

Abstract

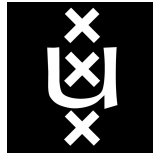
This thesis gives a basic introduction to supersymmetry in quantum mechanics.

Any supersymmetric system contains operators which obey the $sl(1,1)$ algebra

$$\begin{aligned} [H_s, Q] &= [H_s, Q^\dagger] = 0, \\ \{Q, Q^\dagger\} &\equiv QQ^\dagger + Q^\dagger Q = H_s, \\ \{Q, Q\} &= \{Q^\dagger, Q^\dagger\} = 0, \end{aligned}$$

where H_s is the supersymmetric Hamiltonian. Q and Q^\dagger are operators which carry the name supercharges. These supercharges leave the energy of a system invariant but exchange bosonic and fermionic degrees of freedom. The symmetry in the name supersymmetry is due to the fact that the supercharges commute with the Hamiltonian.

By supersymmetry one can see how two different potentials are linked and share the same energy spectrum. To illustrate this property some basic examples from "normal" quantum mechanics are treated, like the infinite square well and the harmonic oscillator. The harmonic oscillator example shows that despite the degeneracy the groundstate contains only a bosonic state. This is true for all unbroken supersymmetry systems. However supersymmetry can also be broken, in this case the groundstate has both a bosonic and a fermionic state. A tool to check whether or not supersymmetry is broken is the Witten index. This index is applicable in both quantum mechanics as well as quantum field theory. The breaking of supersymmetry appears to be important, because it could explain why supersymmetry has not been observed in nature. The supersymmetric Lagrangian and the resulting equations of motion are calculated. This thesis then moves on to superfields which contain anti-commuting "coordinates". These coordinates extend normal space-time and automatically manifest supersymmetry. We move on to the Quantum Calogero Moser model using supersymmetry. The Calogero Moser is a system containing N particles which interact pairwise and can be subjected to three different kinds of potentials. The thesis ends by giving examples from different fields in physics like nuclear physics and the study black holes. Very important in the discovery and development of supersymmetry is particle physics. In this field every boson has a supersymmetric partner particle which is a fermion, and vice versa. It was introduced to solve the hierarchy problem of the Higgs mass. The absence of any data conforming unbroken supersymmetry implies that supersymmetry is either absent in nature or broken some time after the big bang. In any case the mathematics of supersymmetry are useful and elegant.



UNIVERSITY OF AMSTERDAM

BACHELOR THESIS

**A brief introduction to
supersymmetry in quantum
mechanics.**

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October 27, 2011

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1 Introduction

Supersymmetry, abbreviated as SUSY, was introduced by Gel'fand, Likhtman, Ramond, Neveu and Schwartz and later rediscovered by other groups [6]. In theoretical physics it was first introduced to unify bosonic and fermionic sectors in string theory [6]. In particle physics it was interesting to connect bosonic particles with fermionic ones. Wess and Zumino showed how to construct a 3 + 1 dimensional field theory which was invariant under SUSY [6]. The algebra that the operators of a system must obey in order to let the system be supersymmetric, both in Quantum mechanics and quantum field theory, is the super Lie algebra

$$\begin{aligned} [H_s, Q] &= [H_s, Q^\dagger] = 0, \\ \{Q, Q^\dagger\} &\equiv QQ^\dagger + Q^\dagger Q = H_s, \\ \{Q, Q\} &= \{Q^\dagger, Q^\dagger\} = 0, \end{aligned}$$

where Q is an operator called the supercharge and Q^\dagger is its Hermitian conjugate. And H_s is the supersymmetry Hamiltonian. This algebra forms the foundation of supersymmetry. The fact that Q commutes with H produces the symmetry. Unbroken SUSY requires that for every boson a fermionic superpartner particle exists with the same mass, and vice versa. This has until now not been confirmed by experiments. A possible explanation is that SUSY was spontaneously broken some time after the big bang. Despite the seemingly absence of SUSY QM in nature it is a powerful mathematical theory in its own right. For example SUSY connects the energy eigenvalues of Hamiltonians with very different potentials. Also it can be used in many models such as the quantum Calogero-Moser model and the Calogero-Marchioro model [4, 5, 6]. In this paper I will begin by introducing the formalism of SUSY. This is followed by some examples like the infinite square well and the harmonic oscillator. Followed by the introduction of more theory. Broken SUSY is the next topic to be discussed. After this we move on to the Quantum Calogero-Moser model. This model allows the particle to have pair-wise interaction and subject them to an harmonic oscillator. I will end the thesis with a brief description of fields in physics which use SUSY, like nuclear physics and black hole physics.

2 An introduction to supersymmetry

In this section I will introduce the concepts of and the mathematics that go with supersymmetry in quantum mechanics. Some properties of supersymmetry will be shown using basic examples like the infinite square well and the harmonic oscillator. Important concepts in this chapter are the superpotential, the superlie algebra and the supercharges.

2.1 Supersymmetry formalism

2.1.1 The superpotential

As mentioned in the introduction any system that has operators abiding the closed super Lie or $sl(1/1)$ algebra

$$[H_s, Q] = [H_s, Q^\dagger] = 0, \quad (2.1)$$

$$\{Q, Q^\dagger\} = H_s, \quad (2.2)$$

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0, \quad (2.3)$$

is supersymmetric. Where H_s is the Hamiltonian and Q and its hermitian conjugate Q^\dagger are called the supercharges. These supercharges commute with the Hamiltonian. The Hamiltonian is the generator of time, $i\hbar\frac{\partial}{\partial t} = H$, so by commuting with the Hamiltonian the supercharges are constants of motion and because of this they form a symmetry. This sections will illustrate how this algebra is formed by beginning with an example from quantum mechanics. This example is a one dimensional Hamiltonian with a potential $V(x)$. The first step is to connect the groundstate wave function and the Hamiltonian. For simplicity we set $2m = \hbar = 1$ unless stated otherwise. Also for now we assume the groundstate energy to be 0. We Start by giving the Hamiltonian,

$$H_1 = -\frac{d^2}{dx^2} + V_1(x). \quad (2.4)$$

The index of the Hamiltonian H_1 is used to distinct it from its so-called supersymmetry partner Hamiltonian which we will encounter later on. When we can actually speak of fermions and bosons, the index 1 stands for the bosonic part, and the index 2 for the fermionic part.

In terms of the groundstate equation 2.4 reads:

$$H_1\psi_0(x) = -\frac{d^2\psi_0}{dx^2} + V_1(x)\psi_0(x) = 0. \quad (2.5)$$

This allows us to write the potential in terms of the groundstate:

$$V_1(x) = \frac{\psi_0''(x)}{\psi_0(x)}. \quad (2.6)$$

We can factorize the Hamiltonian by making the following Ansatz:

$$H_1 = A^\dagger A, \quad (2.7)$$

with

$$A = \frac{d}{dx} + W(x), \quad A^\dagger = -\frac{d}{dx} + W(x). \quad (2.8)$$

Remembering how operators work this yields

$$H_1 = A^\dagger A = -\frac{d^2}{dx^2} - W'(x) + W^2(x) = -\frac{d^2}{dx^2} + V_1(x). \quad (2.9)$$

Writing V_1 explicitly gives

$$V_1(x) = W^2(x) - W'(x). \quad (2.10)$$

This equation is called the Riccati equation [6]. $W(x)$ is referred to as the superpotential in SUSY QM. By choosing $A\psi_0 = 0$, $H_1\psi_0 = A^\dagger A\psi_0 = 0$. Using this we can write $W(x)$ in terms of the ground state as follows

$$W(x) = -\frac{\psi_0'(x)}{\psi_0(x)}. \quad (2.11)$$

Next we define a second Hamiltonian,

$$H_2 = AA^\dagger = -\frac{d^2}{dx^2} + V_2(x), \quad (2.12)$$

$$V_2(x) = W^2(x) + W'(x). \quad (2.13)$$

V_1 and V_2 are called supersymmetry partner potentials [6]. As I will show, the eigenvalues and eigenfunctions of H_1 and H_2 are related. Also, the eigenvalues of H_1 and H_2 are positive semi-definite ($E_n^{(1,2)} \geq 0$) [6]. To see how the eigenvalues are related we start with

$$H_1|\psi_n^{(1)}\rangle = A^\dagger A|\psi_n^{(1)}\rangle = E_n^{(1)}|\psi_n^{(1)}\rangle. \quad (2.14)$$

Keeping in mind that you can always move an eigenvalue through an expression, in this case $E_n^{(1)}$, we get

$$H_2 A|\psi_n^{(1)}\rangle = AA^\dagger A|\psi_n^{(1)}\rangle = AH_1|\psi_n^{(1)}\rangle = E_n^{(1)} A|\psi_n^{(1)}\rangle. \quad (2.15)$$

Using the same trick on H_2 :

$$H_2|\psi_n^{(2)}\rangle = AA^\dagger|\psi_n^{(2)}\rangle = E_n^{(2)}|\psi_n^{(2)}\rangle, \quad (2.16)$$

gives

$$H_1 A^\dagger|\psi_n^{(2)}\rangle = A^\dagger AA^\dagger|\psi_n^{(2)}\rangle = A^\dagger H_2|\psi_n^{(2)}\rangle = E_n^{(2)} A^\dagger|\psi_n^{(2)}\rangle. \quad (2.17)$$

This means that $A\psi_n^{(1)}$ is an eigenfunction of H_2 and $A^\dagger|\psi_n^{(2)}\rangle$ is an eigenfunction of H_1 , so $|\psi_n^{(2)}\rangle \propto |\psi_m^{(1)}\rangle$ or $|\psi_n^{(2)}\rangle = c|\psi_m^{(1)}\rangle$ with eigenvalue $E_n^{(2)} = E_m^{(1)}$, if the

spectrum is non-degenerate. Later on I will derive the relation between n and m . To determine the constant c we start by normalizing both $|\psi_n^{(2)}\rangle$ and $|\psi_m^{(1)}\rangle$

$$\begin{aligned}
|\psi_n^{(2)}\rangle &= cA|\psi_m^{(1)}\rangle, & c \in \mathbb{C}, \\
1 &= \langle\psi_m^{(1)}|\psi_m^{(1)}\rangle = \langle\psi_n^{(2)}|\psi_n^{(2)}\rangle \\
&= \langle cA\psi_m^{(1)}|cA\psi_m^{(1)}\rangle \\
&= \langle\psi_m^{(1)}|A^\dagger c^* cA\psi_m^{(1)}\rangle = c^* c \langle\psi_m^{(1)}|A^\dagger A|\psi_m^{(1)}\rangle \\
&= c^* c \langle\psi_m^{(1)}|H_1|\psi_m^{(1)}\rangle \\
&= c^* c E_{(m)} \langle\psi_m^{(1)}|\psi_m^{(1)}\rangle = c^* c E_{(m)} \\
&\Rightarrow c^* c = 1/E_{(m)}.
\end{aligned}$$

Because E_m is always positive, we might as well take c real, so $c = \frac{1}{\sqrt{E_m^{(1)}}}$, and hence

$$|\psi_n^{(2)}\rangle = \frac{1}{\sqrt{E_m^{(1)}}} A|\psi_m^{(1)}\rangle. \quad (2.18)$$

Applying the same procedure to find the relation between $|\psi_m^{(1)}\rangle$ and $A^\dagger|\psi_n^{(2)}\rangle$ we find

$$|\psi_m^{(1)}\rangle = \frac{1}{\sqrt{E_n^{(2)}}} A^\dagger|\psi_n^{(2)}\rangle. \quad (2.19)$$

As we saw before every eigenvalue of H_2 is also an eigenvalue of H_1 . Only the groundstate of H_1 is not an eigenvalue of H_2 , because if one inserts $m = 0$, $E_0^{(1)} = 0$, into equation 2.18 the corresponding eigenfunction of H_2 would blow up. This almost one-to-one correspondence, implies that $m = n + 1$ and so for $n \in \{0, 1, 2, \dots\}$

$$E_{n+1}^{(1)} = E_n^{(2)}, \quad (2.20)$$

$$|\psi_n^{(2)}\rangle = \frac{1}{\sqrt{E_{n+1}^{(1)}}} A|\psi_{n+1}^{(1)}\rangle, \quad (2.21)$$

$$|\psi_{n+1}^{(1)}\rangle = \frac{1}{\sqrt{E_n^{(2)}}} A^\dagger|\psi_n^{(2)}\rangle. \quad (2.22)$$

Notice that if $|\psi_{n+1}^{(1)}\rangle$ of H_1 is normalized then the wave function $|\psi_n^{(2)}\rangle$ in equations 2.21 then 2.22 is also normalized and vice versa. From the equations one can conclude that A converts an eigenfunction of H_1 into H_2 and lowers n by one. A^\dagger changes an eigenfunction of H_2 into H_1 and raises n by 1. The groundstate of H_1 is destroyed by A , this means there is no SUSY partner for this state [6]. Both the eigenfunctions and states of H_2 can be found easily when you know those of H_1 . As we shall this means that two entirely different potentials have the same energy spectra. You can even construct a whole hierarchy of related Hamiltonians by taking H_2 , subtract from it the ground-state energy and call this H_1' . Now one can define new operators A' and A'^\dagger

so that $H'_1 = A'^\dagger A$ and define another Hamiltonian $H'_2 = A'^\dagger A$. From this new Hamiltonian another can be constructed and so on until all the bound states are exhausted, if there are finite bound states that is. Every new Hamiltonian has one less state than the one it is constructed from.

In order to get the supersymmetry Hamiltonian we combine the Hilbert space of the two Hamiltonians, $H_s = H_1 \oplus H_2$. This can be realized by writing the supersymmetry Hamiltonian as a 2x2 matrix with H_1 and H_2 on its diagonals

$$H_s = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}. \quad (2.23)$$

The operators Q and Q^\dagger are matrices who's entries contain the operators A and A^\dagger :

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}. \quad (2.24)$$

We can now check if Q and Q^\dagger obey the closed super Lie algebra($sl(1/1)$) which, again, is [6]

$$[H_s, Q] = [H_s, Q^\dagger] = 0, \quad (2.25)$$

$$\{Q, Q^\dagger\} = H_s, \quad (2.26)$$

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0. \quad (2.27)$$

Putting equations 2.23 and 2.24 into equations 2.25 - 2.27 gives

$$H_s = QQ^\dagger + Q^\dagger Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \quad (2.28)$$

$$= \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \quad (2.29)$$

$$[H_s, Q] = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} \quad (2.30)$$

$$= \begin{pmatrix} 0 & 0 \\ AA^\dagger A & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ AA^\dagger A & 0 \end{pmatrix} = 0, \quad (2.31)$$

$$\{Q, Q\} = 2 \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = 0 \quad (2.32)$$

which confirms that that H_s and Q in this representation obey the $sl(1/1)$ algebra and the system is supersymmetric.

2.2 Examples

2.2.1 The infinite square well

This short and easy example shows how supersymmetry can yield the same energy spectra for two entirely different potentials, except for the groundstate. The infinite square well with width a has the following potential

$$V(x) = \begin{cases} 0, & 0 \leq x \leq a, \\ \infty, & -\infty < x < 0, x > a. \end{cases} \quad (2.33)$$

First we want to factorize the Hamiltonian. This means that $H_1\psi_0^{(1)} = 0$ so we subtract off the groundstate energy to get

$$H_1 = -\frac{d^2}{dx^2} - \frac{\pi^2}{a^2} \quad (2.34)$$

So $V_1(x) = -\frac{\pi^2}{a^2}$. The eigenvalues and eigenfunctions depending on n are:

$$E_n^1 = \frac{n(n+2)\pi^2}{a^2}, \quad (2.35)$$

$$\psi_n^{(1)} = \sqrt{\frac{2}{a}} \sin\left(\frac{(n+1)\pi x}{a}\right), \quad 0 \leq x \leq a. \quad (2.36)$$

Now using equation 2.11 to find the superpotential and the partner potential we get:

$$W(x) = -\frac{\pi}{a} \cot\left(\frac{\pi x}{a}\right), \quad (2.37)$$

$$V_2(x) = \frac{\pi^2}{a^2} [2 \operatorname{cosec}^2\left(\frac{\pi x}{a}\right) - 1], \quad (2.38)$$

with $E_n^{(2)} = E_{n+1}^{(1)}$. So you can see that - as I stated before - two very different potentials yield the same energy spectrum, except for the groundstate of H_1 .

2.2.2 The supersymmetry quantum harmonic oscillator

The supersymmetry harmonic oscillator is a combination of bosonic and fermionic oscillators. This example treats only one particle so in a sense there are no bosons and fermions, but I will use the terminology because in the Calogero-Moser model - which will be treated later on - there are actually bosons and fermions. The CM model contains an harmonic oscillator.

For simplicity we set the Hamiltonian H of the harmonic oscillator dimensionless. This means that the operator H is given in terms of ω . So $\mathcal{H} = H\omega$, where \mathcal{H} is the proper dimensional Hamiltonian operator. Also remember that $2m = \hbar = 1$. First we will take a look at the bosonic oscillator which is just the infamous quantum mechanical harmonic oscillator

$$H_b = p^2 + \frac{1}{4}x^2 - \frac{1}{2}, \quad p = -i\frac{d}{dx}, \quad [x, p] = i. \quad (2.39)$$

Now we introduce the ladder operators

$$a = \left(\frac{x}{2} + ip\right), \quad a^\dagger = \left(\frac{x}{2} - ip\right). \quad (2.40)$$

The Hamiltonian can be factorized: $H_b = a^\dagger a$. The groundstate $|0\rangle$ is defined by $a|0\rangle = 0$. This results in a first order differential equation. The eigenstates are

$$|n_b\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \text{ where } |0\rangle \text{ is simply } e^{-x^2}. \quad (2.41)$$

Here we introduce the operators ψ and ψ^\dagger which obey the algebra for the fermionic creation and annihilation operators.

$$\{\psi^\dagger, \psi\} = 1, \quad \{\psi^\dagger, \psi^\dagger\} = \{\psi, \psi\} = 0, \quad (2.42)$$

where the anti-commuting relation for any given operators A and B is $\{A, B\} \equiv AB + BA$. The algebra can be realized by writing the states as an \mathbb{R}^2 vector with the bosonic state as the first element and the fermionic state as the second, $\phi = \begin{pmatrix} \phi_b \\ \phi_f \end{pmatrix}$. Except for the groundstate which is just $|0\rangle = |\phi_b^0\rangle$, the bosonic groundstate and the fermionic vacuum. In this base the fermion operators are [6]

$$\psi = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (2.43)$$

$$\psi^\dagger = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (2.44)$$

The commutation relation for ψ and ψ^\dagger is:

$$[\psi, \psi^\dagger] = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.45)$$

which is the third Pauli matrix. Now we can write down the fermionic harmonic oscillator in the same way as H_b but instead of a and a^\dagger we use ψ and ψ^\dagger .

$$H_f = \psi^\dagger \psi - \frac{1}{2}. \quad (2.46)$$

The supersymmetric Hamiltonian becomes

$$H_s = H_b + H_f = (a^\dagger a + \psi^\dagger \psi). \quad (2.47)$$

The supercharges are

$$Q = a\psi^\dagger, \quad Q^\dagger = a^\dagger\psi, \quad (2.48)$$

which are a combination of fermionic and bosonic operators. The total Hamiltonian, bosonic and fermionic sectors included is [6]

$$H_s = QQ^\dagger + Q^\dagger Q = \left(-\frac{d^2}{dx^2} + \frac{x^2}{4}\right)I + \frac{1}{2}[\psi, \psi^\dagger], \quad (2.49)$$

where I is the Identity matrix. A state is a combination of a bosonic part and a fermionic one, which is labeled $|n_b, n_f\rangle$. The states live in the so called "Fock-space". The groundstate has no fermionic part, its fermion number is zero. The fermion number operator N_f is

$$N_f = \frac{1 - [\psi, \psi^\dagger]}{2}. \quad (2.50)$$

The eigenvalues of N_f , denoted as n_f , are zero and one. This is in agreement with the Pauli principle. The energy states, in units of ω , of the fermionic sectors are:

$$E_F = (n_f - \frac{1}{2}) = \pm \frac{1}{2} \quad (2.51)$$

So the spectrum is degenerate. The operators $a, a^\dagger, \psi, \psi^\dagger, Q, Q^\dagger$ in this Fock space act like [6]:

$$\begin{aligned} a|n_b, n_f\rangle &= |n_b - 1, n_f\rangle, & \psi|n_b, n_f\rangle &= |n_b, n_f - 1\rangle, \\ a^\dagger|n_b, n_f\rangle &= |n_b + 1, n_f\rangle, & \psi^\dagger|n_b, n_f\rangle &= |n_b, n_f + 1\rangle, \\ Q|n_b, n_f\rangle &\approx |n_b + 1, n_f - 1\rangle, & Q^\dagger|n_b, n_f\rangle &\approx |n_b - 1, n_f + 1\rangle. \end{aligned} \quad (2.52)$$

So Q changes a fermion into a boson whilst leaving the energy of the state the same and Q^\dagger changes a boson into a fermion without altering the energy of the state. This result is very important, it shows the fermion-boson degeneracy property of SUSY QM theory. It also shows how the supercharges exchange bosons and fermions.

2.3 More formalism and a short summary

Now that we have seen an example of the fermion-boson degeneracy we can expand this to the more general case and write down the Hamiltonian in terms of a superpotential and the commutation relation of the fermionic creation and annihilation operators. This section also serves as a small summary of the the basic ingredients of supersymmetry.

The bosonic and fermionic Hamiltonian can be factorized by the operators

$$A = \frac{d}{dx} + W(x), \quad A^\dagger = -\frac{d}{dx} + W(x), \quad (2.53)$$

with $W(x) = -\frac{(\psi_0^b)'(x)}{\psi_0^b(x)}$ the superpotential. The fermion creation and annihilation operators ψ and ψ^\dagger and their (anti-)commuting relations are:

$$\psi = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \psi^\dagger = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (2.54)$$

$$\{\psi^\dagger, \psi\} = 1, \quad \{\psi^\dagger, \psi^\dagger\} = \{\psi, \psi\} = 0, \quad (2.55)$$

$$[\psi, \psi^\dagger] = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.56)$$

The supercharges are created by multiplying the operators A and A^\dagger with the fermion creation and annihilation operators like this

$$Q = A\psi^\dagger, \quad Q^\dagger = A^\dagger\psi. \quad (2.57)$$

The (anti-)commutation relation of the supercharges with the Hamiltonian are:

$$[H_s, Q] = [H_s, Q^\dagger] = 0, \quad \{Q, Q^\dagger\} = H_s, \quad \{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0. \quad (2.58)$$

So

$$H_s = (QQ^\dagger + Q^\dagger Q) = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \quad (2.59)$$

$$= \begin{pmatrix} \frac{d^2}{dx^2} + W^2(x) - W'(x) & 0 \\ 0 & \frac{d^2}{dx^2} + W^2(x) + W'(x) \end{pmatrix}. \quad (2.60)$$

Using equation 2.56 we can rewrite the Hamiltonian in another way.

$$H_s = \frac{1}{2} \left(-\frac{d^2}{dx^2} + W^2(x) \right) - W'(x) [\psi, \psi^\dagger] = \frac{1}{2} (p^2 + W^2(x) - \sigma_3 W'(x)) \quad (2.61)$$

It is also possible to construct the groundstate wave function of the system via the formula of the superpotential

$$W(x) = \frac{-\psi'_0(x)}{\psi(x)} \rightarrow \psi_0(x) = ce^{-\int W(x)dx} \quad (2.62)$$

where c is some complex constant, $c \in \mathbb{C}$ which is determined by the normalization criterion for any state ψ : $\langle \psi | \psi \rangle = 1$

Supersymmetry also requires that the Hamiltonian eliminates the groundstate $|0\rangle$ [6]:

$$H_s|0\rangle = 0, \text{ so } Q|0\rangle = Q^\dagger|0\rangle = 0. \quad (2.63)$$

If this is not the case the supersymmetry is broken, which will be discussed further in section 3. Because Q and Q^\dagger commute with H the spectrum is degenerate. Q and Q^\dagger are operators that change bosonic degrees of freedom into fermionic ones and vice versa without changing the energy of the system [6]. They are also the generators of the supersymmetry, because their anti-commuting relation produces the supersymmetric Hamiltonian.

2.3.1 The supersymmetry Lagrangian

Deriving the Lagrangian is always very useful, taking its integral produces the action $S = \int L(q(t), \dot{q}(t), t) dt$, which is a functional whose image is \mathbb{R} . Using the Euler-Lagrange equation one can get the equations of motion.

Lets start by giving the supersymmetric Hamiltonian

$$\mathcal{H} = -\frac{1}{2}p^2 + \frac{1}{2}W^2(x) - \frac{1}{2}W'(x)[\psi, \psi^\dagger]. \quad (2.64)$$

This results in the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 + i\psi^\dagger\dot{\psi} - \frac{1}{2}W^2(x) + \frac{1}{2}W'(x)[\psi, \psi^\dagger]. \quad (2.65)$$

Where we use the convention that $\dot{q}_i \equiv \frac{dq_i}{dt}$, with $q_i \in \{x, \psi, \psi^\dagger\}$.

The action is

$$S = \int \left\{ \frac{1}{2} \dot{x}^2 + \psi^\dagger \dot{\psi} - \frac{1}{2} W^2(x) + \frac{1}{2} W'(x) [\psi, \psi^\dagger] \right\} dt. \quad (2.66)$$

We want to check if this action is real e.g. $S \in \mathbb{R}$. So S should be equal to its Hermitian conjugate S^\dagger . Taking the Hermitian conjugate of for example two anti-commuting variables α and β their order reverses, $(\alpha\beta)^\dagger = \beta^\dagger\alpha^\dagger$. To check if $S = S^\dagger$ we take the Hermitian conjugate of S

$$S^\dagger = \int \left\{ \frac{1}{2} \dot{x}^2 - \dot{\psi}^\dagger \psi - \frac{1}{2} W^2(x) + \frac{1}{2} W'(x) [\psi, \psi^\dagger] \right\} dt. \quad (2.67)$$

$$(2.68)$$

We can see that the only term that changes by taking S^\dagger is $\int \psi^\dagger \dot{\psi} dt$ which becomes $-\int \dot{\psi}^\dagger \psi dt$. So these two should be equal. Taking into account that the boundary conditions are always such that $\int \dot{f} g dt = -\int f \dot{g} dt$ for any nice functions f and g we see that

$$-\int \dot{\psi}^\dagger \psi dt = \int \psi^\dagger \dot{\psi} dt,$$

which is what we wanted to show.

Now we check if this action is indeed supersymmetric, by letting the action vary. The variation of the action should be zero, $\delta S = 0$. $S = \int \mathcal{L} dt$, so $\delta S = \int \delta \mathcal{L} dt$. Thus we can start by letting \mathcal{L} vary. In order to make things clearer we write $\frac{dq_i}{dt}$ instead of \dot{q}_i .

$$\delta \mathcal{L} = \delta \frac{1}{2} \left(\frac{dx}{dt} \right)^2 + \delta \psi^\dagger \frac{d\psi}{dt} - \delta \frac{1}{2} W^2(x) + \delta \frac{1}{2} [\psi, \psi^\dagger] W'(x). \quad (2.69)$$

Beginning with the $\delta \frac{1}{2} \left(\frac{dx}{dt} \right)^2$ term:

$$\delta \frac{1}{2} \left(\frac{dx}{dt} \right)^2 = \frac{1}{2} \left\{ \left(\frac{d(x + \delta x)}{dt} \right)^2 - \left(\frac{dx}{dt} \right)^2 \right\}. \quad (2.70)$$

Working out the first part

$$\frac{1}{2} \left(\frac{d(x + \delta x)}{dt} \right)^2 = \frac{1}{2} \left(\frac{dx + d\delta x}{dt} \right)^2 = \frac{1}{2} \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{d\delta x}{dt} \right)^2 + 2 \frac{dx}{dt} \frac{d\delta x}{dt} \right\}. \quad (2.71)$$

In variational calculus we are only interested in the linear terms of ϵ and thus only linear terms in δ . So the above becomes:

$$\frac{1}{2} \left(\frac{dx}{dt} \right)^2 + \frac{dx}{dt} \frac{d\delta x}{dt}, \text{ so} \quad (2.72)$$

$$\delta \frac{1}{2} \left(\frac{dx}{dt} \right)^2 = \frac{dx}{dt} \frac{d\delta x}{dt}. \quad (2.73)$$

Now we must use this procedure on all the terms of equation 2.69.

$$\delta\iota\psi^\dagger \frac{d\psi}{dt} = \iota[(\psi^\dagger + \delta\psi^\dagger) \frac{d(\psi + \delta\psi)}{dt} - \psi^\dagger \frac{d\psi}{dt}] \quad (2.74)$$

$$= \iota[\psi^\dagger \frac{d\psi}{dt} + (\delta\psi^\dagger) \frac{d\psi}{dt} + \psi^\dagger \frac{d\delta\psi}{dt} + (\delta\psi^\dagger) \frac{d\delta\psi}{dt} - \psi^\dagger \frac{d\psi}{dt}] \quad (2.75)$$

$$= \iota[(\delta\psi^\dagger) \frac{d\psi}{dt} + \psi^\dagger \frac{d\delta\psi}{dt}]. \quad (2.76)$$

The next term is

$$\delta \frac{1}{2} W^2(x) = \frac{1}{2} [W^2(x + \delta x) - W^2(x)]. \quad (2.77)$$

We must use the Taylor expansion of $W(x)$ around x . Only the terms up to the first derivative count,

$$W(x + \delta x) = W(x) + W'(x)(x + \delta x - x) = W(x) + W'(x)\delta x. \quad (2.78)$$

Plugging this into equation 2.77 yields:

$$\delta \frac{1}{2} W^2(x) = \frac{1}{2} (W(x) + W'(x)\delta x)^2 - W^2(x) \quad (2.79)$$

$$= \frac{1}{2} (W(x) + W'(x)\delta x)^2 - W^2(x) \quad (2.80)$$

$$= \frac{1}{2} [W^2(x) + 2W(x)W'(x)\delta x - W^2(x)] = W(x)W'(x)\delta x. \quad (2.81)$$

And now we arrive at the final term. We again make use of the first order Taylor expansion and that non-linear terms in δ are dismissed.

$$\begin{aligned} \delta W'(x)[\psi, \psi^\dagger] &= W'(x + \delta x)[\psi + \delta\psi, \psi^\dagger + \delta\psi^\dagger] - W'(x)[\psi, \psi^\dagger] \\ &= (W'(x) + W''(x)\delta x)\{(\psi + \delta\psi)(\psi^\dagger + \delta\psi^\dagger) - (\psi^\dagger + \delta\psi^\dagger)(\psi + \delta\psi)\} - W'(x)[\psi, \psi^\dagger] \\ &= (W'(x) + W''(x)\delta x)(\psi\psi^\dagger + \delta\psi\psi^\dagger + \psi\delta\psi^\dagger - \psi^\dagger\psi - \psi^\dagger\delta\psi - \delta\psi^\dagger\psi) - W'(x)[\psi, \psi^\dagger] \\ &= \{(W''(x)\delta x)[\psi, \psi^\dagger] + W'(x)([\delta\psi, \psi^\dagger] + [\psi, \delta\psi^\dagger])\}. \end{aligned}$$

Everything together results in

$$\begin{aligned} \delta\mathcal{L} &= \frac{dx}{dt} \frac{d\delta x}{dt} + \iota(\delta\psi^\dagger) \frac{d\psi}{dt} + \psi^\dagger \iota \frac{d\delta\psi}{dt} \\ &\quad - W(x)W'(x)\delta x + \\ &\quad \frac{1}{2} \{W''(x)\delta x[\psi, \psi^\dagger] + W'(x)([\delta\psi, \psi^\dagger] + [\psi, \delta\psi^\dagger])\}. \end{aligned} \quad (2.82)$$

At this point the δq_i terms should be introduced in order to continue. The supersymmetry transformations are [9]

$$\delta x = \epsilon^\dagger \psi + \psi^\dagger \epsilon, \quad (2.83)$$

$$\delta\psi = \epsilon\{\alpha\dot{x} + \beta W(x)\}, \quad (2.84)$$

$$\delta\psi^\dagger = \epsilon^\dagger\{\alpha^* \dot{x} + \beta^* W(x)\}, \quad (2.85)$$

where ϵ and ϵ^\dagger are infinitesimal anti-commuting parameters. α and β are complex constants which will be determined later on. Remember that the goal is to show that the Lagrangian is supersymmetric by showing that if we plug in the supersymmetry transformations into equation 2.82, it will be zero.

The (anti-)commuting rules for all the variables involved are

$$\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = \{\epsilon, \epsilon\} = \{\epsilon^\dagger, \epsilon^\dagger\} = \quad (2.86)$$

$$\{\psi, \psi^\dagger\} = \{\epsilon, \psi\} = \{\epsilon^\dagger, \psi^\dagger\} = \{\epsilon, \psi^\dagger\} = 0, \quad (2.87)$$

$$[x, \psi] = [x, \psi^\dagger] = [x, \epsilon] = [x, \epsilon^\dagger] = 0. \quad (2.88)$$

In the following calculations I make use of the fact that we actually want to calculate $\delta S = \int \delta \mathcal{L} dt$, so in some terms I will use partial integration. When I make use of partial integration the boundary terms vanish so $\int \dot{f} g = -\int f \dot{g}$ for variables f and g which depend on time. Now we plug the transformations into 2.82 and treat them term by term.

The first term

$$\begin{aligned} \dot{x} \frac{d\delta x}{dt} &= \dot{x} \frac{\epsilon^\dagger \dot{\psi} + \dot{\psi}^\dagger \epsilon}{dt} \\ &= \dot{x} (\epsilon^\dagger \dot{\psi} + \dot{\psi}^\dagger \epsilon). \end{aligned}$$

The 2nd term

$$i(\delta\psi^\dagger)\dot{\psi} = i\epsilon^\dagger\dot{\psi}[\alpha^*\dot{x} + \beta^*W(x)].$$

The 3rd one

$$\begin{aligned} \psi^\dagger i \frac{d\delta\psi}{dt} &= \psi^\dagger i \epsilon \frac{d}{dt} [\alpha\dot{x} + \beta W(x)] \\ &= \psi^\dagger i \epsilon [\alpha\ddot{x} + \beta\dot{W}(x)] = -i\dot{\psi}^\dagger \epsilon [\alpha\dot{x} + \beta W(x)]. \end{aligned}$$

Going to the 4th

$$-W(x)W'(x)\delta x = -W(x)W'(x)(\epsilon^\dagger\psi + \psi^\dagger\epsilon).$$

The 5th term is

$$\begin{aligned} \frac{1}{2}W''(x)\delta x[\psi, \psi^\dagger] &= \frac{1}{2}W''(x)(\epsilon^\dagger\psi + \psi^\dagger\epsilon)(\psi\psi^\dagger - \psi^\dagger\psi) \\ &= \frac{1}{2}W''(x)\{(\epsilon^\dagger\psi^2\psi^\dagger + \psi^\dagger\epsilon\psi\psi^\dagger - \epsilon^\dagger\psi\psi^\dagger\psi - \psi^\dagger\epsilon\psi^\dagger\psi)\} = \\ &= W''(x)\psi^\dagger\epsilon\psi\psi^\dagger = -W''(x)\epsilon\psi^\dagger\psi\psi^\dagger \\ &= W''(x)\epsilon\psi\psi^\dagger\psi^\dagger = 0. \end{aligned}$$

The 6th will be

$$\begin{aligned}
\frac{1}{2}W'(x)[\delta\psi, \psi^\dagger] &= \frac{1}{2}W'(x)\{\epsilon(\alpha\dot{x} + \beta W(x))\psi^\dagger - \psi^\dagger\epsilon(\alpha\dot{x} + \beta W(x))\} \\
&= \frac{1}{2}W'(x)[\alpha\dot{x} + \beta W(x)][\epsilon\psi^\dagger - \psi^\dagger\epsilon] \\
&= W'(x)[\alpha\dot{x} + \beta W(x)]\epsilon\psi^\dagger.
\end{aligned}$$

And the final term

$$\begin{aligned}
\frac{1}{2}W'(x)[\psi, \delta\psi^\dagger] &= \frac{1}{2}W'(x)\{\psi\epsilon^\dagger(\alpha^*\dot{x} + \beta^*W(x)) + \epsilon^\dagger(\alpha^*\dot{x} + \beta^*W(x))\psi\} \\
&= \frac{1}{2}W'(x)[\alpha^*\dot{x} + \beta^*W(x)][\psi\epsilon^\dagger - \epsilon^\dagger\psi] \\
&= W'(x)[\alpha^*\dot{x} + \beta^*W(x)]\psi\epsilon^\dagger.
\end{aligned}$$

Everything together yields

$$\begin{aligned}
\delta\mathcal{L} &= \dot{x}(\epsilon^\dagger\dot{\psi} + \dot{\psi}^\dagger\epsilon) + i\epsilon^\dagger\dot{\psi}[\alpha^*\dot{x} + \beta^*W(x)] \\
&\quad - i\dot{\psi}^\dagger\epsilon[\alpha\dot{x} + \beta W(x)] - W(x)W'(x)(\epsilon^\dagger\psi + \psi^\dagger\epsilon) \\
&\quad + W'(x)[\alpha\dot{x} + \beta W(x)]\epsilon\psi^\dagger + W'(x)[\alpha^*\dot{x} + \beta^*W(x)]\psi\epsilon^\dagger,
\end{aligned} \tag{2.89}$$

which is quite a mess. However if we work out the terms in order of $W(x)$ we can create some order [7]. Beginning in 0 order terms of $W(x)$ we get

$$\dot{x}\{\epsilon^\dagger\dot{\psi} + \dot{\psi}^\dagger\epsilon + i\alpha^*\epsilon^\dagger\dot{\psi} - i\alpha\dot{\psi}^\dagger\epsilon\} \tag{2.90}$$

This should be 0, so from this we see that $i\alpha^* = -1 \rightarrow \alpha = -i$.

Now we take all $W'(x)W(x)$ terms

$$W'(x)W(x)\{-\epsilon^\dagger\psi - \psi^\dagger\epsilon + \beta\epsilon\psi^\dagger + \beta\psi\epsilon^\dagger\} = \tag{2.91}$$

$$W'(x)W(x)\{\psi\epsilon^\dagger + \epsilon\psi^\dagger + \beta\epsilon\psi^\dagger + \beta\psi\epsilon^\dagger\} = 0. \tag{2.92}$$

It clearly follows from this that $\beta = -1$.

The remaining terms are

$$\delta\mathcal{L} = i\epsilon^\dagger\dot{\psi}\beta^*W(x) - i\dot{\psi}^\dagger\epsilon\beta W(x) \tag{2.93}$$

$$+ W'(x)\alpha\dot{x}\epsilon\psi^\dagger + W'(x)\alpha^*\dot{x}\psi\epsilon^\dagger \tag{2.94}$$

We can rewrite this using $W'(x)\dot{x} = \frac{dW}{dx}\frac{dx}{dt} = \dot{W}(x)$ and partial integration

$$\delta\mathcal{L} = \dot{W}(x)\{i\beta\psi^\dagger\epsilon - i\beta^*\epsilon^\dagger\psi + \alpha\epsilon\psi^\dagger + \alpha^*\psi\epsilon^\dagger\} \tag{2.95}$$

$$= \dot{W}(x)\{\psi^\dagger\epsilon(i\beta - \alpha) + \epsilon^\dagger\psi(-i\beta^* - \alpha^*)\} = 0. \tag{2.96}$$

So $\beta = -i\alpha$ which is consistent with our findings earlier, namely $\alpha = -i$ and $\beta = -1$. So with the transformations

$$\delta x = \epsilon^\dagger \psi + \psi^\dagger \epsilon, \quad (2.97)$$

$$\delta \psi = \epsilon[-i\dot{x} - W(x)], \quad (2.98)$$

$$\delta \psi^\dagger = \epsilon^\dagger[i\dot{x} - W(x)], \quad (2.99)$$

the variation in S is zero and thus the Lagrangian and action are supersymmetric.

The equations of motion Now that we have shown that the Action and the Lagrangian are correctly defined we can find the equations of motion. The Euler-Lagrange equations are

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0, \quad (2.100)$$

with $q_i \in \{x, \psi, \psi^\dagger\}$. Applying this to the Lagrangian

$$\mathcal{L} = \frac{1}{2} \dot{x}^2 + i\psi^\dagger \dot{\psi} - \frac{1}{2} W^2(x) + \frac{1}{2} W'(x)[\psi, \psi^\dagger],$$

simple calculations yield

$$\ddot{x} = -W(x)W'(x) + \frac{1}{2}W''(x)[\psi, \psi^\dagger], \quad (2.101)$$

$$\dot{\psi}^\dagger = iW'(x)\psi^\dagger, \quad (2.102)$$

$$\dot{\psi} = -iW'(x)\psi. \quad (2.103)$$

The last two equations are Hermite conjugates of each other as expected. To solve these equations one needs to know the superpotential.

We now possess a lot of tools to describe supersymmetry, but there is a lot more to it as we will see in the following chapters.

3 Broken supersymmetry

Nature exhibits many symmetries like rotational symmetry, translational symmetry and the Poincar group symmetries. However supersymmetry has not been observed in nature but why not? A possible explanation is that supersymmetry was spontaneously broken some time after the big bang[13]. Supersymmetry is broken if a system obeys the algebra of 2.58, but does not annihilate the groundstate, $H_s|0\rangle \neq 0$, so $Q|0\rangle = Q^\dagger|0\rangle \neq 0$ [6]. The groundstate in SUSY QM is defined as being solely bosonic and therefore it has fermion number 0. We write the SUSY groundstate as $|0\rangle$. In the unbroken case its energy is 0, $H|0\rangle = 0$, since $H = \frac{1}{2}\{Q, Q^\dagger\}$ this implies that $Q|0\rangle = Q^\dagger|0\rangle = 0$. As mentioned before, the operator Q^\dagger changes a bosonic state into a fermionic one with the same energy, except for the groundstate which is annihilated by this operator. In the case where $E_0 \neq 0$, $H|0\rangle = E_0|0\rangle$ so $\frac{1}{2}\{Q, Q^\dagger\}|0\rangle = E_0|0\rangle$. The supercharges do not annihilate the groundstate in this case. This is strange because the supercharges contain fermionic and bosonic ladder operators and should therefore annihilate the groundstate. There might be a state with lower energy, but that would be against the definition of the groundstate. because $Q^\dagger|0\rangle \neq 0$ there is actually a fermionic counterpart of the bosonic groundstate. Although in this case there is actually a one-to-one correspondence between fermionic and bosonic states, the symmetry is said to be broken. In essence broken supersymmetry implies the absence of a unique normalizable groundstate.

3.1 Determining SUSY breaking

But how can you find out if supersymmetry is broken in a certain case? The most common way is by using the Witten index. This index basically tells you the difference in the number of fermionic and bosonic groundstates,

$$\Delta = n_b^{(E=0)} - n_f^{(E=0)}$$

If $\Delta \neq 0$ the system is unbroken, because then there is at least one state with zero energy. A more formal way to express the Witten index is

$$\Delta = \text{Tr}(-1)^F,$$

where F is the fermion number and the trace is over all bounded and continuum states of the super-Hamiltonian [6]. In quantum field theory the Witten index needs to be regulated to be well defined and a parameter β is added [6],

$$\Delta(\beta) = \text{Tr}(-1)^F e^{-\beta H}.$$

In quantum field theory it is quite hard to determine if SUSY is broken in a non-perturbative way. However in quantum mechanics it is possible to determine the index in a non-perturbative way [6]. The Quantum mechanics Witten index is

$$\Delta(\beta) = \text{Tr}[e^{-\beta H_1} - e^{-\beta H_2}].$$

We take a general SUSY Hamiltonian

$$H = \frac{1}{2}[p^2 + W^2(x) - \sigma_3 W'(x)],$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli matrix and plug into equation 3.1. As mentioned before the trace is over all the states. Both integral variables run from $-\infty$ to ∞ .

$$\Delta(\beta) = Tr \left(\sigma_3 \int \left[\frac{dpdx}{\pi} \right] e^{-\frac{\beta}{2}[p^2 + W^2 - \sigma_3 W'(x)]} \right).$$

Expanding the term proportional to σ_3 and taking the trace this becomes

$$\Delta(\beta) = \int \left[\frac{dpdx}{\pi} \right] e^{-\frac{\beta}{2}[p^2 + W^2]} \sinh\left(\frac{\beta W'(x)}{2}\right).$$

We are interested in the case that $\lim_{\beta \downarrow 0}$ so $\sinh\left(\frac{\beta W'(x)}{2}\right) \approx \frac{\beta W'(x)}{2}$ and the Witten index becomes

$$\Delta(\beta) = \int \left[\frac{dpdx}{\pi} \right] e^{-\frac{\beta}{2}[p^2 + W^2]} \frac{\beta W'(x)}{2}.$$

Solving the p integral gives

$$\Delta(\beta) = \sqrt{\frac{\beta}{2\pi}} \int dx e^{-\frac{\beta}{2}W(x)^2} W'(x).$$

The outcome of this integral depends entirely on the form of $W(x)$, choosing $W(x)$ in the right way, this integral could produce any number. However most superpotentials are not physically realizable and we are interested in superpotentials which produce a Witten index which is an integer. Imagine a function $W(x)$ on the domain $[a, b]$ and its range is $[\pm\infty, \mp\infty]$, in other words $W(a) \rightarrow \pm\infty$ and $W(b) \rightarrow \mp\infty$. From now on this property of a function is called A. An example a function with property A is the tangent function on the domain $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. For functions with property A we can make the following substitution

$$y(W) = W(x), \frac{dy}{dx} = \frac{dW}{dx} \rightarrow dy = W'(x)dx, \quad (3.1)$$

$$\Delta(\beta) = \sqrt{\frac{\beta}{2\pi}} \int_a^b dy e^{-\frac{\beta}{2}y^2} = \sqrt{\frac{\beta}{2\pi}} \sqrt{\frac{2\pi}{\beta}} = \epsilon 1, \quad (3.2)$$

with

$$\epsilon = \begin{cases} -1, & W(a) > W(b) \\ 1, & W(a) < W(b). \end{cases} \quad (3.3)$$

So one finds a Witten index of +1 if $W(x)$ is an increasing function and -1 is a decreasing function. But we can generalize this result in the following way. Let there be some functions f_1, f_2, \dots, f_n whose domains do not overlap and all have the property A. Let the superpotential $W(x)$ be the sum of these functions. It appears that one can add the individual Witten indexes, Δ_{f_i} of these functions to get the total Witten index, $\Delta_{W(x)}$. In other words, $\Delta_{W(x)} = \sum_{i=1}^n \Delta_{f_i}$.

To illustrate this see the images below where some random functions which all have the property A are plotted. The dotted lines indicate that the functions range from $-\infty$ to ∞ .

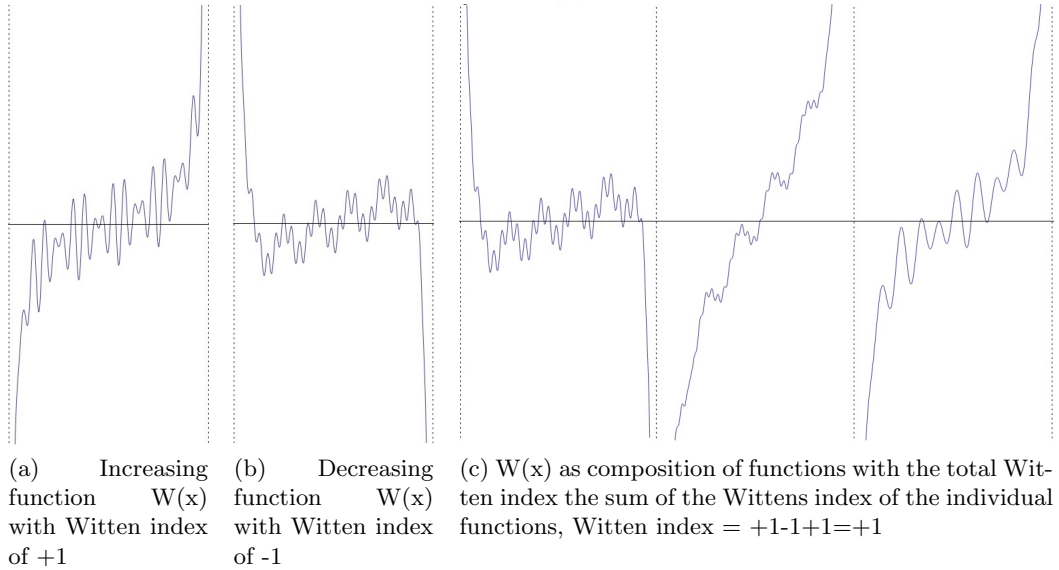


Figure 1: Different possibilities of the Witten index, depending on the slope of the superpotential

So any possible integer is possible as the outcome of the Witten index. In most cases however there is just one groundstate and the Witten index is either 0 or 1 [7]. We have seen how to calculate possible breaking in supersymmetrical quantum mechanical systems. For further reading about the Witten index I suggest reading [?].

4 Superspace

This section treats the geometrical properties of supersymmetry and the quantum Calogero Moser model. Because of the anti-commuting properties of the variables, the square root of a variable, a fact that makes the theory very interesting.

4.1 An introduction of a one dimensional superspace.

We now come to the point where we add geometric properties to supersymmetry. I will treat the N=1 and 0+1 dimension case where 0+1 stands for zero space dimensions and 1 time dimension. From now on the conjugate momentum operator is $\pi = -i\frac{\partial}{\partial\phi}$. The superpotential W is now a function of ϕ . In accordance with equations 2.43 and 2.44 the fermion operators are

$$\psi = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (4.1)$$

$$\psi^\dagger = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (4.2)$$

which satisfy the anti-commuting relations

$$\{\psi^\dagger, \psi\} = 1, \quad \{\psi^\dagger, \psi^\dagger\} = \{\psi, \psi\} = 0. \quad (4.3)$$

Here, the supercharges take the form [1]

$$Q = \psi(W + i\pi), \quad (4.4)$$

$$Q = \psi^\dagger(W - i\pi), \quad (4.5)$$

this results in

$$H = \frac{1}{2}\{Q, Q^\dagger\} = \frac{1}{2}(\pi^2 + W^2 - [\psi^\dagger, \psi]W').$$

ψ and ψ^\dagger obey the Grassmann algebra, which for any Grassmann variables θ, θ_1 and θ_2 , is [1]

$$\{\theta^*, \theta\} = \{\theta^*, \theta^*\} = \{\theta, \theta\} = 0, \quad (4.6)$$

$$(\theta_1\theta_2)^* = \theta_2^*\theta_1^*, \quad (4.7)$$

$$\left\{\frac{\partial}{\partial\theta}, \theta\right\} = \left\{\frac{\partial}{\partial\theta^*}, \theta^*\right\} = 1, \quad (4.8)$$

$$\left\{\frac{\partial}{\partial\theta}, \theta^*\right\} = \left\{\frac{\partial}{\partial\theta^*}, \theta\right\} = 0, \quad (4.9)$$

$$\int d\theta = 0, \quad \int (d\theta\theta) = 1, \quad (4.10)$$

$$d\theta\theta = -\theta d\theta, \quad \frac{\partial}{\partial\theta} = \int d\theta. \quad (4.11)$$

So anti-commuting integration is the same as anti-commuting differentiation! Equation 4.6 states that $(\theta^*)^2 = \theta^2 = 0$. So if we Taylor expand a function f depending on a Grassmann parameter θ it cuts off at the second order term

$$f(\theta) = f(0) + \theta f'(0) + \frac{1}{2}\theta^2 f''(0) + \dots = f(0) + \theta f'(0).$$

This means that any function of a Grassman variable can be reduced to a constant term and a linear term, a property which is rather handy as we will see later on.

Moving back to physics, the supersymmetric Lagrangian on a space is

$$\mathcal{L} = \frac{1}{2}(\dot{\phi}^2 + \imath\psi^*\dot{\psi} - W^2 - W'[\psi^*, \psi]), \quad (4.12)$$

where a dot denotes the time derivative. Notice the analogy with Lagrangian given earlier in the section 2.3.1.

The supersymmetry variation on a field χ is

$$[\epsilon^*Q + \epsilon Q^\dagger, \chi],$$

where ϵ is an infinitesimal anticommuting parameter [1]. The actions of the supersymmetry generators Q and Q^\dagger on the fields ϕ , ψ and ψ^\dagger are [1]

$$[Q, \phi] = \psi, \quad [Q^\dagger, \phi] = \psi^*, \quad (4.13)$$

$$\{Q, \psi\} = 0, \quad \{Q^\dagger, \psi\} = W - \imath\pi, \quad (4.14)$$

$$\{Q, \psi^\dagger\} = W + \imath\pi, \quad \{Q^\dagger, \psi^*\} = 0. \quad (4.15)$$

So the supersymmetry variations are

$$\delta\phi = \epsilon^*\psi - \epsilon\psi^\dagger, \quad (4.16)$$

$$\delta\psi = \epsilon(W - \imath\pi), \delta\psi^* = \epsilon(W + \imath\pi) \quad (4.17)$$

After all the algebra we can finally introduce a superfield Φ , which is a general function on superspace [1]. A superfield is an extension of space-time by including the Grassmann parameters θ and θ^* for each supercharge Q . In the 0+1 dimension we describe the geometric variables are t, θ and θ^* . The superfield is given by

$$\Phi(t, \theta, \theta^*) = \phi(t) + \theta\psi(t) - \theta^*\psi^*(t) + \theta\theta^*F(t). \quad (4.18)$$

This field commutes, so that ϕ and F are commuting fields and ψ is anticommuting. If ϕ and F are real, then $\Phi^* = \Phi$, making the superfield real. ϕ , ψ and F are the components of the superfield. Note that there can be no higher order terms in θ or θ^* because quadratic terms and higher are zero. The Hermitian generator of time is $\mathcal{H} = \imath\frac{\partial}{\partial t}$, we now want to find similar operators acting on superfields which obey the supersymmetry algebra. Before we do this I would like to introduce some more rules concerning the Grassmann variables θ_1 and θ_2 and their complex conjugates.

$$d\theta_1 d\theta_2 = -d\theta_2 d\theta_1, \quad (4.19)$$

$$\left(\frac{\partial}{\partial\theta}\right)^\dagger = \frac{\partial}{\partial\theta^*}. \quad (4.20)$$

And for a general superfield A:

$$\left(\frac{\partial}{\partial\theta}\right)^* = -(-)^A \frac{\partial}{\partial\theta^*} A^*,$$

where $(-)^A = +1$ for a commuting superfield and -1 if A is anticommuting.

We can define differential operators on space which satisfy the supersymmetry algebra

$$\mathcal{Q} = \frac{\partial}{\partial\theta} + \iota\theta^* \frac{\partial}{\partial t}, \quad \mathcal{Q}^\dagger = \frac{\partial}{\partial\theta^*} + \iota\theta \frac{\partial}{\partial t} \quad (4.21)$$

$$\{\mathcal{Q}, \mathcal{Q}^\dagger\} = 2\mathcal{H}, \quad \{\mathcal{Q}, \mathcal{Q}\} = \{\mathcal{Q}^\dagger, \mathcal{Q}^\dagger\} = 0. \quad (4.22)$$

In analogy with equation 4.1 we define the variation on any superfield Φ by

$$\delta\Phi = [\epsilon^* \mathcal{Q} + \epsilon \mathcal{Q}^\dagger, \Phi] = (\epsilon^* \mathcal{Q} + \epsilon \mathcal{Q}^\dagger)\Phi.$$

We can get the variation of the individual components by calculating

$$\mathcal{Q}\Phi = \psi + \theta^*(F + \iota\dot{\phi}) - \iota\theta\theta^*\dot{\psi}, \quad (4.23)$$

$$\mathcal{Q}^\dagger\Phi = \psi^* + \theta(F + \iota\dot{\phi}) - \iota\theta\theta^*\dot{\psi}^*, \quad (4.24)$$

expanding both sides of equation 4.1 yields [1]

$$\delta\phi = \epsilon^*\psi - \epsilon\psi^*, \quad (4.25)$$

$$\delta\psi = \epsilon(F - \iota\dot{\phi}), \quad (4.26)$$

$$\delta F = -\iota(\epsilon^*\dot{\psi} - \epsilon\dot{\psi}^*). \quad (4.27)$$

The last of the above equation shows that the the supersymmetry variation of the $\theta\theta^*$ component of any superfield is a total time derivative. An action like $\int dt F$, where F is the highest component of the field, is automatically supersymmetric invariant. The action of the field Φ is $S = \int dt d\theta d\theta^* \mathcal{L}(\Phi)$, but because of anticommuting integration, which is the same as anticommuting differentiation, $d\theta d\theta^*$ automatically picks out the highest component of \mathcal{L} . So any field of the form in equation 4.18 is automatically supersymmetric invariant.

We can also introduce *superderivatives* on superspace,

$$\mathcal{D} = \frac{\partial}{\partial\theta} - \iota\theta^* \frac{\partial}{\partial t}, \quad \mathcal{D}^\dagger = \frac{\partial}{\partial\theta^*} - \iota\theta \frac{\partial}{\partial t} \quad (4.28)$$

$$\{\mathcal{D}, \mathcal{D}^\dagger\} = -2\mathcal{H}, \quad (4.29)$$

$$\{\mathcal{D}, \mathcal{D}\} = -2\mathcal{H}, \quad (4.30)$$

$$\{\mathcal{D}, \mathcal{D}\} = \{\mathcal{D}^\dagger, \mathcal{D}^\dagger\} = \{\mathcal{D}, \mathcal{Q}^\dagger\} = \{\mathcal{D}, \mathcal{Q}\} = 0, \quad (4.31)$$

Their action on a superfield differs from \mathcal{Q} and \mathcal{Q}^\dagger by taking $t \rightarrow -t$

$$\mathcal{D}\Phi = \psi + \theta^*(F - \iota\dot{\phi}) + \iota\theta\theta^*\dot{\psi}, \quad (4.32)$$

$$\mathcal{D}^\dagger\Phi = -\psi^* - \theta(F + \iota\dot{\phi}) + \iota\theta\theta^*\dot{\psi}^*. \quad (4.33)$$

You would be right to ask what their use is if they look so much like Q and Q^\dagger , well it seems that the superderivative of a superfield also transforms like a superfield under the transformations of supersymmetry:

$$\delta\mathcal{D}\Phi = [\epsilon^*Q + \epsilon Q^\dagger, \mathcal{D}\Phi] = \mathcal{D}[\epsilon^*Q + \epsilon Q^\dagger, \Phi] = \mathcal{D}\delta\Phi \quad (4.34)$$

$$= \mathcal{D}(\epsilon^*Q + \epsilon Q^\dagger)\Phi = (\epsilon^*Q + \epsilon Q^\dagger)\mathcal{D}\Phi, \quad (4.35)$$

the second equality is allowed because Q acts on the field and \mathcal{D} acts on the superspace coordinates, and the last equality holds because \mathcal{D} commutes with Q . Therefor we can see that an arbitrary polynomial function of superfields and their superderivatives transforms in the same way under supersymmetry variations as a superfield does [1].

With all the rules and algebra we can finally calculate the action S and show that it is invariant under supersymmetry variations. The action is

$$S = \int dt d\theta d\theta^* \left[-\frac{1}{2} \mathcal{D}\Phi \mathcal{D}^\dagger\Phi + f(\Phi) \right]. \quad (4.36)$$

Where Φ is a superfield in the form as in equation 4.18 and $f(\Phi)$ is some function [7]. As we have seen before we can expand a function on a field in the anticommuting variables and get a finite series because all quadratic and higher terms vanish. Applying this to $f(\Phi)$ the action becomes

$$S = \int dt d\theta d\theta^* \frac{1}{2} \{ \psi + \theta^*(F - \dot{\phi}) + \iota\theta\theta^*\dot{\psi} \} \{ \psi^* + \theta(F + \dot{\phi}) - \iota\theta\theta^*\dot{\psi}^* \} \quad (4.37)$$

$$+ f + (\theta\psi - \theta^*\psi^* + \theta\theta^*F)f' + \frac{1}{2}(\theta\psi - \theta^*\psi^* + \theta\theta^*F)^2 f''. \quad (4.38)$$

Fortunately this enormous expression can be reduced thanks to the fact that anti commuting integration is the same as anti commuting differentiation and the neat property that quadratic and higher terms of θ and θ^* are 0. So we can basically pick out just a few remaining terms and the integral becomes

$$S = \int dt \left\{ \frac{1}{2} \left(F^2 + \dot{\phi}^2 - \iota(\dot{\psi}\psi^* - \psi^*\dot{\psi}) \right) - Ff'(\phi) + \frac{1}{2}(\psi^*\psi - \psi\psi^*)f''(\phi) \right\}. \quad (4.39)$$

If we want the term inside the parenthesis to be the Lagrangian as in equation 4.12, so that the action is $S = \int dt \mathcal{L}$, we set $F = W = f'$, where W is the superpotential. From this we see that the supersymmetry action on the field is linear.

The extension of superspace to more dimensions is beyond the scope of this thesis.

5 The Calogero-Moser model

The Calogero-Moser model, CM model for short, is a multi particle one dimensional dynamical system. The particles have long range interactions and can be

subjected to various potentials. In addition the particles also oscillate, making this a very interesting model. It can be extended by adding fermions, in this case using supersymmetry is an obvious choice.

I will begin by giving a short introduction to this model and then show how supersymmetry is used to add fermionic degrees of freedom.

The non supersymmetric Hamiltonian looks like [5]:

$$H = \frac{1}{2} \sum_{j=1}^N p_j^2 + \frac{\omega^2}{2N} \sum_{i < j} (x_i - x_j)^2 + g^2 \sum_{i < j}^N V(x_i - x_j), \quad (5.1)$$

$$p_j = -i \frac{\partial}{\partial x_j}, \quad (5.2)$$

$$V(L) = \begin{cases} \frac{1}{L^2} & \text{type I (rational)} \\ \frac{1}{\sinh^2(L)} & \text{type II (hyperbolic)} \\ \frac{1}{\sin^2(L)} & \text{type III (trigonometric)}. \end{cases} \quad (5.3)$$

The first part of the Hamiltonian is basically the momentum operator of all particles plus an harmonic oscillator, the second part represents the a pairwise interaction. g is a coupling constant. If $g > 0$ there is an attractive force between the particles, if $g < 0$ then the force becomes repulsive. In the case where $g = 0$, the Model becomes an N particle bosonic harmonic oscillator. In this thesis I will only discuss the rational, type I force. The Hamiltonian is translation-invariant [5]. One of the most important features is that the system is completely integrable, both at classic and at a quantum level [3, 5, 11, 4]. This means that it is exactly solvable.

The bosonic dynamical variables of the CM model are the coordinates $\{q_i\}$ and their associated canonically conjugate momenta $\{p_j\}$, $q = (q_1, \dots, q_N)$ and $p = (p_1, \dots, p_N)$. So there are $2N$ degrees of freedom in the bosonic variables. The well known canonical commutation relations hold [11, 5]:

$$[q_i, p_k] = i\delta_{jk}, \quad [q_i, q_k] = [p_i, p_k] = 0, \quad j, k \in \{1, \dots, N\} \quad (5.4)$$

At this point we would like to expand the system by adding fermions. This is an $\mathcal{N} = 2$ supersymmetric quantum mechanical system [3, 4] so there are also $2N$ fermionic degrees of freedom

$$\psi = (\psi_1, \dots, \psi_N), \quad \psi^\dagger = (\psi_1^\dagger, \dots, \psi_N^\dagger), \quad j, k \in \{1, \dots, N\}. \quad (5.5)$$

The fermionic variables ψ and ψ^\dagger are annihilation and creation operators respectively and are hermitian conjugates of each others. For them the canonical anti-commutation relations hold

$$\{\psi_j^\dagger, \psi_k\} = \delta_{jk}, \quad \{\psi_j, \psi_k\} = \{\psi_j^\dagger, \psi_k^\dagger\} = 0, \quad j, k \in \{1, \dots, N\}. \quad (5.6)$$

The bosonic and fermionic variables commute with each other

$$[\psi_j^\dagger, q_i] = [\psi_j, q_i] = [\psi_j^\dagger, p_i] = [\psi_j, p_i] = 0 \quad i, j \in \{1, \dots, N\}. \quad (5.7)$$

The momentum operator is:

$$p_j = -i \frac{\partial}{\partial q_j}, \quad j, k \in \{1, \dots, N\} \quad (5.8)$$

The supercharges Q and Q^\dagger take the form [4, 3]:

$$Q = \sum_{j=1}^N \psi_j^\dagger (p_j - iW), \quad Q^\dagger = \sum_{j=1}^N \psi_j (p_j + iW) \quad (5.9)$$

The supersymmetric Hamiltonian is

$$H_s = \frac{1}{2} \{Q, Q^\dagger\} = \frac{1}{2} \sum_{j=1}^N (p_j^2 + W^2) - \frac{1}{2} \sum_{j,k=1, j \neq k}^N [\psi_j^\dagger, \psi_k] \frac{\partial W}{\partial q_j} \quad (5.10)$$

The dynamics of the Hamiltonian is determined by the superpotential $W(q) = W(q_1, \dots, q_N)$. In the CM model the potential takes the form [14]

$$W(q) = \frac{\omega}{N} \sum_{i < j} (x_i - x_j) + g \sum_{i < j} (x_i - x_j)^{-1}, \quad (5.11)$$

so the supercharges are

$$Q = \sum_{j=1}^N \psi_j^\dagger \left(p_j - i \frac{\omega}{N} \sum_i (x_j - x_i) - ig \sum_{i \neq j} (x_j - x_i)^{-1} \right), \quad (5.12)$$

$$Q^\dagger = \sum_{j=1}^N \psi_j \left(p_j + i \frac{\omega}{N} \sum_i (x_j - x_i) + ig \sum_{i \neq j} (x_j - x_i)^{-1} \right). \quad (5.13)$$

Combining equation 5.10 and equation 5.11 and doing some tedious calculations results in [14]

$$H_s = \frac{1}{2} \sum_{j=1}^N p_j^2 + \frac{\omega^2}{2N} \sum_{i < j} (x_i - x_j)^2 - g^2 \sum_{i < j} (x_i - x_j)^{-2} \quad (5.14)$$

$$\frac{\omega^2}{2} gN(N-1) + \frac{\omega}{2N} \sum_{i < j} [\psi_i^\dagger - \psi_j^\dagger, \psi_i - \psi_j] + \frac{1}{2} g \sum_{j,k=1, j \neq k}^N [\psi_i^\dagger, \psi_j] \frac{\partial U_j}{\partial q_i}, \quad (5.15)$$

where

$$U_j \equiv \sum_{k \neq j}^N \frac{1}{x_j - x_k}.$$

The term $\frac{\omega^2}{2} gN(N-1)$ is produced by summing up the term of the form $(x_i - x_j)(x_i - x_k)^{-1}$. The cross terms $(x_i - x_j)^{-1}(x_i - x_k)^{-1}$ with $j \neq k$ cancel out. The first three terms give the bosonic Hamiltonian mentioned in the introduction.

The coupling constant g plays an important role because it determines whether the system is broken or unbroken. If $g < 0$ the groundstate converges when $x_i \rightarrow \infty$ due to the negative exponential power. If $x_i = x_j$ the groundstate also converges so the supersymmetry is conserved [14]. This is not the case if $g > 0$ and $x_i = x_j$, the groundstate diverges and supersymmetry is broken. In the unbroken case there is a unique groundstate which is annihilated by Q and Q^\dagger and is

$$|\phi_0\rangle = e^{\int W(x)dx} |0\rangle = \prod_{i<j}^N |x_i - x_j|^{-g} \exp\left(-\frac{1}{2}\omega \sum_{i<j} (x_i + x_j)^2\right) |0\rangle = |\phi_0^{(B)}, 0\rangle = |\phi_0\rangle,$$

here $|0\rangle$ is the state in the 2^N dimensional Fock space that is annihilated by all the operators ψ_i . We can now introduce the boson and fermion ladder operators A_i , A_i^\dagger and Ψ_i , Ψ_i^\dagger . The A_i 's are of the form $A_i \sim \sum_j^n ((p_j + i\omega q_j)^i + f(a_i))$ where $f(a_i)$ is some complicated function, which is not as important as the first term. Its conjugate adjoint is constructed in the same fashion. the $p_j + i\omega q_j$ part is the ladder operator of the regular harmonic oscillator. The operators Ψ_i and Ψ_i^\dagger are a combination of the ψ_j and ψ_j^\dagger . Ψ_j annihilates the groundstate, $\Psi_j |0\rangle = 0|0\rangle$. These new operators obey the the following algebraic relations

$$\{\Psi_m, \Psi_n\} = [A_m, \Psi_n] = [A_m, A_n] = 0, \quad (5.16)$$

$$\{Q, \Psi_n^\dagger\} = \{Q^\dagger, \Psi_n\} = 0, \quad (5.17)$$

$$\{Q^\dagger, \Psi_n^\dagger\} = A_n^\dagger, \quad (5.18)$$

$$[Q, A_n] = [Q^\dagger, A_n^\dagger] = 0, \quad (5.19)$$

$$[Q, A_n^\dagger] = 2n\omega\Psi_n^\dagger, \quad (5.20)$$

$$[H_s, \Psi_n] = n\omega\Psi_n, \quad (5.21)$$

$$[H_s, A_n] = n\omega A_n. \quad (5.22)$$

The higher wave states can be constructed by applying combinations of the raising operators [14]

$$|\psi_{(n_2, \dots, n_N)}^{(B)}, \psi_{(\nu_2, \dots, \nu_N)}^{(F)}\rangle = A_2^{\dagger n_2} \dots A_N^{\dagger n_N} \Psi_2^{\dagger n_2} \dots \Psi_N^{\dagger n_N} |\psi_0\rangle$$

The energy of the system is

$$E = \omega \sum_{k=2}^N k(n_k + \nu_k),$$

here ν_k is a fermion number and can take the values 0 and 1. k starts at 2 because the groundstate, where $k = 0$ has 0 energy.

If $g > 0$, the symmetry is broken and there is a groundstate doublet e.g. the groundstate contains both a fermion and boson state. The energy of the groundstate doublet is

$$E_0 = (1 + Ng)(N - 1)\omega$$

6 Why supersymmetry is useful

As the title of this section implies, this section is about how supersymmetry is used in different fields of physics, outside that of purely theoretical. SUSY plays an important role in course particle physics, where supersymmetry might be experimentally confirmed. But even if it will not be confirmed, this section will show that the mathematics behind it are very useful. I will not do calculations, but rather explain or show why it is used in these fields of physics.

6.1 SUSY in particle physics

In particle physics supersymmetry has a special place. Supporters of the theory defend it by stating it can solve the hierarchy problem, which will be discussed shortly. Those who oppose it do this because SUSY is not based on experiments but rather a theory created to fix theoretical problems. The supersymmetry theory connecting mesons and baryons was introduced by Hironari Miyazawa in 1966 but was ignored at the time [15]. The first usage of supersymmetry in the Standard Model was introduced by Howard Georgi and Savvas Dimopoulos. It is called the Minimal Supersymmetric Standard Model abbreviated as MSSM. It was proposed to solve the so called hierarchy problem. Supersymmetry in particle physics entails that for every boson and fermion there exists a supersymmetric fermion, boson respectively with the same energy and thus the same mass. The naming of these supersymmetric partner particles are little awe-inspiring. Bosonic superpartner particles are named by putting an 's' in front of name of the original particle. For example selectron, stoposion etc. For the fermions -ino is added at the end of the original names, so for example stopino, gluino etc. Until now no experimental data has confirmed the existence of these superpartner particles. Perhaps the LHC can find these particles. A reason for the absence of the superpartner particles could be that 10^{-35} seconds after the big bang the symmetry was spontaneously broken [13]. Another interesting theory is that the lightest superpartner particle, the sneutrino, contributes to dark matter [13].

6.1.1 The hierarchy problem

And now for the hierarchy problem. Basically a hierarchy problem is a problem in which the fundamental parameters of a Lagrangian are significantly bigger than the experimental value. In particle physics the biggest hierarchy problem is that of the Higgs mass. At the Planck scale, where the distances are in the order of 10^{-35} m one would expect certain corrections in the Higgs mass due to particle loops. For example a Higgs boson may split into a virtual top quark-antiquark pair. These corrections are extremely large at the Planck scale 10^{19} GeV. However estimates from experiments put the Higgs mass much lower at 160 GeV, which is 19 orders of magnitude lower. In a sense there is nothing wrong with these corrections, they just seem unnatural. A solution presented by SUSY is that for every bosonic particle loop there is also a fermionic particle loop, which has a negative contribution to the mass, and thus the contributions

are canceled out. Other solutions also exist, but this seems to be the most elegant. Even if SUSY is broken, the fermionic particle loop contributions would still soften the correction.

6.1.2 The unity of forces

If we look at the three fundamental forces in the standard model - that would be the weak, the strong and the electromagnetic force - and plot their strength against an energy scale the resulting lines would not intersect in one point [7, 12]. However in the Minimal Supersymmetry Standard Model (MSSM) the three lines actually intersect in one point, as is shown in the graph below. So

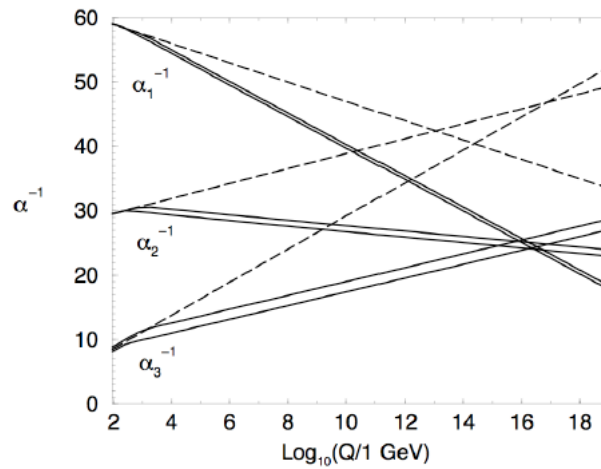


Figure 2: The dashed lines indicate the three fundamental forces in the standard model, the solid the fundamental forces in the Minimal supersymmetric standard model. The horizontal axis represents the energy, the vertical the coupling strength. Illustration from [12]

supersymmetry appears to treat the forces on a more equal ground.

6.2 Orbital energy values of hydrogen atom

Although a system in nature described by supersymmetry may not be supersymmetric in a physical sense, using supersymmetry can make calculations easier or more elegant. The much taught method for solving the energy levels of a hydrogen atom involves a brute force approach. It appears that when Dirac tried to solve the Hamiltonian he used only operator algebra, more akin to the Heisenberg matrix formulation. It is comparable to how the harmonic oscillator usually is solved in the way that makes use of operator algebra. For further reading I suggest the article of reference [10].

6.3 Nuclear physics

Another field where Supersymmetry is applied to is nuclear physics. In the article [2] supersymmetry is used to describe the pairing of many-nucleon systems,

the nuclear structure and fusion/fission reactions below the Coulomb barrier. Even nuclei in explosive environments like supernovae are treated.

6.4 The extreme Reissner-Nordström black hole

The Reissner-Nordström black hole is a black hole which has a charge but no angular momentum. The charge will be denoted as q , as not confuse it with the supercharge Q or \mathcal{Q} . This black hole has two horizons, an event horizon and a Cauchy horizon. The Cauchy horizon is a horizon which separates time-like geodesics and space-like geodesics. In other words it is a singularity beyond which motion cannot be described. It is located closer to the center of the black hole than the event horizon, from which light cannot escape. These horizons are located at $r_{\pm} = \frac{GM}{c^2} \pm \sqrt{\frac{G^2 M^2}{C^4} - \frac{q^2 G}{4\pi\epsilon_0 c^4}}$, in natural units where $c = G = 4\pi\epsilon_0 = 1$ this reduces to $r_{\pm} = M \pm \sqrt{M^2 - q^2}$. In the case where $q = 0$, this reduces to a Schwarzschild black hole and there is just one horizon. If $M = q$, the black hole becomes an extreme Reissner-Nordström black hole. Particles near this horizon can then be described by Super Conformal Quantum mechanics, SCQM for short. In conformal geometry transformations on spaces preserve angles. Like in figure 3. Interested? Read the article of reference [8].

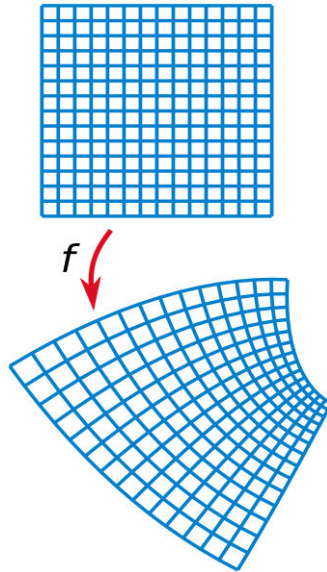


Figure 3: A conformal transformation leaves intersections, and thus angles invariant. Illustration from http://www.enotes.com/topic/Conformal_map

7 Conclusion

A supersymmetric system, be it in quantum mechanics or quantum field theory, must obey the following algebra

$$[H_s, Q] = [H_s, Q^\dagger] = 0, \quad (7.1)$$

$$\{Q, Q^\dagger\} \equiv QQ^\dagger + Q^\dagger Q = H_s, \quad (7.2)$$

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0, \quad (7.3)$$

where Q is an operator called the supercharge and Q^\dagger is its Hermitian conjugate. H_s is the supersymmetric Hamiltonian. The supercharges commute with Hamiltonian and thus produce a symmetry in energy. We have found that the supercharges are operators which do not alter the energy of a state but exchanges fermions with bosons and vice versa. So supersymmetry links fermionic degrees of freedom with bosonic degrees of freedom. We have also seen that supersymmetry can be broken or unbroken. The breaking of supersymmetry can be important in particle physics because it could solve the hierarchy problem. In this field the consequence of supersymmetry is that for every boson a supersymmetric partner fermion exists, and vice versa, with the same mass. This has not been experimentally confirmed yet. But if supersymmetry is broken the masses of two supersymmetric partner particles need not be the same. Supersymmetry can also be realized in a geometric way, giving rise to superfields which depend on time and anti-commuting coordinates. Furthermore we conclude that despite the lack of experimental evidence, supersymmetry is used to describe or simplify models in many fields in physics, like black hole physics and nuclear physics.

References

- [1] Philip C. Argyres, *An introduction to global sypersymmetry*, 2010.
- [2] A. B. Balantekin, *Symmetry and supersymmetry in nuclear physics*, <http://arxiv.org/abs/0711.0710v1> (2007).
- [3] Corrigan E. Bordner A. J. and Sasaki R., *Calogero-moser models: A new formulation*, arXiv:hep-th/9805106v3 (1998).
- [4] Manton N. S. Bordner A. J. and Sasaki R., *Calogero-moser models v: Supersymmetry and quantum lax pairs*, YITP , VOL.;NO.99-47;PAGE.32P (1999).
- [5] Dr. Francesco Calogero, *Calogero-moser system*, http://www.scholarpedia.org/article/Calogero-Moser_system (2011).
- [6] Khare A. Cooper F. and Sukhatme U., *Supersymmetry and quantum mechanics*, World Scientific Publishing Company, ISBN: 9810246056, 2008.
- [7] Prof. dr. Jan de Boer, *Personal conversations*, 2010.
- [8] Piet Claus et. al., *Black holes and superconformal mechanics*, <http://arxiv.org/abs/hep-th/9804177v3> (1998).
- [9] Kentaro Hori and Cumrum Vafa, *Mirror symmetry*, <http://arxiv.org/abs/hep-th/0002222v3> (2000).
- [10] Chad Husko and Brenton Knuffman, *Shape invariance in supersymmetric quantum mechanics and its application to selected special functions of modern physics*, Journal of Young Investigators, <http://www.jyi.org/research/re.php?id=87> (2005).
- [11] Pocklington A.J. Khastgir S. P. and Sasaki R., *Quantum calogermoser models: integrability for all root systems*, Journal of Physics A: Mathematical and General Volume 33 number 49 9033 (2000).
- [12] Andrew Larkoski, *Why supersymmetry is super*, http://www.slac.stanford.edu/slac/sass/talks/Larkoski_Supersymmetry.pdf (2011).
- [13] Hitoshi Murayama, *Introduction to supersymmetry*, 2010.
- [14] T. Wellman, *An introduction to supersymmetry in quantum*, Master's thesis, www.physics.brown.edu/physics/undergradpages/Ftheses/SeniorThesis_Wellman.pdf, 2003.
- [15] Wikipedia, *Supersymmetry*, <http://en.wikipedia.org/wiki/Supersymmetry> (2010).