Surface Charges, Holographic Renormalization, and Lifshitz Spacetime

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M.Sc. Thesis

Abstract

Two distinct holographic methods are introduced and applied in order to study field theories with a quadratic dispersion at strong coupling. The first holographic method is the one introduced by Brown & Henneaux almost 25 years ago, in which the Virasoro algebra is derived from purely classical Einstein gravity. The second method is the holographic renormalization group in the Hamilton–Jacobi canonical formalism, which spawned from the AdS/CFT correspondence. Several fun examples of these methods are used to get a feel for them, which include Strominger's consistency check relating the Bekenstein–Hawking entropy to the Cardy formula of the BTZ black hole. Both methods are finally applied to the study of Lifshitz spacetime, which is a spacetime that has anisotropic scale invariance. The anisotropy is closely related to deviations from a linear dispersion; this thesis focuses on the case of quadratic dispersion $E \sim p^2$.

Although most techniques that are discussed emanated from string theory, they seem to be valid alongside any consistent quantum theory of gravity. Basic knowledge of quantum field theory, general relativity, group theory, and conformal field theory is presumed.

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Enjoy!

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Acknowledgements

I would like to thank my advisors Jan de Boer and Alex Maloney, and the nice people at McGill for the warm welcome. I especially thank Kyriakos Papadodimas, Balt van Rees, and Omid Saremi for answering virtually any physics related question that popped in my head. I also thank Marco Baggio for many useful discussions. Finally, I would like to thank my family for being so supportive.

Cheers!

Introduction

AdS/CFT correspondence. Gauge/gravity dualities such as the celebrated anti-De Sitter/conformal field theory correspondence [1–3] are concrete realizations of 't Hooft and Susskind's holographic principle [4]. Maldacena's AdS/CFT conjecture [1] states that there is an exact equivalence between string theory on an AdS background and a conformally invariant quantum field theory on the conformal boundary of AdS. This correspondence is holographic in nature, as it relates a theory in $d+1$ dimensions to a theory in d dimensions. In order to make the correspondence more tangible, one can take suitable limits such that the string-theory side is well approximated by ordinary (super)gravity, i.e. such that it is weakly coupled. The AdS/CFT correspondence relates the weakly-coupled regime on one side to the strongly-coupled regime on the other, so the corresponding boundary field theory in the 'suitable limits' is necessarily strongly coupled.[i]

Holographic renormalization. It turns out that the radial coordinate on the gravity side is closely related to the renormalization group scale in the boundary theory. This realization has spawned an interesting field of study known as holographic renormalization, cf. [7, 8]. The main idea in holographic renormalization is to find renormalized correlation functions in the strongly-coupled boundary QFT by doing computations on the gravity side of the correspondence. Namely, the QFT correlation functions typically have UV divergences, which are translated to IR (large radius r) divergences on the gravity side. The name of the game is then to remove these large-r divergences, hence to renormalize the QFT correlators holographically.

Anisotropic scaling. Apart from the well-known AdS/CFT correspondence, there has been a lot of interest in studying field theories by building phenomenological models by means of gravity duals. Such studies go by the name AdS/QCD or AdS/condensed matter (for obvious reasons). In this thesis we will discuss a phenomenological gravity dual to a generic field theory at a Lifshitz-like fixed point. Instead of the usual scale-invariance $x^\mu \to \lambda x^\mu$, such theories are invariant under dilatations of the form $(t, x^i) \to (\lambda^z\, t\,,\, \lambda\, x^i)$. Such scaling is known as Lifshitz-like scaling, see e.g. [9]. The parameter z is known as

[i] It should be mentioned that this thesis does not contain an elementary introduction to AdS/CFT. For such a review on AdS/CFT we refer to e.g. [5] and for an in-depth treatment see the classic 'MAGOO' review [6].

the dynamical critical exponent; it can roughly be seen as the exponent in the dispersion relation $E \sim p^z$. The holographic dictionary for these theories is still not very well understood; first steps have been made in [9–14]. We shall focus on the case where the critical dynamical exponent is $z = 2$ (quadratic dispersion).

Structure of this thesis. We will set out to find a holographic description of these Lifshitz-like theories using two distinct methods. One of the first things that one tries to do in these phenomenological approaches is to match the symmetries of the QFT with the Killing symmetries of the proposed gravity dual. In the first method this is done by matching the symmetry algebra of the boundary QFT with the symmetries in the bulk by means of a Poisson-bracket realization in terms of so-called surface charges. The second method is a formulation of holographic renormalization that uses the Hamilton–Jacobi (as opposed to Lagrangian) canonical formalism [8, 15].

This thesis is divided into three chapters. Chapter 1 introduces the surface-charge method introduced by Brown & Henneaux [16, 17] in the mid-80's. This chapter also reviews a result that is very interesting in its own right, which is the matching of the entropy calculated from a 2D CFT (at high temperature) and the Bekenstein–Hawking entropy of its 3D gravity dual.

Chapter 2 reviews the Hamilton–Jacobi formalism in the context of Einstein gravity and subsequently of holographic renormalization. A method of expanding the resulting equations is formulated and a brief review is given of the holographic derivation of the Callan–Symanzik equation. A sample calculation is done to illustrate the power of the method, the outcome of which precisely agrees with the main result of Chapter 1.

Finally, in Chapter 3, we will apply both methods. The application of the first method seeks to find a Poisson-bracket realization of the symmetries of the Lifshitz gravity dual. For the application of the second method, we have the somewhat modest goal of finding a match between counterterms generated with the Hamilton–Jacobi method and a result from the literature [13].

Chapter 1

Surface Charges in AdS_3

3D gravity. At first glance, pure Einstein gravity in three spacetime dimensions seems trivial, because it has no local propagating degrees of freedom. This is can be seen by the following counting argument. Let us consider Einstein gravity in n spacetime dimensions. The induced metric (and its canonical conjugate) has $n(n-1)/2$ independent components. On the other hand, there are n constraint equations that need to be satisfied by physical solutions; in the Hamiltonian formulation these are the Hamilton and momentum constraints. The number of local physical degrees of freedom, i.e. the total number of degrees of freedom minus the number of constraints, is $n(n-3)/2$, which obviously vanishes when $n = 3$.

The most generic form of the Riemann tensor in low-dimensional gravity is rather constrained by its symmetries. The Riemann tensor in any number of dimensions n can be decomposed in terms of the metric tensor $g_{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$ (and its trace), and the conformal Weyl tensor $C_{\mu\nu\rho\sigma}$ (which is traceless and conformally invariant).[i]

$$\begin{aligned} R_{\mu\nu\rho\sigma} = \frac{1}{n-2}\left(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma}\right) \\ - \frac{1}{(n-2)(n-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)R \ + \ C_{\mu\nu\rho\sigma} \end{aligned} \tag{1.1}$$

Now, let us focus on the case of $n = 3$ spacetime dimensions, in which case the conformal Weyl tensor vanishes identically [18]. From this it follows that the Riemann tensor on a solution of the vacuum Einstein equations, $R_{\mu\nu} = \frac{1}{2}(R - 2\Lambda)g_{\mu\nu}$, is maximally symmetric[ii]

$$R_{\mu\nu\rho\sigma} = \Lambda\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \qquad (n = 3,\ \text{on-shell}) \tag{1.2}$$

Note that this is essentially the same expression that one would obtain from the Gauss–Codazzi equations associated with the embedding of an n-dimensional spherical or hy-

[i] See e.g. [18, 19] for a review.

[ii] Taking the trace of the Einstein equation yields $R = 6\Lambda$ when $n = 3$, so that $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$.

perbolic space (depending on the sign of Λ) in $n+1$-dimensional Minkowski space (cf. Appendix C).

If we take $\Lambda = 0$ for the moment, we see another manifestation of the locally trivial property of 3D gravity. Namely, the geodesic equation in the Newtonian limit (with $T^{\mu\nu} \approx u^\mu u^\nu \, \Phi$) in n spacetime dimensions is given by [18]

$$\ddot{\vec{x}} = -2\frac{n-3}{n-2}\vec{\nabla}\Phi \tag{1.3}$$

The right-hand side of (1.3) vanishes when $n = 3$, so there will be no gravitational force acting on our classical particles. In other words, the Newtonian limit is trivial if $\Lambda = 0$.[iii]

In view of these facts, it was rather surprising when it was found that there exists a black-hole solution to pure Einstein gravity (with $\Lambda < 0$) in three dimensions. This black hole is often called the BTZ black hole after the authors of [20]. The reason for the existence of this black hole has to do with the fact that, even though the 3D theory is locally trivial, globally it is not.

Structure of this chapter. In the first section, Section 1.1, we will discuss some basic properties of Lorentzian and Euclidean three-dimensional AdS. We also discuss two other solutions to 3D Einstein gravity with a negative cosmological constant: thermal AdS and the BTZ black hole. One may wonder how these can be non-trivially related to ordinary (Euclidean) AdS if the Riemann tensor is necessarily maximally symmetric (1.2). We basically follow [21, 22] and see that these solutions can be generated by making periodic identifications in the original vacuum AdS solution, which is equivalent to taking a quotient of the group of isometries of ordinary AdS.

In Sections 1.2 and 1.3, we proceed with a discussion of Brown & Henneaux's first hint at the AdS_3/CFT_2 correspondence dating back to 1986 [16]. Section 1.2 introduces the basic notion of so-called asymptotic symmetries and surface-charge generators. Section 1.3 will then discuss Brown & Henneaux's main result, which basically states that these surface-charge generators of the asymptotic symmetries of AdS_3 form two copies of the Virasoro algebra at the boundary. They even found an explicit expression for the central charge, namely $c = 3\ell/2G$. Needless to say, this is a very nice result indeed.

This will lead us on to studying the BTZ black hole in Section 1.4. We will compute the black hole entropy both from the gravity side through the Bekenstein–Hawking entropy [23, 24] as well as from the field theory side through Cardy's formula [25]. Cardy's formula gives the leading-order contribution to the entropy in terms of the central charge. We will study Strominger's result [26], where he found that the two calculations of the entropy are in precise agreement when one plugs the Brown–Henneaux central charge into Cardy's formula. In the process of deriving Cardy's formula, we will briefly go over modular invariance as well, since it is an essential requisite.

[iii] In order to get a non-trivial Newtonian limit, one must add extra terms to the action, see e.g. [18].

1.1 Global Symmetries of AdS_3

In the following two chapters we will be dealing with three-dimensional anti-De Sitter spacetime AdS_3 quite extensively. It is therefore useful to have a solid understanding of what it really is and what its symmetries are. We construct some often-used metrics from the hyperbolic embedding constraint and we will analyze their global symmetries (isometries). We will use the notion of a Killing–Cartan metric, which is introduced in some detail in Appendix A.

Lorentzian AdS_3. Let us start with ordinary Lorentzian AdS_3. It is easiest to think of AdS_3 as being embedded in $\mathbb{R}^{2,2}$. Let $x^\mu \in \mathbb{R}^{2,2}$ be the coordinate on the full space and y^a be the coordinate on the embedded hypersurface[i]. An embedding of a hypersurface can be described by a constraint $\Phi(x) = constant$ as well as by parametric relations $x^\mu(y)$, cf. Appendix C. The embedding of AdS_3 into $\mathbb{R}^{2,2}$ is given by

$$-(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2 = -\ell^2 \qquad \Leftrightarrow \qquad \begin{cases} x^0 = \ell \cosh\rho \cos\frac{t}{\ell} \\ x^1 = \ell \cosh\rho \sin\frac{t}{\ell} \\ x^2 = \ell \sinh\rho \cos\varphi \\ x^3 = \ell \sinh\rho \sin\varphi \end{cases} \tag{1.4}$$

The parameter ℓ is the three-dimensional AdS curvature radius and it is related to the cosmological constant through $\Lambda = -1/\ell^2$. The coordinates $y^a = (t, \varphi, \rho)$ on the hypersurface range over $t \in [0, 2\pi\ell)$, $\varphi \in [0, 2\pi)$ and $\rho \in [0, \infty)$. The induced metric on this three-dimensional hypersurface is the AdS_3 metric in global coordinates,

$$ds^2 = -\cosh(\rho)^2 dt^2 + \ell^2 \sinh(\rho)^2 d\varphi^2 + \ell^2 d\rho^2 \tag{1.5}$$

Another set of coordinates that is often used is obtained by defining a new radial coordinate $r = \ell \sinh\rho$, which yields

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right) dt^2 + r^2 d\varphi^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 \tag{1.6}$$

This will turn out to be a very useful set of coordinates. The time coordinate is periodic, which violates causality. This is usually taken care of by taking the universal cover instead, whose full domain is $t \in (-\infty, \infty)$, $\varphi \in [0, 2\pi)$, and $r \in [0, \infty)$.[ii]

To find the isometries of AdS_3, we recast the aforementioned embedding constraint in the following suggestive form.

$$\det g = 1, \qquad \text{where} \qquad g \equiv \ell^{-2} \begin{pmatrix} x^0 - x^2 & -x^1 + x^3 \\ x^1 + x^3 & x^0 + x^2 \end{pmatrix} \in SL_2(\mathbb{R}) \tag{1.7}$$

[i] For this codimension-one embedding, the indices run over $\mu = 0, 1, 2, 3$ and $a = 0, 1, 2$.

[ii] See e.g. Appendix A.5 of Carlip's book [18] for a brief review of the concept of covering spaces.

The metric on the group manifold of $SL_2(\mathbb{R})$ is given by the Killing–Cartan metric (cf. Appendix A)

$$ds^2 = \frac{1}{2}\mathrm{tr}\left(g^{-1}\,dg\right)^2 \tag{1.8}$$

The Killing–Cartan metric is invariant under group actions both from the left $g \to k_{\mathrm{L}}\,g$ and the right $g \to gk_{\mathrm{R}}$, where $k_{\mathrm{L}}, k_{\mathrm{R}} \in SL_2(\mathbb{R})$ are rigid. The left- and right-actions act independently of one another, so the isometry group of Lorentzian AdS_3 consists of two copies denoted $SL_2(\mathbb{R})_{\mathrm{L}} \times SL_2(\mathbb{R})_{\mathrm{R}}$.

Euclidean $\boldsymbol{AdS_3}$. We get a Euclidean signature by Wick-rotating Lorentzian AdS_3, so the metric is an analytic continuation of the one above.

$$ds^2 = \left(1 + \frac{r^2}{\ell^2}\right) dt^2 + r^2 d\varphi^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 \tag{1.9}$$

The isometry group of this space is a little different though. Similar to the Lorentzian case, Euclidean AdS_3 can be embedded in $\mathbb{R}^{1,3}$ by the constraint

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = -\ell^2 \tag{1.10}$$

Notice the important change of sign coming from the Wick rotation $x^1 \to -ix^1$. To find the isometries of Euclidean AdS_3, we write the constraint as $\det g = 1$ with

$$g \equiv \ell^{-2} \begin{pmatrix} x^0 - x^2 & ix^1 + x^3 \\ -ix^1 + x^3 & x^0 + x^2 \end{pmatrix} \in SL_2(\mathbb{C})/SU(2) \tag{1.11}$$

In this case we mod out by $SU(2)$, because the above g is Hermitian, which means that it can always be written as $g = hh^\dagger$ for some $h \in SL_2(\mathbb{C})$. The $SU(2)$ factor is most easily exposed by noticing the invariance under $h \to hu$, with $u \in SU(2)$. The metric on the coset $SL_2(\mathbb{C})/SU(2)$ is given by the Killing–Cartan metric $ds^2 = \frac{1}{2}\mathrm{tr}\left(g^{-1}\,dg\right)^2$. This time, it is invariant under actions from the left on h, not g itself. In other words, $h \to k_{\mathrm{L}}\,h$ with $k_{\mathrm{L}} \in SL_2(\mathbb{C})$ leaves the metric invariant. The group element itself transforms as $g \to k_{\mathrm{L}}\,gk_{\mathrm{L}}^\dagger$ under $h \to k_{\mathrm{L}}\,h$. Notice that, in a sense, the right-action on g is no longer independent of the left-action, namely $k_{\mathrm{R}} = k_{\mathrm{L}}^\dagger$. We conclude that the isometry group of Euclidean AdS_3 is a single copy of $SL_2(\mathbb{C})$.

Generating excitations by taking quotients of the isometry group. We should always keep in mind that 3D Einstein gravity is only non-trivial globally, which means that the permitted excitations should be topological. One could think of the vacuum as the space with the maximal amount of symmetry. Seen in this light, one could generate excitations by effectively reducing the amount of symmetry. This can be done by making isometry-group identifications, i.e. by taking a quotient. What this statement means is most easily understood by looking at specific examples.

In the two examples that follow, we continue working in Euclidean signature. In that case, the metric is given by the Killing–Cartan metric on the $SL_2(\mathbb{C})/SU(2)$ coset in terms of g. In both examples, we start out with the following element in the coset (cf. [27])

$$g = \begin{pmatrix} \rho + \dfrac{z\bar{z}}{\rho} & \dfrac{z}{\rho} \\ \dfrac{\bar{z}}{\rho} & \dfrac{1}{\rho} \end{pmatrix} \in SL_2(\mathbb{C})/SU(2) \tag{1.12}$$

which is Hermitian, so it is indeed restricted to the coset. The identifications that will be made are $g \sim kgk^\dagger$ with some matrix $k \in SL_2(\mathbb{C})$, which shall be specified shortly. The corresponding Killing–Cartan metric is given by

$$ds^2 = \frac{1}{2}\mathrm{tr}\left(g^{-1}\,dg\right)^2 = \frac{d\rho^2 + dz\,d\bar{z}}{\rho^2} \tag{1.13}$$

with $\rho \in (0, \infty)$, $z \in \mathbb{C}$ and $\bar{z}$ is its complex conjugate.

Thermal AdS_3. The first excitation that we generate with this method is so-called thermal AdS. We wish to reduce the isometry group by modding out by $g \sim kgk^\dagger$ for some arbitrary $k \in SL_2(\mathbb{C})$. The matrix k itself is defined up to conjugation with $SL_2(\mathbb{C})$, so we may choose k to be diagonal without loss of generality. We will thus mod out by

$$g \sim kgk^\dagger \quad \text{with} \quad k = \begin{pmatrix} e^{i\pi\,\tau} & 0 \\ 0 & e^{-i\pi\,\tau} \end{pmatrix} \qquad \Leftrightarrow \qquad (z, \rho) \sim \left(e^{2\pi i\,\tau}\, z\,,\; e^{i\pi(\tau - \bar{\tau})}\,\rho\right) \tag{1.14}$$

Notice that the exponent in the ρ-identification reduces to $i\pi(\tau - \bar{\tau}) = -2\pi\,\mathrm{Im}(\tau)$. The quantity $\tau \in \mathbb{C}$ is known as a modular parameter, which is discussed more elaborately in Section 1.4. In order to see what happens physically when we mod out by (1.14), we must relate the coordinates $(z, \bar{z}, \rho)$ to the (t, φ, r) from the Euclidean-AdS metric (1.9). Let us define[iii]

$$z = e^{i(\varphi + it)} \tag{1.15}$$

Notice that this specific relation between $(z, \bar{z})$ and (t, φ) is not obviously justified; we will give the justification for it shortly. The relation (1.15) yields the following periodic identification on t and φ.

$$(t, \varphi) \sim (t + \beta\,,\; \varphi + \theta) \qquad \text{and} \qquad \varphi \sim \varphi + 2\pi \tag{1.16}$$

where we decomposed the complex modular parameter as $2\pi\tau = \theta + i\beta$. The parameter $\beta \in \mathbb{R}$ should be interpreted as the inverse temperature and $\theta \in \mathbb{R}$ is the so-called angular

[iii]We choose our units such that $\ell = 1$ for the moment. Factors of ℓ can be restored by dimensional analysis at any point.

potential. The second identification in (1.16) is the usual one for a polar angle and naturally follows from $z = e^{2\pi i} z$.

We will now check that (1.15) is indeed justified. Plugging (1.15) into the metric (1.13) gives us a metric in terms of t, φ, and ρ (not r).

$$ds^2 = \frac{d\rho^2 + e^{-2t}(dt^2 + d\varphi^2)}{\rho^2} \tag{1.17}$$

We want to work towards getting the metric (1.9), so we identify r^2 with whatever multiplies $d\varphi^2$, i.e. $r \equiv e^{-t}/\rho$. Notice that the above identifications $(t,\rho) \sim (t+\beta, e^{-\beta}\rho)$ does not affect the proper radial coordinate r. At this point, we have the following metric in terms of (t,φ,r).

$$ds^2 = \left(1+r^2\right) dt^2 + r^2 d\varphi^2 + \frac{dr^2}{r^2} + \frac{2}{r} dt\, dr \tag{1.18}$$

To get rid of the off-diagonal piece g_{tr}, one can shift the time coordinate $t \to t + f(r)$, where $f(r) = \frac{1}{2}\log(1+r^{-2})$ follows from setting g_{tr} to zero. Plugging this into the above metric gives us exactly what we were looking for, namely

$$ds^2 = \left(1+r^2\right) dt^2 + r^2 d\varphi^2 + \left(1+r^2\right)^{-1} dr^2 \tag{1.19}$$

Then, after restoring factors of ℓ, we indeed get the same metric as Euclidean global AdS (1.9)

$$ds^2 = \left(1+\frac{r^2}{\ell^2}\right) dt^2 + r^2 d\varphi^2 + \left(1+\frac{r^2}{\ell^2}\right)^{-1} dr^2$$

but with the crucial difference that now we have introduced an additional identification $(t,\varphi) \sim (t+\beta, \varphi+\theta)$ on top of just $\varphi \sim \varphi + 2\pi$. In conclusion, we have given AdS_3 a finite temperature and angular momentum by taking a quotient with respect to some arbitrary constant matrix $k \in SL_2(\mathbb{C})$.

The BTZ black hole. The BTZ black hole metric can be obtained by taking another set of coordinates in (1.12) with the *same* identifications (1.14), cf. [21, 22].

$$\begin{aligned} z &= \left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right)^{1/2} \exp\left[\frac{r_+ + r_-}{\ell}(\varphi' + it'/\ell)\right] \\ \rho &= \left(\frac{r_+^2 - r_-^2}{r^2 - r_-^2}\right)^{1/2} \exp\left[\frac{r_+\varphi' + r_- it'/\ell}{\ell}\right] \end{aligned} \tag{1.20}$$

The metric (1.13) in terms of these (t', φ', r) coordinates is the Euclidean BTZ metric

$$ds^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2\ell^2} dt'^2 + \frac{r^2\ell^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2\left(d\varphi' + \frac{r_+|r_-|}{r^2\ell} dt'\right)^2 \tag{1.21}$$

The outer horizon r_+ is real-valued, while the inner horizon r_- is imaginary, $r_- = i|r_-|$, which is a corollary of Wick-rotating to Euclidean signature. We mod out by some $k \in SL_2(\mathbb{C})$ as before, so that the periodicities in terms of t' and φ' become

$$(t', \varphi') \sim (t' + \beta' \,,\ \varphi' + \theta') \quad \text{and} \quad \varphi' \sim \varphi' + 2\pi \tag{1.22}$$

with

$$\theta' + i\beta' \;=\; 2\pi\tau' \;=\; \frac{2\pi i \ell}{r_+ + r_-} \tag{1.23}$$

The second identification, $\varphi' \sim \varphi' + 2\pi$, follows directly from noticing that $z = e^{2\pi i} z$.

These are the metrics that we will work with in this chapter. For a more complete treatment of this subject, see e.g. Carlip's book [18], Kraus' lecture notes [27], or the BTZ follow-up paper [21].

Comparing modular parameters. We saw that thermal AdS_3 and the BTZ black hole can be derived from the same Killing–Cartan metric on the coset $SL_2(\mathbb{C})/SU(2)$. We will now relate the modular parameters τ and τ', which respectively correspond to thermal-AdS and BTZ. In order to do this, we must find out what the periodicities are of the cylindrical coordinates (t, φ) and (t', φ'). It is useful to work in complex coordinates,

$$w \equiv \varphi + i\frac{t}{\ell} \qquad \text{and} \qquad w' \equiv \varphi' + i\frac{t'}{\ell} \tag{1.24}$$

in terms of which the periodic identifications are[iv]

$$w \;\sim\; w + 2\pi \;\sim\; w + 2\pi\tau \qquad \text{and} \qquad w' \;\sim\; w' + 2\pi \;\sim\; w' + 2\pi\tau' \tag{1.25}$$

We would like to see what the unprimed periodicities are in terms of the primed ones, so we need to relate w to w'. When we compare the two realizations of z in (1.15) and (1.20), we see that

$$w' = \frac{r_+ + r_-}{\ell} w + \frac{1}{2} \log\left(\frac{r^2 - r_+^2}{r^2 - r_-^2}\right) \tag{1.26}$$

Plugging the unprimed periodicities $w \sim w + 2\pi \sim w + 2\pi\tau$ into this relation gives

$$w' \;\sim\; w' + 2\pi\tau' \;\sim\; w' + 2\pi\tau'\tau \tag{1.27}$$

It follows from the second identification that the product of the two parameters must be $\tau'\tau = \pm 1$ in order to restore $w' \sim w' + 2\pi$. We thus find a relation between the two modular parameters.

$$\tau' = -\frac{1}{\tau} \tag{1.28}$$

where we have chosen the negative sign for later convenience. We will put this result to good use in Section (1.4).

[iv]Remember that $2\pi\tau = \theta + i\beta/\ell$ and $2\pi\tau' = \theta' + i\beta'/\ell$. Also, τ' was given explicitly in terms of the inner and outer horizon in (1.23).

1.2 Asymptotic Symmetries and Surface Charges

We will now formally derive the notion of asymptotic symmetries and surface charges. In this section, we will explain the following two statements. Firstly, in the context of general relativity, asymptotic symmetries are deformations of spacetime that act non-trivially at infinity. Secondly, surface charges are generators of asymptotic deformations and they form a projective representation of the algebra of asymptotic symmetries.[i]

In order to correctly explain these statements, we must first specify what 'spacetime at infinity' is. We can then define the notion of surface charges as generators of arbitrary deformations. After that, we can define asymptotic symmetries, because by that time we will know what 'acting non-trivially at infinity' means. Finally, we discuss how the surface charges of the asymptotic symmetries form a projective representation of the asymptotic symmetry algebra. Let us get started.

Asymptotic boundary conditions. Before we can talk about how deformations act on the asymptotic structure of a space, we need to pin down what we actually mean by 'asymptotic structure of a space'. We follow Henneaux & Teitelboim's definition of an asymptotic structure. In their paper [28], they give a natural definition of (four-dimensional) 'asymptotically anti-De Sitter space' by posing invariance under the isometry group of AdS_4, namely $O(2,3)$. They figured out what the most lenient boundary conditions are that close under $O(2,3)$ actions. We expect that this strategy can, in principle, be applied to other spaces as well by simply replacing the AdS_4 isometry group with the isometry group of the space in question.

The boundary conditions are basically restrictions on the allowed finite deformations of some background geometry. Let us denote the metric on the background geometry by $\bar{g}_{\mu\nu}$ and the deformation by $h_{\mu\nu}$, so that the deformed metric is $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. The boundary conditions are of the form

$$h_{\mu\nu} \sim O(r^n) \tag{1.29}$$

where r is some well-defined radial coordinate and n is some integer. By the above notation we actually mean that each component of the deviation has its own r-dependence, i.e. $h_{00} \sim O(r^{n_1})$, $h_{01} \sim O(r^{n_2})$, etc.

Surface charges. Consider an asymptotic deformation ζ^μ that acts on the metric as $g_{\mu\nu} \to g_{\mu\nu} + \mathscr{L}_\zeta g_{\mu\nu}$. Such a deformation can be generated by a corresponding charge denoted Q_ζ. In order to find an explicit expression for such a charge Q_ζ, we will work in the Hamiltonian formulation of general relativity (GR), which is introduced is considerable detail in Appendix D. Other, more sophisticated, covariant formulations have been

[i]Loosely speaking, 'asymptotic' is jargon for 'at infinity'. In the cases discussed in this thesis, the asymptotic region is located at spatial infinity.

developed by people such as Barnich, Brandt, and Compère, cf. [29–31].

We have seen in Appendix D that the GR Hamiltonian generically has the form

$$H = \int_{\Sigma_t} d^d y \sqrt{q} \left\{ N \hat{\mathcal{H}} + N^a \hat{\mathcal{H}}_a \right\} + \oint_{\partial\Sigma_t} d^{d-1}\theta \sqrt{\gamma} \left\{ N \hat{\mathcal{H}}^{\text{bndy}} + N^a \hat{\mathcal{H}}_a^{\text{bndy}} \right\} \tag{1.30}$$

The Hamiltonian generates time translations in the standard canonical formalism. The above Hamiltonian is a slight generalization of this idea, because it actually generates a flow along the flow vector $t^\mu = N n^\mu + N^a e_a^\mu$.[ii] This becomes more obvious when we go from the normal/tangent basis to the full-spacetime basis. We thus write

$$\hat{\mathcal{H}} = n^\mu \mathcal{H}_\mu \qquad \text{and} \qquad \hat{\mathcal{H}}_a = e_a^\mu \mathcal{H}_\mu \tag{1.31}$$

as the normal and tangent component of the same $(d+1)$-dimensional object $\mathcal{H}_\mu$. We do the same for the boundary quantities

$$\hat{\mathcal{H}}^{\text{bndy}} = n^\mu \mathcal{Q}_\mu \qquad \text{and} \qquad \hat{\mathcal{H}}_a^{\text{bndy}} = e_a^\mu \mathcal{Q}_\mu \tag{1.32}$$

One should keep in mind that the latter two quantities are not constraints, i.e. they do not necessarily vanish on shell. The reason for using the letter $\mathcal{Q}$ instead of something like $\mathcal{H}^{\text{bndy}}$ will become apparent shortly. Writing things on this basis gives the more intuitive

$$H = \int_{\Sigma_t} d^d y \sqrt{q}\, \mathcal{H}_\mu t^\mu + \oint_{\partial\Sigma_t} d^{d-1}\theta \sqrt{\gamma}\, \mathcal{Q}_\mu t^\mu \tag{1.33}$$

It now seems natural to define other generators by replacing the flow vector t^μ by a generic vector ζ^μ, thus defining the generator of $\mathscr{L}_\zeta$ (Lie transport) as

$$Q_\zeta = \int_{\Sigma_t} d^d y \sqrt{q}\, \mathcal{H}_\mu \zeta^\mu + \oint_{\partial\Sigma_t} d^{d-1}\theta \sqrt{\gamma}\, \mathcal{Q}_\mu \zeta^\mu \tag{1.34}$$

in such a way that $Q_t = H$. This charge Q_ζ depends on the fields and their canonically conjugate momenta. One is typically interested in charges of solutions to the field equations, for which the constraints vanish, $\mathcal{H}_\mu = 0$. The piece that remains is the one coming from the boundary.

$$Q_\zeta = \oint_{\partial\Sigma_t} d^{d-1}\theta \sqrt{\gamma}\, \mathcal{Q}_\mu \zeta^\mu \qquad \text{(on-shell)} \tag{1.35}$$

Because this on-shell charge contains only the surface piece, it is often called the 'surface charge' itself. The sole purpose of the quantity $\mathcal{Q}_\mu$ is to make sure that we have a well-defined variational principle from the Hamiltonian (1.33). More precisely, the $\mathcal{Q}_\mu$ is defined

[ii] Note that this notation for the tangent basis vector e_a^μ may be confusing; it is not a Vielbein. See (C.7) for the definitions of n^μ and e_a^μ.

in such a way that its variation exactly cancels the unwanted surface terms from the variation of the bulk constraints $\mathcal{H}_\mu$. In order to identify what comprises this $\mathcal{Q}_\mu$, however, we cannot blindly rely on the analysis in Appendix D, for there it is assumed that $\delta q_{ab} = 0$ at $\partial\Sigma_t$. We wish to be less restrictive and roughly allow for $\delta q_{ab} \sim h_{ab}$, which is dictated by the boundary conditions (1.29). This means that the surface terms in the Hamiltonian (D.20) will not cancel all surface terms coming from the variation of the bulk constraints $\mathcal{H}_\mu$ with respect to the metric. We thus need to vary the bulk piece and keep all surface terms that emerge.

The variation of the bulk piece of (1.34) can be computed relatively easily, which will be done explicitly in the next section. The generic expression for the bulk variation looks something like[iii]

$$\delta \int_{\Sigma_t} d^d y \sqrt{q}\, \mathcal{H}_\mu \zeta^\mu \;=\; \int_{\Sigma_t} d^d y \sqrt{q} \Big\{ (\,\dots\,)^{ab}\, \delta q_{ab} + (\,\dots\,)_{ab}\, \delta p^{ab} \Big\} \;+\; \oint_{\partial\Sigma_t} d^{d-1}\theta \sqrt{\gamma}\, \{\,\dots\,\} \tag{1.36}$$

The surface term $\mathcal{Q}_\mu$ is then defined though its variation, which must precisely cancel the above surface term, i.e.

$$\delta \int_{\Sigma_t} d^d y \sqrt{q}\, \mathcal{Q}_\mu \zeta^\mu \;=\; - \oint_{\partial\Sigma_t} d^{d-1}\theta \sqrt{\gamma}\, \{\,\dots\,\} \tag{1.37}$$

Using the on-shell expression (1.35), we conclude that

$$\delta Q_\zeta = - \oint_{\partial\Sigma_t} d^{d-1}\theta \sqrt{\gamma}\, \{\,\dots\,\} \qquad\qquad \text{(on-shell)} \tag{1.38}$$

In order to find Q_ζ itself, we need to (functionally) integrate this relation somehow. This is where the story get interesting. Notice first of all that, because of this integration, there must be a constant of integration sitting in Q_ζ. The appearance of this constant of integration gives rise to the a possible realization of a central term in the algebra of the Q_ζ's. This is great, but it comes at a price, because integrating (1.38) is not an easy task. In fact, there is no general way of doing the integration for arbitrary spacetimes. All the known cases tackle this problem by linearizing in terms of the deviation $h_{\mu\nu}$ from some background $\bar{g}_{\mu\nu}$.

Asymptotic symmetry group. A deformation ζ^μ whose surface charge vanishes is called a *trivial* deformation and 'acts trivially at infinity'. An asymptotic symmetry is defined to be a *non-trivial* deformation that respects the boundary conditions (1.29). The asymptotic symmetries form an algebra, which is often loosely called the 'asymptotic symmetry group'.

[iii] Remember from Appendix C and D that q_{ab} is the induced metric on the hypersurface Σ_t and p^{ab} is its conjugate momentum.

The asymptotic symmetry group of a space $\mathcal{M}$ will be denoted $ASG(\mathcal{M})$. In conclusion, we have

$$ASG(\mathcal{M}) \;:=\; \left\{ \, \xi^\mu \;\middle|\; \mathscr{L}_\xi \text{ respects b.c.} \quad \textbf{and} \quad Q_\xi \neq 0 \, \right\} \tag{1.39}$$

which clearly depends heavily on the asymptotic boundary conditions (1.29). The asymptotic symmetry group contains the isometry group as a subgroup. Notice that the adjective 'asymptotic' in all of this has to do with the fact that the on-shell charge only contains a surface term $\sim\partial\Sigma_t$ at spatial infinity, see Figure D.1.

Surface-charge algebra. The integrability issue mentioned above also affects the validity of the surface charges' being proper representations of the asymptotic symmetry group. For the case of asymptotically AdS_3 spaces, Brown & Henneaux [17] showed that the surface charges indeed form a proper representation (under appropriate boundary conditions). In fact, they showed that it is a *projective* representation of the asymptotic symmetry algebra $\{\zeta\}$; it allows for a central extension. In other words, their Poisson brackets (or rather Dirac brackets) are[iv]

$$\{\, Q_\zeta[g] \,,\, Q_\eta[g] \,\} = Q_{[\zeta,\eta]}[g] + C_{\zeta\eta} \tag{1.40}$$

where $C_{\zeta\eta}$ is a central charge and the surface-charge generators are defined such that

$$Q_\zeta[\bar{g}] = 0 \quad \text{for any} \quad \zeta^\mu \qquad\qquad Q_0[g] = 0 \quad \text{for any} \quad g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{1.41}$$

We find the Brown–Henneaux central charges by evaluating (1.40) on the background, i.e. $h_{\mu\nu} = 0$.

$$C_{\zeta\eta} \;=\; Q_\zeta[\mathscr{L}_\eta \bar{g}] \;=\; -Q_\eta[\mathscr{L}_\zeta \bar{g}] \tag{1.42}$$

where we used the canonical relation $\{Q_\zeta, Q_\eta\} = \delta_\eta Q_\zeta$ and we also used the fact that the variation of the metric is a Lie derivative, $\delta_\eta \bar{g}_{\mu\nu} = \mathscr{L}_\eta \bar{g}_{\mu\nu}$.[v]

This introduction of surface charges has been rather formal thus far. In the next section we will move straight on to the best known example of this, for which we will explicitly calculate an algebra of surface charges. There is an omission that must be mentioned. We will not prove the fact that the Poisson bracket algebra of surface charges is a proper representation of the asymptotic symmetry group, see [17].

[iv]We mentioned above that the surface charges functionally depend on the geometry $g_{\mu\nu}$, which is now explicitly indicated by $Q_\zeta[g]$.

[v]The variation of the other asymptotic transformation, $\delta_\eta\zeta = [\zeta,\eta]$, leads to $Q_{[\zeta,\eta]}[\bar{g}]$, which vanishes by (1.41). Other contributions are subleading in $1/r$, which vanish as the limit $r \to \infty$ is taken.

1.3 Brown–Henneaux 1986

In this section we review Brown & Henneaux's seminal paper [16] in considerable detail. The main idea of [16] is to study the asymptotic symmetry group of AdS_3 and to find the surface charge representation explicitly.

We have seen in Section 1.1 that the isometry group of global AdS_3 is $SL_2(\mathbb{R})_{\rm L} \times SL_2(\mathbb{R})_{\rm R}$. The asymptotic symmetry group turns out to be an infinite-dimensional extension of the corresponding isometry algebra $sl_2(\mathbb{R})_{\rm L} \times sl_2(\mathbb{R})_{\rm R}$. Namely, the asymptotic symmetries comprise the conformal algebra in two dimensions, i.e. Virasoro with no central charge. We will see that the surface-charge representation of the asymptotic symmetry group will turn out to be projective with central charge $c = 3\ell/2G$.

Surface charges for pure Einstein gravity on AdS_3. Let us work in the (t, φ, r) coordinates from the global-AdS metric (1.6), which we write here again for convenience.

$$\bar{g}_{\mu\nu}\, dx^\mu dx^\nu = -\left(1 + \frac{r^2}{\ell^2}\right) dt^2 + r^2 d\varphi^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 \tag{1.43}$$

This is the background on which we will define our surface charges. The boundary conditions (1.29) for asymptotically AdS_3 spaces in these coordinates are

$$(h_{\mu\nu}) = \begin{pmatrix} h_{tt} & h_{t\varphi} & h_{tr} \\ & h_{\varphi\varphi} & h_{tr} \\ & & h_{rr} \end{pmatrix} = \begin{pmatrix} O(1) & O(1) & O(r^{-3}) \\ & O(1) & O(r^{-3}) \\ & & O(r^{-4}) \end{pmatrix} \tag{1.44}$$

Geometries are said to be (locally) asymptotically AdS_3 when they respect these boundary conditions. It is more convenient to go back to the normal/tangent basis, cf. (1.31), when doing actual calculations in the Hamiltonian formalism. The bulk term from (1.34) for pure gravity is

$$\mathcal{H}_\mu\, \zeta^\mu \;=\; \hat{\mathcal{H}}\, \hat{\zeta} \;+\; \hat{\mathcal{H}}_a\, \hat{\zeta}^a \tag{1.45}$$

where we have written the deformation on the normal/tangent basis as well, $\zeta^\mu = \hat{\zeta}\, n^\mu + \hat{\zeta}^a\, e^\mu_a$. The GR Hamiltonian is given in (D.20), in which the Hamilton and momentum constraints are found to be given by

$$\hat{\mathcal{H}} = -\frac{R - 2\Lambda}{2\kappa} + 2\kappa\left(p_{ab}p^{ab} - \frac{1}{d-1}p^2\right) \qquad\qquad \hat{\mathcal{H}}_a = -2\nabla^b p_{ab} \tag{1.46}$$

The variation of the bulk piece of the surface charge (1.34), $\delta \int d^d y \sqrt{q}\, \mathcal{H}_\mu \zeta^\mu$, can be straightforwardly computed and is given by

$$\int_{\Sigma_t} d^d y \sqrt{q} \Big\{ (\;\dots\;)^{ab}\, \delta q_{ab} + (\;\dots\;)_{ab}\, \delta p^{ab} \Big\}$$
$$- \oint_{\partial\Sigma_t} d\sigma_c \left\{ \frac{1}{2\kappa} G^{abcd} \left(\hat{\zeta}\, \nabla_d \delta q_{ab} - \nabla_d \hat{\zeta}\, \delta q_{ab} \right) + \left(2\hat{\zeta}^a p^{bc} - \hat{\zeta}^c p^{ab} \right) \delta q_{ab} + 2\hat{\zeta}_a\, \delta p^{ac} \right\} \tag{1.47}$$

where the surface element is $d\sigma_a = d^{d-1}\theta \sqrt{\gamma}\, r_a$ and we introduced $G_{abcd} \equiv q_{c(a} q_{b)d} - q_{ab} q_{cd}$. The surface-charge density $\mathcal{Q}_\mu$ is then defined in such a way that the variation of the surface piece in (1.34) precisely cancels the surface term that emerges from varying the bulk piece, so that

$$\delta Q_\zeta = \oint_{\partial\Sigma_t} d\sigma_a \left\{ \frac{1}{2\kappa} G^{abcd} \left(\hat{\zeta}\, \nabla_b \delta q_{cd} - \nabla_b \hat{\zeta}\, \delta q_{cd} \right) + \left(2\hat{\zeta}^b p^{ac} - \hat{\zeta}^a p^{bc} \right) \delta q_{bc} + 2\hat{\zeta}_b\, \delta p^{ab} \right\} \tag{1.48}$$

As we mentioned on page 18, the trick is now to integrate this equality. Using the boundary conditions (1.44), Henneaux & Teitelboim [28] did this integration and obtained

$$Q_\zeta[g] = \oint_{\partial\Sigma_t(\infty)} d\sigma_a \left\{ \frac{1}{2\kappa} \bar{G}^{abcd} \left[\bar{\nabla}_b h_{cd} - h_{cd} \bar{\nabla}_b \right] \hat{\zeta} + 2p^{ab}\, \hat{\zeta}_b \right\} \; + \; O(h^2) \tag{1.49}$$

for any $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$ that respects (1.44). By $\partial\Sigma_t(\infty)$ we mean the $r \to \infty$ limit of $\partial\Sigma_t$. The barred quantities depend on the background metric $\bar{g}_{\mu\nu}$ from (1.43). The momentum p^{ab} is the canonical conjugate of the induced metric $q_{ab} = e^\mu_a e^\nu_b\, g_{\mu\nu}$, where $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ is the deformed metric.[i] In other words, p^{ab} contains the deformation.

Interestingly, the above charge Q_ζ contains the charge defined by Brown & York in [32].

$$Q^{\text{BY}}_\zeta \;\equiv\; \oint_{\partial\Sigma_t} d\sigma_a\, \tau^{ab}\, \hat{\zeta}_b \;=\; \oint_{\partial\Sigma_t} d\sigma_a\, 2p^{ab}\, \hat{\zeta}_b \tag{1.50}$$

where we use the definition of the so-called Brown–York stress tensor

$$\tau^{ab} \;\equiv\; \frac{2}{\sqrt{q}} \frac{\delta S_{\text{cl}}}{\delta q_{ab}} \bigg|_{\partial\Sigma_t} \;=\; 2p^{ab} \Big|_{\partial\Sigma_t} \tag{1.51}$$

The Hamilton–Jacobi canonical formalism is used to relate the derivative with respect to the surface metric to the canonical momentum, see also Appendix B and Section 2.1. Thus, the charge (1.49) is more general than the Brown–York charge, as it does not only generate deformations along the leaf $\sim\hat{\zeta}^a$ but also in the direction of the temporal flow $\sim\hat{\zeta}$.

[i] Note that the notation is a little awkward, because we use the letter h for the deviation $h_{\mu\nu}$ as well as its pull-back onto Σ_t, i.e. $h_{ab} = e^\mu_a e^\nu_b\, h_{\mu\nu}$. Again, see (C.7) for the definition of e^μ_a.

The asymptotic symmetry group of AdS_3. Solving the Killing equations $\mathscr{L}_\xi g_{\mu\nu} = 0$ asymptotically gives the general form of the asymptotic symmetries $\xi = \xi^\mu \partial_\mu$.

$$\begin{aligned} \xi^t &= \ell\left(T+\bar{T}\right) + \frac{\ell^3}{2r^2}\left(\partial^2 T + \bar{\partial}^2\bar{T}\right) + O(r^{-4}) \\ \xi^\varphi &= \ \left(T-\bar{T}\right) + \frac{\ell^2}{2r^2}\left(\partial^2 T - \bar{\partial}^2\bar{T}\right) + O(r^{-4}) \\ \xi^r &= -r\left(\partial T + \bar{\partial}\bar{T}\right) + O(r^{-1}) \end{aligned} \tag{1.52}$$

where we introduced the following notation.

$$\begin{aligned} T &= T(t/\ell + \varphi) & \partial &\equiv \frac{1}{2}\left(\ell\partial_t + \partial_\varphi\right) \\ \bar{T} &= \bar{T}(t/\ell - \varphi) & \bar{\partial} &\equiv \frac{1}{2}\left(\ell\partial_t - \partial_\varphi\right) \end{aligned} \tag{1.53}$$

The asymptotic Killing vector $\xi^\mu\partial_\mu$ can be decomposed into a T-dependent term and a $\bar{T}$-dependent one, i.e.

$$\xi = \lambda[T] + \bar{\lambda}[\bar{T}] \ + \ (\text{subleading in } \tfrac{1}{r}) \tag{1.54}$$

where

$$\begin{aligned} \lambda[T] &= \left(2T + \frac{\ell^2}{r^2}\partial^2 T\right)\partial - \partial T\, r\partial_r \\ \bar{\lambda}[\bar{T}] &= \left(2\bar{T} + \frac{\ell^2}{r^2}\bar{\partial}^2 \bar{T}\right)\bar{\partial} - \bar{\partial}\bar{T}\, r\partial_r \end{aligned} \tag{1.55}$$

It is convenient to write out the deformations on a Fourier basis. We define, for all $n \in \mathbb{Z}$,

$$\lambda_n \equiv \lambda[e^{in(t/\ell+\varphi)}] \qquad\qquad \bar{\lambda}_n \equiv \bar{\lambda}[e^{in(t/\ell-\varphi)}] \tag{1.56}$$

which must span the asymptotic symmetry algebra. These modes λ_n and $\bar{\lambda}_n$ obey the conformal algebra

$$[\lambda_m, \lambda_n] = i(m-n)\lambda_{m+n} \qquad [\bar{\lambda}_m, \bar{\lambda}_n] = i(m-n)\bar{\lambda}_{m+n} \qquad [\lambda_m, \bar{\lambda}_n] = 0 \tag{1.57}$$

The algebra of *global* symmetries of AdS_3 consists of two copies of the Möbius algebra $sl_2(\mathbb{R}) = \{\lambda_{-1}, \lambda_0, \lambda_1\}$. Thus, the asymptotic symmetry algebra that corresponds to the boundary conditions (1.44) is an infinite-dimensional extension of the isometry algebra. Let us denote the surface charge that generates λ_n ($\bar{\lambda}_n$) at infinity by L_n ($\bar{L}_n$), i.e.

$$L_n \equiv Q_{\lambda_n} \qquad\qquad \bar{L}_n \equiv Q_{\bar{\lambda}_n} \tag{1.58}$$

From [17], we know that the algebra of surface charges is isomorphic to the algebra of asymptotic deformations up to a central extension, so the central charge is the only thing that remains to be computed.[ii]

[ii] We will not prove this statement here, cf. [17].

Brown–Henneaux central charge of AdS_3. The central charge is computed via (1.42). We find that

$$
\begin{aligned}
C_{\lambda_m \lambda_n} &= \frac{\ell}{8G} m(m^2-1)\delta_{m+n,0} \\
C_{\bar{\lambda}_m \bar{\lambda}_n} &= \frac{\ell}{8G} m(m^2-1)\delta_{m+n,0} \\
C_{\lambda_m \bar{\lambda}_n} &= 0
\end{aligned}
\tag{1.59}
$$

which means that we now end up with the full Virasoro algebra

$$
\begin{aligned}
\{L_m, L_n\} &= (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \\
\{\bar{L}_m, \bar{L}_n\} &= (m-n)\bar{L}_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \\
\{L_m, \bar{L}_n\} &= 0
\end{aligned}
\tag{1.60}
$$

where the central charge is

$$
c = \frac{3\ell}{2G}
\tag{1.61}
$$

This is the famous result by Brown and Henneaux. We have effectively gone from a purely classical theory in three dimensional (curved) spacetime to a quantum mechanical description of a CFT in two (flat) dimensions. This is closely connected to the AdS/CFT correspondence [1–3].

Some calibration. All charges were defined with respect to global AdS. In order to make the conventions fit the analysis in the next section, we shift L_0 by $c/24$, i.e. we redefine

$$
L_0 \;\equiv\; L_0^{\text{Brown–Henneaux}} - \frac{c}{24}
\tag{1.62}
$$

This means in particular that $L_0 = -c/24$ on global AdS.

1.4 Strominger's Application of Cardy's Formula

In this section we will go over Strominger's paper [26], in which it is shown that the Bekenstein–Hawking entropy of a three-dimensional black hole is in precise agreement with the entropy that is obtained by basic CFT reasoning.

Bekenstein–Hawking entropy of the BTZ black hole. The metric of a BTZ black hole is given by [i]

$$ds^2_{\text{BTZ}} = -\frac{(r^2-r_+^2)(r^2-r_-^2)}{r^2\ell^2}dt^2 + \frac{r^2\ell^2}{(r^2-r_+^2)(r^2-r_-^2)}dr^2 + r^2\left(d\varphi + \frac{r_+r_-}{r^2\ell}dt\right)^2 \tag{1.63}$$

where r_+ and r_- are related to the mass M and the angular momentum J of the black hole via

$$\frac{r_+ - r_-}{\sqrt{8\ell G}} = \sqrt{\ell M + J} \qquad\qquad \frac{r_+ + r_-}{\sqrt{8\ell G}} = \sqrt{\ell M - J} \tag{1.64}$$

We have chosen $r_- \leq r_+$ without loss of generality. In order to find the Bekenstein–Hawking entropy, we need to find the radius at which the horizon is located, which we get by setting the lapse $N = 1/\sqrt{-g^{tt}}$ to zero and solving for r.[ii] The lapse vanishes at $r = r_\pm$ and since $r_- \leq r_+$, we find that the horizon is located at $r = r_+$. The Bekenstein–Hawking entropy [23, 24] is then

$$S \;=\; \frac{2\pi r_+}{4G} \;=\; 2\pi\sqrt{\frac{\ell}{8G}(\ell M + J)} \;+\; 2\pi\sqrt{\frac{\ell}{8G}(\ell M - J)} \tag{1.65}$$

CFT ground state energies. The Virasoro zero-modes (1.58) of the BTZ black hole are

$$L_0[g_{\text{BTZ}}] = \frac{1}{2}(\ell M + J) \qquad \text{and} \qquad \bar{L}_0[g_{\text{BTZ}}] = \frac{1}{2}(\ell M - J) \tag{1.66}$$

There is an important distinction to be made between a BTZ black hole with $M = J = 0$ and global AdS_3, which can be viewed as a BTZ black hole with $J = 0$ and $M = -1/8G$. We see that[iii]

$$ds^2_{\text{BTZ}} = -\frac{r^2}{\ell^2}dt^2 + \frac{\ell^2}{r^2}dr^2 + r^2 d\varphi^2 \qquad \text{(for } J=0 \text{ and } M=0\text{)}, \tag{1.67a}$$

$$ds^2_{\text{BTZ}} = -\left(1+\frac{r^2}{\ell^2}\right)dt^2 + \left(1+\frac{r^2}{\ell^2}\right)^{-1}dr^2 + r^2 d\varphi^2 \quad \text{(for } J=0 \text{ and } M=-\frac{1}{8G}\text{)} \tag{1.67b}$$

[i] See (1.21) for the Euclidean version. The inner and outer radii r_- and r_+ are both real-valued in Lorentzian signature.

[ii] Note that, just like for a Kerr black hole, setting $g_{tt} = 0$ would yield the ergosphere radius instead of the proper horizon radius.

[iii] Note that for $M = J = 0$ we have $r_+ = r_- = 0$, while for $J = 0$ and $M = -1/8G$ we have $r_- = 0$ and $r_+ = \sqrt{-\ell^2}$

The first one looks like the AdS_3 metric on the Poincaré patch, apart from the fact that φ is not periodically identified in proper Poincaré coordinates. The second is AdS_3 in global coordinates (1.6). These different "ground states" have a nice interpretation on the CFT side. Namely, in a theory with superconformal symmetry it is known that the Ramond ground state and the Neveu–Schwarz ground state have different L_0 and $\bar{L}_0$ eigenvalues. In our convention, the Ramond ground state has vanishing L_0 and $\bar{L}_0$ eigenvalues, which corresponds to the $M = 0$ BTZ black hole, i.e. Poincaré-AdS. In constrast, the NS ground state has $L_0 |0_{\text{NS}}\rangle = \bar{L}_0 |0_{\text{NS}}\rangle = -\frac{c}{24} |0_{\text{NS}}\rangle$, which corresponds to a BTZ black hole with $\ell M = -c/12 = -\ell/8G$, i.e. global AdS. We used Brown & Henneaux's result $c = 3\ell/2G$ from the previous section. In short, we should remember the following slogan for non-rotating BTZ black holes.

$$\begin{aligned} \text{R vacuum} &\sim \text{Poincaré-}AdS_3 \qquad (M = 0) \\ \text{NS vacuum} &\sim \text{global } AdS_3 \qquad (M = -\frac{1}{8G}) \end{aligned} \tag{1.68}$$

Modular invariance. The conformal boundary of asymptotically AdS_3 geometries is a cylinder. We can define complex coordinates on this cylinder analogous to the barred and unbarred notation that we introduced in (1.53); they are related by means of a Wick rotation $t \to it$.

$$\begin{aligned} w &= \varphi + i\frac{t}{\ell} \qquad & \partial &= \frac{\partial}{\partial w} = \frac{1}{2}(\partial_\varphi - i\ell\partial_t) \\ \bar{w} &= \varphi - i\frac{t}{\ell} \qquad & \bar{\partial} &= \frac{\partial}{\partial \bar{w}} = \frac{1}{2}(\partial_\varphi + i\ell\partial_t) \end{aligned} \tag{1.69}$$

Since φ is periodic, we should identify $w \sim w + 2\pi$. We give the system a finite temperature by making the imaginary-time direction periodic as well, so that the identifications are

$$w \sim w + 2\pi \sim w + 2\pi\tau, \qquad \text{with} \quad \text{Re}(\tau) = \frac{\theta}{2\pi} \quad \text{and} \quad \text{Im}(\tau) = \frac{\beta}{2\pi\ell} \tag{1.70}$$

where $\beta = 1/T$ is to be interpreted as the inverse temperature (the factors of 2π are conventional). The quantity θ is the canonical conjugate of the angular momentum J, just as β is H's conjugate; θ is sometimes called the *angular potential*, in analogy with a chemical potential. The complex number τ is called the *modular parameter*. There are certain transformations that change the modular parameter in such a way that the shape of the torus remains unchanged. These are known as *modular transformations* and they form a group that is isomorphic to $SL_2(\mathbb{Z})/\mathbb{Z}_2$ (we will not prove this statement here). An example of modular transformations are so-called Dehn twists depicted in Figure 1.1. The two Dehn twists $\mathcal{T} : \tau \mapsto \tau + 1$ and $\mathcal{U} : \tau \mapsto \tau/(\tau + 1)$ can be combined to generate all modular transformations. However, a more convenient choice of independent generators that is usually chosen is

$$\mathcal{T} : \tau \mapsto \tau + 1 \qquad \text{and} \qquad \mathcal{S} : \tau \mapsto -\frac{1}{\tau} \tag{1.71}$$

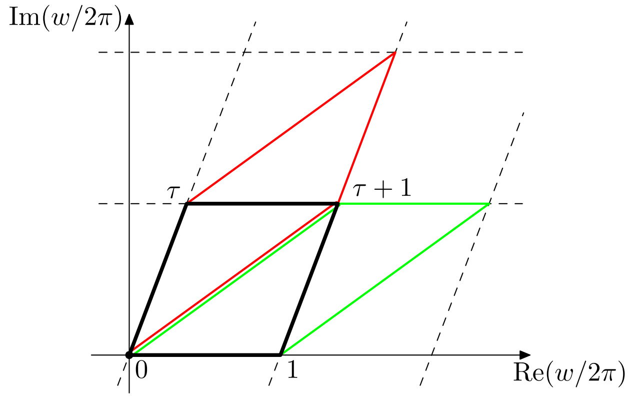


Figure 1.1: The shape of the torus is invariant under Dehn twists in both directions. The green (light grey) twist is generated by $\mathcal{T}$, whereas the red (dark grey) one is generated by $\mathcal{U}$.

where $\mathcal{S} = \mathcal{T}^{-1}\mathcal{U}\mathcal{T}^{-1}$. The transformation $\mathcal{S}$ can roughly be interpreted as interchanging the roles of the spatial and temporal coordinates. The invariance under $\mathcal{S}$ is crucial in deriving Cardy's formula. We will not go into full detail on the subject of modular transformations, because the subject is simply too vast to do it justice as a subsection. For a more thorough treatment of modular transformations and their relation to studying conformal field theories, see e.g. [33–35].

We will proceed with the following strategy. We compute the low-temperature NS partition function, which we can relate to the high-temperature regime via $\mathcal{S}$ (this is Cardy's key insight). We can then compute the entropy of the high-temperature CFT in a saddle-point approximation, which we finally compare with the Bekenstein–Hawking entropy (1.65).

CFT partition function at *low* temperature. The NS partition function Z is generically given by

$$Z[\tau,\bar{\tau}] \;=\; \mathrm{tr}\, e^{2\pi i\,(\tau L_0 - \bar{\tau}\bar{L}_0)} \;=\; \langle 0|\, e^{2\pi i\,(\tau L_0 - \bar{\tau}\bar{L}_0)}\, |0\rangle \;+\; (\text{sum over excited states}) \tag{1.72}$$

At low temperatures, $T = 1/\mathrm{Im}(\tau) \ll 1$, the above trace is well approximated by

$$Z_{\text{low-}T}[\tau,\bar{\tau}] = e^{2\pi i\,\frac{c}{24}\,(-\tau+\bar{\tau})} \;+\; O(e^{-\mathrm{Im}(\tau)}) \tag{1.73}$$

where we used the fact that the eigenvalue of L_0 and $\bar{L}_0$ in the NS vacuum is $-c/24$, cf. (1.62). The minus sign on $-c/24$ is responsible for making the leading-order piece of order $e^{+\mathrm{Im}(\tau)}$.

CFT partition function at *high* temperature. Let us denote the high-temperature L_0 ($\bar{L}_0$) eigenvalue by ℓ_0 ($\bar{\ell}_0$). At high temperature, we can do a saddle-point approximation, such that the leading-order behavior is given by[iv]

$$\log Z_{\text{high-}T}[\tau,\bar\tau] \simeq S(\ell_0,\bar\ell_0) + 2\pi i\left(\tau\ell_0 - \bar\tau\bar\ell_0\right) \tag{1.74}$$

where ℓ_0 and $\bar\ell_0$ are functions of τ and $\bar\tau$, respectively, that extremize the right-hand side. This is an example of a Legendre transformation, which in this case replaces dependence on ℓ_0 ($\bar\ell_0$) by dependence on τ ($\bar\tau$).

Entropy. In order to get an expression for the entropy, one can do the inverse Legendre transformation

$$S(\ell_0,\bar\ell_0) \;\simeq\; \log Z_{\text{high-}T}[\tau,\bar\tau] \;-\; 2\pi i\left(\tau\ell_0 - \bar\tau\bar\ell_0\right) \tag{1.75}$$

In this case, τ and $\bar\tau$ are functions of ℓ_0 and $\bar\ell_0$ that extremize the right-hand side.

Like we mentioned above, the derivation of Cardy's formula hinges on the fact that we have modular invariance. This means explicitly that we can relate the above general expression for the high-T partition function to the one that is obtained from the modularly transformed low-T partition function (1.73).

$$\log Z_{\text{high-}T}[\tau,\bar\tau] \;\simeq\; \log Z_{\text{low-}T}[-\tfrac{1}{\tau},-\tfrac{1}{\bar\tau}] \;=\; 2\pi i\,\frac{c}{24}\left(\frac{1}{\tau}-\frac{1}{\bar\tau}\right) \tag{1.76}$$

When we plug this into (1.75) we get

$$S(\ell_0,\bar\ell_0) \;\simeq\; 2\pi i\,\frac{c}{24}\left(\frac{1}{\tau}-\frac{1}{\bar\tau}\right) \;-\; 2\pi i\left(\tau\ell_0 - \bar\tau\bar\ell_0\right) \tag{1.77}$$

Cardy's formula. The last step in deriving Cardy's formula is to extremize (1.77) with respect to τ and $\bar\tau$ and then to plug the result back into (1.77). We find the extremal values for τ and $\bar\tau$.

$$\tau(\ell_0) = i\sqrt{\frac{c}{24\ell_0}} \qquad\qquad \bar\tau(\bar\ell_0) = -i\sqrt{\frac{c}{24\bar\ell_0}} \tag{1.78}$$

We have chosen the signs for the roots such that the temperature $1/\mathrm{Im}(\tau)$ is positive. When we plug these back into (1.77) we finally arrive at *Cardy's formula.*

$$S \;\simeq\; 2\pi\sqrt{\frac{c}{6}\ell_0} \;+\; 2\pi\sqrt{\frac{c}{6}\bar\ell_0} \tag{1.79}$$

[iv] To see how the entropy S enters the story, it is nice to think of this trace in the following classical setting. When one diagonalizes the Hamiltonian H, the trace is just a sum over energy-eigenvalues ε. The phase-space degeneracy is accounted for by the density of states $\rho(\varepsilon) = e^{S(\varepsilon)}$, i.e. $\rho(\varepsilon)$ counts the number of states with energy ε, which is exactly the logarithm of the entropy $S(\varepsilon)$. Thus, the entropy enters the exponent through $\operatorname{tr} e^{-\beta H} = \int d\varepsilon\, e^{S-\beta\varepsilon}$.

This formula, which was first found by Cardy [25], gives the entropy of a CFT in the *high-temperature* regime. The key in obtaining the Cardy formula is the fact that we have modular invariance, which tells us in particular that we can relate a torus with modular parameter τ to one that has a modular parameter $-1/\tau$, thus relating a theory at low temperature $\mathrm{Im}(\tau) \sim \infty$ to one at high temperature $\mathrm{Im}(\tau) \sim 0$.

Strominger's application of Cardy's formula. In Section 1.1 we saw that thermal AdS_3 and the BTZ black hole can be derived from the same Killing–Cartan metric on the coset $SL_2(\mathbb{C})/SU(2)$, cf. (1.12). Moreover, we saw in (1.28) that the modular parameters of their (conformal) boundaries are related by $\mathcal{S} : \tau \mapsto -1/\tau$, which is precisely the same relation that we used in the above derivation of Cardy's formula! All we need to do now is simply plug in the Brown–Henneaux central charge $c = 3G/2\ell$ and the L_0 and $\bar{L}_0$ eigenvalues of BTZ from (1.66), $\ell_0 = \ell M + J$ and $\bar{\ell}_0 = \ell M - J$. The entropy from Cardy's formula (1.79) is

$$S \;=\; 2\pi\sqrt{\frac{\ell}{8G}(\ell M + J)} \;+\; 2\pi\sqrt{\frac{\ell}{8G}(\ell M - J)} \tag{1.80}$$

which is exactly the same as the Bekenstein–Hawking entropy from (1.65).

In conclusion, we find precise agreement between the entropy calculated from the conformal field theory side and from the gravity side.

$$\boxed{S_{\text{Cardy}} \;=\; S_{\text{Bekenstein–Hawking}}} \tag{1.81}$$

This result is striking mainly because of the vastly different way the separate sides are obtained. It hints at a full duality between quantum gravity in asymptotically AdS_3 spaces and (super)conformal field theories defined on its conformal boundary, although such a complete duality at the level of partition functions is not (yet) proved, see e.g. [36].

Chapter 2

Holographic Renormalization in Hamilton–Jacobi Formalism

Holographic renormalization. In this chapter we will review various aspects of holographic renormalization. As the title of the chapter suggests, we will work in the Hamilton–Jacobi canonical formalism. As we discussed in the Introduction on page 7, the idea of holographic renormalization is basically to compute correlators in the strongly-coupled field theory on the boundary and remove its UV divergences by removing the corresponding IR (large-r) divergences on the gravity side of the correspondence.

One can move away from the conformally-invariant fixed point by turning on finite (relevant or marginal) perturbations on the boundary. According to the holographic dictionary, such perturbations are sourced by normalizable modes of the classical fields in the bulk. In practice, one adds matter fields like scalars, vectors etc. (depending on the desired perturbation) to the gravitational action. The boundary is then put at a finite radius $r = r_0$, so that the divergences are exposed in terms of the regulator r_0. The divergences are subtracted by appropriate counterterms, so that finally the limit $r_0 \to \infty$ can be taken.

Structure of this chapter. We start off with a brief review of the application of the Hamilton–Jacobi formalism to GR in Section 2.1. This includes a nice side track that closely resonates with the definition of the ADM mass at the end of Appendix D, which is based on a seminal paper by Brown & York [32].

In Section 2.2, we review De Boer, Verlinde, and Verlinde's Hamilton–Jacobi formulation of holographic renormalization [15], which includes a method for expanding the Hamilton constraint. As a nice warp-up example for the real stuff in Section 3.2, we calculate Brown & Henneaux's AdS_3 central charge $c = 3G/2\ell$ with the freshly acquired techniques.

There is a close connection between the radial coordinate r and the energy scale μ in the field theory on the (cut-off) boundary. In this sense, we expect that the radial flow in the bulk translates to an RG flow in the space of couplings on the boundary. This will be the topic of Section 2.3.

2.1 Hamilton–Jacobi Formulation of Gravity

We will now introduce the Hamilton–Jacobi canonical formalism applied to gravity. We have included the Hamilton–Jacobi (HJ) formalism for a point-particle in Appendix B to refresh (or introduce) the reader's knowlegde on the subject.

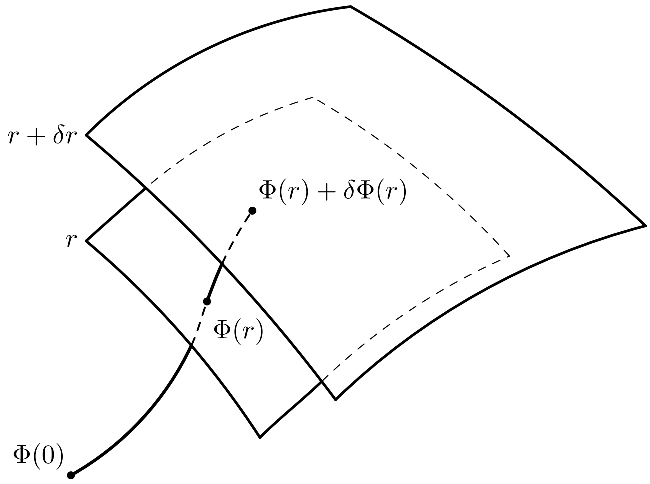


Figure 2.1: In this figure, Φ denotes the collection of fields like the metric tensor, form fields etc. The (active) variations of the fields $\Phi(r) \to \Phi(r) + \delta\Phi(r)$ are induced by a (passive) variation of the foliation parameter $r \to r + \delta r$.

Hamilton–Jacobi equation for gravity (temporal foliation). The HJ formalism appears to be tailor-made for describing bulk dynamics in terms of data on the boundary (and vice versa). A schematic view of the setup is shown in Figure 2.1. The Hamilton–Jacobi equations of motion for a classical point-particle are

$$H = -\frac{\delta S_{\text{cl}}}{\delta t} \tag{2.1a}$$

$$p_a = \frac{\delta S_{\text{cl}}}{\delta q^a} \tag{2.1b}$$

All the above quantities are evaluated at the varying endpoint t, cf. Appendix B. In general relativity, the above HJ equations (2.1) are generalized to

$$H = 0 \tag{2.2a}$$

$$p^{ab} = \frac{1}{\sqrt{q}} \frac{\delta S_{\text{cl}}}{\delta q_{ab}} \tag{2.2b}$$

which is depicted in Figure 2.2. The generalization of the second HJ equation (2.2b) is rather obvious and taken from Brown & York's paper [32]. In generalizing the first HJ equation, one might have expected something like $H = -\delta S_{\text{cl}}/\delta t$. The foliation parameter t, however, does not have an absolute meaning because of general covariance (as opposed to the classical case). This means that $\delta S_{\text{cl}}/\delta t$ must vanish for generally covariant theories and we are left with the constraint $H = 0$.

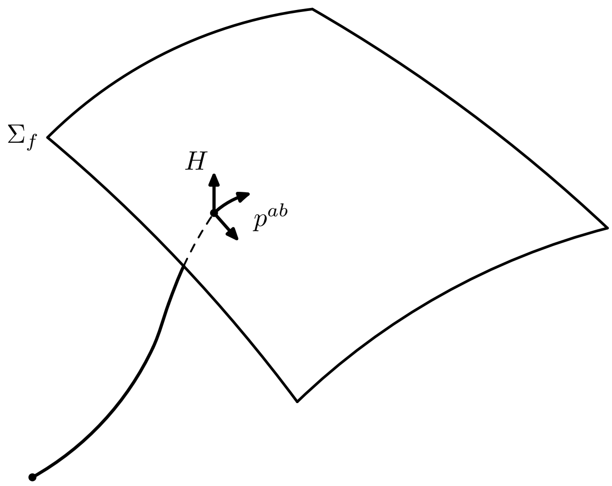


Figure 2.2: A pictorial motivation of the generalized HJ equations; $f = t$ for a temporal foliation and $f = r$ for a radial one (cf. Appendix D). The momentum p^{ab} generates transformations tangent to Σ_f, while H generates (in principle) a flow along the flow vector f^μ. This is matches the nature of the point-particle's HJ formalism, cf. Appendix B.

A radial foliation. We now turn to De Boer, Verlinde, and Verlinde's (dBVV) radial foliation [15], cf. Appendix D. In dBVV's Hamilton–Jacobi method of holographic renormalization, the radial coordinate gets the special role that is usually laid out for the time coordinate. So, instead of studying the dynamics of time-like hypersurfaces like ADM did, dBVV studied radial evolution of equal-r slices. Note that by an 'equal-r slice', we mean a hypersurface whose embedding constraint, $\phi =$ constant, is given by $\phi(x^\mu) = r$.[i] A generic metric is then written as

$$ds^2 = N^2 dr^2 + q_{ab}(dy^a + N^a dr)(dy^b + N^b dr) \tag{2.3}$$

where q_{ab} is the induced metric along the equal-r hypersurface Σ_r, which generally depends on both the foliation parameter r and the intrinsic coordinates y^a. We may always fix the gauge by setting the lapse and shift to some specified values, the most common one being $N = 1$ and $N^a = 0$. The radial dBVV Hamiltonian will then correspond to a radial flow $r^\mu = N\, n^\mu + N^a\, e^\mu_a$ instead of the standard ADM Hamiltonian's time-like flow t^μ. In most cases, the radial coordinate (foliation parameter) r is chosen to be the logarithm of a

[i]See Appendix D if this does not sound familiar.

'proper' radial coordinate, such that the AdS metric is given in terms of Gaussian normal coordinates. At large r, the metric tends to

$$ds^2 = dr^2 + e^{2r/\ell}\,\eta_{ab}\,dy^a dy^b \tag{2.4}$$

which means that shifting $r \mapsto r + a$ acts as a rigid scaling on the hypersurface and the dBVV Hamiltonian can be seen as the generator of such shifts.

The Hamilton–Jacobi equations. Let us formally denote the collection of fields by Φ. The way the HJ equations are usually written down is obtained by plugging the second HJ equation into the first, i.e.

$$H(\Phi, \tfrac{\delta S_{\rm cl}}{\delta \Phi}, r) = 0 \qquad \text{with} \qquad H = \int_{\Sigma_t} d^d y \sqrt{q}\,\Big(N\,\mathcal{H} + N^a\,\mathcal{H}_a\Big) \tag{2.5}$$

Solving the HJ equation comes down to solving the radial Hamilton and momentum constraints, $\mathcal{H} = 0$ and $\mathcal{H}_a = 0$, cf. Appendix D. The momentum constraint is exactly the requirement that the boundary (Brown–York) stress tensor (1.51) is conserved on shell,[ii]

$$0 \;=\; \mathcal{H}_a \;=\; 2\nabla^b\, p_{ab} \;=\; \nabla^b\,\tau_{ab} \tag{2.6}$$

In other words, the momentum constraint requires that the on-shell action is invariant under local diffeomorphisms on the hypersurface Σ_r, which can be automatically satisfied by choosing a suitable Ansatz. The name of the game is to solve the Hamilton constraint,

$$\mathcal{H}(\Phi, \tfrac{\delta S_{\rm cl}}{\delta \Phi}, r) = 0 \tag{2.7}$$

In principle, there are still surface terms that need to be taken into account. However, we assume the hypersurface Σ_r to be compact ($\partial\Sigma_r = \varnothing$) throughout this chapter, so that we need not worry about surface terms.[iii]

[ii] The momentum contraint contains more terms in the presence of matter fields, see e.g. (D.26) and (D.33).

[iii] For example, we could assume that the induced metric q_{ab} has Euclidean signature. The topology of the d-dimensional hypersurface Σ_r ($r \sim \infty$) at finite temperature is $S^1 \times S^{d-1}$. In Lorentzian signature, we would get a surface term proportional to the trace of the extrinsic curvature on $\partial\Sigma_r$ at $t = \pm\infty$, which is closely related the ADM mass term in (D.16). There is, of course, a whole array of additional issues when one works in Lorentzian signature, cf. [37].

2.2 Expanding The Hamilton Constraint

We will now discuss De Boer, Verlinde, and Verlinde's (dBVV) approach to holographic renormalization [8, 15], which was refined and extended in e.g. [38–40].

Splitting up the on-shell action. We will now proceed with dBVV's analysis of solving the Hamilton–Jacobi equation (2.5) in the holographic context. An important assumption that we will make is one may split up the on-shell action as $S_{\text{cl}} = S_{\text{loc}} + \Gamma$, where S_{loc} contains only local power-law divergent terms and Γ contains the remainder. This non-local remainder will turn out to comprise the effective action of the theory on the cut-off boundary. Γ has no power-law divergences (by definition), but it may still be logarithmically divergent. The regularization of Γ is fairly straightforward and, in fact, it turns out to be inessential for determining the conformal anomaly.

Premises for a derivative expansion. It is very useful to do a derivative expansion in order for the calculation to be manageable. We expand the local piece in the on-shell action, such that[i]

$$S_{\text{cl}} = S_{\text{loc}}^{(0)} + S_{\text{loc}}^{(2)} + \cdots + S_{\text{loc}}^{(d)} + \Gamma \tag{2.8}$$

where $d+1$ is the number of spacetime dimensions. The term $S_{\text{loc}}^{(d)}$ is only there when d is even.[ii] We shall use the following premises for expanding the Hamilton constraint.

1. $S_{\text{loc}}^{(n)}$ contains all terms that involve n derivatives ∂_a.[iii]
2. S_{loc} consists of only local covariant (and gauge-invariant) expressions in terms of the fields and their derivatives at the cut-off boundary.
3. S_{loc} contains all power-law divergences S_{cl}.
4. S_{loc} is universal, i.e. it has the same form for any solution with the same asymptotics.

These are basically Martelli & Mück's expansion premises [39], except for the first one. Their first premise reads "$S_{\text{loc}}^{(n)}$ *has exactly* $n/2$ *inverse (induced) metrics*". The reason for

[i] There is a subtle difference between our convention for what Γ is, which comes from [39], and the convention from dBVV's original paper [15]. The two are related through $\Gamma_{\text{dBVV}} = \Gamma + S_{\text{loc}}^{(d)}$.

[ii] The reason why the expansion of S_{loc} is only done up to level d has to with the fact that we implicitly assume that our space tends to AdS at $r \sim \infty$. Namely, the volume form (read: $\sqrt{q}$) asymptotically goes as $e^{d\,r/\ell}$ for AdS_{d+1}. For odd d, S_{loc}^{d-1} is the last term.

[iii] This premise may seem too restrictive for us to also apply to anisotropically-scaling spacetimes. However, in the next section we will explicitly show that there is no real obstruction when the space asymptotes to Lifshitz spacetime.

not picking this method of expanding is simply that it does not reproduce the required result when we consider asymptotically Lifshitz spacetimes in Section 3.2.

The (cut-off) boundary momenta $\Pi(x)$, conjugate to the fields $\Phi(x)$ at the boundary, are given in terms a derivative of the on-shell action with respect to the fields themselves (not $\dot{\Phi}$).

$$\Pi\Big|_{\Sigma_r} = \frac{1}{\sqrt{q}}\frac{\delta S_{\text{loc}}}{\delta \Phi}\Bigg|_{\Sigma_r} \tag{2.9}$$

which is similar to (B.5). The above expansion naturally induces an expansion for Π,

$$\Pi = \Pi^{(0)} + \Pi^{(2)} + \cdots + \Pi^{(d)} + \Pi^{(\Gamma)} \tag{2.10}$$

To be more specific, the fields that one typically encounters (apart from fermions) are the induced metric q^{ab}, a scalar ϕ, and a one-form A_a. Their respective canonical momenta are

$$p_{ab} = \frac{1}{\sqrt{q}}\frac{\delta S_{\text{cl}}}{\delta q^{ab}} \qquad \pi = \frac{1}{\sqrt{q}}\frac{\delta S_{\text{cl}}}{\delta \phi} \qquad E^a = \frac{1}{\sqrt{q}}\frac{\delta S_{\text{cl}}}{\delta A_a} \tag{2.11}$$

The full Hamilton constraint is given in (D.34), which can also be expanded with the above premises,

$$\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(2)} + \cdots + \mathcal{H}^{(d)} + \mathcal{H}^{(\Gamma)} \tag{2.12}$$

The contribution at level $n = 2k$ is given by

$$\mathcal{H}^{(n)} = \sum_{i+j=n} \left\{ -2\kappa\Big(q^{ac}q^{bd} - \tfrac{1}{d-1}q^{ab}q^{cd}\Big)\, p^{(i)}_{ab}p^{(j)}_{cd} \; - \; \frac{1}{2}\pi^{(i)}\pi^{(j)} \; - \; \frac{1}{2}\, E^{(i)a}E^{(j)}_a \right\} - \mathcal{L}^{(n)} \tag{2.13}$$

where $\mathcal{L}^{(n)}$ is the n^{th}-level term in the expansion for the Lagrangian restricted to the boundary. This boundary Lagrangian plays the role of the potential in the (radial) Hamiltonian; it is defined to contain all of the non-'kinetic' terms (no radial derivatives or their associated momenta).

$$\mathcal{L} \; = \; \frac{1}{2\kappa}\Big(R - 2\Lambda\Big) \; + \; \mathcal{L}_{\text{matter}} \tag{2.14}$$

Check out (D.34) to see what $\mathcal{L}_{\text{matter}}$ is for a self-interacting scalar and a massive vector. Keep in mind that, unlike in the original ADM picture, the 'dynamics' is actually in terms of radial (not temporal) evolution. One should beware of the fact that the crossterms in (2.13), i.e. terms with $i \neq j$, appear twice.

The last thing that we need is an expression for $\mathcal{H}^{(\Gamma)}$ from (2.12). We use the fact that Γ has the same power-law behavior as $S^{(d)}_{\text{loc}}$ by construction (even though it may also be logarithmically divergent). We thus get $\mathcal{H}^{(\Gamma)}$ by combining Γ and $S^{(0)}_{\text{loc}}$ in (2.13), i.e.

$$\mathcal{H}^{(\Gamma)} \;=\; -4\kappa\Big(q^{ac}q^{bd} - \frac{1}{d-1}q^{ab}q^{cd}\Big)\,p^{(0)}_{ab}\,p^{(\Gamma)}_{cd} \;-\; \pi^{(0)}\pi^{(\Gamma)} \;-\; E^{(0)\,a}E^{(\Gamma)}_{a} \tag{2.15}$$

It should be clear that there is no such thing as $\mathcal{L}^{(\Gamma)}$.

Warm-up example: pure gravity in 3 spacetime dimensions. Let us work out a simple example to see how the formalism that we have just laid out works in practice. The Ansatz for S_{loc} for pure gravity in $d+1=3$ spacetime dimension that is compatible with the above premises is (in the pure-radial gauge $N=1$, $N^a=0$)

$$S_{\text{loc}} = \frac{1}{2\kappa}\int\limits_{\Sigma_{r'}} d^2y\sqrt{q}\;\{c_1 + c_2\,R + \ldots\} \tag{2.16}$$

where $c_i \in \mathbb{R}$. The boundary Lagrangian is simply $\mathcal{L} = (R-2\Lambda)/2\kappa$. The expansion for S_{loc} and $\mathcal{L}$ is

$$S^{(0)}_{\text{loc}} = \frac{1}{2\kappa}\int d^2y\sqrt{q}\,c_1 \qquad\qquad \mathcal{L}^{(0)} = -\frac{\Lambda}{\kappa} \tag{2.17a}$$

$$S^{(2)}_{\text{loc}} = \frac{1}{2\kappa}\int d^2y\sqrt{q}\,c_2\,R \qquad\qquad \mathcal{L}^{(2)} = \frac{R}{2\kappa} \tag{2.17b}$$

The only canonical momentum that we need to compute is p^{ab}, whose expansion follows from plugging the expanded S_{cl} into (2.11).

$$p^{(0)}_{ab} = -\frac{c_1}{4\kappa}q_{ab} \qquad\qquad p^{(2)}_{ab} = \frac{c_2}{2\kappa}\left(R_{ab} - \frac{1}{2}q_{ab}R\right) \tag{2.18}$$

When we plug these into the expanded Hamilton constraint (2.13), we get

$$\mathcal{H}^{(0)} = \frac{1}{2\kappa}\left(\frac{c_1^2}{2} + 2\Lambda\right) \qquad\qquad \mathcal{H}^{(2)} = -\frac{1}{2\kappa}\,R \tag{2.19}$$

The Hamilton constraint $\mathcal{H}=0$ is satisfied at level zero if we let $c_1 = -2\sqrt{-\Lambda} = -2/\ell$, where ℓ is the AdS curvature radius like we had before.[iv] At second level, though, we immediately see a problem arising. Namely, there is no dependence on c_1 or c_2 in $\mathcal{H}^{(2)}$, which means that we cannot put it to zero in the same way. This is where the effective boundary action Γ comes into play. Notice that the full expansion in 3 dimensions is

$$\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(2)} + \mathcal{H}^{(\Gamma)} \tag{2.20}$$

[iv] We chose the negative root so that S_{cl} has a negative cosmological constant.

so we note that Γ should be such that $\mathcal{H}^{(\Gamma)} = -\mathcal{H}^{(2)}$, which implies that[v]

$$q^{ab}\frac{2}{\sqrt{q}}\frac{\delta\Gamma}{\delta q^{ab}} \;=\; -\frac{\ell}{16\pi G}\,R \tag{2.21}$$

We recognize the trace of our (unregulated) boundary stress tensor on the left-hand side, so we see that this is the familiar expression of the Weyl anomaly in a two-dimensional CFT. We get the central charge c by relating this to the standard form $T^a{}_a = -(c/24\pi)R$, so that we find

$$\boxed{c = \frac{3\ell}{2G}} \tag{2.22}$$

which is the exact same central charge that we got from the Brown–Henneaux procedure in the previous chapter, which is a nice consistency check.

Logarithmic divergence. We were a little hasty in calling Γ the effective boundary action. We should have made sure that it has no logarithmic divergence first.[vi] To reveal any logarithmic dependence in Γ, one can do an infinitesimal rescaling $r \to r + \varepsilon$, see e.g. [39]. The difference is

$$\begin{aligned}
\Gamma(r+\varepsilon) - \Gamma(r) &= \varepsilon \int d^2y\ (\partial_r q_{ab})\,\frac{\delta\Gamma}{\delta q_{ab}} + O(\varepsilon^2) \\
&= \frac{\varepsilon}{\ell}\int d^2y\, 2q_{ab}\frac{\delta\Gamma}{\delta q_{ab}} + O(\varepsilon^2) \\
&= -\varepsilon \int d^2y\sqrt{q}\,\mathcal{H}^{(\Gamma)} + O(\varepsilon^2) \qquad \Rightarrow \qquad \partial_r\Gamma \;\simeq\; -\int d^2y\sqrt{q}\,\mathcal{H}^{(\Gamma)}
\end{aligned} \tag{2.23}$$

We used that the induced metric asymptotes to AdS at $r \sim \infty$, from which it follows that $\partial_r q^{ab} \simeq (2/\ell)\, q^{ab}$. The anomalous piece of the Hamilton constraint $\mathcal{H}^{(\Gamma)}$ is proportional to the Ricci scalar and it is well known that the combination $\sqrt{q}R$ is Weyl invariant in two dimensions, thus also independent of r (to leading order).[vii] The above expression can then be integrated to get

$$\Gamma \;=\; -r\int d^2y\sqrt{q}\,\mathcal{H}^{(\Gamma)} \;+\; (\text{finite piece}) \tag{2.24}$$

The renormalized effective boundary action is obtained by

$$\Gamma_{\text{ren}} = \lim_{r\to\infty}\left\{\ \Gamma \;+\; r\int d^2y\sqrt{q}\,\mathcal{H}^{(\Gamma)}\ \right\} \tag{2.25}$$

[v] We use the notation $\kappa \equiv 8\pi G$, cf. Appendix C.

[vi] By 'logarithmic', we mean logarithmic in $e^{r/\ell}$, which is the same as power-law in r. The transformation $r \to r + \lambda$ looks like a translation in terms of r, but it is of course a dilatation when exponentiated.

[vii] The anomalous Hamilton constraint has a different form in a higher number of dimensions, but the fact that $\sqrt{q}\,\mathcal{H}^{(\Gamma)}$ is Weyl invariant remains true.

The metric and the anomalous Hamilton constraint can be renormalized by rescaling it as

$$q_{ab} \to q_{ab}^{\text{ren}} = \lim_{r\to\infty} e^{-2r/\ell}\, q_{ab} \qquad \text{and} \qquad \mathcal{H}^{(\Gamma)} \to \mathcal{H}^{(\Gamma)}_{\text{ren}} = \lim_{r\to\infty} e^{2r/\ell}\, \mathcal{H}^{(\Gamma)} \tag{2.26}$$

The proper Weyl anomaly is then given by

$$q^{ab}_{\text{ren}} \frac{2}{\sqrt{q_{\text{ren}}}} \frac{\delta \Gamma_{\text{ren}}}{\delta q^{ab}_{\text{ren}}} \;=\; -\frac{\ell}{16\pi G}\, R[q_{\text{ren}}] \tag{2.27}$$

which gives the same central charge as before. The reason for the coincidental answers of the renormalized and unrenormalized anomalies can be seen related to the fact that both sides of the unrenormalized anomaly are precisely equally divergent, a fact that may not be generically true.

2.3 From Radial to RG Flow

The study of holographic RG flows relies on the identification of two seemingly distinct notions of flow: the *radial flow* in the bulk gravity theory and the *RG flow* in the effective (cut-off) boundary theory. This identification follows from the standard AdS/CFT dictionary, which in particular states that the IR of the bulk theory corresponds to the UV on the boundary (and vice versa) [7]. In other words, the RG scale of the effective theory that lives on Σ_r goes as $\mu \sim e^{r/\ell}$ at $r \sim \infty$. In this section we will make a precise connection between the radial flow in the bulk and RG flow on the boundary.

Domain-wall geometries. We will now study the radial evolution of so-called domain wall solutions. The metric is a generalization of the AdS metric (2.4) given in terms of some function $f(r)$.

$$ds^2 = dr^2 + e^{f(r)}\,\eta_{ab}\,dy^a dy^b \tag{2.28}$$

AdS obviously corresponds to $f(r) = 2r/\ell$. The above metric is a solution to the field equations that come from varying the action

$$S = S_{\text{grav}} + S_\phi = \int\limits_{\mathcal{M}} d^{d+1}x\sqrt{g}\left\{\frac{1}{2\kappa}\left(\tilde{R} - 2\Lambda\right) - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right\} + \frac{1}{2\kappa}\oint\limits_{\partial\mathcal{M}} d^d y\sqrt{q}\,2K \tag{2.29}$$

Given that the potential can be written in terms of some other function of the scalar $U(\phi)$ according to

$$V(\phi) = (\ldots)\,U'(\phi)^2 - (\ldots)\,U(\phi)^2 \tag{2.30a}$$

the field equations are

$$\dot{\phi} = (\ldots)\,U'(\phi) \qquad\qquad \dot{f} = (\ldots)\,U(\phi) \tag{2.30b}$$

The ellipses represent numerical factors and by the dot on e.g. $\dot{\phi}$ we mean Lie differentiation along the radial flow vector $r^\mu\partial_\mu$, which is just (logarithmic) radial differentiation ∂_r in the pure-radial gauge $N = 1$, $N^a = 0$. We will now show these equations are easily obtained with the HJ method laid out in the previous two sections.

Solving the Hamilton constraint. We set out to solve the Hamilton constraint coming from the above action (2.29). The boundary Lagrangian is $\mathcal{L} = \mathcal{L}_{\text{grav}} + \mathcal{L}_\phi$, which are given in (D.34) by

$$\mathcal{L}_{\text{grav}} = \frac{1}{2\kappa}\int\limits_{\Sigma_r} d^d y\,\sqrt{q}\,\Big\{R - 2\Lambda\Big\} \tag{2.31a}$$

$$\mathcal{L}_\phi = -\int\limits_{\Sigma_r} d^d y\sqrt{q}\left\{\frac{1}{2}\partial^a\phi\,\partial_a\phi + V(\phi)\right\} \tag{2.31b}$$

The Ansatz for the local part of the on-shell action follows from the premises on page 34.

$$S_{\text{loc}} = \int\limits_{\Sigma_r} d^d y \sqrt{q}\, \{U + \Phi\, R + M\, \partial_a\phi\, \partial^a\phi + \ldots\} \tag{2.32}$$

where $U(\phi)$, $\Phi(\phi)$, and $M(\phi)$ are ordinary functions of ϕ. We truncated the series in such a way that we can read off the first two levels in the derivative-like expansion. This does not mean that we have all the information that we need in any number of dimensions d, but it will suffice for deriving (2.30). The expansion for the boundary Lagrangian (2.31) and the local on-shell action S_{loc} also follow from the premises.

$$S^{(0)}_{\text{loc}} = \int d^d y \sqrt{q}\, U \qquad\qquad \mathcal{L}^{(0)} = -\frac{\Lambda}{\kappa} + V(\phi) \tag{2.33a}$$

$$S^{(2)}_{\text{loc}} = \int d^d y \sqrt{q} \left\{\Phi\, R + M\, \partial^a\phi \partial_a\phi\right\} \qquad\qquad \mathcal{L}^{(2)} = \frac{R}{2\kappa} - \frac{1}{2}\partial_a\phi\, \partial^a\phi \tag{2.33b}$$

The canonical momenta at lowest (most divergent) level are

$$p^{(0)}_{ab} = -\frac{1}{2} q_{ab}\, U \qquad\qquad \pi^{(0)} = U' \tag{2.34}$$

and at second level, they are

$$\begin{aligned} p^{(2)}_{ab} &= \Phi\left(R_{ab} - \tfrac{1}{2} q_{ab}\, R\right) + \big(M - \Phi''\big)\partial_a\phi\, \partial_b\phi + \big(-\tfrac{1}{2}M + \Phi''\big) q_{ab}\, \partial_c\phi\, \partial^c\phi \\ &\qquad + \Phi'\big(q_{ab}\nabla_c\nabla^c - \nabla_{(a}\nabla_{b)}\big)\phi \\ \pi^{(2)} &= \Phi'\, R + M'\, \partial_a\phi\, \partial^a\phi + 2M\, \nabla_a\nabla^a\phi \end{aligned} \tag{2.35}$$

When we plug these into the expanded Hamilton constraint (2.13), up to level two, we get

$$\mathcal{H}^{(0)} = \tfrac{d}{2d-2}\kappa\, U^2 - \tfrac{1}{2}(U')^2 + V(\phi) \tag{2.36a}$$

$$\begin{aligned} \mathcal{H}^{(2)} = &\left[\tfrac{d-2}{d-1}\kappa\, U\Phi - U'\Phi' - \tfrac{1}{2\kappa}\right] R \\ &+ \left[-2\kappa\, U\Phi' - 2U'M\right] \nabla_a\nabla^a\phi \\ &+ \left[\tfrac{d-2}{d-1}\kappa\, UM - 2\kappa\, U\Phi'' - U'M' + \tfrac{1}{2}\right] \partial_a\phi\, \partial^a\phi \end{aligned} \tag{2.36b}$$

We see that the level-zero Hamilton constraint $\mathcal{H}^{(0)} = 0$ yields the (fake) superpotential relation (2.30a). The field equations (2.30b) are directly obtained from the Hamilton equation $\dot{q} = \partial H/\partial p$ at the boundary, i.e. with the momenta given by (2.11).

$$\dot{\phi} = \pi \approx U' \qquad\qquad \dot{q}_{ab} = -4\kappa\left(p_{ab} - \tfrac{1}{d-1} q_{ab}\, p\right) \approx -2\kappa\, U\, q_{ab} \tag{2.37}$$

The first field equation in (2.30b) is indeed already sitting there. The second one comes from noticing that the domain-wall metric (2.28) implies $\dot{q}_{ab} = \dot{f}\, q_{ab}$.

The Callan–Symanzik equation. In order to complete the connection between the radial flow and boundary RG flow we will now review a derivation of the Callan–Symanzik equation from purely holographic reasoning. Because it is not essential for the remainder of this text, our treatment will be rather shallow, see [8] for a more complete analysis. The derivation requires the 'AdS/CFT master equation' (see e.g. [2]), which relates the effective action Γ to the generating functional of the boundary CFT. It formally looks like

$$\Gamma[\Phi] = \left\langle e^{\int d^d y \sqrt{q}\, \Phi\cdot\mathcal{O}} \right\rangle_{\text{bndy}} \tag{2.38}$$

This Γ is the same one as $\Gamma = S_{\text{cl}} - S_{\text{loc}}$ that we saw before. In [8], the notion of a physical scale $\mu = \exp f(r)$ is introduced, such that $q_{ab} = \mu\, \delta_{ab}$. Via (2.37) we see that $\dot{\mu} = -2\kappa\, U\, \mu$. The beta function for ϕ is then defined as the usual logarithmic derivative

$$\beta_\phi \;\equiv\; \mu \frac{\partial}{\partial \mu}\phi \;=\; \frac{1}{2\kappa}\frac{U'}{U} \tag{2.39}$$

where we assumed that $f(r)$ is invertible. We use (2.38) together with the commutation relation $[\Phi_I, \delta/\delta\Phi_J] = -\delta_{IJ}$ to arrive at the Callan–Symanzik equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_\phi \frac{\partial}{\partial\phi}\right)\langle \mathcal{O}(y_1)\cdots\mathcal{O}(y_n)\rangle + \sum_{i=1}^{n} \gamma_i \,\langle \mathcal{O}(y_1)\cdots\mathcal{O}(y_i)\cdots\mathcal{O}(y_n)\rangle \tag{2.40}$$

where γ_i is the anomalous dimension of the operator $\mathcal{O}(y_i)$ which is sourced by $\phi(y_i)$. Admittedly, we have been rather sloppy in this last paragraph. For instance, we have not properly defined the partial derivatives. Again, we refer the reader to De Boer's lecture notes on HJ holography [8] for a more thorough treatment of this topic. In these lecture notes, an arbitrary number of scalars is taken, which is closer to the nature of the renormalization group, since one typically has a space of couplings that has more than one dimension. The result is very nice nonetheless; we explicitly see a strong result from ordinary RG arise by just plugging in the level-zero Hamilton constraint. Of course, the 'AdS/CFT master equation' appeared somewhat mysteriously and has not been justified. However, a satisfactory justification of it is beyond the scope of this thesis.

Chapter 3

Lifshitz/Schrödinger Asymptotics

Schrödinger spacetime. Over the past two years or so, there has been a lot of interest in extending the well-known AdS/CFT correspondence [1–3] to condensed matter applications, see e.g. [41–43] for an overview. A subclass of these so-called AdS/CMT set-ups consists of using holographic duals to study theories at a Schrödinger-invariant UV fixed point.[i] By Schrödinger-invariant we mean invariant under the group of symmetries of the free Schrödinger equation $i\partial_t = -\nabla^2/2m$. This so-called Schrödinger group is introduced in Section 3.1. Son [10] and (Koushik) Balasubramanian & McGreevy [11] published their papers on so-called Schrödinger spacetime around the same time. The metric on Schrödinger spacetime typically looks like

$$ds^2 = -r^4\, dt^2 + \frac{dr^2}{r^2} + r^2 \left(2d\xi\, dt + d\vec{x}^2\right) \tag{3.1}$$

which can be obtained by some discrete light-cone quantization (DLCQ) procedure [44]. The isometry group of (3.1) is the Schrödinger group. In fact, the metric (3.1) was derived from the observation that the Schrödinger group is the DLCQ of the Poincaré group [10]. One of the standard properties of holographic dualities is that they relate theories of co-dimension one. Because of this DLCQ procedure, the duality is between two theories of co-dimension two (the ξ coordinate gets interpreted as the particle number density operator M). This is reminiscent of the study of null hypersurfaces in standard GR. In some cases such as [11], a generic value for the so-called dynamical critical exponent z is allowed. The critical dynamical exponent gives us a measure for anisotropy in the scaling behavior. In other words, a system that scales anisotropically with $z \neq 1$ is invariant under simultaneously rescaling $x^i \mapsto \lambda x^i$ and $t \mapsto \lambda^z t$. The critical dynamical exponent roughly enters the dispersion relation as $H \sim P^z$; we will focus on the case where $z = 2$. The story of Schrödinger spacetimes was further developed in e.g. [12, 45, 46].

[i]Keep in mind that holographic dualities typically relate the UV of the boundary theory to the IR of the bulk theory.

This chapter is a first step towards a new way of looking at Schrödinger holography. The main idea is the following.[ii] In a conformal field theory, the conformal algebra picks up a finite central charge upon quantization. This central charge for AdS_3 was obtained in the previous two chapters using two distinct methods, which both led to $c = 3\ell/2G$. Similarly, in a theory with Schrödinger symmetry, the Schrödinger algebra also has a central charge. This time, the central charge is the mass density operator, which for example can be the mass m that appears in the Schrödinger equation $i\partial_t = -\nabla^2/2m$. An important implication of viewing the mass density operator as a central charge is that it removes the need for having the extra ξ dimension.

The first attempt towards finding a finite central charge M is by use of the Brown–Henneaux procedure from Chapter 1, which is pursued in Section 3.1. The second attempt, in Section 3.2, is made by using holographic renormalization techniques from Chapter 2. In both cases, we will not work with Schrödinger spacetime, but with so-called Lifshitz spacetime [9], given by

$$ds^2 = -r^4\, dt^2 + \frac{dr^2}{r^2} + r^2\, d\vec{x}^2 \tag{3.2}$$

i.e. discarding the ξ direction and thus turning the holographic duality back into a codimension-one duality. In contrast to Schrödinger spacetime (3.1), the Lifshitz geometry does not have Galilean boosts $K : x \mapsto x + vt$ as an isometry.

Why Lifshitz spacetime? The reason for using Lifshitz instead of Schrödinger has to do with the fact that we do not need (or want) the Galilean boosts to be exact symmetries of our background. Namely, as we will discuss in Section 3.1, one can only obtain a central charge c from $[A, B] = c$ if both A and B are <u>not</u> exact isometries. The central charge that we will be looking for comes from the commutator of a Galilean boost K and a spatial translation P, i.e. $[K, P] = M$. This means that both K and P cannot be exact Killing symmetries of (3.2). Hence, losing Galilean boosts when choosing Lifshitz in favor of Schrödinger is not a bad thing; it is required. We must, of course, also break spatial-translation invariance. Moreover, it must also be shown that K and P are still approximate (asymptotic) symmetries. We will discuss these issues in Section 3.1.

A disclaimer. The topics in this chapter are work in progress. The notes in red are the points that I'm still working on. Even though I have not finished the analysis, I would still like to share some of my thoughts on these subjects.

[ii]This is Prof. Maloney's idea.

3.1 Lifshitz Spacetime with Schrödinger Symmetry

In this section we will extend the Brown–Henneaux analysis to a different group of asymptotic symmetries. We are interested in finding a surface-charge representation of the Schrödinger group, which is the group of symmetries of the non-relativistic Schrödinger equation. Just like for the conformal/Virasoro case, the Schrödinger algebra picks up a finite central charge upon quantization. In this case, however, the central charge is the mass/particle number density operator M. The main idea in this section is to construct a surface-charge representation of the Schrödinger algebra and find a central extension in the form of a finite M.

The Schrödinger algebra. The so-called Galilei algebra is probably the most basic example of an algebra with a central charge: the mass/particle number density. A system with Galilean invariance has a quadratic dispersion, $H \propto P^2$. This means that the dynamical critical exponent is $z = 2$. Other examples of gauge/gravity dualities have taught us that we need some sort of scale invariance at the boundary, so we extend the Galilei algebra to the Schrödinger algebra (with $z = 2$), denoted Schr_d. The non-vanishing commutators of Schr_d are:

$$
\begin{aligned}
[D, C] &= -2C & \qquad [D, K_i] &= -K_i \\
[D, H] &= 2H & \qquad [D, P_i] &= P_i \\
[C, H] &= D & \qquad [H, K_i] &= -P_i \\
[K_i, P_j] &= M\delta_{ij} & \qquad [C, P_i] &= K_i
\end{aligned}
\tag{3.3}
$$

where $i = 1, \ldots, d-1$ and for $d > 2$ we should also include rotations

$$
\begin{aligned}
[R_{ij}, R_{kl}] &= \delta_{ik}R_{jl} - \delta_{il}R_{jk} - \delta_{jk}R_{il} + \delta_{jl}R_{ik} \\
[R_{ij}, K_k] &= \delta_{ik}K_j - \delta_{jk}K_i \\
[R_{ij}, P_k] &= \delta_{ik}P_j - \delta_{jk}P_i
\end{aligned}
\tag{3.4}
$$

We see that M appears in the algebra only as a central charge. A typical representation of the Schrödinger generators is

$$
\begin{aligned}
&\text{(mass) density} & M &= m \\
&\text{time translations} & H &= \partial_t \\
&\text{spatial translations} & P_i &= \partial_i \\
&\text{Galilei boosts} & K_i &= t\partial_i + m x_i \\
&\text{rotations} & R_{ij} &= x_i\partial_j - x_j\partial_i \\
&\text{scaling transformation} & D &= 2t\partial_t + x_i\partial_i - l \\
&\text{special conformal transformations} & C &= t^2\partial_t + t x_i\partial_i + \tfrac{m}{2}x^2 - lt
\end{aligned}
\tag{3.5}
$$

with $m, l \in \mathbb{R}$.

Asymptotically Lifshitz spacetimes. We are looking for a spacetime that, first of all, is physical. In other words, it should consistently solve the field equations of some theory. On top of that, the asymptotic symmetry algebra of the metric should contain the Schrödinger algebra. Like we mentioned before, we are looking to find the (mass) density operator from the Poisson (Dirac) bracket $\{Q_{K_i}, Q_{P_j}\} = M\delta_{ij}$. As was noted in [16], there is an important restriction to the metrics that we are allowed to choose. If the asymptotic symmetries are exact isometries of the background space $\bar{\mathcal{M}}$, all central charges vanish exactly. In other words, if either ζ or η sits in $iso(\bar{\mathcal{M}})$, all $C_{\zeta\eta}$ vanish. To see this, we note that the central charge from (1.42),

$$C_{\zeta\eta} \;=\; Q_\zeta[\mathscr{L}_\eta \bar{g}] \;=\; -Q_\eta[\mathscr{L}_\zeta \bar{g}]$$

vanishes because the Lie derivative of the metric with respect to an isometry vanishes identically (by definition).

We will thus look for a background geometry $\bar{\mathcal{M}}$ such that the algebra of its isomorphisms is a proper subalgebra of the asymptotic symmetry algebra, i.e. $iso(\bar{\mathcal{M}}) \subset asg(\bar{\mathcal{M}})$. Another thing we are looking for is the property that the asymptotic symmetry algebra covers the Schrödinger algebra, i.e. $Schr_d \subseteq asg(\bar{\mathcal{M}})$. In summary, we would like to have the following hierarchy of algebras.

$$iso(\bar{\mathcal{M}}) \;\subset\; Schr_d \;\subseteq\; asg(\bar{\mathcal{M}}) \tag{3.6}$$

and the generators whose brackets ought to give the central charge must sit in the coset

$$Schr_d / iso(\bar{\mathcal{M}}) \tag{3.7}$$

The Lifshitz background Ansatz. Motivated by the above criteria, we choose a slightly adjusted version of Kachru, Liu, and Mulligan's [9] Lifshitz spacetime solution for our background.

$$\bar{g}_{\mu\nu}\, dx^\mu dx^\nu = -\left(r^4 + \frac{r^2}{x^2+y^2}\right) dt^2 + \frac{dr^2}{r^2} + r^2\left(dx^2 + dy^2\right) \tag{3.8}$$

We adjustment is the second term in g_{tt}, which was put there in order to break the symmetry under spatial translations generated by P. The isometries of (3.8) are $\{H, D, C\}$, which span a subgroup of the Schrödinger group. Interestingly, this subgroup is isomorphic to the Möbius group $SL_2(\mathbb{R})$. The metric (3.8) should be a solution to the Einstein–Proca action $\int dr\,(L_{\text{grav}} + L_A)$ (cf. (D.16) and (D.30)), together the corresponding Proca field

$$\bar{A}_\mu\, dx^\mu = r^2\, dt \tag{3.9}$$

These don't solve the field equations! Maybe drop the r^4 term in g_{tt}? (this would most likely change the ASG too)

Killing's equation is solved in the asymptotic region $r \sim \infty$ by any ξ that is of the form

$$\xi = f^i{}_{,i}\,\partial_t + f^i{}_{,t}\,\partial_i - \frac{1}{2} f^i{}_{,it}\, r\partial_r \tag{3.10}$$

where f is an arbitrary function of $x^i = (x, y)$ and t only. In order to see that the asymptotic symmetry group contains the Schrödinger algebra, we write $f(x,t)$ as a power series in x

$$f^i(x,t) = f^i(0,t) + f^i{}_{,j}(0,t)\, x^j + \frac{1}{2!} f^i{}_{,jk}(0,t)\, x^j x^k + \dots \tag{3.11}$$

In the temporal foliation that will follow, we can choose to put our hypersurface Σ_t at any value of t. We expand the function $f(x,t)$ around $t = 0$ up to second order. We see that the piece linear in x^i already contains all Schrödinger generators. We use the coefficients such that the connection with the corresponding Schrödinger generators (3.5) is clear. For example, the coefficient $H \in \mathbb{R}$ corresponds to the generator $H = \partial_t$.[i]

$$f^i(x,t) = \text{const.} + P^i\, t + K^i\, t^2 + \left[H + (D\,\delta^{ij} + R\,\epsilon^{ij})t + (C\,\delta^{ij} + ..\,\epsilon^{ij})t^2\right] x_j + \dots \tag{3.12}$$

The main thing that should be taken from this expression is that the asymptotic symmetry group contains the Schrödinger group.[ii] We see in particular that the Galilean boosts and spatial translations emerge as symmetries at the asymptotic level, even though they are not isometries. This means that the criteria (3.6) are indeed satisfied.

Asymptotically Lifshitz boundary conditions. Let us define (locally) asymptotically Lifshitz spacetime to be a spacetime whose metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ respect the asymptotic boundary conditions,

$$(h_{\mu\nu}) = \begin{pmatrix} h_{tt} & h_{tr} & h_{tx} & h_{ty} \\ & h_{rr} & h_{rx} & h_{ry} \\ & & h_{xx} & h_{xy} \\ & & & h_{yy} \end{pmatrix} = \begin{pmatrix} O(r^2) & O(r^{-1}) & O(r^2) & O(r^2) \\ & O(r^{-2}) & O(r^{-1}) & O(r^{-1}) \\ & & O(r^2) & O(r^2) \\ & & & O(r^2) \end{pmatrix} \tag{3.13}$$

Comparing these to the background metric itself at $r \sim \infty$,

$$(g_{\mu\nu}) \approx \begin{pmatrix} -r^4 & 0 & 0 & 0 \\ & r^{-2} & 0 & 0 \\ & & r^2 & 0 \\ & & & r^2 \end{pmatrix} \tag{3.14}$$

we see that the deviations are not necessarily subleading. This is no real obstruction, for example, this is also the case in Strominger et al.'s Kerr/CFT paper [47]. These boundary

[i] To be more precise, $\xi \propto \partial_t$ is obtained by setting all coefficients but $H \in \mathbb{R}$ to zero.

[ii] For those who are interested, we see in (3.10) that $\xi^i = \dot{f}^i(0,t) + \dot{f}(0,t)^i{}_{,j} x^j + \dots$ The matrix $\dot{f}(0,t)^{i,j}$ is then decomposed into its trace $\sim \delta^{ij}$, anti-symmetric $\sim \epsilon^{ij}$, and traceless-symmetric $\sim \sigma^{ij}$ pieces. The latter of which does not contribute, because it only appears contracted with the vector x_i. (By the way, I don't know what the $..\,\epsilon^{ij}$ in the last term corresponds to (its corresponding generator is $tx_i\partial_j - tx_j\partial_i$).)

conditions are chosen such that any ξ of the form (3.10) is an asymptotic symmetry (at $t \sim 0$), which obviously includes the Schrödinger transformations (3.12). The boundary conditions are obtained by relating $h_{\mu\nu} \sim \mathscr{L}_\xi \bar{g}_{\mu\nu}$. This however, does not yield a boundary condition for h_{rr}, because $\mathscr{L}_\xi \bar{g}_{rr}$ vanishes identically for all ξ. Using the fact that the asymptotic symmetries should map the deviations onto themselves, $h_{\mu\nu} \sim \mathscr{L}_\xi h_{\mu\nu}$, will then give the h_{rr} boundary condition.

Since Lifshitz spacetime is not a solution to pure Einstein, but rather Einstein–Proca, we need to do the same for the Proca field as well. The boundary conditions for the deviations $\alpha_\mu = A_\mu - \bar{A}_\mu$ read

$$(\alpha_\mu) = \begin{pmatrix}\alpha_t & \alpha_r & \alpha_x & \alpha_y\end{pmatrix} = \begin{pmatrix}O(r^2) & O(r^3) & O(r^2) & O(r^2)\end{pmatrix} \tag{3.15}$$

where, similar to the metric, the boundary condition for α_t was obtained by requiring $\alpha_\mu \sim \mathscr{L}_\xi \alpha_\mu$.

Einstein–Proca surface charges on the Lifshitz background. We must add the contribution from the Proca field A_μ to the surface charges $Q_\zeta(h)$. Using the Proca Hamiltonian (D.33), our best guess at what the surface charge looks like is

$$Q_\zeta(h) = \oint_{\partial\Sigma_t(\infty)} d\sigma_a \left\{ \frac{1}{2\kappa} \bar{G}^{abcd} \left[\bar{\nabla}_b h_{cd} - h_{cd} \bar{\nabla}_b \right] \hat{\zeta} + 2p^{ab}\, \hat{\zeta}_b + E^a \mathcal{N}\, \hat{\zeta} + E^a A_b \hat{\zeta}^b \right\} \tag{3.16}$$

Before I start computing charges, I need to get the following issue out of the way. Namely, there is something terribly wrong with the above boundary conditions: they allow for deviations h_{tt} from the background (3.8) at order r^2, which means that the term that we need in order to break the invariance under spatial translations can be 'gauged away'. To be more specific, the integration $\delta Q \to Q$ allows us to pick any background that is consistent with the boundary conditions (3.13). Consequently, the (possible) central charge should not depend on which one of these backgrounds we pick. In particular, we can pick KLM's Lifshitz spacetime (which is P_i-invariant), so the central charge M must be zero.

I have been playing around trying to fix this issue by making sure that P-invariance is broken a leading order, but I have not yet found such a metric (+ Proca field) with this property that solve the field equations (and still has an ASG that contains the Schrödinger group). Though I feel like it's not impossible...

3.2 Holographic Renormalization of Lifshitz

In view of the previous section, one would like to find a term in the on-shell action that would source the mass density operator proportional to $[K_i, P_j]$. However, for now we will be satisfied with finding out the structure of the local counterterms sitting in S_{loc} for the Einstein–Proca action $\int dr\,(L_{\text{grav}} + L_A)$. We will compare a simple result with that of Ross & Saremi [13] as a consistency check.

An additional effective dimension. The expansion truncates at marginal level $S^{(d)}_{\text{loc}}$ in the case of AdS_{d+1}, (d even). This has to do with the fact that the volume form of AdS_{d+1} scales as $\sqrt{g} \sim e^{d\,r}$ (the lapse scales as 1) in the asymptotic region. For KLM's d-dimensional Lifshitz spacetime [9] with $z = 2$, henceforth denoted Lif_d,

$$ds^2/\ell^2 \;=\; -r^4\,dt^2 + \frac{dr^2}{r^2} + r^2\,d\vec{x}^2 \qquad \text{with} \qquad \vec{x} \in \mathbb{R}^{d-2} \tag{3.17}$$

or in Gaussian normal coordinates with $\ell = 1$,

$$ds^2 \;=\; dr^2 - e^{4r}\,dt^2 + e^{2r}\,d\vec{x}^2 \qquad \text{with} \qquad \vec{x} \in \mathbb{R}^{d-2} \tag{3.18}$$

the volume form on Lif_d scales as $\sqrt{g} \sim e^{d\,r}$, which is the same as AdS_{d+1}. We will focus on Lif_4, which means that the expansion is truncated at $S^{(4)}_{\text{loc}}$, just like in AdS_5. For this reason, we could expect possible anomalous terms to appear, even though we are working in an even number of spacetime dimensions.

Notice that the field content on this background inherits this additional effective dimension, so the analysis need not be altered in a very crude way. One may wonder, however, whether first premise of the isotropic expansion on page 34 is justified. It turns out that it is, which we explain below.

Equivalence of isotropic and anisotropic expansion. The derivative expansion (page 34) is inherently *isotropic*, i.e. it treats ∂_t and ∂_i on equal footing. Lifshitz spacetime, on the other hand, does not possess this isotropic property. We must thus show that expanding isotropically will yield the exact same equations as expanding *an*isotropically.

Let us temporarily alter the first expansion premise from page 34, such that it reflects Lifshitz's anisotropy. The anisotropic premise would read

$S^{(n)}$ contains n^{s} spatial and n^{t} temporal derivatives in such a way that $n^{\text{s}} + 2n^{\text{t}} = n$.

Let us give a sketch of the reason why this anisotropic premise would still yield the same equations in the expansion as the old isotropic one. This is most easily explained using the example from Section 2.3. For conciseness, let us focus on the kinetic term of the scalar field only.[i]

$$\partial_a\phi\,\partial^a\phi = q^{tt}\,(\partial_t\phi)^2 + q^{ij}\,\partial_i\phi\,\partial_j\phi \tag{3.19}$$

These separate terms will respectively contribute at level 4 and 2 in the anisotropic expansion. The Ansatz for the local piece of the on-shell action was given in (2.32). Let us instead write down an Ansatz that is *not* necessarily isotropic,

$$S_{\text{loc}} \;=\; \ldots \;+\; M_1(\phi)\,\partial_t\phi\,\partial^t\phi \;+\; M_2(\phi)\,\partial_i\phi\,\partial^i\phi \;+\; \ldots$$

In turns out that the Hamilton constraint then contains two identical coefficients at different levels involving $M_1(\phi)$ and $M_2(\phi)$. The expanded Hamilton constraint that corresponds to the above S_{loc} looks like[ii]

$$\begin{aligned}\mathcal{H}^{(2)} &= \ldots + \left(\tfrac{d-2}{d-1}\kappa\,UM_2 - 2\kappa\,U\Phi'' - U'M_2' + \tfrac{1}{2}\right)\partial_i\phi\,\partial^i\phi \;+\; \ldots\\ \mathcal{H}^{(4)} &= \ldots + \left(\tfrac{d-2}{d-1}\kappa\,UM_1 - 2\kappa\,U\Phi'' - U'M_1' + \tfrac{1}{2}\right)\partial_t\phi\,\partial^t\phi \;+\; \ldots\end{aligned}$$

See (2.36b) for the same Hamilton constraint in the isotropic case. The main point is this. We see that both M_1 and M_2 must solve the exact same constraint. Thus, restoring isotropy/Lorentz-invariance by setting $M_1 = M_2$ is obviously consistent and most likely inevitable. So even if one were to choose an anisotropic Ansatz, Lorentz invariance is restored by solving the Hamilton constraint.

There are two conclusions from this: the anisotropic expansion yields the same constraints as the old isotropic one *and* Lorentz invariance pops up even if we allow for an anisotropic Ansatz. The fact that we have Lorentz invariance is not very surprising, because at finite r bulk covariance is still preserved. It is only in the limiting case $r \to \infty$ that the (asymptotic) symmetry group changes. This is also related to Hořava's idea of a radially varying speed of light. For example, the metric on a radial hypersurface that is induced by pulling back (3.18) is invariant under Lorentz transformations where the speed of light is $c = e^r$, which is finite at finite r.

Solving the Hamilton constraint for *Lif*$_4$. We will now move on to the main topic of this section. We would like to solve the Hamilton constraint coming from the Einstein–

[i] We assume that we may consistently set the embedded shift to zero, $N^i = -q^{it}/q^{tt} = 0$.

[ii] In order to arrive at this result, one needs to do an embedded temporal foliation, e.g. the Gauss–Codazzi equations yield ${}^{(d)}R = {}^{(d-1)}R - k^2 + k_{ij}k^{ij}$ where $k_{ij} \sim \partial_t\gamma_{ij}$ is the embedded extrinsic curvature.

Proca action.[iii]

$$S_{\text{grav}} + S_{\text{A}} = \int\limits_{\mathcal{M}} d^{d+1}x\sqrt{g}\left\{\frac{\tilde{R}-2\Lambda}{2\kappa} - \frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{m^2}{2}\tilde{A}_\mu\tilde{A}^\mu\right\} + \frac{1}{2\kappa}\oint\limits_{\partial\mathcal{M}} d^dy\sqrt{q}\,2K \tag{3.20}$$

whose corresponding boundary Lagrangian is given by

$$\mathcal{L}_{\text{grav}} + \mathcal{L}_{A} \;=\; \frac{1}{2\kappa}\big(R-2\Lambda\big) - \frac{1}{4}F_{ab}F^{ab} - \frac{m^2}{2}A_aA^a \tag{3.21}$$

Let us introduce $\alpha \equiv -A_aA^a - 1$, which is a deviation from the background of order $\alpha \sim O(e^{-2r})$, [13]. The deviation is defined such that $\alpha = 0$ on the Lifshitz background. The most general Ansatz for the local-term action in terms of the canonical variables is

$$S_{\text{loc}} = \int\limits_{\Sigma_r} d^dy\sqrt{q}\,\Big\{F(\alpha) + (\text{derivative terms})\Big\} \tag{3.22}$$

where we only wrote down the term that contributes to the Hamilton constraint level zero.[iv]

A consistency check. We will do a consistency check with a basic result from Ross & Saremi (RS) [13]; we check if their $F(\alpha)$ is compatible with ours. RS's function $F(\alpha)$ (from Section 2 of [13]) is given by

$$F(\alpha) \;=\; -4 - 2\sqrt{-A^2} \;=\; -4 - 2\sqrt{\alpha+1} \;=\; -6 - \alpha + \frac{1}{4}\alpha^2 + \ldots \tag{3.23}$$

RS's conventions are such that $\kappa = 1/2$, $\Lambda = -5$, and $m = 2$, which we will adopt henceforth.

At level zero (most divergent), we have

$$S^{(0)}_{\text{loc}} \;=\; \int d^dy\sqrt{q}\,F(\alpha) \qquad \text{and} \qquad \mathcal{L}^{(0)} \;=\; -\frac{\Lambda}{\kappa} \;+\; \frac{m^2}{2}(\alpha+1) \tag{3.24}$$

The expanded momenta that follow from the above expansion are

$$p^{(0)}_{ab} = -\frac{1}{2}F(\alpha)\,q_{ab} - A_aA_b\,F'(\alpha) \qquad \text{and} \qquad E^{(0)}_a = -2A_a\,F'(\alpha) \tag{3.25}$$

[iii]We get rid of the $\mathcal{N}$ and ∇_aE^a terms in (D.34) by first integrating out $\mathcal{N}$, which comes down to substituting $\mathcal{N} = \nabla_aE^a/m^2$. We choose our Ansatz for S_{loc} to be manifestly covariant, such that $\nabla^b\tau_{ab} = 2\nabla^bp_{ab} = 0$. This is nice, because the momentum constraint yields

$$0 = \mathcal{H}_a = -2\nabla^bp_{ab} - A_a\nabla_bE^b = -A_a\nabla_bE^b \qquad \Rightarrow \qquad \nabla_aE^a = 0$$

I'm not very confident that this reasoning is valid.

[iv]The coefficients of the other terms are all functions of the Lorentz-scalar quantity α.

Instead of turning straight to the Hamilton constraint, we will start with the Hamilton equation $\dot{q}_{ab} = \delta H/\delta p^{ab}$ at level zero,[v]

$$\dot{q}^{\text{Lif}}_{ab} \;\simeq\; -2p^{(0)}_{ab} + p^{(0)}\, q^{\text{Lif}}_{ab} \tag{3.26}$$

where q^{Lif}_{ab} is the induced metric on the Lifshitz background, i.e. $q^{\text{Lif}}_{ab}\, dy^a dy^b = -e^{4t}\, dt^2 + e^{2t}\, d\vec{x}^2$. The tt-component and ij-components on the Lifshitz background ($\alpha = 0$) respectively give

$$c_0 + 2c_1 + 8 = 0 \qquad \text{and} \qquad c_0 - 2c_1 + 4 = 0 \tag{3.27}$$

which indeed return the expected values $c_0 = -6$ and $c_1 = -1$. The HJ method finally produces

$$F(\alpha) = -6 - \alpha + c_2\, \alpha^2 \tag{3.28}$$

which agrees with (3.23). Remember that the deviations are of order $\alpha \sim O(e^{-2r})$, while the volume form is of order $\sqrt{q} \sim O(e^{4r})$. Thus, the term quadratic in α is marginal, which means that fixing c_0 and c_1 suffices to remove the divergences. RS defined their $F(\alpha)$ for this purpose only, so it would have sufficed to determine $F(\alpha)$ only up to linear order. In other words, one can freely add to the action any term proportional to $\alpha^2 = (A_a A^a + 1)^2$.

How about the level-zero Hamilton constraint? We use the recipe outlined in Section 2.2 to get

$$\mathcal{H}^{(0)} \;=\; \frac{3}{8}F^2 - \frac{1}{2}(\alpha+1)F'F - \frac{1}{2}(\alpha^2 - 2\alpha - 3)(F')^2 - 2\alpha - 12 \;=\; 0 \tag{3.29}$$

We assume that we may write the function as a power series,

$$F(\alpha) = c_0 + c_1\, \alpha + c_2\, \alpha^2 + \ldots \tag{3.30}$$

which we plug back into (3.29). At lowest level, we have

$$\mathcal{H}^{(0)} \;=\; \frac{3}{8}c_0^2 - \frac{1}{2}c_0\, c_1 + \frac{3}{2}c_1^2 - 12 \;+\; \left(-(c_0 - 6c_1)c_2 + \frac{1}{4}c_0\, c_1 + \frac{1}{2}c_1^2 - 2\right)\alpha \;+\; O(\alpha^2) \tag{3.31}$$

We now notice that when we plug in the values $c_0 = -6$ and $c_1 = -1$, the constant and linear terms vanish identically.

$$\mathcal{H}^{(0)} \;=\; O(\alpha^2) \tag{3.32}$$

[v] The reason for doing this is that the Hamilton constraint turns out *not* to fix the value of c_0. We will check the consistency of the level-zero Hamilton constraint shortly though.

So the Hamilton constraint is consistent with the answers that we got from the Hamilton equation $\dot{q}_{ab} = \delta H/\delta p^{ab}$. In principle, one can determine c_2, c_3, etc. from this, but like we just mentioned this is not needed for removing divergences. Nevertheless, it would be interesting to exactly obtain the square-root behavior from (3.23). However, one finds two possible values $c_2 = -\frac{1}{4}$ and $c_2 = \frac{5}{12}$, which both do not fit RS's $c_2 = \frac{1}{4}$.

Concluding remarks. We have managed thus far to construct a relatively systematic way to find the counterterms that cancel the UV divergences in a class of strongly coupled QFT's at a fixed point that exhibit Lifshitz-like scaling. The analysis is of course far from complete and there is much to be done. For example, one should continue the above program at level two and four.

One could also try to see if one can get an actual RG flow by adding a scalar field to the Proca–Einstein theory, which should give something analogous to (2.37) already at level zero.

The initial objective to find an operator that corresponds to $[K, P]$ with a finite expectation value is also to be explored. It would be very interesting to find a Lifshitz analogue of AdS's conformal anomaly.

Appendix

A Killing–Cartan Metrics

The metric on a compact simple Lie group manifold can be obtained from combining group elements into the product of two so-called bi-invariant Cartan one-forms. It should be noted that this section is not intended as a thorough mathematical treatment of the above statement. It mainly serves as a motivation for obtaining the generalized form of the (thermal) AdS_3 and BTZ metrics discussed and their isometries in Section 1.1. The definitions in this section are mostly taken from Chapter 18 and 21 of Frankel's book [48].

Generalized action. There is a generalized notion of group-action on elements of the (co)tangent space about some point g on the group manifold. Such actions are sometimes referred to as induced group-actions. Generalized left-action by some group element $h \in G$ denoted as L_h. Note that the left-action is simply $L_h(g) = hg$ when $g \in G$. However, instead of left-actions on the group itself, we can also consider the left-action on the tangent space in g, so that $L_{h*} : T_gG \to T_{hg}G$. This $*$ notation is used to keep in mind that it acts as a push-forward. Similarly, $L_h^* : T_g^*G \to T_{hg}^*G$ acts on cotangent spaces as a pull-back. The right-action R_h is defined in an obviously similar fashion.

Maurer–Cartan forms. Maurer–Cartan forms are defined to be left-invariant objects. They are defined as $\theta : T_gG \to T_eG$ such that, for $v \in T_gG$ a tangent vector at g, $\theta(v) = L_{g^{-1}*}(v)$. In other words, a Maurer–Cartan form takes a tangent vector $v \in T_gG$ for any g and pushes it forward to the Lie algebra $\mathfrak{g} \equiv T_eG$. Left-invariance, $L_g^*(\theta) = \theta$, then simply follows by using the definition for θ plus the fact that L_g^* is a pull-back, so that for any $v \in T_gG$

$$L_h^*(\theta)(v) = \theta\big(L_{h*}(v)\big) = \big(L_{(hg)^{-1}} \circ L_h\big)_*(v) = L_{g^{-1}*}(v) = \theta(v) \tag{A.1}$$

The Maurer–Cartan form is not right-invariant though; it transforms in the adjoint representation, $R_h^*(\theta) = ad_{h^{-1}}(\theta)$. This follows from the fact that the right-transformed Maurer–Cartan form acts as a conjugation $R_h^*(\theta) : T_gG \to T_{h^{-1}gh}G$. Let $v \in T_gG$, then

$$R_h^*(\theta)(v) = \theta\big(R_{h*}(v)\big) = \big(L_{(gh)^{-1}} \circ R_h\big)_*(v) = \big(L_{h^{-1}} \circ L_{g^{-1}} \circ R_h\big)_*(v) = \mathrm{ad}_{h^{-1}}(\theta)(v) \tag{A.2}$$

In the last equality we used that left- and right-actions commute.

The Killing–Cartan metric. A metric tensor is defined as the inner product of basis vectors on some tangent space T_gG. For instance, on a chart with coordinates x^μ, the basis vectors are the generators ∂_μ, so that[i]

$$g_{\mu\nu} \equiv \langle \partial_\mu, \partial_\nu \rangle_g \tag{A.3}$$

So what we need is a well-defined inner product like the one above. We obtain this by first defining an inner product on the algebra $\mathfrak{g} \equiv T_eG$, which in turn induces a metric at any point g by pushing it forward to the algebra with the Maurer–Cartan form. For this, we use so-called Killing–Cartan inner product, which is defined for any matrix group G as

$$\langle u, v \rangle_e \equiv \frac{1}{2}\mathrm{tr}(u \cdot v) \qquad \text{with} \qquad u, v \in T_eG \tag{A.4}$$

where the dot $\cdot$ stands for matrix multiplication. In particular, we can let u and v be the generators T^a and T^b, then it follows that

$$\left\langle T^a, T^b \right\rangle_e = \frac{1}{2}\sum_{cd} f^{acd} f^{bdc} \tag{A.5}$$

with f^{abc} the structure constants. This induces an inner product at a general $g \in G$ by use of the Maurer–Cartan form θ.

$$\langle u, v \rangle_g = \langle \theta(u), \theta(v) \rangle_e \tag{A.6}$$

When we take $v = u$ and $\theta(u) = g^{-1}dg$ a Maurer–Cartan matrix, then we get the metric

$$ds^2 = \frac{1}{2}\mathrm{tr}[\theta^2] = \frac{1}{2}\mathrm{tr}\left[(g^{-1}dg)^2\right] \tag{A.7}$$

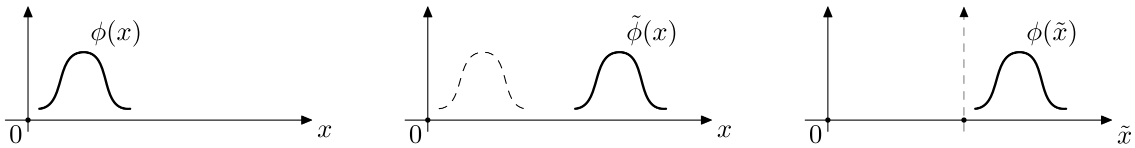
This metric is automatically left-invariant under G, since it is made out of Maurer–Cartan forms. If G is a simply connected (compact and connected) group manifold, then the metric is also right-invariant. Such a metric is called bi-invariant.

The above form of the Killing–Cartan metric is the one that we will use in practice. It gives a neat connection between a metric and its underlying isometry group. We apply this result in particular in Section 1.1.

[i] Note that we use the letter g for both the metric tensor and the group element.

B Hamilton–Jacobi Formalism

In order to refresh our knowledge of the Hamilton–Jacobi formalism we will consider the simplest example, namely that of a classical point particle in one dimension. Note that we will not discuss anything profound in this section; the mere aim is to remind ourselves of the formal structure of the Hamilton–Jacobi formalism in classical mechanics. The main concept that we will use in Chapter 2 of the main text is the possibility of expressing the dynamics in terms of boundary data. Most of this section is taken from the book by Goldstein et al. [49].



(a) Initial situation. (b) Active transformation. (c) Passive transformation.

Figure B.1: The active transformation $\phi(x) \to \tilde{\phi}(x)$ in (b) is equivalent to the passive one $\phi(x) \to \phi(\tilde{x})$ in (c), which means that $\tilde{\phi}(x) = \phi(\tilde{x})$. Notice that actively shifting the field to the right corresponds to doing a passive transformation to the left, i.e. active and passive transformations work in each other's opposite direction.

Active versus passive transformations. Before we start, it is nice to get an important thing straight, and that is the distinction between active and passive transformations. Suppose we have some function $\phi(x)$ that we wish to transform. There are two ways to go about doing this, namely we can actively move it or to to (passively) change the underlying coordinate frame. In Figure B.1, we see that actively transforming in one direction is equivalent to passively transforming in the opposite direction. Two equivalent ways of infinitesimally transforming the function ϕ are then

$$x \to \tilde{x} = x + \delta x \qquad \text{or} \qquad \phi(x) \to \tilde{\phi}(x) = \phi(x - \delta x) \tag{B.1}$$

In terms of infinitesimal changes of the function itself, i.e. $\phi \to \phi + \delta\phi$, we see that

$$\delta\phi_{\text{active}} = -\delta\phi_{\text{passive}} \tag{B.2}$$

We will use this last property in the derivation of the Hamilton–Jacobi equation below.

Derivation of the Hamilton–Jacobi equation of motion. Let us start out with the following action describing the propagation from a time $t' = 0$ to $t' = t$.

$$S[q] = \int_0^t dt' \, L(q_{(t')}, \dot{q}_{(t')}, t') \tag{B.3}$$

where the dependence on time t' is explicit. In the more familiar canonical formalisms, like the Lagrangian or Hamiltonian formalisms, the variations at the endpoints $t' = 0$ and $t' = t$ are taken to vanish, i.e. $\delta q(0) = \delta q(t) = 0$. In obtaining the Hamilton–Jacobi equations of motion, we take a different route. We still let $\delta q(0) = 0$, but we will allow for a generic (infinitesimal) variation at $q(t) \to q(t) + \delta q(t)$ as is illustrated in Figure B.2.

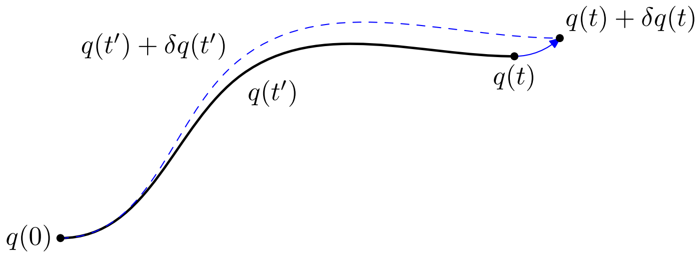


Figure B.2: One of the endpoints gets shifted in the Hamilton–Jacobi approach. Notice that the whole path gets shifted $q(t') \to q(t') + \delta q(t')$ upon shifting the endpoint $q(t)$.

Let us denote the action evaluated on the classical path by S_{cl}. The variation of S_{cl} around the classical path $q(t')$ is given by

$$\begin{aligned}\delta S_{\text{cl}} &= \int_0^t dt' \left\{ \frac{\partial L_{\text{cl}}}{\partial q}\,\delta q + \frac{\partial L_{\text{cl}}}{\partial \dot q}\,\delta \dot q + \frac{\partial L_{\text{cl}}}{\partial t'}\,\delta t' \right\} \\ &= \int_0^t dt' \left\{ \frac{\partial L_{\text{cl}}}{\partial q} - \frac{\partial}{\partial t'}\left(\frac{\partial L_{\text{cl}}}{\partial \dot q}\right)\right\}\delta q \;+\; \left(\frac{\partial L_{\text{cl}}}{\partial \dot q}\,\delta q + L_{\text{cl}}\,\delta t'\right)\Bigg|_0^t \\ &= \frac{\partial L_{\text{cl}}}{\partial \dot q}(t)\,\delta q(t) + L_{\text{cl}}\,\delta t \end{aligned} \tag{B.4}$$

where we use the fact that the classical path $q(t')$ satisfies the Euler–Lagrange equation. This is already a very interesting result, because it means that we may express the canonical momentum $p \equiv \partial L/\partial \dot q$ in terms of a (functional) derivative of the classical action with respect to the position at the endpoint,

$$\boxed{p(t) = \frac{\delta S_{\text{cl}}}{\delta q(t)}} \tag{B.5}$$

We will call this one the *second Hamilton–Jacobi equation*. In order to derive the first Hamilton–Jacobi equation, we go one step further and pick up where we left off in (B.4). In order to relate the two terms in the last line, we rewrite the active transformation $q \to q + \delta q$ in terms of the passive transformation $t \to t + \delta t$, i.e.

$$\delta q(t) \;=\; q(t - \delta t) - q(t) \;=\; -\dot q(t)\delta t \tag{B.6}$$

Plugging this into (B.4) yields *first Hamilton–Jacobi equation*

$$\delta S_{\text{cl}} = (-p\,\dot q + L_{\text{cl}})\,\delta t \qquad \Leftrightarrow \qquad \boxed{H_{\text{cl}}(q(t), p(t), t) = -\frac{\delta S_{\text{cl}}}{\delta t}} \tag{B.7}$$

One often combines the two HJ equations into a single equation,

$$\frac{\delta S_{\rm cl}}{\delta t} = -H_{\rm cl}\big(q(t), \tfrac{\partial S_{\rm cl}}{\partial q}(t), t\big) \tag{B.8}$$

The Hamilton–Jacobi equation depends only on the 'boundary' data, a fact that will be very useful in the context of the holographic renormalization group, cf. Chapter 2. Furthermore, the Hamilton–Jacobi equation should be read as a single functional differential equation for the on-shell action $S_{\rm cl}$.

Functional dependence of the classical action. Let us get another possible source of confusion out of the way. It is clear from (B.4) that the on-shell action is a functional that depends explicitly on both $q(t)$ and t. The first HJ equation follows from the total t variation and the second one follows from the q variation.

$$S_{\rm cl} = S[q_{\rm cl}(t), t] \tag{B.9}$$

To emphasize that $S_{\rm cl}$ depends on the *classical* path, we adorned the path $q(t)$ with the subscript 'cl'. We did not do the same throughout the above derivation in order to keep the formulas tidy.

C Gauss–Codazzi Equations

In this section we will review the established method of describing space- and time-like hypersurfaces in a covariant manner. The cherry on top of the hypersurface cake is definitely the Gauss–Codazzi equations. Namely, they relate the degrees of freedom of the embedding space to those intrinsic (and perpendicular) to a hypersurface in a concise and generally covariant way. Most of what is treated in this section is taken from Poisson's book [50].[i]

Convention and notation. Unless stated otherwise, we work in Euclidean signature to avoid some obnoxious subtleties. Covariant derivatives of vectors V^μ and one-forms A_μ are given by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \qquad \text{and} \qquad \nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda \tag{C.1}$$

where the Christoffel symbols

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(-\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda}\right) \tag{C.2}$$

comprise the affine connection, in terms of which the Riemann tensor is given by

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - (\mu \leftrightarrow \nu) \tag{C.3}$$

The Ricci tensor is $R_{\mu\nu} \equiv R^\lambda{}_{\mu\lambda\nu}$ and the Ricci scalar is $R \equiv g^{\kappa\lambda} R_{\kappa\lambda}$, so that the Einstein tensor is $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. The Einstein field equations are

$$G_{\mu\nu} = \kappa T_{\mu\nu} \qquad \text{where} \qquad T^{\mu\nu} \equiv \frac{2}{\sqrt{g}} \frac{\delta S_{\text{matter}}}{\delta g_{\mu\nu}} \qquad \text{and} \qquad \kappa \equiv 8\pi G \tag{C.4}$$

which comes from varying the action

$$S \;=\; \frac{1}{2\kappa} \int_{\mathcal{M}} d^{d+1}x \, \sqrt{g}\, R \;+\; \frac{1}{\kappa} \oint_{\partial\mathcal{M}} d^d y \, \sqrt{q}\, K \;+\; S_{\text{matter}} \tag{C.5}$$

The boundary integral of the trace of the extrinsic curvature $K = q^{ab} K_{ab}$ is the Gibbons–Hawking term needed for $\mathcal{L}_{\text{grav}}$ to be a proper Lagrangian density in the canonical sense. The extrinsic curvature K_{ab}, the induced metric q_{ab}, and the coordinates y^a will be defined shortly. It must be noted that the variation of the Ricci tensor is generically given by

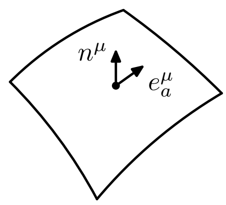
$$g^{\mu\nu}\, \delta R_{\mu\nu} = \left[-g^{\mu\nu} \nabla^\lambda \nabla_\lambda + \nabla^\mu \nabla^\nu\right] \delta g_{\mu\nu} \tag{C.6}$$

[i] We will not go into null hypersurfaces, since they involve some additional subtleties (and we will not encounter them in this work). See e.g. Chapter 3 of Poisson's book.

which is related to the Gibbons–Hawking term through Gauss' law (see e.g. §4.1.4 of [50]). The variation of the metric is not a proper tensor, $\delta g_{\mu\nu} = -g_{\mu\kappa} g_{\nu\lambda} \delta g^{\kappa\lambda}$, which requires some caution.

We use the following notation for our indices. Greek indices run over all spacetime coordinates $\mu, \nu, .. = 1, .., d+1$, roman indices from the beginning of the alphabet run over the coordinates on the hypersurfaces $a, b, .. = 1, .., d$ in the first foliation. We say 'first foliation', because we can also have a second (embedded) foliation; such quantities are labeled by roman indices from the middle of the alphabet and run over $i, j, .. = 1, .., d-1$.

Induced metric and extrinsic curvature. A hypersurface can be written in two equivalent ways: as a constraint $\Phi(x^\mu) = 0$ or as a set of parametric equations $x^\mu = x^\mu(y^a)$. The constraint is often used to get the unit vector n_μ normal to the hypersurface, whereas the parametric relations give the vector(s) e^μ_a tangent to hypersurface. The normal and tangent vectors are depicted in the figure on the right and they are given by



$$n_\mu \propto \partial_\mu \Phi \qquad \text{and} \qquad e^\mu_a = \frac{\partial x^\mu}{\partial y^a} \tag{C.7}$$

The 'constant' of proportionality in n_μ is $|g^{\mu\nu}\partial_\mu \Phi \partial_\nu \Phi|^{-1/2}$, which is obtained from the normalization condition $g^{\mu\nu} n_\mu n_\nu = \mp 1$. The red/grey minus-plus sign indicates whether the hypersurface has a time-like unit normal $(-)$, like in ADM's temporal foliation, or a space-like one $(+)$, as in dBVV's radial foliation. We will also encounter terms in our calculations that carry a $\pm$, by which we mean $-\mp$. We always have orthogonality between the unit normal and the tangent vectors, $n_\mu e^\mu_a = 0$ for all a. Note that the tangent vectors do not have unit length in general. The tangent vectors can be used to project, or pull back, a $(d+1)$-tensor onto the hypersurface, turning it into a d-tensor. The most basic instance of such a projection is the pull-back of the metric itself onto the hypersurface.

$$q_{ab} = e^\mu_a e^\nu_b g_{\mu\nu} \tag{C.8}$$

This q_{ab} is known as the induced metric, since the existence of $g_{\mu\nu}$ induces a metric on the hypersurface. The completeness relation that is also obeyed gives the full (inverse) metric in terms of a tangent piece involving q_{ab} and a normal piece involving n_μ.

$$g^{\mu\nu} = q^{ab}\, e^\mu_a e^\nu_b \mp n^\mu n^\nu \tag{C.9}$$

where q^{ab} is the inverse of the induced metric and $n^\mu = g^{\mu\nu} n_\nu$. All quantities intrinsic to the hypersurface can be obtained from q_{ab} in a way that is similar to how we calculated quantities from the full spacetime metric $g_{\mu\nu}$. However, in order to relate these intrinsic quantities like the Ricci scalar to the full-spacetime Ricci scalar we need some more information. This information should involve how the hypersurface is embedded into the full spacetime, i.e. extrinsic quantities. This is described by the extrinsic curvature, which is given by the amount of change in q_{ab} as one moves in the normal direction. More precisely, the extrinsic curvature is given by (1/2 times) the Lie derivative of the metric in the direction of $n = n^\mu \partial_\mu$, pulled back onto the hypersurface.

$$K_{ab} = \frac{1}{2} e_a^\mu e_b^\nu \, \mathscr{L}_n g_{\mu\nu} \qquad \text{or equivalently} \qquad K_{ab} = e_a^\mu e_b^\nu \nabla_{(\mu} n_{\nu)} \tag{C.10}$$

The latter may not be obviously symmetric under $\mu \leftrightarrow \nu$, though this symmetry is easily exposed by using the Leibniz rule and $n_\mu e_a^\mu = 0$. In mathematical lingo, the induced metric and extrinsic curvature are called the first and second fundamental forms respectively.

The Gauss–Weingarten equation. To find the equations relating the intrinsic and extrinsic quantities in terms of the full-spacetime quantities we start out with the useful identity known as the Gauss–Weingarten equation, which gives an explicit expression for the derivative of a tangent basis vector projected along the tangent direction, $e_a^\lambda \nabla_\lambda e_b^\mu$. It may come as some surprise that this quantity will not be purely tangential, even though it is 'projected' along the hypersurface. The non-tangential piece will turn out to be proportional to the extrinsic curvature K_{ab}. In order to keep the reasoning clear, let us introduce some generic vector $\tilde{V}^\mu$ that is tangent to the hypersurface, i.e. $n_\mu \tilde{V}^\mu = 0$ and $\tilde{V}^\mu = e_a^\mu V^a$. Of course, at some point we will let $\tilde{V}^\mu \to e_a^\mu$, but because the tangent basis vectors are also used for projecting onto the hypersurface it is easy to lose the overview of the calculation.[ii] The derivative of $\tilde{V}^\mu$ along the hypersurface is given by

$$e_a^\lambda \nabla_\lambda \tilde{V}^\mu = \left[q^{bc} e_b^\mu e_c^\lambda \mp n^\mu n^\lambda\right] e_a^\kappa \nabla_\kappa \tilde{V}_\lambda = \left[q^{bc} e_a^\kappa e_c^\lambda \nabla_\kappa \tilde{V}_\lambda\right] e_b^\mu \mp \left[e_a^\kappa n^\lambda \nabla_\kappa \tilde{V}_\lambda\right] n^\mu \tag{C.11}$$

So the tangent derivative of a tangent vector is not necessarily tangent as well, since it apparently has a component along n^μ. The covariant derivative ∇_μ induces a covariant derivative that is purely intrinsic to the hypersurface by pulling it back onto the hypersurface. This induced covariant derivative is denoted by ∇_a. For any $\tilde{V}^\mu$ (not just tangent vectors) we have

$$\nabla_a V_b \equiv e_a^\kappa e_b^\lambda \nabla_\kappa \tilde{V}_\lambda \tag{C.12}$$

where V^a is the tangent piece in the decomposition $\tilde{V}^\mu = V^a e_a^\mu \mp V n^\mu$. This induced covariant derivative can equivalently be defined analogous to (C.1) and (C.2) in terms q_{ab} instead of $g_{\mu\nu}$. The first term on the right-hand side of (C.11) is simply $(\nabla_a V^b) e_b^\mu$. Let us turn to the second term. We use the Leibniz rule and the fact that we chose $\tilde{V}^\mu$ to be tangent to the hypersurface so that $n_\mu \tilde{V}^\mu = 0$ in order to write $n^\lambda \nabla_\kappa V_\lambda = -V^\lambda \nabla_\kappa n_\lambda$. Through the tangential property $\tilde{V}^\mu = e_b^\mu V^b$ we recognize the extrinsic curvature $e_a^\lambda e_b^\nu \nabla_\lambda n_\nu = K_{ab}$, so that we end up with

$$e_a^\lambda \nabla_\lambda \tilde{V}^\mu = \left(\nabla_a V^b\right) e_b^\mu \pm \left(K_{ab} V^b\right) n^\mu \tag{C.13}$$

[ii] As a notational note, in cases where it may otherwise lead to confusion, we give the full $(d+1)$-vectors a tilde while we leave d-vectors (on the tangent basis) untouched.

So the non-tangent (normal) piece of the tangent derivative of a tangent vector is proportional to the extrinsic curvature K_{ab}. We note that $\{e^\mu_1, .., e^\mu_d\}$ is a collection of d tangent vectors, so we simply let $\tilde{V}^\mu \to e^\mu_a$ in the above relation. Equivalently, we can let $V^b \to \delta^b_a$. We finally end up with

$$e^\lambda_a \nabla_\lambda e^\mu_b = \Gamma^c_{ab} e^\mu_c \pm K_{ab} n^\mu \tag{C.14}$$

This is known as the Gauss–Weingarten equation. The extrinsic curvature is sometimes defined to be the normal component of $e^\lambda_a \nabla_\lambda e^\mu_b$.

The Gauss–Codazzi equations. We are interested in relating the intrinsic Riemann tensor to the full-spacetime one. The Riemann tensor is the measure for the failure of the covariant derivative of a (constant) vector $\tilde{V}^\mu$ to be parallelly transported.

$$[\nabla_\mu, \nabla_\nu]\tilde{V}^\rho = \tilde{R}^\rho{}_{\sigma\mu\nu}\tilde{V}^\sigma \qquad \text{and} \qquad [\nabla_a, \nabla_b]V^c = R^c{}_{dab}V^d \tag{C.15}$$

So we should take the second tangent derivative and subtract by some permutation like $a \leftrightarrow b$. We can take the tangent derivative of (C.14) and then use the Gauss–Weingarten equation itself again to rewrite the resulting equation into something useful. This 'something useful' will be in fact the Gauss–Codazzi equations. So let us start with the tangent derivative of the Gauss–Weingarten equation (C.14).

$$e^\kappa_a \nabla_\kappa \left(e^\lambda_b \nabla_\lambda e^\mu_c\right) = e^\kappa_a \nabla_\kappa \left(\Gamma^d_{bc} e^\mu_d \pm K_{bc} n^\mu\right) \tag{C.16}$$

Let us first rewrite the left-hand side first. The pieces that will be rewritten by use of the Gauss–Weingarten equation (C.14) are underlined.

$$\begin{aligned} e^\kappa_a \nabla_\kappa \left(e^\lambda_b \nabla_\lambda e^\mu_c\right) &= e^\kappa_a e^\lambda_b \nabla_\kappa \nabla_\lambda e^\mu_c + \underline{e^\kappa_a \nabla_\kappa e^\lambda_b} \nabla_\lambda e^\mu_c \\ &= e^\kappa_a e^\lambda_b \nabla_\kappa \nabla_\lambda e^\mu_c + \underline{\nabla_\lambda e^\mu_c \left(\Gamma^d_{ab} e^\lambda_d\right.} \pm K_{ab} n^\lambda\big) \\ &= e^\kappa_a e^\lambda_b \nabla_\kappa \nabla_\lambda e^\mu_c + \Gamma^d_{ab} \left(\Gamma^e_{cd} e^\mu_e \pm K_{cd} n^\mu\right) \pm K_{ab} n^\lambda \nabla_\lambda e^\mu_c \end{aligned} \tag{C.17}$$

Now, let us turn to the right-hand side of (C.16).

$$\begin{aligned} e^\lambda_a \nabla_\lambda \left(\Gamma^d_{bc} e^\mu_d \pm K_{bc} n^\mu\right) &= \partial_a \Gamma^d_{bc} e^\mu_d + \Gamma^d_{bc} \underline{e^\lambda_a \nabla_\lambda e^\mu_d} \pm \partial_a K_{bc} n^\mu \pm K_{bc} e^\lambda_a \nabla_\lambda n^\mu \\ &= \partial_a \Gamma^d_{bc} e^\mu_d + \Gamma^d_{bc} \left(\Gamma^e_{ad} e^\mu_e \pm K_{ad} n^\mu\right) \pm \partial_a K_{bc} n^\mu \pm K_{bc} e^\lambda_a \nabla_\lambda n^\mu \end{aligned} \tag{C.18}$$

which combine into

$$\begin{aligned} e^\kappa_a e^\lambda_b \nabla_\kappa \nabla_\lambda e^\mu_c = \left(\partial_a \Gamma^d_{bc} + \Gamma^e_{bc}\Gamma^d_{ae} - \Gamma^e_{ab}\Gamma^d_{ce}\right) e^\mu_d &\pm \left(\partial_a K_{bc} + \Gamma^d_{bc} K_{ad} - \Gamma^d_{ab} K_{cd}\right) n^\mu \\ &\pm K_{bc} e^\lambda_a \nabla_\lambda n^\mu \mp K_{ab} n^\lambda \nabla_\lambda e^\mu_c \end{aligned} \tag{C.19}$$

Like we mentioned above, the Riemann tensor may be obtained through the commutator of covariant derivatives. So we should replace $\nabla_\kappa \nabla_\lambda$ in (C.19) by the commutator $[\nabla_\kappa, \nabla_\lambda]$, which translates to subtracting a permutation $a \leftrightarrow b$ from the right-hand side. It then follows that

$$e_a^\kappa e_b^\lambda e_c^\nu \tilde{R}^\mu{}_{\nu\kappa\lambda} = e_d^\mu R^d{}_{cab} \mp (\nabla_b K_{ac} - \nabla_a K_{bc}) n^\mu \mp \left(K_{ac} e_b^\lambda \nabla_\lambda n^\mu - K_{bc} e_a^\lambda \nabla_\lambda n^\mu\right) \tag{C.20}$$

This expression relates the Riemann tensor of the full spacetime to the intrinsic Riemann tensor and the extrinsic curvature. We get the Gauss–Codazzi equations by decomposing (C.20) into its tangent and normal components. By 'decomposing' we mean contracting with $e_{e\,\mu}$ and n_μ, respectively. We then end up with the Gauss–Codazzi equations.

$$e_a^\kappa e_b^\lambda e_c^\mu e_d^\nu \tilde{R}_{\kappa\lambda\mu\nu} = R_{abcd} \mp (K_{ad} K_{bc} - K_{ac} K_{bd}) \tag{C.21a}$$

$$n^\kappa e_a^\lambda e_b^\mu e_c^\nu \tilde{R}_{\kappa\lambda\mu\nu} = \nabla_b K_{ac} - \nabla_c K_{ab} \tag{C.21b}$$

To get the first Gauss–Codazzi equation we used $e_d^\mu e_a^\nu \nabla_\nu n_\mu = K_{ad}$ and for the second one we used $n_\mu \nabla_\lambda n^\mu = \nabla_\lambda (n_\mu n^\mu)/2 = 0$, because n^μ is normalized to one. We have also relabeled some indices.

Relation between the Ricci scalars. It is useful to relate the Ricci scalars of the embedding space and the hypersurface, since that is the quantity that appears in the Einstein–Hilbert action. The Ricci scalar can be written using the completeness relation $g^{\mu\nu} = q^{ab} e_a^\mu e_b^\nu \mp n^\mu n^\nu$.

$$\tilde{R} = g^{\kappa\lambda} g^{\mu\nu} \tilde{R}_{\kappa\mu\lambda\nu} = \left(q^{ab} q^{cd} e_a^\kappa e_b^\lambda e_c^\mu e_d^\nu \mp 2 q^{ab} n^\kappa n^\lambda e_a^\mu e_b^\nu\right) \tilde{R}_{\kappa\mu\lambda\nu} \tag{C.22}$$

We immediately see that the first term in the parentheses can easily be rewritten using the first Gauss–Codazzi equation (C.21a).

$$q^{ab} q^{cd} e_a^\kappa e_b^\lambda e_c^\mu e_d^\nu \tilde{R}_{\kappa\mu\lambda\nu} = R \mp \left(K^{ab} K_{ab} - K^2\right) \tag{C.23}$$

The second term, on the other hand, requires some work. We use the completeness relation again in order to get

$$\begin{aligned} 2q^{ab} n^\kappa n^\lambda e_a^\mu e_b^\nu \tilde{R}_{\kappa\mu\lambda\nu} &= 2 n^\kappa n^\lambda (g^{\mu\nu} \pm n^\mu n^\nu) \tilde{R}_{\kappa\mu\lambda\nu} \\ &= 2 n^\kappa n^\lambda \tilde{R}^\mu{}_{\kappa\mu\lambda} \\ &= 2 n^\lambda (\nabla_\kappa \nabla_\lambda - \nabla_\lambda \nabla_\kappa) n^\kappa \\ &= 2\nabla_\kappa \left(n^\lambda \nabla_\lambda n^\kappa - n^\kappa \nabla_\lambda n^\lambda\right) - 2\left(\nabla_\kappa n_\lambda \nabla^\lambda n^\kappa - \nabla_\kappa n^\kappa \nabla_\lambda n^\lambda\right) \\ &= 2\nabla_\kappa \left(n^\lambda \nabla_\lambda n^\kappa - n^\kappa \nabla_\lambda n^\lambda\right) - 2\left(K^{ab} K_{ab} - K^2\right) \end{aligned} \tag{C.24}$$

We used the Leibniz rule in the fourth line and $\nabla_\kappa n^\kappa \nabla_\lambda n^\lambda = K^2$ and $\nabla_\kappa n_\lambda \nabla^\lambda n^\kappa = K_{ab} K^{ab}$ in the last line. Combining these two terms we end up with the relations between the Ricci scalars

$$\tilde{R} = R \mp \left(K^2 - K^{ab} K_{ab}\right) \mp 2\nabla_\kappa \left(n^\lambda \nabla_\lambda n^\kappa - n^\kappa \nabla_\lambda n^\lambda\right) \tag{C.25}$$

The last term on the right-hand side becomes a boundary term in the action, which is closely related to the Gibbons–Hawking term. This is the starting point of the next section.

D Temporal and Radial Gravitational Hamiltonian

The Arnowitt–Deser–Misner (ADM) formalism [51] is a Hamiltonian approach to general relativity. Constructing a Hamiltonian is useful for finding generalized notions of conserved quantities such as the ADM mass and angular momentum. The Hamiltonian formulation of gravity depends a key role throughout this entire thesis. Spacetime is foliated into an infinite stack of constant-time hypersurfaces, where a time-like flow brings us from one hypersurface to the next. This picture is somewhat altered in Hamilton–Jacobi gravity. In HJ gravity, we foliate spacetime along a *radial* flow instead. These two situations are depicted in Figure D.1 and D.2 and we will treat them in parallel. Just like the previous section is most of this is taken from Poisson's book [50].

The ADM decomposition. In ADM's picture, spacetime $\mathcal{M}$ is foliated into a stack of 'constant-time' hypersurfaces Σ_t, in other words $\mathcal{M} = \mathbb{R}\times\Sigma_t$, cf. Figure D.1. In the De Boer, Verlinde, Verlinde's (dBVV) picture spacetime is foliated in terms of 'constant-radius' hypersurfaces Σ_r as $\mathcal{M} = \mathbb{R}\times\Sigma_r$.[i] We can define a scalar function $\Phi(x^\mu) = f$, which describes every hypersurface through $\Phi =$ constant. The parameter $f \in \{t, r\}$ is called the foliation parameter. Unlike we had before, the parametric relations $x^\mu = x^\mu(f, y^a)$ depend on $d+1$ parameters instead of d. This has to do with the fact that we are foliating spacetime instead of describing only a single hypersurface. Similar to before, the normal and tangent vectors to the hypersurface Σ_f are

$$n_\mu = N\partial_\mu\Phi \qquad \text{and} \qquad e^\mu_a = \frac{\partial x^\mu}{\partial y^a}\bigg|_{f\ \text{fixed}} \tag{D.1}$$

The normalization factor of the normal vector $N \equiv |g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi|^{-1/2}$ is known as the lapse function. In the case of a foliation, there is another interesting vector, which is known as the flow vector.

$$f^\mu = \frac{\partial x^\mu}{\partial f}\bigg|_{y^a\ \text{fixed}} \tag{D.2}$$

This vector points in the direction of increasing 'time' $f = t$ or 'radius' $f = r$ and it is normalized according to $f^\mu\partial_\mu\Phi = 1$. Notice the abuse of notation, using f for both the foliation parameter and the flow vector $f = f^\mu\partial_\mu$. However, it will be clear from the context which one of the two we mean. It is important to realize that f^μ does not necessarily point in same direction as the unit normal n^μ.

$$f^\mu = Nn^\mu + N^a e^\mu_a \tag{D.3}$$

The tangent piece N^a is called the shift function. Note that the normal component N follows from the definitions (D.2) and (D.1), which yield $f^\mu n_\mu = \mp N$. Combining the

[i]Note that this puts a rather strong condition on the topological properties of $\mathcal{M}$. In dBVV's case it suffices to assume that this foliation is accurate in the asymptotic region $r \sim \infty$.

above expressions, we see that $dx^\mu = e^\mu_a dy^a + f^\mu df$ span the cotangent space. The metric will thus be written as

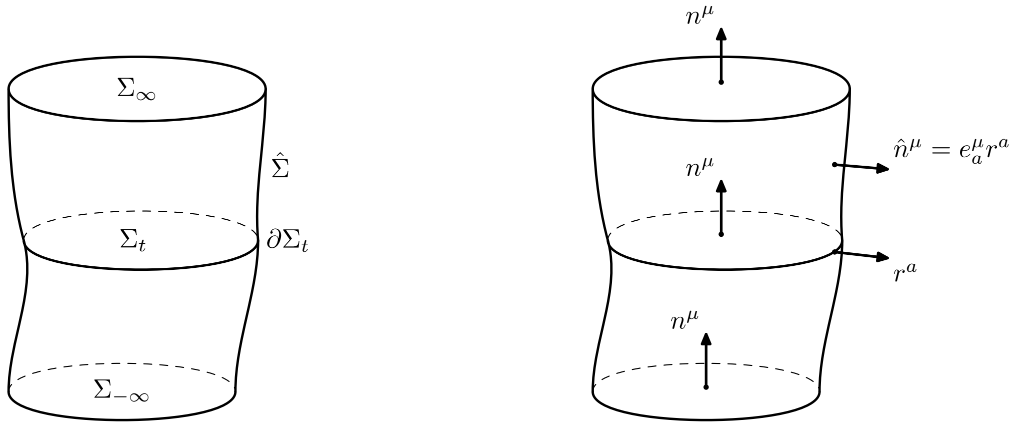
$$ds^2 = N^2 df^2 + q_{ab}\left(dy^a + N^a df\right)\left(dy^b + N^b df\right) \tag{D.4}$$

In matrix notation, the metric and its inverse are given by

$$(g_{\mu\nu}) = \begin{pmatrix} N^2 + N_c N^c & N_a \\ N_b & q_{ab} \end{pmatrix} \qquad \text{and} \qquad (g^{\mu\nu}) = N^{-2}\begin{pmatrix} 1 & N^a \\ N^b & q^{ab}N^2 + N^a N^b \end{pmatrix} \tag{D.5}$$

Some may introduce the ADM decomposition starting out with this metric.

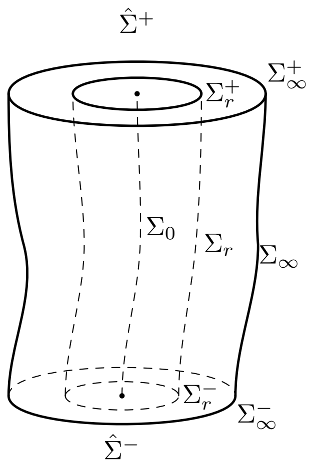
Foliations and notations. It is also useful to define the following cascade of embeddings.



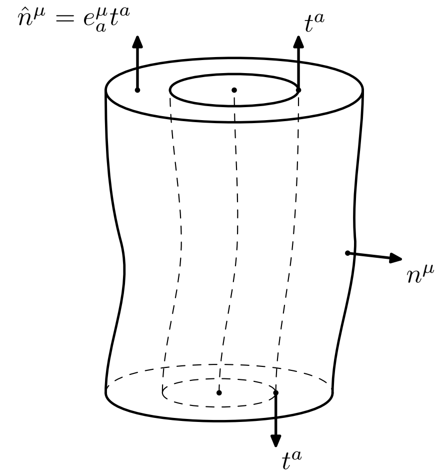
(a) Spatial hypersurfaces. (b) Normal vectors.

Figure D.1: The normal vector at $t = -\infty$ points inward instead of outward, which must be compensated for by an extra minus sign. The normal vector $\hat{n}^\mu$ on $\hat{\Sigma} = \mathbb{R} \times \partial\Sigma_t$ is orthogonal to n^μ on Σ_t at any value of t. Also note that in n spacetime dimensions, there are $n-3$ suppressed dimensions in this figure.

ADM foliation	manifold	metric	intr. curv.	extr. curv.	normal	tangent
full spacetime	$\mathcal{M}$	$g_{\mu\nu}$	$\tilde{R}^\mu{}_{\nu\kappa\lambda}$			
spatial hypersurface	Σ_t	q_{ab}	$R^a{}_{bcd}$	K_{ab}	n^μ	e^μ_a
spatial boundary	$\hat{\Sigma}$	$\hat{q}_{ab}$		$\hat{K}_{ab}$	$\hat{n}^\mu$	$\hat{e}^\mu_a$
hypersurf. boundary	$\partial\Sigma_t$	γ_{ij}		k_{ij}	r^a	e^a_i



(a) Radial hypersurfaces.



(b) Normal vectors.

Figure D.2: The normal vector $\hat{n}^\mu$ on $\hat{\Sigma}^\pm = \mathbb{R}\times\Sigma^\pm_r$ is orthogonal to n^μ on Σ_r at any value of r. Also note that in n spacetime dimensions, there are again $n-3$ suppressed dimensions in this figure.

dBVV foliation	manifold	metric	intr. curv.	extr. curv.	normal	tangent
full spacetime	$\mathcal{M}$	$g_{\mu\nu}$	$\tilde{R}^\mu{}_{\nu\kappa\lambda}$			
radial hypersurface	Σ_t	q_{ab}	$R^a{}_{bcd}$	K_{ab}	n^μ	e^μ_a
time-like hypersurf.	$\hat{\Sigma}^\pm$	$\hat{q}_{ab}$		$\hat{K}^\pm_{ab}$	$\hat{n}^\mu$	$\hat{e}^\mu_a$
hypersurface edges	$\Sigma^\pm_r$	γ_{ij}		$k^\pm_{ij}$	t^a	e^a_i

The gravitational Lagrangian. A Hamiltonian is generally obtained by taking the Legendre transform of the Lagrangian that removes the dependence on the (generalized) velocity in favor of a canonical momentum, e.g. $H = p\dot{q} - L$. So we must first specify what the generalization of the velocity $\dot{q}$ is in the gravitational action. The gravitational equivalent of a point-particle's position q is the induced metric q_{ab} on the constant-time or constant-radius hypersurface Σ_f. We define the generalized velocity $\dot{q}_{ab}$ as the Lie derivative along the flow vector $f^\mu\partial_\mu$.

$$\dot{q}_{ab} \equiv \mathscr{L}_f q_{ab} \tag{D.6}$$

We have $\mathscr{L}_f q_{ab} = e^\mu_a e^\nu_b \mathscr{L}_f g_{\mu\nu}$, because the tangent vectors can be parallelly transported along f^μ, so they commute with Lie differentiation. Notice then that $\dot{q}_{ab}$ looks very much like the extrinsic curvature $K_{ab} = \frac{1}{2}e^\mu_a e^\nu_b \mathscr{L}_n g_{\mu\nu}$, except for a factor of two and, more importantly, the fact that the derivative is taken along f^μ instead of n^μ. There is an extra factor of N and a completely separate piece coming from N^a in (D.3), which follows from

$$\begin{aligned}\mathscr{L}_f g_{\mu\nu} &= \nabla_\mu f_\nu + \nabla_\nu f_\mu \\ &= \nabla_\mu (N n_\nu) + \nabla_\mu (e^\mu_a N^a) + (\mu \leftrightarrow \nu) \\ &= \nabla_\mu N n_\nu + N \nabla_\mu n_\nu + \nabla_\mu (e^\mu_a N^a) + (\mu \leftrightarrow \nu)\end{aligned} \tag{D.7}$$

Just like we had in the previous section, ∇_μ is the covariant derivative in the full spacetime, while ∇_a is its induced cousin. Projecting along $e^\mu_a e^\nu_b$ gives

$$\dot{q}_{ab} = 2NK_{ab} + \nabla_a N_b + \nabla_b N_a \tag{D.8}$$

We saw that we can use the Gauss–Codazzi equations split up the Riemann tensor in terms of intrinsic quantities $\sim q_{ab}$ and extrinsic quantities $\sim K_{ab}$. Since we now know that the generalized velocity $\dot{q}_{ab}$ and the extrinsic curvature K_{ab} are very closely related, the first step towards finding the gravitational Hamiltonian is plugging in the contracted Gauss–Codazzi equation (C.25) into the Einstein–Hilbert Lagrangian. This will yield a proper Lagrangian as a function of $(q_{ab}, \dot{q}_{ab})$.

$$\begin{aligned}2\kappa\, S_{\text{EH}} &= \int\limits_{\mathcal{M}} d^{d+1}x\sqrt{g}\,\left(\tilde{R} - 2\Lambda\right) \\ &= \int\limits_{\mathcal{M}} d^{d+1}x\sqrt{g}\,\left(R + K^2 - K^{ab}K_{ab} - 2\Lambda\right) + 2\oint\limits_{\partial\mathcal{M}} \left(n^\lambda \nabla_\lambda n_\kappa - n_\kappa \nabla_\lambda n^\lambda\right) d\sigma^\kappa\end{aligned} \tag{D.9}$$

We must be careful with describing the boundary $\partial\mathcal{M}$. Let us focus on the boundary term in (D.9) for the moment. When we split up the spacetime boundary as $\partial\mathcal{M} = \Sigma_{-\infty} \cup \Sigma_\infty \cup \hat{\Sigma}$ (temporal) or $\partial\mathcal{M} = \Sigma_\infty \cup \hat{\Sigma}^+ \cup \hat{\Sigma}^-$ (radial), we can write the surface integral using the normal vectors depicted in Figure D.1. The surface elements for the temporal foliation are

$$d\sigma^\kappa = \begin{cases} n^\kappa \sqrt{q}\, d^d y & \text{on } \Sigma_\infty \\ -n^\kappa \sqrt{q}\, d^d y & \text{on } \Sigma_{-\infty} \\ \hat{n}^\kappa \sqrt{\hat{q}}\, d^d \hat{y} & \text{on } \hat{\Sigma} \end{cases} \qquad \text{(temporal foliation)} \tag{D.10}$$

and the surface elements in the radial foliation are

$$d\sigma^\kappa = \begin{cases} n^\kappa \sqrt{q}\, d^d y & \text{on } \Sigma_\infty \\ \hat{n}^\kappa \sqrt{\hat{q}}\, d^d \hat{y} & \text{on } \hat{\Sigma}^\pm \end{cases} \qquad \text{(radial foliation)} \tag{D.11}$$

Remember that $2n^\kappa \nabla_\lambda n_\kappa = \nabla_\lambda (n^\kappa n_\kappa) = 0$, $\nabla_\lambda n^\lambda = K$ and $\hat{n}^\kappa n_\kappa = 0$, so that

$$\begin{aligned}&2\oint\limits_{\partial\mathcal{M}} \left(n^\lambda \nabla_\lambda n_\kappa - n_\kappa \nabla_\lambda n^\lambda\right) d\sigma^\kappa \\ &= \begin{cases} \int\limits_{\Sigma_{-\infty}} d^d y \sqrt{q}\; 2K - \int\limits_{\Sigma_\infty} d^d y \sqrt{q}\; 2K - 2\int\limits_{\hat{\Sigma}} d^d \hat{y} \sqrt{\hat{q}}\; n^\kappa n^\lambda \nabla_\kappa \hat{n}_\lambda & \text{(temporal)} \\ -\int\limits_{\Sigma_\infty} d^d y \sqrt{q}\; 2K - 2\int\limits_{\hat{\Sigma}^\pm} d^d \hat{y} \sqrt{\hat{q}}\; n^\kappa n^\lambda \nabla_\kappa \hat{n}_\lambda & \text{(radial)} \end{cases}\end{aligned} \tag{D.12}$$

By $\int_{\hat{\Sigma}^\pm}$ we mean the sum of $\int_{\hat{\Sigma}^+}$ and $\int_{\hat{\Sigma}^-}$. We now wish to relate these surface terms to the Gibbons–Hawking term, which we must also split up into integrals over the same three pieces of $\partial\mathcal{M}$. We use the same surface elements (D.10). Because of the sign issue coming from the fact that n^μ points inward at $t = -\infty$, we have $K = \nabla_\lambda n^\lambda_{\text{outward}} = -\nabla_\lambda n^\lambda$ at $t = -\infty$. This sign issue is absent at $t = \infty$. The Gibbons–Hawking term is somewhat different at the spatial boundary $\hat{\Sigma}$, namely instead of K we have $\hat{K}$, where $\hat{K}_{ab} = \hat{e}^\mu_a \hat{e}^\nu_b \nabla_\mu \hat{n}_\nu$. The Gibbons–Hawking term thus becomes

$$2\kappa\ S_{\text{GH}} = \begin{cases} -\displaystyle\int\limits_{\Sigma_{-\infty}} d^dy\sqrt{q}\ 2K + \int\limits_{\Sigma_\infty} d^dy\sqrt{q}\ 2K + \int\limits_{\hat{\Sigma}} d^d\hat{y}\sqrt{\hat{q}}\ 2\hat{K} & \text{(temporal)} \\ \displaystyle\int\limits_{\Sigma_\infty} d^dy\sqrt{q}\ 2K + \int\limits_{\hat{\Sigma}^\pm} d^d\hat{y}\sqrt{\hat{q}}\ 2\hat{K}^\pm & \text{(radial)} \end{cases} \tag{D.13}$$

Notice that we did not have the above sign issue in the radial foliation. Adding this Gibbons–Hawking term to the Einstein–Hilbert term (D.9) makes all of the unhatted terms cancel, but the hatted ones do not exactly cancel. Let us rewrite the (temporal) $\hat{\Sigma}$ integrand of the Einstein–Hilbert term plus the Gibbons–Hawking term (times 1/2) as follows.

$$\begin{aligned} \hat{K} - n^\mu n^\nu \nabla_\mu \hat{n}_\nu &= \left(\hat{q}^{ab}\hat{e}^\mu_a \hat{e}^\nu_b - n^\mu n^\nu\right)\nabla_\mu \hat{n}_\nu \\ &= \left(g^{\mu\nu} - \hat{n}^\mu \hat{n}^\nu - n^\mu n^\nu\right)\nabla_\mu \hat{n}_\nu \\ &= \left(q^{ab} e^\mu_a e^\nu_b - \hat{n}^\mu \hat{n}^\nu\right)\nabla_\mu \hat{n}_\nu \\ &= q^{ab} e^\mu_a e^\nu_b\ \nabla_\mu \hat{n}_\nu \\ &= q^{ab}\nabla_a r_b \\ &= \left(\gamma^{ij} e^a_i e^b_j + r^a r^b\right)\nabla_a r_b \\ &= \gamma^{ij} e^a_i e^b_j \nabla_a r_b = k \end{aligned} \tag{D.14}$$

We can repeat these steps in a similar fashion for the radial foliation, for which we obtain two boundary remainders $k^\pm$.

The Lagrangian we have obtained in this way is still a proper Lagrangian, because the embedded extrinsic curvatures k and $k^\pm$ should not be viewed as canonical variables, cf. the discussion in [32]. The gravitational Lagrangian is defined through writing the action as an integral over f.

$$S_{\text{grav}} = S_{\text{EH}} + S_{\text{GH}} = \int_{-\infty}^{\infty} df\ L_{\text{grav}} \tag{D.15}$$

such that the gravitational Lagrangian is

$$2\kappa\, L_{\text{grav}} = \int\limits_{\Sigma_t} d^d y\, N\sqrt{q}\left\{R - 2\Lambda \mp \left(K^2 - K^{ab}K_{ab}\right)\right\} + \begin{cases} \oint\limits_{\partial\Sigma_t} d^{d-1}\theta\, N\sqrt{\gamma}\; 2k & \text{(temporal)} \\ \oint\limits_{\Sigma_r^\pm} d^{d-1}\theta\, N\sqrt{\gamma}\; 2k^\pm & \text{(radial)} \end{cases} \tag{D.16}$$

where we used the contracted Gauss–Codazzi equation (C.25) that relates the Ricci scalars and $\sqrt{g} = N\sqrt{q}$ and $\sqrt{q} = N\sqrt{\gamma}$. We have succeeded in writing the Lagrangian in the proper canonical form. We can use this form of the action to define the Hamiltonian by means of a Legendre transformation, which we will do now.

The gravitational Hamiltonian. The Legendre transformation in the gravitational case is a direct generalization of the 1D point particle's $H(p,x) = p\dot{x} - L(\dot{x},x)$.

$$H_{\text{grav}}(q,p) = \int\limits_{\Sigma_t} d^d y\,\sqrt{q}\; p^{ab}\dot{q}_{ab} \;-\; L_{\text{grav}}(q,\dot{q}) \qquad \text{with} \qquad p^{ab} \equiv \frac{1}{\sqrt{q}}\frac{\delta L_{\text{grav}}}{\delta \dot{q}_{ab}} \tag{D.17}$$

Recall that $\dot{q}_{ab}$ was defined as the Lie derivative of the induced metric along the flow vector f^μ in (D.6), i.e. $\dot{q}_{ab} = \mathscr{L}_f q_{ab}$. The lapse and the shift are Lagrange multipliers, because they have no canonical conjugate like $\dot{N}$ and $\dot{N}^a$. The gravitational Lagrangian (D.16) clearly depends on q_{ab}, but its dependence on $\dot{q}_{ab}$ is not so obvious; it depends on K_{ab} instead. We must thus express the derivative in the definition of p^{ab} as

$$\frac{\delta}{\delta \dot{q}_{ab}} = \frac{\partial K_{cd}}{\partial \dot{q}_{ab}}\frac{\delta}{\delta K_{cd}} = \frac{1}{2N}\frac{\delta}{\delta K_{ab}} \tag{D.18}$$

so that the canonical momentum is

$$p^{ab} = \pm\frac{1}{2\kappa}\left(K^{ab} - q^{ab}K\right) \tag{D.19}$$

After plugging this into (D.17) we finally end up with the gravitational Hamiltonian

$$H_{\text{grav}} = H_{\text{bulk}} + H_{\text{bndy}} \tag{D.20a}$$

which is more explicitly given by

$$H_{\text{grav}} = \int\limits_{\Sigma_f} d^d y \sqrt{q}\Big(N\mathcal{H} + N^a\mathcal{H}_a\Big) \;+\; \oint\limits_{\partial\Sigma_t \text{ or } \Sigma_r^\pm} d^{d-1}\theta\sqrt{\gamma}\Big(N\mathcal{H}^{\text{bndy}} + N^a\mathcal{H}_a^{\text{bndy}}\Big) \tag{D.20b}$$

The so-called Hamilton and momentum constraint functions are respectively

$$\mathcal{H} = -\frac{R-2\Lambda}{2\kappa} \pm 2\kappa\big(p^{ab}p_{ab} - \frac{1}{d-1}p^2\big) \qquad \text{and} \qquad \mathcal{H}_a = -2\nabla^b p_{ab} \tag{D.20c}$$

and the boundary terms are

$$\mathcal{H}^{\text{bndy}} = \begin{cases} k/\kappa & (\text{temp.}) \\ k^{\pm}/\kappa & (\text{rad.}) \end{cases} \qquad \text{and} \qquad \mathcal{H}_a^{\text{bndy}} = \begin{cases} 2r^b p_{ab} & (\text{temp.}) \\ 2t^b p_{ab} & (\text{rad.}) \end{cases} \tag{D.20d}$$

The equations of motion consist of the Hamilton constraint $\mathcal{H} = 0$ and the momentum constraints $\mathcal{H}_a = 0$ together with the Hamiltonian evolution equations $\dot{q}_{ab} = \delta H_{\text{grav}}/\delta p^{ab}$ and $\dot{p}^{ab} = -\delta H_{\text{grav}}/\delta q_{ab}$.

The variation of the above boundary terms cancel all boundary terms that arise from varying the bulk piece when one assumes $\delta N = \delta N^a = \delta q_{ab} = 0$ at the boundary (with no restriction on δp^{ab} at the boundary as it is conjugate to q_{ab}). Just like in the Gibbons–Hawking story, the derivatives of these variations are not fixed.

The matter-field Hamiltonians. In Chapter 2 we will need the Hamiltonian for the matter fields as well. In the literature, the gauge $N = 1$, $N^a = 0$ is often picked before going through the Legendre transformation etc., though there is nothing that tells us that this is justified. For example, the lesson one may learn from the gravitational case is that, if one had chosen the mass gauge $N = 1$, $N^a = 0$, the 'ADM' angular momentum [52] would have been impossible to compute. Let us repeat the above steps for a scalar field ϕ and a one-form A_μ, whose Lagrangians are respectively given by

$$L_\phi = -\int d^d y \sqrt{q}\, N \left\{ \frac{1}{2} \partial^\mu \phi \, \partial_\mu \phi + V(\phi) \right\} \tag{D.21a}$$

$$L_A = -\int d^d y \sqrt{q}\, N \left\{ \frac{1}{4} \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} + \frac{m^2}{2} \tilde{A}^\mu \tilde{A}_\mu \right\} \tag{D.21b}$$

The scalar is often called self-interacting because of $V(\phi)$ and the massive one-form is known as a Proca field. The respective phase-space variables in terms of which these Lagrangians are defined are $(\phi, \dot{\phi})$ and $(A_a, \dot{A}_a)$. The dot again represents Lie differentiation along the flow vector $f^\mu \partial_\mu$ and $A_a \equiv e^\mu_a \tilde{A}_\mu$ is the pull-back of the $(d+1)$-dimensional one-form onto the hypersurface. The first step is to rewrite the above Lagrangians in such a way that the dependence on the above phase-space variables becomes explicit. We choose to follow the gravitational case rather closely, so we will take an intermediate step first. In this step we first write all quantities out on the normal/tangent basis $\{n^\mu, e^\mu_a\}$ before going to the flow/tangent basis $\{f^\mu, e^\mu_a\}$.

The scalar-field Hamiltonian. By using the definition of the induced derivative (C.12) and the completeness relation $g^{\mu\nu} = q^{ab}\, e^\mu_a e^\nu_b \mp n^\mu n^\nu$ we immediately see that the kinetic term splits up into

$$\partial^\mu \phi \, \partial_\mu \phi = \partial^a \phi \, \partial_a \phi \mp (n^\mu \partial_\mu \phi)^2 \tag{D.22}$$

On the other hand, we have

$$\dot{\phi} \equiv \mathscr{L}_t \phi = N\, n^\mu \partial_\mu \phi + N^a\, e^\mu_a \partial_\mu \phi \qquad \Rightarrow \qquad n^\mu \partial_\mu \phi = \frac{1}{N} \left(\dot{\phi} - N^a\, \partial_a \phi \right) \tag{D.23}$$

which implies that

$$L_\phi = -\int d^d y \sqrt{q} \left\{ \mp \frac{1}{2N} \left(\dot{\phi} - N^a \, \partial_a \phi \right)^2 + N \left(\frac{1}{2} \partial^a \phi \, \partial_a \phi - V(\phi) \right) \right\} \tag{D.24}$$

This formulation enables us to calculate ϕ's canonical momentum π.

$$\pi \equiv \frac{1}{\sqrt{q}} \frac{\delta L_\phi}{\delta \dot{\phi}} = \pm \frac{1}{N} \left(\dot{\phi} - N^a \, \partial_a \phi \right) \qquad \Rightarrow \qquad \dot{\phi} = \pm N \pi + N^a \partial_a \phi \tag{D.25}$$

Putting this all together will return the Hamiltonian for ϕ.

$$H_\phi = \int d^d y \sqrt{q} \, \pi \dot{\phi} \; - L_\phi \; = \int d^d y \sqrt{q} \left\{ N \left(\pm \frac{1}{2} \pi^2 + \frac{1}{2} \partial^a \phi \partial_a \phi \right) + N^a \Big(\pi \partial_a \phi \Big) + V(\phi) \right\} \tag{D.26}$$

The one-form Hamiltonian. Let us repeat the above story one more time. We again use the completeness relation to rewrite the Lagrangian on the normal/tangent basis.

$$\tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} = F^{ab} F_{ab} \mp 2 \Big(\mathscr{L}_n A_a - \partial_a \mathcal{N} \Big) \Big(\mathscr{L}_n A^a - \partial^a \mathcal{N} \Big) \tag{D.27}$$

where we abbreviated the normal component of the one-form by $\mathcal{N} \equiv n^\mu \tilde{A}_\mu$ and we previously defined $A_a = e^\mu_a \tilde{A}_\mu$ as its tangent component. The first thing that we should notice is that the normal derivative of this normal component $\mathscr{L}_n \mathcal{N}$ does not appear in the Lagrangian. This implies that there will be no canonical conjugate to $\mathcal{N}$, so it must be non-dynamical. Let us continue.

$$\dot{A}_a \equiv \mathscr{L}_t A_a = N \left(\mathscr{L}_n A_a - \partial_a \mathcal{N} \right) + \partial_a \left(N \mathcal{N} + N^b A_b \right) \tag{D.28}$$

or conversely,

$$\mathscr{L}_n A_a - \partial_a \mathcal{N} \; = \; \frac{1}{N} \left[\dot{A}_a - \partial_a \left(N \mathcal{N} + N^b A_b \right) \right] \tag{D.29}$$

which is now ready to be plugged back into (D.27) to get the Lagrangian that is explicitly written in terms of $(A_a, \dot{A}_a)$.

$$L_A = -\int d^d y \sqrt{q} \left\{ \mp \frac{1}{2N} \left[\dot{A}_a - \partial_a \left(N \mathcal{N} + N^b A_b \right) \right]^2 + N \left[\frac{1}{4} F^{ab} F_{ab} + \frac{m^2}{2} \left(A_a A^a \mp \mathcal{N}^2 \right) \right] \right\} \tag{D.30}$$

By the square of the first square-bracket term, contraction of the index a is implied. Like we mentioned above, we see that $\mathcal{N}$ is non-dynamical, because the Lagrangian indeed has no $\dot{\mathcal{N}}$-dependence. The canonical momentum conjugate to A_a is

$$E^a \equiv \frac{1}{\sqrt{q}} \frac{\delta L_A}{\delta \dot{A}_a} \; = \; \pm \frac{1}{N} \left[\dot{A}^a - \partial^a \left(N \mathcal{N} + N^b A_b \right) \right] \tag{D.31}$$

or again, conversely,

$$\dot{A}_a = \pm N\, E_a + \partial_a \left(N\, \mathcal{N} + N^b\, A_b \right) \tag{D.32}$$

We use this expression in the last step, where we do the actual Legendre transformation and get the Hamiltonian.

$$\begin{aligned}
H_{\scriptscriptstyle A} &= \int d^d y \sqrt{q}\, E^a \dot{A}_a \;\; - L_{\scriptscriptstyle A} \\
&= \int\limits_{\Sigma_t} d^d y \sqrt{q} \left\{ N \left(\pm \frac{1}{2} E_a E^a + \frac{1}{4} F^{ab} F_{ab} + \frac{m^2}{2} \left(A_a A^a \mp \mathcal{N}^2 \right) - \mathcal{N} \nabla_a E^a \right) + N^a \left(-A_a \nabla_b E^b \right) \right\} \\
&\quad + \begin{cases} \displaystyle\oint\limits_{\partial\Sigma_t} d^{d-1}\theta \sqrt{\gamma} \Big\{ N \left(\mathcal{N} r_a E^a \right) + N^a \left(A_a r_b E^b \right) \Big\} & \text{(temporal)} \\ \displaystyle\oint\limits_{\Sigma_r^\pm} d^{d-1}\theta \sqrt{\gamma} \Big\{ N \left(\mathcal{N} t_a E^a \right) + N^a \left(A_a t_b E^b \right) \Big\} & \text{(radial)} \end{cases}
\end{aligned} \tag{D.33}$$

The full Hamilton constraint. For future reference, let us write out explicitly the Hamilton constraint for the Einstein–Hilbert action in the presence of matter fields.

$$\mathcal{H} \;=\; \mathcal{T} \;-\; \mathcal{L} \;+\; \text{(surface terms)} \tag{D.34a}$$

where the respective kinetic and potential pieces are given by[ii]

$$\begin{aligned}
\mathcal{T} \;&=\; \pm \left[\; 2\kappa \big(p^{ab} p_{ab} - \frac{1}{d-1} p^2 \big) \;+\; \frac{1}{2} \pi^2 \;+\; \frac{1}{2} E_a E^a \;\right] \\
\mathcal{L} \;&=\; \frac{1}{2\kappa} \big(R - 2\Lambda \big) \;-\; \frac{1}{2} \partial^a \phi \partial_a \phi \;-\; \frac{1}{4} F^{ab} F_{ab} \;-\; \frac{m^2}{2} \big(A_a A^a \mp \mathcal{N}^2 \big) \;+\; \mathcal{N} \nabla_a E^a
\end{aligned} \tag{D.34b}$$

Remember that the $+$ ($-$) sign corresponds to a time-like (space-like) foliation.

[ii]The potential piece is actually $-\mathcal{L}$, whose unusual choice of sign is chosen such that $\mathcal{L}$ may be interpreted as a Lagrangian (it is the Lagrangian restricted to the hypersurface).

E Hamilton–Jacobi surface charges.

In this section I'd like to mention a thought that I had while writing this thesis. It should be mentioned beforehand that the following analysis is rather rough around edges...

It seems to me that the Hamilton–Jacobi formalism is more natural for defining surface charges than Regge & Teitelboim's Hamiltonian approach [16, 17, 52] or the fully covariant Lagrangian approach [29–31]. The main advantage is that obtaining the 'integrated' surface charge Q_ζ does not rely heavily on the specific boundary conditions, as I will show here.

The gravitational generalization to the HJ equations of motions can be written as ($\delta S_{\text{cl}}/\delta t = 0$ in the classical case)

$$H = \left.\frac{\delta S_{\text{cl}}}{\delta t}\right|_{\Sigma_t} \tag{E.1a}$$

$$p^{ab} = \left.\frac{1}{\sqrt{q}}\frac{\delta S_{\text{cl}}}{\delta q_{ab}}\right|_{\Sigma_t} \tag{E.1b}$$

I'd like to view the Hamiltonian H, which can be seen as the generator of an infinitesimal flow along the flow vector t^μ, as a special case of a generator Q_ζ that generates a flow along a generic vector ζ^μ.

$$Q_\zeta[g] \equiv \zeta^\mu \frac{\delta}{\delta x^\mu} S[q(y^a,t),t] = \int\limits_{\Sigma_t} d^d y\, (\mathscr{L}_\zeta q_{ab})\frac{\delta S}{\delta q_{ab}} + \int dt \int\limits_{\Sigma_t} d^d y\, \zeta^\mu \frac{\partial \mathcal{L}}{\partial x^\mu} \tag{E.2}$$

where I have made the distinction between implicit spacetime dependence (first term) and explicit spacetime dependence (second term). The induced metric $q_{ab} = e^\mu_a e^\nu_b\, g_{\mu\nu}$ and the metric $g_{\mu\nu}$ itself are taken to be in the class of metrics of the form $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ obey the Brown–Henneaux-like asymptotic boundary conditions (1.29). The barred metric $\bar{g}_{\mu\nu}$ is a solution to the field equations; it is the analogue of a point particle's classical path. The first term in the square brackets can be simplified by noticing that there is only an explicit dependence on the foliation parameter t, so that

$$\zeta^\mu \frac{\partial L}{\partial x^\mu} = \zeta^\mu \frac{\partial t}{\partial x^\mu}\frac{\partial L}{\partial t} = N^{-1} n_\mu\, \zeta^\mu \frac{\partial L}{\partial t} \tag{E.3}$$

where used the definition of the unit normal, (D.1). The charge is defined in such a way that $Q_t = H$, namely[i]

$$
\begin{aligned}
Q_t[g] &= \int\limits_{\Sigma_t} d^d y \, (\mathscr{L}_t q_{ab}) \frac{\delta S}{\delta q_{ab}} - \int dt \int\limits_{\Sigma_t} d^d y \, \frac{\partial \mathcal{L}}{\partial t} \\
&= \int\limits_{\Sigma_t} d^d y \sqrt{q} \, \dot{q}_{ab} \, p^{ab} - L\big|_{\Sigma_t} \\
&= H\big|_{\Sigma_t}
\end{aligned}
\tag{E.4}
$$

This Hamiltonian is zero on shell, so Q_t vanishes weakly whenever $g = \bar{g}$.

Equivalence to the Brown–Henneaux charge. For the case of pure Einstein gravity on asymptotically AdS_3 spaces, Brown & Henneaux [16] found the following surface charge through Regge & Teitelboim's method [52].

$$
Q_\zeta[g] = \oint\limits_{\partial\Sigma_t(\infty)} d\sigma_a \left\{ \frac{1}{2\kappa} \bar{G}^{abcd} \left[\bar{\nabla}_b h_{cd} - h_{cd} \bar{\nabla}_b \right] \hat{\zeta} + 2p^{ab} \hat{\zeta}_b \right\} + O(h^2) \tag{E.5}
$$

This expression was given before in (1.49). This result was obtained by a careful integration that takes δQ_ζ to Q_ζ, which is done by taking into account the specific asymptotic boundary conditions for asymptotically AdS_3 spaces, cf. Section 1.3.

The implicit-dependence term: Brown–York charge. I will now derive this precise form of the charge in the HJ formalism, which does not require such an integration and in fact follows quite straightforwardly from linearized gravity. Let us start by showing that the first piece in the definition (E.2) reduces to the p^{ab}-term in (E.5). I also show that this charge is just the quasi-local charge defined by Brown & York [32]. The so-called Brown–York stress tensor is defined as

$$
\tau^{ab} \equiv \frac{2}{\sqrt{q}} \frac{\delta S_{\text{cl}}}{\delta q_{ab}}\bigg|_{\Sigma_t} = 2p^{ab} \tag{E.6}
$$

in terms of which the Brown–York charge, which generates deformations along the hypersurface Σ_t (cf. Figure D.1), is defined as

$$
Q^{\text{BY}}_{\hat{\zeta}} \equiv \int\limits_{\Sigma_t} d\sigma_a \, \tau^{ab} \, \hat{\zeta}_b \tag{E.7}
$$

We give the deformation vector a hat to fit the notation of Chapter 1 and the surface element is $d\sigma_a = d^{d-1}\theta \sqrt{\gamma} \, r_a$ like before (see also Appendix D). The first term in the Hamilton–Jacobi charge (E.2) can be rewritten as

$$
\int\limits_{\Sigma_t} d^d y \, (\mathscr{L}_\zeta q_{ab}) \frac{\delta S_{\text{cl}}}{\delta q_{ab}} = \int\limits_{\Sigma_t} d^d y \sqrt{q} \, \tau^{ab} \, \nabla_a \hat{\zeta}_b = \oint\limits_{\Sigma_t} d\sigma_a \, \tau^{ab} \, \hat{\zeta}_b \tag{E.8}
$$

[i] Remember that $n_\mu t^\mu = -N$.

where we used (E.1b) together with the definition of the Brown–York tensor as well as the fact that it is conserved on shell, $\nabla_a \tau^{ab} = 0$.[ii] We also did some straightforward rewriting,

$$\mathscr{L}_\zeta q_{ab} \;=\; e^\mu_a e^\nu_b \,\mathscr{L}_\zeta g_{\mu\nu} \;=\; e^\mu_a e^\nu_b \left(\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu\right) \;=\; \nabla_a \hat{\zeta}_b + \nabla_b \hat{\zeta}_a \tag{E.9}$$

Check out Appendix C if this does not look familiar. In conclusion, we indeed find that the implicit-dependence term in the Hamilton–Jacobi charge (E.2) reduces to the Brown–York charge, which is exactly the term proportional to $2p^{ab} = \tau^{ab}$ sitting in the Brown–Henneaux charge (E.5).

The explicit-dependence term. Let us take the standard gauge $N = 1$, $N^a = 0$, such that the zero-component of x^μ is precisely the foliation parameter, i.e. $x^0 = t$.[iii] Because $S[q(t)]$ depends on t only, the deformation gets projected onto the normal component $\hat{\zeta} \equiv \zeta^\mu n_\mu$, i.e.

$$\begin{aligned}
\int\limits_{\Sigma_t} d^d y \,(\mathscr{L}_\zeta q_{ab})\,\frac{\delta S}{\delta q_{ab}} &= \int\limits_{\Sigma_t} d^d y \sqrt{q}\,\hat{\zeta}\,\frac{\partial \mathcal{L}}{\partial t} \\
&= \frac{1}{2\kappa}\int\limits_{\Sigma_t} d^d y \sqrt{q}\,\hat{\zeta}\, R^{(1)} + O(h^2) \\
&= \frac{1}{2\kappa}\int\limits_{\Sigma_t} d^d y \sqrt{q}\,\hat{\zeta}\, \bar{G}^{abcd}\,\nabla_a \nabla_b h_{cd} + O(h^2) \\
&= \frac{1}{2\kappa}\oint\limits_{\Sigma_t} d\sigma_a\, \bar{G}^{abcd}\left(\hat{\zeta}\,\nabla_b h_{cd} - h_{cd}\,\nabla_b \hat{\zeta}\right) + O(h^2) + \text{another bulk term}
\end{aligned} \tag{E.10}$$

where $G_{abcd} \equiv q_{a(c}q_{d)b} - q_{ab}q_{cd}$ like before. We used the linearized Ricci scalar, $R^{(1)} = \bar{G}^{abcd}\nabla_a \nabla_b h_{cd}$, because $q_{ab} = \bar{q}_{ab} + h_{ab}$. This gives the first term in the Brown–Henneaux charge (E.5).

Concluding remark. In conclusion, the Hamilton–Jacobi charge reduces to the Brown–Henneaux charge when we replace the Einstein metric $\bar{g}_{\mu\nu}$ to a class of linearly deformed metrics $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$.

[ii] This is just the momentum constraint in the Hamiltonian picture, i.e. $0 = \mathcal{H}_a = 2\nabla^b p_{ab}$, cf. (D.20).

[iii] Remember the difference between the *foliation parameter* of the constraint surface $\phi(x^\mu) = t$ and the *flow vector* $t^\mu = N\,n^\nu + N^a\,e^\mu_a$, where $e^\mu_a \equiv \partial x^\mu/\partial y^a$ with y^a the coordinates intrinsic to Σ_t; e^μ_a is <u>not</u> a Vielbein.

I don't yet know what to do with the 'another bulk term', which is

$$\frac{1}{2\kappa}\int\limits_{\Sigma_t} d^d y \sqrt{q}\, \bar{G}^{abcd}\, h_{cd}\, \nabla_a \nabla_b\, \hat{\zeta} \tag{E.11}$$

Maybe one could assume that the deformation is a massless scalar field or something?? (I haven't given this much thought.)

I also discarded the ADM term, which would most likely contribute too.

F Code For Computing Surface Charges

Brown-Henneaux Charges

Kristian Holsheimer

note: Don' t use the symbols d, g, i, k, l, m, or n.

- **Define a metric and an appropriate background (like Minkowski or (anti-)De Sitter):**

```
$Assumptions = And[t ∈ Reals, ϕ ∈ Reals, r ∈ Reals, 0 ≤ ϕ ≤ 2 π, r > 0, α ∈ Reals, A ∈ Reals, R ∈ Reals, R > 0];
coordinates = {t, ϕ, r};
```

$$\texttt{metricoriginal} = \left\{\left\{-\left(\alpha^2 + \frac{r^2}{R^2}\right), A\,\alpha, 0\right\}, \{A\,\alpha, r^2, 0\}, \left\{0, 0, \left(\alpha^2 + \frac{r^2 - A^2}{R^2}\right)^{-1}\right\}\right\};$$

```
metric = metricoriginal;
```

$$\texttt{background} = \left\{\left\{-\left(1 + \frac{r^2}{R^2}\right), 0, 0\right\}, \{0, r^2, 0\}, \left\{0, 0, \left(1 + \frac{r^2}{R^2}\right)^{-1}\right\}\right\};$$

```
MatrixForm[metric]
MatrixForm[background]
```

$$\begin{pmatrix} -\frac{r^2}{R^2} - \alpha^2 & A\,\alpha & 0 \\ A\,\alpha & r^2 & 0 \\ 0 & 0 & \frac{1}{\frac{-A^2 + r^2}{R^2} + \alpha^2} \end{pmatrix}$$

$$\begin{pmatrix} -1 - \frac{r^2}{R^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & \frac{1}{1 + \frac{r^2}{R^2}} \end{pmatrix}$$

- **Evaluate this cell!!!**

```
(* definition of the 'normal' metric *)
invmetric := FullSimplify[Inverse[metric]];
d = Length[metric[[1]]];

totalmetric =
  Table[
   Table[
    KroneckerDelta[m, 2 d - n + 1]
     + If[(m - (d + 1/2)) (n - (d + 1/2)) < 0,
      0,
      If[m ≤ d,
       metric[[m, n]],
       invmetric[[d - n, d - m]]
      ]
     ],
    {m, 1, 2 d}],
   {n, 1, 2 d}];

g[m_, n_] := totalmetric[[-m, -n]];


(* definition of the global/asymptotic metric *)
global = background;
invglobal := FullSimplify[Inverse[global]];

totalglobal =
  Table[
   Table[
    KroneckerDelta[m, 2 d - n + 1]
     + If[(m - (d + 1/2)) (n - (d + 1/2)) < 0,
      0,
      If[m ≤ d,
       global[[m, n]],
       invglobal[[d - n, d - m]]
      ]
     ],
    {m, 1, 2 d}],
   {n, 1, 2 d}];

glob[m_, n_] := totalglobal[[-m, -n]];


(* definition of the induced 'normal' metric *)
inducedmetric := Table[metric[[i, j]], {i, 2, d}, {j, 2, d}];
invinducedmetric := Inverse[inducedmetric];

totalinducedmetric =
  Table[
```

```
    Table[
     KroneckerDelta[m, 2 (d - 1) + 1 - n]
      + If[(m - (d - 1/2)) (n - (d - 1/2)) < 0,
       0,
       If[m ≤ (d - 1),
        inducedmetric[[m, n]],
        invinducedmetric[[(d - 1) - n, (d - 1) - m]]
       ]
      ],
     {m, 1, 2 (d - 1)}],
    {n, 1, 2 (d - 1)}];

gind[m_, n_] := If[(m n < 0) ⋁ (Abs[m] == 1 ⋁ Abs[n] == 1),
   Print["Sum runs over invalid values of the indices of gij."]; Abort[],
   If[m < 0,
    totalinducedmetric[[-m - 1, -n - 1]],
    totalinducedmetric[[-m + 1, -n + 1]]
   ]
  ];

(* induced global metric *)
inducedglobal := Table[global[[i, j]], {i, 2, d}, {j, 2, d}];
invinducedglobal := Inverse[inducedglobal];

totalinducedglobal =
  Table[
   Table[
    KroneckerDelta[m, 2 (d - 1) + 1 - n]
     + If[(m - (d - 1/2)) (n - (d - 1/2)) < 0,
      0,
      If[m ≤ (d - 1),
       inducedglobal[[m, n]],
       invinducedglobal[[(d - 1) - n, (d - 1) - m]]
      ]
     ],
    {m, 1, 2 (d - 1)}],
   {n, 1, 2 (d - 1)}];

globind[m_, n_] := If[(m n < 0) ⋁ (Abs[m] == 1 ⋁ Abs[n] == 1),
   Print["Sum runs over invalid values of the indices of gij."]; Abort[],
   If[m < 0,
    totalinducedglobal[[-m - 1, -n - 1]],
    totalinducedglobal[[-m + 1, -n + 1]]
   ]
  ];

(* differentiation and connections *)
pd[n_, func_] := If[n < 0,
   D[func, coordinates[[-n]]],
   Print["Lower the index on ∂m manually."]; Abort[]
  ];
```

```
gamma[k_, l_, m_] := If[k > 0 ∧ l < 0 ∧ m < 0,
   Sum[1/2 g[k, i] (pd[l, g[m, -i]] + pd[m, g[l, -i]] - pd[-i, g[l, m]]), {i, 1, d}],
   Print["Sum runs over invalid values of the indices of Γ^i_jk. (g)"]; Abort[]
  ];

gammainduced[k_, l_, m_] := If[k > 0 ∧ l < 0 ∧ m < 0,
   Sum[1/2 gind[k, i] (pd[l, gind[m, -i]] + pd[m, gind[l, -i]] - pd[-i, gind[l, m]]), {i, 2, d}],
   Print["Sum runs over invalid values of the indices of Γ^i_jk. (gind)"]; Abort[]
  ];

gammaglob[k_, l_, m_] := If[k > 0 ∧ l < 0 ∧ m < 0,
   Sum[1/2 glob[k, i] (pd[l, glob[m, -i]] + pd[m, glob[l, -i]] - pd[-i, glob[l, m]]), {i, 1, d}],
   Print["Sum runs over invalid values of the indices of Γ^i_jk. (glob)"]; Abort[]
  ];

gammaglobind[k_, l_, m_] := If[k > 0 ∧ l < 0 ∧ m < 0,
   Sum[1/2 globind[k, i] (pd[l, globind[m, -i]] + pd[m, globind[l, -i]] - pd[-i, globind[l, m]]), {i, 2, d}],
   Print["Sum runs over invalid values of the indices of Γ^i_jk. (globind)"]; Abort[]
  ];
(* these definitions only works for Γ^i_jk,
no raised or lowered indices (which make no sense anyway). i.e. the
 2nd and 3rd indices should always be summed over negative values. *)

(* define the canonical conjugate to some metric with some induced metric-tensor *)
pitotal = Table[
   Sum[Sqrt[-Det[metric]] gamma[1, -k, -l] gind[i, k] gind[j, l], {k, 2, d}, {l, 2, d}]
    - Sum[gamma[1, -m, -n] gind[m, n] gind[-k, -l] gind[i, k] gind[j, l]
     , {k, 2, d}, {l, 2, d}, {m, 2, d}, {n, 2, d}]
   , {i, 2, d}, {j, 2, d}];

pi[m_, n_] := If[(m < 0 ∨ n < 0) ∨ (Abs[m] == 1 ∨ Abs[n] == 1),
   Print["Sum runs over invalid values of the indices of π^ij."]; Abort[],
   pitotal[[m - 1]][[n - 1]]
  ];

(* define adm's G^ijkl *)
Gtotal = Table[1/2 Sqrt[Det[inducedglobal]]
    (globind[i, k] globind[j, l] + globind[i, l] globind[j, k] - 2 globind[i, j] globind[k, l])
   , {i, 2, d}, {j, 2, d}, {k, 2, d}, {l, 2, d}];
G[i_, j_, k_, l_] :=
  If[(i < 2 ∨ j < 2 ∨ k < 2 ∨ l < 2), Print["Sum runs over invalid values of the indices of G^ijkl."]; Abort[],
   Gtotal[[i - 1]][[j - 1]][[k - 1]][[l - 1]]
```

```
];
```

```
(* induced deformation vectors *)
vecind[vec_] := Join[{vec[[1]]/Sqrt[-g[1, 1]]},
   Table[vec[[j]] + vec[[1]] Sum[g[-1, -i] gind[i, j], {i, 2, d}], {j, 2, d}]];


(* integrand of the charge, i.e. surface integral *)
integrand[vec_] := (
   Sum[
     G[i, j, k, d] (
       vecind[vec][[1]] (
          pd[-k, gind[-i, -j]]
           - Sum[gammaglobind[m, -k, -i] gind[-m, -j] + gammaglobind[m, -k, -j] gind[-m, -i], {m, 2, d}]
         )
        + pd[-k, vecind[vec][[1]]] (globind[-i, -j] - gind[-i, -j])
      ), {i, 2, d}, {j, 2, d}, {k, 2, d}] +
    2 Sum[gind[-i, -j] vecind[vec][[i]] pi[j, d], {i, 2, d}, {j, 2, d}]
  );


(* Lie derivative of a metric (or any other (0,2)-tensor) *)
lieD[vec_, metric_] := Table[Sum[
    vec[[l]] pd[-l, metric[[m, n]]]
     + pd[-m, vec[[l]]] metric[[l, n]]
     + pd[-n, vec[[l]]] metric[[l, m]]
    , {l, 1, d}], {m, 1, d}, {n, 1, d}];
```

- **The ADM-like charges of `metricoriginal':**

ADM's mass:

```
ξ = {1, 0, 0};
Integrate[Limit[integrand[ξ], r → ∞], {ϕ, 0, 2 π}]
```

$$2\pi\left(1+\frac{A^2}{R^2}-\alpha^2\right)$$

Regge-Teitelboim's angular momentum:

```
ξ = {0, -1, 0};
Integrate[Limit[integrand[ξ], r → ∞], {ϕ, 0, 2 π}]
```

$$-4A\pi\alpha$$

These are exactly the charges (4.12) of the Brown-Henneaux paper. Let' s see if we can get the central charges from (5.6).

- **Give the asymptotic Killing vectors:**

```
aa[n_] := {R (1 - n^2 R^2/(2 r^2)) Cos[n t/R] Cos[n ϕ], -(1 + n^2 R^2/(2 r^2)) Sin[n t/R] Sin[n ϕ], r n Sin[n t/R] Cos[n ϕ]};
bb[n_] := {R (1 - n^2 R^2/(2 r^2)) Sin[n t/R] Sin[n ϕ], -(1 + n^2 R^2/(2 r^2)) Cos[n t/R] Cos[n ϕ], r n Cos[n t/R] Sin[n ϕ]};
cc[n_] := {R (1 - n^2 R^2/(2 r^2)) Sin[n t/R] Cos[n ϕ], (1 + n^2 R^2/(2 r^2)) Cos[n t/R] Sin[n ϕ], r n Cos[n t/R] Cos[n ϕ]};
dd[n_] := {R (1 - n^2 R^2/(2 r^2)) Cos[n t/R] Sin[n ϕ], (1 + n^2 R^2/(2 r^2)) Sin[n t/R] Cos[n ϕ], r n Sin[n t/R] Sin[n ϕ]};

$Assumptions = $Assumptions ⋀ And[p ∈ Integers, q ∈ Integers];
```

- **In order to compute the central charge we deform the background metric with respect to the above asymptotic symmetries (see discussion on page 222 of B-H paper):**

```
metricnew = FullSimplify[background - lieD[aa[p], background]] /. t → 0;
metric = metricnew;
MatrixForm[metric]
```

$$\begin{pmatrix} -1-\frac{r^2}{R^2} & \frac{p R \left(2\left(-1+p^2\right) r^2+p^2 R^2\right) \operatorname{Sin}[p\,\phi]}{2 r^2} & \frac{p^2 R^3 \left(2 r^2+R^2\right) \operatorname{Cos}[p\,\phi]}{r^3\left(r^2+R^2\right)} \\ \frac{p R \left(2\left(-1+p^2\right) r^2+p^2 R^2\right) \operatorname{Sin}[p\,\phi]}{2 r^2} & r^2 & 0 \\ \frac{p^2 R^3 \left(2 r^2+R^2\right) \operatorname{Cos}[p\,\phi]}{r^3\left(r^2+R^2\right)} & 0 & \frac{R^2}{r^2+R^2} \end{pmatrix}$$

- **Before continuing, run the `Evaluate this cell!!!' cell again.**

This should give the commutator [J(A) , J(C)] in Eq. (5.6) of the B-H paper.

```
ξ = cc[p] /. t → 0
Normal[Series[integrand[ξ], {r, ∞, 0}]] // Refine
Integrate[%, {ϕ, 0, 2 π}]
```

$$\left\{0,\ \left(1+\frac{p^2 R^2}{2 r^2}\right) \operatorname{Sin}[p\,\phi],\ p\, r \operatorname{Cos}[p\,\phi]\right\}$$

$$-2 p R \operatorname{Sin}[p\,\phi]^2 + 2 p^3 R \operatorname{Sin}[p\,\phi]^2$$

$$2 p \left(-1+p^2\right) \pi R$$

... and it does!

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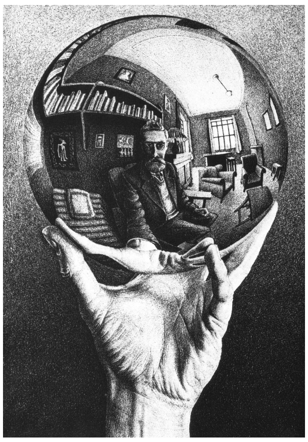
Samenvatting

Deze doctoraalscriptie behandelt een bepaalde klasse van fysische systemen op twee verschillende holografische methoden. Zoals de naam al aanduidt, berusten deze holografische methoden op het holografisch principe. Dit principe luidt grofweg als volgt. De informatie van een fysisch systeem (zoals bijvoorbeeld de positie en snelheden van moleculen) kan worden gerepresenteerd op een (denkbeeldig) vlies dat het systeem omsluit. Dit idee werd afgeleid in de context van zwarte gaten door de Amerikaanse natuurkundige Leonard Susskind en de Nederlandse Nobelprijswinnaar Gerard 't Hooft. De uitspraak dat de hoeveelheid informatie evenredig is met het oppervlak van het omsluitende vlies is bepaald niet triviaal; men zou immers verwachten dat de informatie van een systeem zou schalen met het volume.

De verreweg meest concrete uiting van het holografisch principe is dat van zogenaamde holografische dualiteiten in snarentheorie. De eerste en meest bekende dualiteit wordt vaak aangeduid met de AdS/CFT correspondentie, geïntroduceerd door de Argentijnse natuurkundige Juan Maldacena in 1998. AdS staat voor de anti-De Sitter ruimte, hetgeen een ruimte is



die op een vrij specifieke manier gekromd is. De correspondentie relateerd een zwaartekrachtstheorie in deze anti-De Sitter ruimte aan een zogenaamde schaalonafhankelijke kwantumveldentheorie *zonder zwaartekracht* op de rand van dezelfde ruimte. Een schaalonafhankelijke veldentheorie heet in natuurkundig vakjargon corformal field theory, hetgeen vaak wordt afgekort met CFT. De schaal waar het hier over gaat is ruwweg de energieschaal. Een CFT is dus een theorie waarin alle grootheden onveranderd blijven onder het verschuiven van de energieschaal. We komen terug op het concept van schaalinvariantie in de volgende paragraaf. De gemakkelijkste manier van je deze holografische correspondentie voor te stellen is als een grote bol. De vorm en de inhoud van de bol wordt beschreven door de zwaartekrachtstheorie in AdS, terwijl de duale theorie op de rand niets voelt van de zwaartekracht. De rand is namelijk nagenoeg vlak,[i] net zoals het

[i]Einsteins algemene relativiteitstheorie zegt dat zwaartekracht niets anders is dan de kromming van de

aardoppervlak vlak lijkt te zijn, vanwege de enorme afmeting van de aardstraal. Hoewel de theorie op de rand zelf geen zwaartekracht voelt, ondervindt deze tòch een effect van de zwaartekrachtstheorie in het inwendige van de bol. Sterker nog, de twee theoriën zijn volkomen equivalent, oftewel elkaars dualen. Het is alsof een van de theorieën in een lachspiegel kijkt waarbij het niet duidelijk is welke van de twee het origineel is en welke de reflectie.

Stel je eens een biljarttafel voor en werp er vervolgens drie knikkers een voor een op. Je kunt je voorstellen dat de kans dat de knikkers met elkaar botsen vrij klein is. Als je dan de beweging van de knikkers in dit systeem zou willen beschrijven is het een goede eerste benadering om de botsingskans te verwaarlozen. De beweging van een van de drie knikkers wordt in zo'n geval niet beïnvloed door de andere twee. De beschrijving, ook wel theorie, van de drie knikkers bestaat dan simpelweg uit drie 'kopieën' van eenzelfde eenvoudig te verkrijgen theorie. Zulke beschrijvingen worden in de natuurkunde vrije theorieën genoemd. De stap naar een algemenere theorie die wel degelijk interactie tussen de knikkers onderling toelaat is vervolgens verkregen door de vrije theorie een klein beetje te vervormen. Dit vervormen is vaak nogal technisch, maar het idee is duidelijk: het dominante gedrag de knikkers is een vrije theorie. Je kunt vervolgens wel nagaan dat het verwaarlozen van de botsingskans, ook wel interactie genoemd, niet altijd gerechtvaardigd is. Zo ligt het er bijvoorbeeld aan hoe hard je de knikker op de tafel gooit of. We kunnen ook, in plaats van knikkers, drie biljartballen over de tafel strooien. De botsingkans is evenredig met de afmeting van de ballen. Kortom, de interactie tussen deze biljartballen is in zulke voorbeelden niet meer zomaar te verwaarlozen. In de natuurkunde praat men dan over sterke versus zwakke interactie. Langzame knikkers interageren zwakker dan snelle knikkers en knikkers interageren zwakker dan biljartballen bij dezelfde snelheid. De vuistregel in de meeste natuurkundige modellen is dat een sterke interactie vaak erg lastig te beschrijven is, omdat de houvast van de vrije theorie dan niet binnen handbereik is. Om de analogie met algemene natuurkundige theorieën volledig te maken zullen we ons taalgebruik even scherpstellen. In de natuurkunde gebruikt men het woord 'interactie' vaak als de 'effectieve afmeting van de ballen', hetgeen afhangt van de 'energieschaal' oftewel 'de snelheid van de ballen'. In andere woorden, de mate van interactie tussen snelle knikkers kan evengoed beschreven worden door langzame biljartballen.[ii]

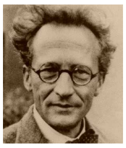
Het leuke van de AdS/CFT correspondentie is niet alleen dat het twee ogenschijnlijk volledig verschillende theorieën aan elkaar relateert, maar vooral ook dat de ene theorie een *sterke* interactie beschrijft, terwijl de ander een *zwakke* interactie heeft. Om preciezer te zijn, heeft de zwaartekrachtstheorie een zwakke interactie en de schaalonafhankelijke theorie juist een sterke. We hebben zojuist besproken dat een schaalonafhankelijke theorie

ruimte, dus 'geen zwaartekracht' is synoniem voor 'de ruimte is vlak'.

[ii] Iedere analogie kent haar grenzen en als je deze laatste zin te letterlijk zou opvatten is deze grens duidelijk bereikt. Toch is het handig om er op deze manier tegenaan te kijken, zodat je nog wat aan je boeren verstand hebt.

per definitie niet van haar energieschaal afhangt. In termen van het knikker voorbeeld vertaald zich dat naar de eigenschap dat de effectieve afmeting niet afhangt van hoe hard de knikkers bewegen, hetgeen uiteraard vrij lastig voor te stellen is. De meeste realistische systemen vertonen echter niet of nauwelijks schaalonafhankelijkheid. De beschrijving van zo'n theorie met sterke interactie is met het vergeven van de schaalonafhankelijkheid daarentegen vrijwel onmogelijk geworden. Nu komt AdS/CFT om de hoek kijken. De sterk interagerende theorie die erg lastig te beschrijven is kunnen we nu interpreteren als de theorie op de rand van de grote bol. We kunnen dan de holografische dualiteit toepassen en overgaan naar de duale theorie in het inwendige van de bol die slechts zwak interageert. In deze stap hebben we sterke interactie zonder zwaartekracht ingeruild voor zwakke interactie met zwaartekracht. De duale beschrijving is weliswaar niet altijd eenvoudig, het is wel haalbaar. Dit is precies de kracht van deze holografische theorieën; het vertaalt een onmogelijke taak naar een haalbare. Het vervormen van een theorie door middel van zo'n holografische methode valt onder de noemer 'holografische renormalisatie'.

In mijn scriptie zoek ik voor een holografische beschrijving van een klasse van theorieën die



dezelfde symmetrieën hebben als de Schrödinger vergelijking. Dit doe ik enerzijds door middel van holografische renormalisatie, zoals net uitgelegd, en anderzijds gebruikmakend van een andere methode die conceptueel iets lastiger uit te leggen is, maar in essentie hetzelfde resultaat beoogt.