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The Bell Inequalities

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Abstract: Bell's inequality provided a measurable condition any Local Hidden Variable theory had to obey. According to this reasoning it quantum mechanics could not contain any hidden variables and so it was either wrong, or hidden variables didn't exist. Experiments on Bell's inequality where in favor of quantum mechanics, and therefore seemingly disproving the existence of Local Hidden Variables (LHV). In this paper three no-hidden-variables theories will be discussed and compared, namely the original Bell's theorem, the Greenberger, Horne and Zeilinger (GHZ) theorem and the Bell, Kochen and Specker (BKS) theorem. Each theorem will be attempted to be extended for more particles and for higher spin. These attempts are more successful in the case of Bell's theorem than the GHZ and BKS theorems. A link that exists between the GHZ theorem and the BKS theorem will also be exploited to extend the BKS theorem.

Samenvatting: De wereld van de quantum mechanica betreft de aller kleinste deeltjes, denk aan fotonen, elektronen of protonen. Die deeltjes hebben veel verschillende eigenschappen, ook eigenschappen waar er in de wereld om ons heen geen parallellen van zijn. Een van die eigenschappen is de spin van een deeltje. Je kan je hierbij voorstellen dat een deeltje heel snel om zijn eigen as draait, net als een tol. Maar de punt van deze tol kan of naar boven, of naar beneden wijzen, dan tolt hij de andere kant op. Dit noemen we dan spin up en spin down. Deze spin kunnen we meten met speciale apparaten, maar ook voorspellen aan de hand van de theorie. Er is alleen iets geks aan de hand met die theorie, we kunnen namelijk niet met 100% zekerheid zeggen wat de meting zal gaan geven. Sterker nog, de kans is 50% om spin up te vinden en 50% om spin down te vinden. In eerste instantie zou je denken dat er gewoon nog informatie ontbreekt, er zijn nog variabelen nodig die moeten helpen in de voorspelling van de juiste spin. Net als bijvoorbeeld de windrichting en gewicht van een bal helpen om te bepalen waar hij neerkomt als we hem weg schoppen. Er zou dus nog meer informatie nodig zijn om de quantum theorie compleet te maken, dit zouden de zogenaamde *hidden variables* (verborgen variabelen) zijn. Echter bleek dit niet zo te zijn. John. S. Bell publiceerde in 1966 een theorie waarin hij liet zien dat quantum mechanica geen *hidden variables* kon bevatten, en als deze er wel zouden zijn, dat de quantum theorie helemaal fout was. Bell's theorema werd getoetst en het bleek dat de quatum mechanica toch goed zat. Dit betekend dat de spin van het deeltje niet bepaald is, totdat hij gemeten word. Deze ontdekking leidde er toe dat meerdere mensen opzoek gingen naar andere soorten bewijzen tegen de *hidden variables*. In mijn onderzoek kijk ik naar drie van deze bewijzen. Ten eerste het origineel van Bell, ten tweede een theorema van Greenberger, Horne en Zeilinger en ten slotte een van Kochen en Specker, en Bell zelf. Bij elk van deze theorema's kijk in hoeverre ze nog steeds geldig zijn voor systemen met bijvoorbeeld meerdere deeltjes of voor hogere spin (als de deeltjes nog meer kanten op kunnen tollen). Het blijkt dat Bell's orginele ongelijkheid het meest algemeen is en ook het minste aanneemt van zowel de quantum theorie als de LHV's.

Contents

1 Introduction

In quantum mechanics one can make predictions about the properties of particles. For example their position, momentum and spin. However the predictions (in many cases) can't be made with a 100% certainty. One can only say something about the chance that the particle is in position X or the odds that we find a particle has a spin up. Many who are first introduced to quantum theory think that the percentages involved come from the fact that we do not have all the information necessary to make an accurate prediction. There is still information missing. Einstein, Podolsky and Rosen argued the same. They had an issue with the indeterministic nature of quantum mechanics and so they published a gedankenexperiment that showed why they believed why there was still information missing from the theory. By adding this informations one would be able to make accurate predictions about the system without uncertainty. They believed the particles' properties already had certain values, and where not determined by the measuring process. They called this missing information the hidden variables. In an initial attempt to construct some properties of these hidden variables, almost 30 years later John Bell published a paper containting a theorem that showed that quantum mechanics and hidden variables did not match at all. He provided an inequality that a theory containing hidden variables had to obey, and which quantum mechanics violated. The inequality showed that there where two options. The first was that quantum mechanics was correct which meant that it was complete and hidden variables didn't excist. This also meant that nature was non-local, since the signal between the two particles had to go faster than the speed of light. The second option was that hidden variables did exist, which in turn meant quantum mechanics was wrong and nature would obey locality. In either case, the two (LHV and quantum mechanics) could not work together. When the inequality was tested by Aspect et al, it proved to be in favor of quantum mechanics. Eventhough now the results of this test have been questioned [17] the initial response to the results was that it had some major consequences. Not only would this disprove realism but it would also show that nature was fundamentally non-local. It seemed that indeed the particles knew what its counterpart were doing instantaneously, therefore the signal between them would go faster than the speed of light. This posed to be quite the issue, after all that would be in direct violation of relativity. If information can travel faster than the speed of light this would mean that it could travel backwards in time in certain situations. However non-local correlations cannot be used to transfer information faster that the speed of light and therefore be used to send messages [13, p. 421]. Non the less it shows that the topic is well worth exploring. Consequently there have been more attempts at making both LHV theories and theorems disproving them.

In this thesis three major no-hidden-variables theories will be discussed and compared. We will mostly be focusing on the technical derivation of several forms of Bell inequalities, and will cover some interpretation issues will in the discussion. At first the EPR paradox is explained, making clear the issue that started the no-hidden-variables theories. After that in each section a theorem will be explained in detail after which they will be extended for both higher amounts of spin as well as more particles. Finally the assumptions and with that the validity of the theorems will be discussed. The experiments done to test them are briefly mentioned along with their results.

2 Einstein, Podolsy and Rosen

The goal when creating a theory, any theory whether in biology, maths, sociology or physics, is to describe the world we see around us using formulas, definitions, variables and so forth. It aims to attach to each physical element of the reality we see, a theoretical counterpart which describes it. These elements of the theory then work together to predict the outcome of the system that is being described. One could argue that such a theory cannot be complete until each physical element is represented in the theory, and thus the behavior of each physical element can be predicted. In quantum mechanics this isn't always the case. Wave-functions are used to describe quantum mechanical systems, but they give information about the probability distribution of a particles properties. For the most part they cannot predict anything about a particle with a 100% certainty. Furthermore there are so called incompatible observables, with non-commuting operators, of which we cannot know both values simultaneously. In 1935, Einstein Podolsky and Rosen (EPR) argued that because of this fundamental uncertainty in quantum mechanics, the theory wasn't complete. It couldn't be called complete until each physical element was represented by, what they called, an element of reality. The one condition defining this element of reality is the possibility of predicting it with certainty without disturbing the system [8, p. 777]. Einstein, Podolsky and Rosen then conducted a thought experiment to prove that quantum mechanics is incomplete and that some 'elements of reality' have to exist that would enable the theory to make correct predictions without disturbing the system. This set-up is best known as the EPR-paradox (or theorem [19]) and will be explained below. Where the original paper uses momentum, we will be following an example using spin as introduced by Bohm [6].

2.1 The EPR theorem

The set-up is as follows. A source emits two particles [1], A and B, which are correlated. Because the momentum is conserved both particles will propagate in opposite directions, resulting in them moving away from each other to remote locations in space. Both particles will undergo measurements, typically by Stern-Gerlach devices, which are set under certain parameters or settings. In the case of the Stern-Gerlach magnets this simply means measuring the x, y or z component of the particle's spin. When measured, each particle gives one of two results, either +1 or -1. It is also assumed that when the same settings are used, the results of the measurements will be the same.

Let's now perform measurements on some of the emitted particles with no particular settings on the devices. (Not to be confused with random settings, they can in fact be the same with each measurement, or not. The point is that settings simply do not matter yet.) Before the measurements are performed, there is no way of knowing what the outcome will be and thus the results may appear to be completely random. This random behaviour can have 3 different explanations [19].

- There are uncontrollable fluctuations on a microscopic level which influence the results of the measurements.
- The process itself is fundamentally indeterministic.
- The particles contain some microscopic property determining the outcome of the experiments, and we simply don't know what they are yet.

As of yet there is no way of telling which explanation is the right one, so we need to involve a concept called locality. Locality roughly says that no influence or signal can move faster than the speed of light, and is therefore only applicable in its immediate surroundings. In terms of the two particles from the set-up this would mean that the measurement on A would in no way affect the measurement on B, provided they are far enough away from each other. In quantum mechanics the results of the measurement does not depend on the distance or times at which the spins are measured. This means that if A and B are measured using the same settings the results will be the same for both particles. Let's say, for example, we set both our Stern-Gerlach magnets to measure the y-component of the spin. Then if we measure A to be +1, we know that B is also +1(or -1 if the particles are anti-correlated). Furthermore, because thanks to locality, any random fluctuations on A could not have any affect at all on B, we can eliminate the first explanation. We've already established that when the settings are the same for A and B, the results are the same. So this means that the process cannot be indeterministic, which eliminates explanation number two. The only option left is that the particles must carry with them all the information needed to provide the given result. Some intrinsic property exists that determines their outcome. EPR called these the 'elements of reality'.

It is important to note that EPR does not say that quantum mechanics is wrong, they indeed seem to assume that quantum mechanics is correct in its predictions of the spin measurements, but that it is

[1] Specifically the decay of a neutral pi meson into an electron and a positron. $\pi^0 \rightarrow e^- + e^+$

yet incomplete. That would mean that quantum mechanics only gives a statistical interpretation of the actual state of the system. It is therefore that they believe that the 'elements of reality' exist which are the information attached to the system that determines the outcome. The elements of reality would complete the theory. In all this the notion of locality is very important. It dictates that the elements of reality are attached to the region of space where the experiment is done. They cannot fluctuate or suddenly change because of something happening in another distant region of space, they only evolve locally. Laloë [19] shows that the EPR conclusions can be summed up in the the following points:

(a) The system contains some elements of reality that influence the outcome of the experiments performed on the system.

(b) The functions that predict the results are well defined and therefore not indeterministic.

(c) So called 'incompatible' observables are now able to have a well defined value do to the fact that they can be predicted. The 'incompatible' observables are therefore only a result of the incompleteness of the theory.

EPR conclude their article saying that they do not know whether a description of physical reality that satisfies the conditions of a complete theory exists, but they believe it is possible [8, p. 780]. There have been suggestions for local hidden variable theories after the release of EPR and even Bell intended to find out what what such a theory would look like. More specifically what the elements of reality would look like. Following EPR's reasoning he wanted to find out more about the properties of the elements of reality. What he found was quite the opposite.

3 The Bell inequalities

Bell's initial goal was to try to create a description for the hidden variables. Basically continuing where EPR left off and introducing the hidden variables into the mathematics [19]. He uses λ to describe all the extra information attached to a system necessary to predict the outcome of a measurement. Going through the calculations incorporating λ result in a inequality that needs to be satisfied by all systems similar to that of the EPR paradox. This thus provided a measurable condition which the LHV (Local Hidden Variables) theories had to obey. In the end it is evident that quantum mechanics does not obey this condition. So instead of showing what the hidden variables would look like inside the quantum theory, he disproved the possibility of quantum mechanics containing any local hidden variables at all.

In this section we will discuss Bell's proof as he did in his original article [4]. After that we will see how the Bell theorem works for all non product states, and for an arbitrary large spin. This will be followed by an example of a bell's inequality for spin-1. The final extension of the Bell's theorem comes from Mermin [21], Ardehali [1], Belinskii and Klyshko [3] that shows how one can obtain an inequality for N spin-1/2 particles.

3.1 The original Bell inequality

Bell [4] starts off by making a generalization of the EPR experiment where he measured the two particles along arbitrary directions instead of the same. We will call the direction along which the first particle is measured $\vec{a}$, and the second $\vec{b}$. The particles can then be measured and their values will always be either +1 or -1 (when measured in units of $\frac{\hbar}{2}$). Bell then calculates the expectation value of the product of these two result. Quantum mechanics predicts:

$$E(\vec{a},\vec{b}) = -\vec{a}\cdot\vec{b}. \tag{3.1}$$

Note that when the two directions are parallel (or anti-parallel) as in the original EPR article, one obtains:

$$\begin{aligned} E(\vec{a},\vec{a}) &= -1 \\ E(\vec{a},\vec{-a}) &= +1. \end{aligned} \tag{3.2}$$

EPR calls for a complete description of the particles. So Bell introduces the variable λ to characterize the system. λ can be anything from a single variable to a set of functions describing the system, but for simplicity we'll write it as a single variable. Two functions are defined, $A(\vec{a},\lambda)$ and $B(\vec{b},\lambda)$ which give the results for respectively the first and the second particle. Remembering the EPR requirement of locality, A cannot be influenced by B and vice versa. Both functions can only attain the values +1 or -1. Take $\rho(\lambda)$ to be the probability distribution of λ, where $\int \rho(\lambda)d\lambda = 1$. Then the expectation value of the product of $A(\vec{a},\lambda)$ and $B(\vec{b},\lambda)$ becomes:

$$E(\vec{a},\vec{b}) = \int \rho(\lambda)A(\vec{a},\lambda)B(\vec{b},\lambda)d\lambda, \tag{3.3}$$

which must be equal to (3.1). We know from (3.2) that (3.3) has its maximum at 1 and its minimum at -1. When it equals -1 we know that $\vec{a} = \vec{b}$ and $A(\vec{a},\lambda) = -B(\vec{b},\lambda)$. Therefore (3.3) becomes

$$E(\vec{a},\vec{b}) = -\int \rho(\lambda)A(\vec{a},\lambda)A(\vec{b},\lambda)d\lambda. \tag{3.4}$$

Introducing $\vec{c}$ as another unit vector we can see that

$$\begin{aligned} E(\vec{a},\vec{b}) - E(\vec{a},\vec{c}) &= -\int \rho(\lambda)[A(\vec{a},\lambda)A(\vec{b},\lambda) - A(\vec{a},\lambda)A(\vec{c},\lambda)]d\lambda \\ &= \int \rho(\lambda)A(\vec{a},\lambda)A(\vec{b},\lambda)[A(\vec{b},\lambda)A(\vec{c},\lambda) - 1]d\lambda, \end{aligned} \tag{3.5}$$

but since $A(\vec{a},\lambda)$ and $B(\vec{b},\lambda)$ are both ± 1 we know that

$$-1 \leq [A(\vec{a},\lambda)A(\vec{b},\lambda)] \leq +1 \quad \text{and} \quad \rho(\lambda)[1 - A(\vec{b},\lambda)A(\vec{c},\lambda)] \geq 0, \tag{3.6}$$

and therefore

$$|E(\vec{a},\vec{b}) - E(\vec{a},\vec{c})| \leq \int \rho(\lambda)[1 - A(\vec{b},\lambda)A(\vec{c},\lambda)]d\lambda. \tag{3.7}$$

Here the second term on the right is equal to $E(\vec{b},\vec{c})$, so that

$$1 + E(\vec{b},\vec{c}) \geq |E(\vec{a},\vec{b}) - E(\vec{a},\vec{c})|. \tag{3.8}$$

This result is the famous Bell inequality which is valid for any LHV theory. All that remains is to show that there are situations in which the inequality is incompatible with the quantum mechanical predictions. For example if $\vec{a}$, $\vec{b}$ and $\vec{c}$ have the angles 0, $\frac{\pi}{4}$ and $\frac{\pi}{2}$ respectively, one would obtain:

$$E(\vec{a},\vec{b}) = 0, \quad E(\vec{a},\vec{c}) = E(\vec{b},\vec{c}) = -\frac{\sqrt{2}}{2} \approx -0.707, \tag{3.9}$$

which is in disagreement with the inequality [13, p.426]

$$1 - 0.707 = 0.293 \ngeq 0.707. \tag{3.10}$$

Bell thus concluded that quantum mechanics and any local hidden variable theory cannot co-exist. Either locality is preserved, there are hidden variables and quantum mechanics is incorrect, or quantum mechanics is correct and locality is violated. It was now only a matter of actually doing the measurements. The theorem was tested experimentally by Aspect et al [2], which showed to be in favor of quantum mechanics. The results of the experiment has been questioned however recently by Khrennikov[17]. The details do not concern us here, but it is sufficient to say that in his article he shows that the inequality is only violated when one chooses certain settings. The author argues that under certain conditions the inequality is actually not violated. If however the violation of the inequality is indeed confirmed experimentally this would mean that the quantum mechanical predictions where correct. The theorem makes very little assumptions and therefore provides quite a strong argument against local hidden variable theories. The theorem does however assume that the distribution of hidden variables is the same for all measurements, and this assumption will be talked about further in the discussion.

There is another version of Bell's theorem called the CHSH form (or even BCHSH) that is more commonly used in experiments to test the inequality [7]. Using the same notation $A(a,\lambda) = A$, $A(a',\lambda) = A'$, $B(b,\lambda) = B$ and $B(b',\lambda) = B'$, which are either +1 or -1 when measured. If we then look at the following product:

$$M = AB + AB' - A'B + A'B' = (A - A')B + (A + A')B'. \tag{3.11}$$

We can see that it is always equal to either +2 or -2 because either $(A - A')$ or $(A + A')$ is always zero. Then when averaging M over a large number of experiments, we obtain:

$$-2 \leq \; < M > \; \leq +2. \tag{3.12}$$

This is a very clean and general inequality. As said before it is valid to all systems with the same set-up as the EPR paradox. As long as the outcome is the result of fluctuations and influences that act locally. When using quantum mechanical notation to predict the outcome, the highest attainable value for M (and thus $< M >$) is $2\sqrt{2}$. The factor $\sqrt{2}$ is a difference of more than 40% [19].

3.2 Extensions of the Bell inequality

While the original version and also the CHSH form work with the same state, namely a singlet state for two particles, the inequality is also valid for other states. Gisin [9] proved that the Bell inequality is violated for all non-product states. He utilized a general form of a non-product state:

$$|\Psi\rangle = c_1 |\uparrow\downarrow\rangle + c_2 |\downarrow\uparrow\rangle . \tag{3.13}$$

He then proceeds to show that the value M (3.11) from the BCHSH inequality can also be written as $P(\vec{a},\vec{b}) - P(\vec{a},\vec{b'}) + P(\vec{a'},\vec{b}) + P(\vec{a'},\vec{b'})$ and is equal to

$$2(1 + 4|c_1c_2|)^{-1/2},$$

for c_1 and c_2 greater than zero this is always greater than 2 thus violating the inequality [9].

A year later Gisin [10] published a paper showing how the Bell inequality is also valid for an arbitrary large spin, and so expanding the proof even more. This result was later used to create inequalities for different types of systems, instead of the two spin-1/2 particles [10]. Instead of looking at Gisin's proof that show the possibility of expanding Bell's theorem for higher spin, we will look at an example of this by Wu et al [28].

3.2.1 Bell for spin-1

An example of a proof for higher spin particle systems is that of Wu et al [28] which deals with two spin-1 particles. In their proof they use probabilities more directly compared to the original Bell inequalities, which also leads to a contradiction. Here we will discuss their proof briefly. The state that they use is:

$$|\Phi\rangle = \frac{1}{\sqrt{3}}(|1,-1\rangle - |00\rangle + |-1,1\rangle). \tag{3.14}$$

We can see here that the eigenvalues of the eigenvector $|m_i\rangle$ of the spin operator along the z-axis $S_z = 1, 0, -1$ for particles $i = 1, 2$. Let $|m_i'\rangle$ be the eigenvector of the spin operator $S(\beta_i)$. The connection between these two vectors is given by:

$$|m_i'\rangle = \sum_{j=1}^{3} D_{ji}(\beta)\,|m_j\rangle\,, \tag{3.15}$$

and the rotation matrix

$$D_{ji} = \begin{vmatrix} \frac{1+\cos(\beta)}{2} & \frac{\sin(\beta)}{\sqrt{2}} & \frac{1-cos(\beta)}{2} \\ \frac{sin(\beta)}{\sqrt{2}} & \cos(\beta) & -\frac{1-\cos(\beta)}{\sqrt{2}} \\ \frac{1-\cos(\beta)}{2} & \frac{\sin(\beta)}{\sqrt{2}} & \frac{1+\cos(\beta)}{2} \end{vmatrix}. \tag{3.16}$$

The particles propagate along the y-axis, one in the positive direction and one in the negative direction, where they will both meet Stern-Gerlach analyzers after some time. One of the two analyzers will give the results for spin projection along a direction β_1 and the other along β_2. The authors then proceed change the base of (3.14) using (3.15) to rewrite it as:

$$\begin{aligned} \frac{1}{\sqrt{3}}(\sin(\frac{\beta_1-\beta_2}{2})\,|11\rangle - \frac{1}{\sqrt{2}}\sin(\beta_1-\beta_2)\,|10\rangle + \\ \cos^2(\frac{\beta_1-\beta_2}{2})\,|1,-1\rangle + \frac{1}{\sqrt{2}}\sin(\beta_1-\beta_2)\,|01\rangle - \cos(\beta_1-\beta_2)\,|00\rangle \\ - \frac{1}{\sqrt{2}}\sin(\beta_1-\beta_2)\,|0,-1\rangle + \cos^2(\frac{\beta_1-\beta_2}{2})\,|-1,1\rangle \\ + \frac{1}{\sqrt{2}}\sin(\beta_1-\beta_2)\,|-1,0\rangle + \sin(\frac{\beta_1-\beta_2}{2})\,|-1,-1\rangle). \end{aligned} \tag{3.17}$$

Using the following definition for joint probabilities:

$$P_{m_1,m_2} = |\langle\Phi|\, m_1 m_2\rangle|^2\,, \tag{3.18}$$

where the values of the particles are $m_1, m_2 = 1, 0, -1$. For (3.17) the authors obtain the following probabilities according to quantum mechanics:

$$P_{11} = \frac{1}{3}\sin\frac{\beta_1-\beta_2}{2}^2, \tag{3.19}$$

$$P_{00} + P_{0,-1} + P_{-1,0} + P_{-1,-1} = \frac{1}{3}(1 + \sin\frac{\beta_1-\beta_2}{2}^4). \tag{3.20}$$

Now it needs to be shown that for this correlation LHV models cannot be sufficient. As with Bell's theorem, we introduce λ to represent the hidden variables. The same probabilities will be calculated with incorporation of λ with

$$P_{m1,m2}(\beta_1,\beta_2) = \int p_{m1}(\beta_1,\lambda)q_{m2}(\beta_2,\lambda)d\lambda, \tag{3.21}$$

where p_{m1} is the result of measuring the first particle along β_1 and q_{m2} is the result of measuring the second particle along β_2. Here too $\rho(\lambda)$ is normalized. The authors then make use of the following algebraic theorem which is true for any six real numbers x, x', X, y, y' and Y where $0 \le x$, $x' \le X$, $0 \le y$, and $y' \le Y$ then

$$-XY \le xy - xy' + x'y + x'y' - x'Y - Xy \le 0. \tag{3.22}$$

By saying that $x = p_1(\beta_1,\lambda)$, $x' = p_1(\beta_1',\lambda), y = q_1(\beta_2,\lambda)$, $y' = q_1(\beta_2',\lambda)$ and $X = Y = 1$ they obtain the inequality

$$S = P_{11}(\beta_1,\beta_2) - P_{11}(\beta_1,\beta_2') + P_{00}(\beta_1',\beta_2) + P_{0,-1}(\beta_1',\beta_2') + P_{-1,0}(\beta_1',\beta_2) + P_{-1,-1}(\beta_1',\beta_2) \le 1. \tag{3.23}$$

This is violated for the angles $\beta_1 = 0$, $\beta_1' = 2\beta_2$, $\beta_2' = 3\beta_2$ and $\beta_2 = 147,7\,\text{deg}$. This gives $S = 1.12 \not\le 1$. Wu et al hereby provide an example of a Bell-type inequality for higher spin.

3.2.2 Bell for N spin-1/2 particles

The extension of the Bell theorem for a N number of particles was given by Mermin [21], Ardehali [1] and by Belinskii and Klyshko [3] separately. It is therefore also known as the MABK theorem. In this section we will discuss Mermin's and Ardehali's articles.

Their starting point is a N-particle system in a GHZ state (which we will come to later). In the GHZ proof of the Bell inequalities they use extreme values (always either +1 or -1), whereas in this proof they assume that the measurements are imperfect, and therefore that the measured correlations do not attain their extreme values [1].

Both Mermin and Ardehali start out with the correlated (GHZ) state:

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\cdots\uparrow\rangle + i\,|\downarrow\downarrow\cdots\downarrow\rangle). \tag{3.24}$$

Mermin then makes use of the operator A which is:

$$A = \frac{1}{2i}\left[\prod_{j=1}^{n}(\sigma_x^j + i\sigma_y^j) - \prod_{j=1}^{n}(\sigma_x^j - i\sigma_y^j)\right]. \tag{3.25}$$

Taking the diagonal matrix elements of this operator can be used as separate observables. When used on Φ and expanded this gives:

$$\left\langle\Phi\right|\sigma_y^1\sigma_x^2\cdots\sigma_x^N\left|\Phi\right\rangle + \cdots \tag{3.26}$$

$$\begin{aligned}
&- \left\langle\Phi\right|\sigma_y^1\sigma_y^2\sigma_y^3\sigma_x^4\cdots\sigma_x^N\left|\Phi\right\rangle - \cdots\\
&+ \left\langle\Phi\right|\sigma_y^1\cdots\sigma_y^5\sigma_x^6\cdots\sigma_x^N\left|\Phi\right\rangle + \cdots\\
&- \left\langle\Phi\right|\sigma_y^1\cdots\sigma_y^7\sigma_x^8\cdots\sigma_x^N\left|\Phi\right\rangle - \cdots\\
&+ \cdots,
\end{aligned} \tag{3.27}$$

where each line contains all the possible permutations of the subscripts. We are thus considering the set of all possible permutations of x and y subscripts, containing an uneven amount of y subscripts. It is this set of 2^{n-1} terms that will be used to model for LHV. If one would look at the case where each term attains their extreme value of either ± 1 (3.26) would be equal to 2^{n-1}. Doing this would lead to a direct contradiction which is shown in the GHZ case in section 4. As stated before the authors consider the case where this does not happen, and each term does not in general attain it's extreme value. The author then defines the expectation value of A within the EPR frame to be:

$$E = \int \rho(\lambda)\frac{1}{2i}\left[\prod_{j=1}^{n}(E_x^j + iE_y^j) - \prod_{j=1}^{n}(E_x^j - iE_y^j)\right] d\rho, \tag{3.28}$$

where $\rho(\lambda)$ is of course again the normalized distribution function and $E_\mu^j = p_\mu^j(\uparrow,\lambda) - p_\mu^j(\downarrow,\lambda)$. Making the calculations will result in

$$E \leq \begin{cases} 2^{n/2} & \text{n even} \\ 2^{(n-1)/2} & \text{n odd} \end{cases}. \tag{3.29}$$

These are the expected values of A within the EPR reasoning, in other words, assuming locality. The next step is to look at what quantum mechanics predicts about the same situation and to see if there is a violation. When we use quantum mechanics to calculate the expectation value one obtains

$$E = \langle\Phi|\,A\,|\Phi\rangle = 2^{n-1}, \tag{3.30}$$

which is obviously always larger for $N > 3$ than the predicted values using the EPR reasoning. It is also clear that the violation becomes even stronger when the number of particles is increased.

As stated before Ardehali has a very similar argument which works for the same state, although in the articles the state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\cdots\uparrow\rangle - |\downarrow\downarrow\cdots\downarrow\rangle)$ is used instead. The operator B is defined as $B = B_1 + B_2$ where

$$B_1 = (-\sigma_x^1\sigma_x^2\cdots\sigma_x^{n-1} + \sigma_y^1\sigma_y^2\sigma_x^3\cdots\sigma_x^{n-1} + \cdots - \sigma_y^1\sigma_y^2\sigma_y^3\sigma_y^4\sigma_x^5\cdots\sigma_x^{n-1} - \cdots + \sigma_y^1\cdots\sigma_y^6\sigma_x^7\cdots\sigma_x^{n-1} + \cdots - \cdots)(\sigma_a^n - \sigma_b^n), \quad (3.31)$$

$$B_2 = (\sigma_y^1\sigma_x^2\cdots\sigma_x^{n-1} + \cdots - \sigma_y^1\sigma_y^2\sigma_y^3\sigma_x^4\cdots\sigma_x^{n-1} - \cdots + \sigma_y^1\cdots\sigma_y^5\sigma_x^6\cdots\sigma_x^{n-1} + \cdots - \sigma_y^1\cdots\sigma_y^7\sigma_x^8\cdots\sigma_x^{n-1} - \cdots + \cdots)(\sigma_a^n + \sigma_b^n). \quad (3.32)$$

We can see that the set of observables that is considered here contains not only the uneven permutations of x subscripts but also the even amount. The first equation contains all distinct permutations of even subscripts of y, the second equation contains all the permutations of odd subscripts of y. As we can see particles $1, 2, 3, \ldots n-1$ are measured along either the x or y-axis. The final particle is measured along either vector a, which is defined as being in the xy-plane and makes a 45 degree angle with the x-axis or vector b which makes a 135 degree angle with the x-axis while also being in the xy-plane.

Ardehali then proceeds to calculate the expectation value of B using the probability distribution function $P_{x_1,y_1,\cdots a_n,b_n}(m_1,\cdots m_n)$ where the m's denote the results of measured particles and are therefore either $\uparrow$ or $\downarrow$. The author proceeds to define the expectation value of B under local realism:

$$E = \sum P_{x_1,y_1,\cdots a_n,b_n}(m_1,\cdots m_n)(M_1 + M_2), \quad (3.33)$$

where

$$M_1 = -Re\left[\prod_{j=1}^{j=n-1}(m_x^j + im_y^j)(m_a^n - m_b^n)\right], \quad (3.34)$$

$$M_2 = Im\left[\prod_{j=1}^{n-1}(m_x^j + im_y^j)(m_a^n + m_b^n)\right]. \quad (3.35)$$

For the ones that are interested in the exact calculations of the next part, one could look at the original article [1], since the calculations are simply to lengthy to repeat here. Doing these calculations will result in E being bounded by the following values under assumption of local realism:

$$E \leq \begin{cases} 2^{n/2} & \text{n even} \\ 2^{(n+1)/2} & \text{n odd,} \end{cases} \quad (3.36)$$

which is a very similar result to that of (3.29). The expectation value of B according to quantum mechanics results in:

$$\langle\Psi|\, B\, |\Psi\rangle = 2^{(n-1/2)}. \quad (3.37)$$

Here we can see that the violation of the inequality is much larger for even n in this case, and much larger for odd n in the case of (3.29).

Although it might be difficult to actually measure this in an experiment it is still an interesting theoretical result when it comes to Bell inequalities. It shows that the result does not become less valid for systems containing a larger number of particles, but rather it makes the argument even stronger. Gisin accurately noted that these results provide us with a proof that the classical behavior of systems does not automatically come from the fact that they are large. By this he meant not only a large spin, but also a large number of particles contained in the system.

In this chapter we have discussed the original Bell theorem, followed by a number of extensions made over the years. We have seen that the inequality is violated for not only the state of two spin-1/2 particles used by Bell himself, but also for other non-product states and for systems containing a higher spin.

Furthermore we have seen that the violation actually becomes stronger for a higher number of particles, rather than weaker. Bell's theorem assumes very little about both quantum mechanics and of course the LHV's. The only assumption it makes is that $\rho(\lambda)$ is independent of the measuring process. So the proof can only exclude those LHV theories that assume that the measurements do not affect the distribution of hidden variables [16]. The experiment preformed by Aspect et al [2] indeed showed a violation of the inequality. Furthermore the MABK theorem has been tested and here too was the inequality violated [30].

4 GHZ experiment

In 1989 Greenberger, Horne and Zeilinger [12] published a book containing several aspects of quantum theory. One of them was Bell's theorem. A year later they published an article, together with A. Shimony, containing one of the theories discussed in that book. The article, appropriately called "Bell's theorem without inequalities" [11] contained a theorem much like that of Bell, but indeed without an inequality. Instead it had a direct contradiction. Just like Bell's theorem, their starting point was the EPR reasoning, so their contradiction is also based upon the assumption of locality. Their proof relies on more assumptions than Bell's however, but this does result in a stronger contradiction. In the same year David Mermin published an article based on their theorem where he shows a much more elegant version. For convenience from now on we will refer to this version as the GHZM theorem. In the next section we will explain this version of the theorem by Mermin [24] which is for three particles with spin-1/2. After that we will use this version and expand it for a system of an arbitrary number of particles.

4.1 The proof

As with the Bell theorem we will assume the hidden variables exist and try to see if they work within the quantum theory. The proof starts of with three spin-1/2 particles which are emitted by a source in three different directions. The three particles are in the following state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle). \tag{4.1}$$

All three particles are sent towards three devices each with a switch with two settings. The device, which contains a Stern-Gerlach magnet, will measure the x-component when the switch is set to 1, (as seen in figure 1) and will measure the y-component when set on 2. The device will then light up either red (for spin up) or green (for spin down) when hit by a particle. If we look at the situation where one device is set on 1 and the other two on 2, this corresponds to measuring one in the x direction and the other two in the x direction, we can use the following operators:

$$\sigma_x^1\sigma_y^2\sigma_y^3, \qquad \sigma_y^1\sigma_x^2\sigma_y^3, \qquad \sigma_y^1\sigma_y^2\sigma_x^3, \tag{4.2}$$

where σ_x and σ_y are the Pauli matrices, and σ_x^1 works on the first particle, σ_y^2 on the second and so on. Since

$$\begin{aligned} \sigma_x |\uparrow\rangle &= |\downarrow\rangle \qquad & \sigma_y |\uparrow\rangle &= +i\,|\downarrow\rangle \\ \sigma_x |\downarrow\rangle &= |\uparrow\rangle \qquad & \sigma_y |\downarrow\rangle &= -i\,|\uparrow\rangle\,, \end{aligned} \tag{4.3}$$

we can let the operators of (4.2) work on the state (4.1). All three of these will give back the original state with an eigenvalue of 1.

$$\begin{aligned} \sigma_x^1\sigma_y^2\sigma_y^3\,|\Psi\rangle &= +1\,|\Psi\rangle \\ \sigma_y^1\sigma_x^2\sigma_y^3\,|\Psi\rangle &= +1\,|\Psi\rangle \\ \sigma_y^1\sigma_y^2\sigma_x^3\,|\Psi\rangle &= +1\,|\Psi\rangle\,. \end{aligned} \tag{4.4}$$

Thus $|\Psi\rangle$ is an eigenstate of these operators. For these equations to be true it must be that either all particles are measured in the up position (and therefore give the value +1), or one is up, and two are down. So the detectors will always give an odd number of red flashes (one or three).

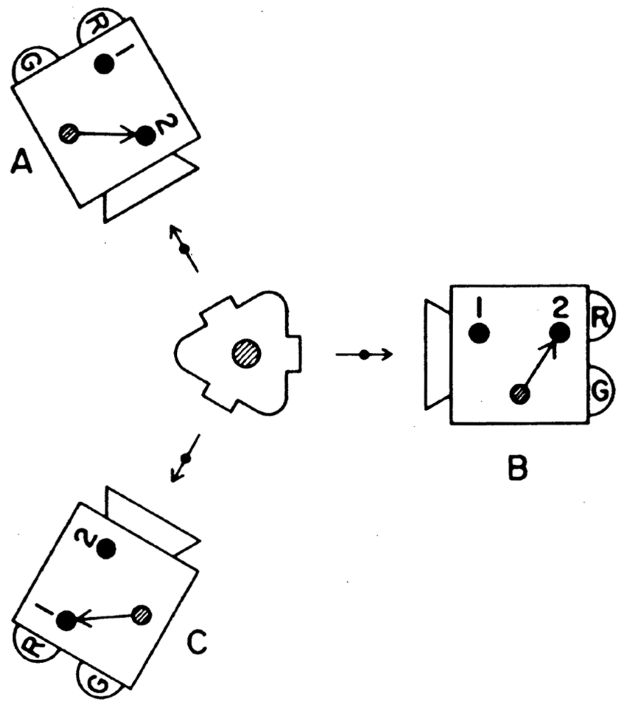

Figure 1: Mermin's set-up of three detectors A, B and C with a source in the middle. Here the settings correspond to 122, 211 or 212 which will result in an odd number of red flashes. From [24, p. 731].

Next we will look at the situation where all the devices are set to 1. In terms of the operators this will be:

$$\sigma_x^1 \sigma_x^2 \sigma_x^3 \left|\Psi\right\rangle = -1 \left|\Psi\right\rangle . \tag{4.5}$$

This would result in an odd number of green(-1) flashes, which also means that one would *never* find an odd number of red flashes.

The next step is to look at the situation from a EPR point of view and therefore assign the hidden variables to the observables. The hidden variables would determine the value of the observables pre-measurement and because of this we can simply assign the values to the observables. The assigned values can obviously be either +1 or -1. We will call M_x^1 the result of the x-component spin measurement of the first particle, M_x^2 for the second particle, etc. Here GHZ makes the assumption that the values of the observables are subject to the same conditions as the observables themselves are. This means that the values we assign to the observables of (4.4) and (4.5) have to satisfy

$$\begin{aligned} M_x^1 M_y^2 M_y^3 &= 1, \\ M_y^1 M_x^2 M_y^3 &= 1, \\ M_y^1 M_y^2 M_x^3 &= 1, \end{aligned} \tag{4.6}$$

and

$$M_x^1 M_x^2 M_x^3 = -1. \tag{4.7}$$

According to the EPR theorem a cause can only create an effect locally. Because of this we know the measurement on one of the particles should not influence the outcome of the other two measurements. Their outcome is already determined beforehand, meaning the particles already *have* a certain value before they even arrive at the apparatus. There should therefore be no objection to reusing the result of the measurement on one of the particles in one setting, in another. So after you've send the particles on their way, with the setting 122 for example, you flip two of the switches to 1. This is after the particles have left the source, but before they arrive at the detectors. The same set of particles that gave +1 for 122, 212 and 221 should also give -1 for 111. Because we saw before that this always happens when setting all the devices to 1. That is why we can simply reuse the resulting values of the measurements of (4.4) to (re)construct the situation of (4.5). To do this we calculate the product of the three values (4.6) which then becomes:

$$\begin{gathered}(M_x^1 M_y^2 M_y^3)(M_y^1 M_x^2 M_y^3)(M_y^1 M_y^2 M_x^3) = 1 \cdot 1 \cdot 1 \\ (M_x^1 M_x^2 M_x^3)(M_y^1 M_y^2 M_y^3)^2 = 1.\end{gathered} \tag{4.8}$$

The second part of the right hand side is squared and since $(\pm 1)^2 = 1$ one can easily see that the first part $(M_x^1 M_x^2 M_x^3)$ must also equal one. But this is in contradiction with (4.7)! Here it is evident that the contradiction resulting from the GHZM thought experiment is much stronger than Bell's inequality. The reason this 'works' (or why we obtain a change in sign) is because σ_x^1 and σ_y^1 do not commute. We therefore cannot interchange them without repercussions. This is exactly what we did at (4.8). So this is why quantum mechanics and hidden variables do not work well together, or rather, not at all.

It is interesting to see that the same contradiction can be explained very simply without needing any knowledge of quantum mechanics. In the same article Mermin shows one can also prove the contradiction using a simple thought experiment using the devices mentioned earlier. As before, the devices set up to measure the particles flash either red or green, the results are then respectively +1 and -1. (Interestingly enough the contradiction can also be shown even without knowing which color stands for which value.) We again assume locality and that since there is no connection between the devices, there is no way for the particle to know which color it *should* flash, just from what the others did. Also it can't know what setting the device is set to until it arrives there. Therefore it must carry all the information with it necessary to know what color to flash for either of the two settings. Mermin uses the notation $\genfrac{}{}{0pt}{}{1}{2}$ for the information the particle must have for either setting 1 or 2. Each particle would then have $\genfrac{}{}{0pt}{}{\mathrm{R}}{\mathrm{R}}$, $\genfrac{}{}{0pt}{}{\mathrm{R}}{\mathrm{G}}$, $\genfrac{}{}{0pt}{}{\mathrm{G}}{\mathrm{R}}$ or $\genfrac{}{}{0pt}{}{\mathrm{G}}{\mathrm{G}}$ as an option.

For the devices to give an odd number of flashes for the settings being, 122, 212 or 221 we would get the following combinations of particle sets:

Setting 1	RRR	RGG	GRG	GGR
Setting 2	RRR	RGG	GRG	GGR

and

Setting 1	RGG	RRR	GGR	GRG ,
Setting 2	GRR	GGG	RRG	RGR

where the top row of each six letter block corresponds to setting 1 and the bottom row of each block to setting 2. It is easy to check these are correct [24].

Now we will look at the situation where we set all the devices to 1. As we saw before at (4.5), that resulted in a negative eigenvalue, or in other words an odd number of green flashes. Even if we flip two of the switches after the particles have already left the result would still be an odd number of green flashes. In that situation we know one of the possible sets given above left the source and in a 122 (etc) situations would give the expected result. But when we flipped two of the switched half way and still obtained an odd number of green flashes it means that the sets of particles given above should also work for the setting with all the switches set to 1. But if we look at the first row of each set *none* of them give an odd number of green flashes. So here too we get a contradiction.

It is quite clear that the GHZM version of the Bell's theorem is a much more elegant version. The beauty of it is that it doesn't depend on statistics of gathered data by performing many measurements, but instead is a direct contradiction. As Laloë [19] put it, it is "... 100% contradiction obtained with 100% certainty." And as Mermin has shown it is even possible to understand the experiment without any knowledge of quantum mechanics whatsoever. Even though the version is alluring, the reasoning has been questioned [19]. For one it is presumed that measuring each observable separately will give the same result as measuring them at the same time. This is true in quantum mechanics for commuting observables. But we can see that the operators σ_x and σ_y of course do not commute. This would prevent us from knowing all the six values of (4.8) at the same time.

4.2 Extensions of the GHZ theorem

4.2.1 Extension to N particles

As before with the Bell inequality, we want to try and expand the GHZM theorem for more particles and a higher spin. In this section an attempt has been made to expand the GHZM thought experiment to one for N particles. We will try to maintain the same method as before, and keep the proof as simple as possible. Later in the section it is shown how one can also expand the use of Mermin's hypothetical devices for N particles. We start with the function:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1\uparrow_2 \dots \uparrow_N\rangle - |\downarrow_1\downarrow_2 \dots \downarrow_N\rangle). \tag{4.9}$$

As before we will use the Pauli matrices (4.3), to construct the following operators for measuring these N particles.

$$\begin{aligned} &\sigma_x^{(1)}\sigma_x^{(2)}\dots\sigma_x^{(N)}|\Psi\rangle = -|\Psi\rangle \text{ for } N \in 1,2,3... \\ &\sigma_y^{(1)}\sigma_y^{(2)}\dots\sigma_y^{(N)}|\Psi\rangle = \begin{cases} -|\Psi\rangle & \text{for } N = 4n \\ +|\Psi\rangle & \text{for } N = 2(2n+1) \end{cases} \quad n \in 0,1,2,3... \end{aligned} \tag{4.10}$$

These two operators correspond to using N devices either all set to 1 or all set to 2. Again an eigenvalue of +1 would result in an even number of green flashes and the eigenvalue -1 in an odd number of green flashes. Using these two equations we can construct the following:

$$\sigma_x^{(1)}\sigma_x^{(2)}\dots\sigma_x^{(j)}\sigma_y^{(j+1)}\dots\sigma_y^{(N)}|\Psi\rangle = \begin{cases} -|\Psi\rangle & \text{for } N-j = 4n \\ +|\Psi\rangle & \text{for } N-j = 2(2n+1) \end{cases} \quad n \in 0,1,2,3... \tag{4.11}$$

We can see that these operators all have (4.9) as an eigenfunction. The eigenvalue is either +1 or -1, depending on the number of particles.

In order to obtain our contradiction, we must look at these results from a localist point of view. From now on we will call the spin measurement of the first particle in either the x or y direction $M^1_{x,y}$, of the second particle $M^2_{x,y}$ and so on. From the third line of (4.11) we see that the following is true for $N-j=2$:

$$M_y^1 M_y^2 M_x^3 M_x^4 M_x^5 \dots M_x^N = 1 \tag{4.12}$$
$$M_y^1 M_x^2 M_y^3 M_x^4 M_x^5 \dots M_x^N = 1 \tag{4.13}$$
$$M_y^1 M_x^2 M_x^3 M_y^4 M_x^5 \dots M_x^N = 1 \tag{4.14}$$

$$\vdots$$

with $N-1$ permutations, where the measurement along the y-axis moves from one particle to the next. Likewise for $N-j=4$ we obtain:

$$M_y^1 M_y^2 M_y^3 M_y^4 M_x^5 \dots M_x^N = -1 \tag{4.15}$$
$$M_y^1 M_y^2 M_y^3 M_x^4 M_y^5 \dots M_x^N = -1 \tag{4.16}$$

$$\vdots$$

According to the localists, the measurement on M^1 could have had no influence on the results of the measurements on the other particles M^2, M^3, etc. Therefore we can simple re-use the values of M^1, M^2 etc, for the predictions on the experiment where we measure the first four particles in the x-directions, which is $M_x^1 M_x^2 M_x^3 M_x^4 M_y^5 \ldots M_y^N$. To obtain this we simply multiply the first three equations

$$\begin{aligned}(M_y^1 M_y^2 M_x^3 M_x^4 M_x^5 \ldots M_x^N)(M_y^1 M_x^2 M_y^3 M_x^4 M_x^5 \ldots M_x^N)(M_y^1 M_x^2 M_x^3 M_y^4 M_x^5 \ldots M_x^N) &= 1 \cdot 1 \cdot 1 \\ (M_y^1 M_y^2 M_y^3 M_y^4 M_x^5 \ldots M_x^N)(M_y^1 M_x^2 M_x^3 M_x^4 M_x^5 \ldots M_x^N)^2 &= 1.\end{aligned} \tag{4.17}$$

Since the second part of this equation is squared, we know this is always equal to one. Therefore the first part must also equal one. However, from (4.15) we obtained that it should equal minus one. This gives us a contradiction.

Of course this would also work using the devices as in the 3 particle proof. To easily show this we will start with an example of 4 particles and build from there. In a system of 4 particles we would use the operators:

$$\sigma_x^{(1)}\sigma_x^{(2)}\sigma_y^{(3)}\sigma_y^{(4)}|\Psi\rangle = +|\Psi\rangle \qquad \sigma_y^{(1)}\sigma_y^{(2)}\sigma_y^{(3)}\sigma_y^{(4)}|\Psi\rangle = -|\Psi\rangle\,, \tag{4.18}$$

which is analog to doing one run with all devices set to 2 and another run with two devices set to 1 and two set to 2. When we look at the first run we need to construct all possible sets which will have an **even** number of red (and green) flashes. (The eigenvalue must be +1.)

Setting 1	RRRR	RGGR	GRGR	GGRR
Setting 2	RRRR	RGGR	GRGR	GGRR
Setting 1	GGGG	RRGG	RGRG	RGGR
Setting 2	GGGG	RRGG	RGRG	RGGR
Setting 1	RGGG	RRRR	GGRR	GRGR
Setting 2	GRRR	GGGG	RRGG	RGRG
Setting 1	RRGG	GGGG	RGGR	RGRG
Setting 2	GGRR	RRRR	GRRG	GRGR

Table 1: All allowed sets for $N = 4$ that would produce an even number of green and red flashes for the settings 1122, 1212, 1221, 2121, 2211 and 1221. They are constructed using the sets for $N = 3$ with adding one particle. The letters with a gray box refer to the added particle. The exception to this are the first box on the second row and the second box on the fourth row. These are added completely since their 3-particle counterpart doesn't exist (or rather wasn't allowed).

As shown in table 1 these would indeed give an even number of red (and green) lights for the runs 1122, 1212, 2112, 2121, 2211 and 1221. The run with all devices set to 2 must produce an odd number of red and green flashes (otherwise it wouldn't produce a negative eigenvalue). Looking at the sets above we see that none of them would do that however, thus again the contradiction. If we want to apply this to N particles we should be able to do so in a similar manner, however the number of possible sets doubles with each new particle. N particles would have 2^N sets so writing them all down for say $N = 6$ would already be a hell of a task. Fortunately, utilizing the results for $N = 3$ and $N = 4$ it is quite easy to convince oneself that:

$$\begin{aligned}&\text{If } N = \text{odd} \longrightarrow \begin{cases} \text{odd number of red flashes} & \text{for two devices set to 2} \\ \text{never odd number of red flashes} & \text{for four devices set to 2} \end{cases} \\ &\text{If } N = \text{even} \longrightarrow \begin{cases} \text{never odd number of red flashes} & \text{for two devices set to 2} \\ \text{odd number of red flashes} & \text{for four devices set to 2} \end{cases},\end{aligned} \tag{4.19}$$

which concludes our extension of the GHZM theorem. The proof is a nice addition to the original, although I believe it would mainly be useful on paper. It does give a nice indication of what to look for should the hypothetical devices ever become a reality. The GHZ experiment is hard to conduct just because creating an entangled state for more than 2 particles is very difficult and the difficulty increases with N. Nevertheless Pan et al [26] did the experiment in 2000, proving in favor (once again) for quantum mechanics. Another experiment using four photons also proved the violation of locality by use of GHZ entangled states [30].

4.2.2 Extension to higher spin

In order to find out whether the GHZ experiment can be extended to particles with a higher spin, we must first look into what is necessary. Why does it work for the spin-1/2 particles? As is evident from (4.4) we need operators which share a eigenstate, but with different eigenvalues. In this section we will study the case of 3 spin-1 particles. For spin-1 particles the Pauli matrices are:

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad S_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.20}$$

We will only be using S_x and S_y. The eigenvalues of the matrices S_x and S_y are -1, 0 and 1. Now we square S_x and S_y, the new matrices are:

$$S_x^2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad S_y^2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad S_x S_y = \frac{1}{2} \begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix}. \tag{4.21}$$

Using these two and S_x and S_y we can construct three operators which share an eigenstate:

$$|\Psi\rangle = |000\rangle \,, \tag{4.22}$$

so that:

$$S_{x1}^2 S_{x2}^2 S_{x3}^2 \, |\Psi\rangle = |\Psi\rangle \tag{4.23}$$

$$S_{y1}^2 S_{y2}^2 S_{y3}^2 \, |\Psi\rangle = |\Psi\rangle \tag{4.24}$$

$$S_{x1} S_{y1} S_{x2} S_{y2} S_{x3} S_{y3} \, |\Psi\rangle = 0. \tag{4.25}$$

When we again try to assign hidden variables to the observables we can rewrite the above equations as:

$$\begin{gathered} M_{x1}^2 M_{x2}^2 M_{x3}^2 = 1 \\ M_{y1}^2 M_{y2}^2 M_{y3}^2 = 1 \\ M_{x1} M_{y1} M_{x2} M_{y2} M_{x3} M_{y3} = 0. \end{gathered} \tag{4.26}$$

From here it is evident that there is a contradiction. The last equation tells us that at least one of the M's is equal to zero. But according to the other 2 equations they can't possibly be zero.

There are several problems with this proof. First of all the contradictions do not come from any correlations between the 3 spin-1 particles, which is what the whole EPR reasoning is based on. The fact that the correlation between two far apart particles cannot be explained other then by hidden variables attached to each particle. It does show however that even in cases where there is no apparent *need* for hidden variables, they still don't work along with quantum mechanics. The second issue is again that not all operators commute. Yes it is true that S_x^2, S_y^2 and $S_x S_y$ commute but S_x and S_y separately do not. Furthermore it could be quite difficult to try test this particular situation in an experiment, also because of the unusual operator $S_x S_y$. Lastly this proof only work for these settings of systems. The 3 spin-1 particle example (which could work with any number of particles) is only applicable for spin-1 particles.

4.3 Conclusion

In this chapter we've discussed the GHZ theorem [11] in form presented by Mermin [24]. His proof was for three spin-1/2 particles send in opposite directions and measured in remote regions of space. We then tried to expand the theorem by making a similar proof for N particles. We saw a contradiction appear that is quite like the proof of Mermin himself. When trying to extend to higher spin a problem arises. For the proof to work a state vector is needed that is an eigenvector for the used observables (operators). Finding a set of operators that share an eigenstate is not easy, and resulted in a set of observables that where not measurable (4.21). Although the proof has some issues, its simplicity and elegance can be well used to explain the problem, even to people without a background in quantum mechanics. Doing an experiment on the GHZ theorem is hard because of the difficulty of creating a entangled state for $N > 2$, for which the difficulty rises with N. However it was performed by Pan et al in 2000 [26], and in favor of quantum mechanics.

If we look more precisely at the proof some flaws can be found in its basic assumptions. First off it is assumed that the results of the measurements are independent of the experimental set-up. Of course this does not have to be the case, for who says that the hidden variables do not differ for each experimental set-up? As mentioned before in section 2.1 the elements of reality are attached to the region of space where the experiment is done. Furthermore GHZ assumes the values of the operators satisfy the same conditions that the operators themselves do. This is true for commuting operators in quantum mechanics, which have been used in the first part of the proof. If we look more closely however, we can see that the individual operators σ_x and σ_y do not commute. For this reason we cannot possibly know all 6 values, that where used in the second part (4.8), simultaneously. It would be better to assume as little as possible about quantum theory and the underlying LHV theory, as does the original Bell theorem.

5 The Bell Kochen and Specker theorem

The last no hidden variable theorem that will be discussed is a theorem by Kochen and Specker [18] and separately by Bell [5] himself. The BKS theorem is quite unlike the other two theorems we have discussed, in that this one doesn't have locality at its base. The BKS theorem revolves around the concept of contextuality. If a system contains variables which are dependent of other variables of that same system, they are called contextual. In quantum theory these correspond to commuting observables and can be measured at the same time. When variables are non-contextual, they do not depend on the other variables in the system, they only depend on the quantities they describe. Logically these then correspond to non-commuting observables. In this section the BKS theorem will be explained. First the original arguments will be discussed after which a newer more intuitive version by Mermin [22, 23] and Peres [27]. After that the extensions to higher spin and more particles is sought and discussed. Finally a link will be made between this theorem and that of GHZ after which some conclusions can be made about the assumptions and set up of the theorem.

5.1 The proof

The proof presented by Kochen and Specker published in 1967 [18] uses the example of one spin-1 particle with -1, 0 or 1 as possible values for measurements on S_x, S_y or S_z. The three components do not commute, but for this particular case of a particle of spin 1, their squares do. The sum of these squared components is

$$S_x^2 + S_y^2 + S_z^2 = 2. \tag{5.1}$$

(With the value of $\hbar$ taken to be 1.) Since the squared components commute, they can be measured at the same time. If we now assign certain hidden variables to each observable, they will determine whether their values will be 1 or 0. The values of the observables must satisfy the same equations that their observables satisfy, in other words they must satisfy (5.1). From here it follows that at least two of the measurements

$$\begin{array}{ccc} \sigma_x^1 & \sigma_x^2 & \sigma_x^1\sigma_x^2 \\ \sigma_y^2 & \sigma_y^1 & \sigma_y^1\sigma_y^2 \\ \sigma_x^1\sigma_y^2 & \sigma_x^2\sigma_y^1 & \sigma_z^1\sigma_z^2 \end{array}$$

Figure 2: Six sets of mutually commuting observables. Each row and column is equal to 1 except for the last row which is -1. [23]

should give 1 as a result. From now on we call the values of the squared observables $M_{x,y,z}$, then:

$$M_x = \left\{ \begin{array}{l} 1 \\ 0 \end{array} \right. , M_y = \left\{ \begin{array}{l} 1 \\ 0 \end{array} \right. , M_z = \left\{ \begin{array}{l} 1 \\ 0 \end{array} \right. . \tag{5.2}$$

Furthermore when we add them together, they cannot be larger than 2, according to (5.1): $M_x + M_y + M_z = 2$. This goes not only for x,xand z but for any set of three orthogonal vectors. So two of these orthogonal vectors need to be 1 and the last one 0. To further illustrate the problem the values assigned to the directions can be color coded, red for +1 and blue for 0. One can now rotate these vectors around with a certain angle. If one maintains this angle then continuously rotating with that angle will result in a set of vectors rotated around a sphere. At some point you will have found all possible rotations using that angle. What we now need is to find a set like that for which it is impossible to color the vectors red or blue is such a way that any combination of orthogonal vectors in this set still satisfy the condition given by (5.1). Kochen and Specker showed that this is the case for when the angle between two vectors is less than $\tan^{-1}(0.5) \approx 26.565$ degrees [23, 18].

An easier and more illustrative example of the theorem comes from Mermin [23] who presented the case of two spin-1/2 particles. Considering the following nine observables one can create the following diagram.

Each row and column corresponds to a set of mutually commuting (contextual) operators. We then need to show that it is impossible to assign values to each observable, as it will lead to a contradiction. We know that all the observables in each row and each column are mutually commuting. Also the product of the three observables of the right column are -1, the product of the three observables in all the other columns and all the rows is 1. The values of the observables must also satisfy the equations which the observables themselves satisfy. Therefore, for example, $M_x^1 M_y^2 M_x^1 M_y^2 = 1$ (first column) using the notation introduced above. Trying to satisfy all the observables in figure 5.1 leads to an inevitable contradiction.

Another way to show this is by use of the figure 3. Here each dot represents a direction, and each triangle a set of orthogonal directions which need to satisfy the same identities as given by figure 5.1. When trying to color each dot with either red (+1) or blue (0) one would see that it is impossible to satisfy these conditions.

The final example for the BKS theorem is one that involves three spin-1/2 particles, creating an eight dimensional problem. The set that is considered here can be represented in a 5-star diagram where each line of the star is a set of orthogonal observables. Here the product of all the lines are equal to 1 except for the horizontal line where the product is equal to -1. Here the contradiction is even more obvious. If we multiply the values of the sets of observables, the identities demand it to be equal to -1. However each observable is on a intersection with two lines, and thus appears twice in the multiplication and they must all be 1. This makes the contradiction even easier to see than with the previous example.

Bell himself wasn't completely convinced with his reasoning. One of the most important assumptions he made was that the value of the measured observables was independent of which set was used to measure it in. His problem with this was that not all additional observables in each set commuted with the additional observables in the other set. If we look at figure 2 we can see that indeed the observables of each row and column commute, but for example $\sigma_x^1\sigma_x^2$ does not commute with $\sigma_x^2\sigma_y^1$. It is clear that these observables are essentially non-contextual and so incompatible with each other. Bell [5] argued that since they where non-contextual, there should actually be no reason to assume that the results of measuring in either setting

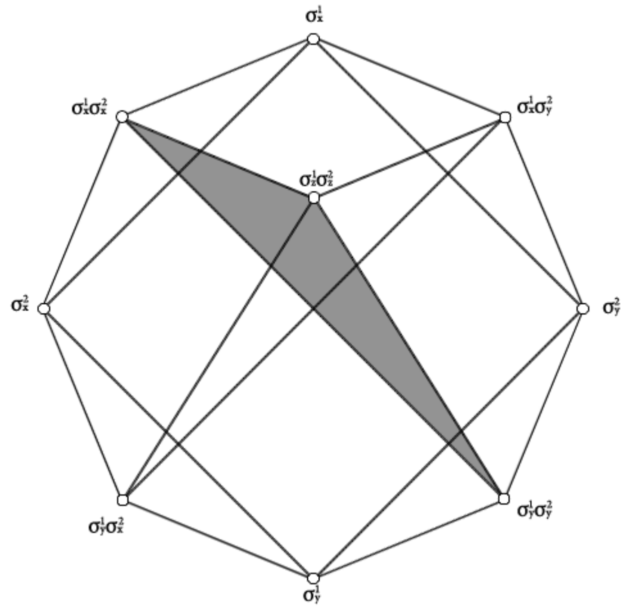

Figure 3: A visual representation of the coloring paradox as given by Cabello [20]

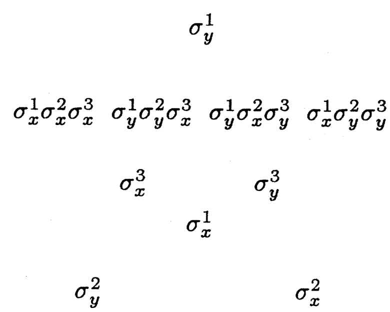

Figure 4: Ten observables whose values cannot be colored accordingly with quantum mechanical predictions. The product of the observables on the horizontal line of the star needs to be -1. The other legs of the star should be +1. Taken from [23, P.811]

should be the same. It thus seems that this non hidden variable proof is less strong than the locality proofs we discussed earlier. Mermin argues however, that in the eight-dimensional case one could exchange the (non-)contextuality argument with the locality argument and reinventing it in a GHZ like form [23]. His arguments are as following:

Consider an observable A which we will measure. We'll measure the observable in two different settings. The first is where we measure A along with mutually commuting operators B, C, D,..etc. Secondly we will measure A along with mutually commuting observables K, L, M, ...etc. However, as with the problem mentioned before, not all observables B, C, D, ... commute with the observables K, L, M ... Next we assume that all the observables are measured by separate devices far away from each other. While measuring A in the first setting (along with B, C, D, ...etc) we replace the devices set to measure B, C, D, ... with devices made to measure K, L, M, ... etc. If we now assume that because of the distance the devices have from each other they do not influence one another, we basically assume locality. To sum up, non-contextual observables are a problem since we can't measure them at the same time, yet we use them for the proof. If however the argument can be replaced by assuming locality, it is no longer an issue. The change of apparatus does not influence the measurement on A.

σ_x^1	σ_x^2	$\sigma_x^1\sigma_x^2$	C
σ_y^2	σ_y^1	$\sigma_y^1\sigma_y^2$	B
$\sigma_x^1\sigma_y^2$	$\sigma_x^2\sigma_y^1$	$\sigma_z^1\sigma_z^2$	A
L	K		

Figure 5: Here figure 2 is used to illustrate the issue of mutually commuting sets containing observables that do not commute with all the observables from other sets. For example C ($\sigma_x^1\sigma_x^2$) does not commute with L ($\sigma_x^1\sigma_y^2$) or K ($\sigma_x^2\sigma_y^1$). This issue can be resolved using locality, where one assumes that the change of apparatus of B and C to K and L doesn't influence the measurement on A.

Mermin then shows that this step works for the 8 dimensional BKS case, which is the star. We can easily see that the four observables on the horizontal line of the star are the same observables that we used for the GHZ example of 3 spin-1/2 particles. In order to convert the BKS example to a the GHZ example all we need is a state $|\Phi\rangle$ for which the first observable has an eigenvalue -1, and the other three an eigenvalue of 1. It works because all the four observables are non-local and they commute with each other. If we however try to convert the two spin-1/2 particle example to a GHZ form we find a there is a problem [22]. The four non-local observables found in the 2 spin-1/2 particles example are $\sigma_x^1\sigma_x^2$, $\sigma_x^1\sigma_y^2$, $\sigma_x^2\sigma_y^1$ and $\sigma_y^1\sigma_y^2$. These observables however do not all commute with each other. We therefore cannot find a state which is an eigenstate for all these observables.

5.2 Extensions of the BKS theorem

As with the previous two theorems we want to see if the BKS theorem can be extended for a) a higher (or rather, a different) spin and b) for a higher number of particles. As we have seen above however, this has already been done in some degree. The original proof worked with one spin-1 particle and Mermin's example with two and three spin-1/2 particles. Expanding this version to a higher spin is quite hard. The reason this type of proof works so well is in the fact that the multiplication of the Pauli matrices in figure 2 and 4 results in the identity matrix I. For spin 1, 3/2, 2 etc, this is not necessarily the case. To find a contradiction for these higher spins one would need to find a relation for the observables used. When one would then try to assign values to the observables it would lead to a contradiction assuming the values satisfy those same conditions. There is no systematic approach to expanding this theorem to a higher spin.

To expand the theorem to more particles can also be a little tricky. Here we can make use of the link Mermin made between the GHZ proof and BKS. BKS can be transformed into a GHZ locality-type proof if one can find a set of non-local mutually commuting observables which share an eigenstate. We already have a set of non-local mutually commuting observables with an eigenstate, namely (4.11). To illustrate how one could go about finding a BKS type proof with these operators, let's look at the case for 4 spin-1/2 particles.

Using the operators

$$\sigma_y^1\sigma_y^2\sigma_x^3\sigma_x^4 \quad \sigma_y^1\sigma_x^2\sigma_y^3\sigma_x^4 \quad \sigma_y^1\sigma_x^2\sigma_x^3\sigma_y^4 \quad \sigma_y^1\sigma_y^2\sigma_y^3\sigma_y^4, \tag{5.3}$$

we see that the multiplication of these results in the identity matrix. So the multiplication of the values should give 1. If we now combine these with the observable $\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4$ we can obtain a contradiction once again.

This might not be as elegant as figure 2 and 4, but it'll do. When we try to color each value blue(+1) or red(-1) so that the multiplications of the values obey the same relations as the observables, we'll again obtain a contradiction. (In the end it looks like all the values should equal -1, which doesn't work for the first three lines of (5.2).) Something similar can be done for even higher spin. One would only have to look for it. There seems to be a lack of consistency in these proofs and so a generalization of it is not easy

$\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4$	$\sigma_y^1\sigma_y^2\sigma_x^3\sigma_x^4$	-1
$\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4$	$\sigma_y^1\sigma_x^2\sigma_y^3\sigma_x^4$	-1
$\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4$	$\sigma_y^1\sigma_x^2\sigma_x^3\sigma_y^4$	-1
$\sigma_x^1\sigma_x^2\sigma_x^3\sigma_x^4$	$\sigma_y^1\sigma_y^2\sigma_y^3\sigma_y^4$	1
	1	

Table 2: The 5 mutually commuting observables that will lead to a contradiction, where the first column is all the same observable. The first three rows are equal to -1 when multiplied, the last one is equal to 1. Same goes for the last column where we multiply the observables obtained from the GHZ example (4.11)

to find. Rather it seems that it is more a matter of trial and error to find a contradiction of this kind for any arbitrary system of particles.

The BKS theorem tries to assign values to observables. The extra information one adds to these observables help determine their values without measuring them. This extra informations is what they consider the hidden variables. It can assign values to any observables not just the ones that need to contain hidden variables due to locality as with the GHZ theorem. The BKS theorem is based upon the same rules of quantum mechanics as the GHZ theorem. Namely that the values of the observables have to obey the same rules and conditions as the observables themselves. The problem with this assumption is that it is only true for mutually commuting observables, which not all of the observables used in the proof are. The problem could be solved by looking only at those who do, an so transferring the assumption of contextuality towards one of locality. This transforms the proof of BKS to a GHZ form, and makes the link between the two theorems clear.

In conclusion we saw that the BKS proof in its original form is a quite complex geometrical proof constructed around one spin-1 particle. A condition placed upon the observables of the particle is used, and it is shown that it is impossible to 'color' the values in such a way that the condition is fulfilled and valid for all sets of orthogonal vectors. The proof brought forward by Peres [27] and build on by Mermin [22, 23] gives us more insight into the theorem and makes it easier to understand. Made with two or three spin-1/2 particles the coloring problem became more insightful. There was a problem with contextuality however, which was solved by introducing locality in its stead. This didn't work in all cases however, only for 3 spin-1/2 particles and up. The proof could be extended for more particles but was difficult to expand for higher spin due to the trial-and-error nature of finding suitable observables. Experiments to test the theorem have been done for single particles [29, 14] as well as two qubit systems [15]. All results confirm quantum mechanics and can be viewed as support for absence of local realism.

6 Discussion and conclusion

The EPR paradox required the existence of hidden variables, to complete the quantum theory. No-hidden-variables theories have shown that quantum mechanics and LHV cannot co-exist. In this paper three major no-hidden-variables theorems have been discussed.

The first was the original Bell theorem considering two entangled spin-1/2 particles. Bell was the first to provide a testable condition that would determine whether quantum mechanics was correct or not and so if a LHV theory was necessary. This theorem has been shown to be valid for all non-product states, and even for an arbitrarily large spin. Another extension of the proof was the MABK theorem, which showed the validity of the proof for N qubits. The original inequality has been tested by Aspect et. al [2] but their results have recently been questioned [17]. The MABK theorem has been tested experimentally as well and shown to be in favor of quantum mechanics. Bell's inequality needs very little assumptions to prove, and is therefore very general. It does however assume that the distribution of hidden variables is not influenced by the measurement. However one could argue that the results of the values, $A(\vec{a},\lambda)$ and $B(\vec{b},\lambda)$ used in the original proof and/or (3.11) cannot be compared at all [25]. This is because different settings are used to measure particles A and B. It it therefore that one cannot use the same distribution of hidden variables for both particles. The results of the measurements are therefore essentially not the same thing, and it would be like comparing apples to oranges. Jammer [16] argues here that only LHV theories that assume one can assign the same distribution to all particles have been excluded by the theorem [16]. Nieuwenhuizen argues that this assumption means that the Bell inequality cannot say anything at all about nature being non-local or not [25].

The Greenberger-Horne-Zeilinger (and Mermin) theorem dealt with three spin-1/2 particles. The observables of these qubits (the Pauli matrices) shared an eigenstate. The eigenvalue of these observables combined with the fact that σ_x and σ_y do not commute, resulted in a direct contradiction. This proof was then extended for N qubits, where $N > 3$. When trying to extend the proof to higher spin a problem arises. A contradiction can still be found on paper, but testing it might be difficult due to the strange observables used. The GHZ theorem relies on a lot of properties of quantum mechanics, especially the fact that values of observables satisfy the same relations as the observables themselves do. As this is only true for commuting observables, and not all observables used in the EPR part of the proof commute, the reliability of the proof can be questioned. Testing of the GHZ theorem is difficult because one needs to create an entangled state for 3 (or more) particles. The experiment conducted by Pan et al [26] was in confirmation with quantum mechanics.

The last proof that was discussed was the BKS theorem. The original proof used intrinsic geometrical arguments to show the incompatibility of quantum theory with hidden variables. They showed that it was impossible to assign values to observables in a way that the restrictions placed on those observables (and thus their values) where still obeyed. This no-coloring theorem involved sets of orthogonal vectors that couldn't be given a certain value (or color corresponding to a value) in a way that the sum $S_x^2 + S_y^2 + S_z^2 = 2$ was still valid. An improvement to the proof was given by Peres and Mermin, where they used two or three spin-1/2 particles. By using sets of mutually commuting observables and their relations, they where able to construct a more intuitive non-coloring proof. The main issue with this proof was that not all observables in a certain set, commuted with all the observables in another set used in the same proof. This problem was called non-contextuality. However, by assuming that when the measuring apparatus are places far enough away from each other, and so that the devices measuring $B, C, \ldots$ cannot influence the measurement on A. And so it doesn't matter if we replace the devices measuring $B, C, \ldots$ with others that measure $L, M, \ldots$ instead. By doing this one assumes locality which can replace the more troublesome assumption of contextuality. The BKS theorem can therefore be changed into a GHZM type proof. The results of experiments performed on the BKS theorem have where also in confirmation with quantum mechanics.

In conclusion all the no-hidden-variable theories discussed have been tested, where most where confirmed. However recent studies by Khrennikov into the Aspect et al experiment have raised questions regarding the interpretation of the results [17]. Where the Bell theorem assumes very little about the form or context of the hidden variables, or the quantum theory they are based on, both BKS and GHZ have more issues with their assumptions. It seems that the original Bell theorem remains the most general in what types of LHV theories it excludes. The beauty and elegance of the GHZ theorem make it easy (easier) to understand and explain to others. Of the three theorems, Bell's original proof appears to be the most suitable for testing. When trying to extend the theorems we where most successful with the original Bell inequality. Bell type inequalities can be found for all non-product states, higher spin and N particles. For the GHZ theorem a generalization could be made for more particles, but a generalization for higher spin could not be found. For the BKS theorem no generalization was found (with exception of the original argument using one spin-1 particle and geometrical proof, which was valid for any angle under approximately 26 degrees). However a few examples have been found for multiple spin-1/2 particles.

In this paper we have not discussed in depth any of the loopholes involved in with the no-hidden-variable theorems, such as the contextuality loophole [25]. Of course we cannot write off LHV's without having closed these. Further research into the subject will eventually show whether this can be accomplished or not, and thus whether or not locality prevails.

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