cookie crumbles.*

Erg intuïtief is kwantummechanica in ieder geval niet, want zoals Niels Bohr[38], één van de pioniers van de kwantummechanica, al zei:

Anyone who is not shocked by quantum theory has not understood it.[39]

[37] Er zijn hierbij een aantal details in overweging te nemen, de relatie is niet direct een staat tot een.

[38] Niels Bohr (1885 - 1962), Deens natuurkundige, speelde een belangrijke rol in de ontwikkeling van de kwantummechanica.

[39] Uit *Alice in Quantumland*, Robert Gilmore, Sigma Press, 1994, Wilmslow.