Holographic Quantum Phase Transitions

MSc thesis (Physics, track: Theoretical Physics)

Marianne Hoogeveen

Supervisor: Prof. J. de Boer
Coordinator MSc track: Prof. B. Nienhuis

Universiteit van Amsterdam
Instituut voor Theoretische Fysica

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Chapter 1

Introduction

In the field of theoretical condensed matter physics, which is the study of the effective behaviour of many particles together, simplifying assumptions are made to model the physics of a certain material under specified conditions (e.g. at a certain temperature or in a magnetic field). The strength of this method is that the "relevant" physics can be separated from the "irrelevant", meaning that we do not need to keep track of all the constituents of a material (electrons, nuclei) and their interactions, but keep only the interactions that will turn out to affect the properties of the material as a whole.

Two cornerstones of condensed matter theory are Fermi liquid theory and Landau's theory of phase transitions. The former describes weakly interacting electron systems at low temperatures in terms of noninteracting degrees of freedom which behave like electrons, but with different parameters (mass, charge). Many problems in condensed matter that are not well understood have to do with the fact that such a weakly coupled description is not valid, and perturbation theory fails. The latter is an effective theory in terms of an *order parameter*[1] describing the behaviour of a system near a continuous phase transition which becomes scale invariant at the *critical point*[2].

Both theories run into difficulties when a material is made to undergo a phase transition while remaining at strictly zero temperature. In the vicinity of such a "quantum phase transition", the material typically exhibits all kinds of exotic behaviour (such as unconventional superconductivity and metallic behaviour that is uncharacteristic of a Fermi liquid description). A naive extension of Landau's theory of phase transitions, due to [50, 61], uses a time-dependent order parameter to construct an effective description. However, quantum phase transitions turn out to be much more difficult than their classical counterparts, as the effective degrees of freedom are typically strongly coupled. In some cases, it hasn't even been possible to identify the effective degrees of freedom, and Landau's theory of phase transitions seems to break down. Clearly, different tools are needed to understand the relevant physics of these new types of phase transitions.

One possible tool comes from an unexpected side: black hole physics. In trying to un-

[1] see section 2.2

[2] A *critical point* is the point at which a continuous phase transition occurs.

derstand quantum mechanics in the presence of black holes, it was discovered in the 1970's that these are thermodynamic objects with en entropy proportional to the area of their event horizon. Since gravitating matter can be made to collapse and form a black hole, this implies that the entropy of matter in some region of space scales with the area of the surface surrounding that space, and not with the volume, as would be expected if we were to model our gravitating matter in terms of some model used in condensed matter physics. Gravity thus seems to demand a drastic reduction in the degrees of freedom. This led 't Hooft in 1993 to conjecture his *holographic principle*[73], which is the speculative idea that a theory of gravity in a volume of space can equivalently be described by a theory without gravity on the boundary of that space[3]. The holographic principle is very general and does not tell us which theory of gravity is equivalent to which field theory. Neither does it tell us which spacetime to use, and how observables in one theory can be mapped to observables in the other theory. A concrete example of such a duality was found in 1998, when Maldacena [59] compared two dual description of D-branes[4], and conjectured that type IIB superstring theory (which contains gravity) in five-dimensional anti-de Sitter spacetime ("AdS_5" in short) is equivalent to a supersymmetric Yang-Mills gauge theory in four-dimensional Minkowski spacetime. This is often called the "AdS/CFT" correspondence, since the gravity theory lives in anti-de Sitter spacetime, and the dual field theory is a so-called "Conformal Field Theory" (or "CFT", in short). A precise mapping between observables was proposed in the same year by Gubser, Klebanov and Polyakov [37], and separately by Witten [76]. Many new examples of such "gauge/gravity dualities" were subsequently derived from different string theory (and M theory) setups. A feature that makes these dualities difficult to prove is that the strongly coupled regime of the gauge theory corresponds to the weakly coupled regime of the gravity (string) theory, and vice versa. Therefore, perturbation theory can never be used on both sides of the duality and one must find exact results to compare. This feature does however make these gauge/gravity dualities a potential tool in studying physics at strong coupling in a weakly coupled "holographic dual".

In order to use this *holographic duality* as a tool to study condensed matter phenomena at strong coupling, more complicated examples of a duality must be found. This is not an easy task, as the theories describing condensed matter phenomena usually do not have all the symmetries (such as supersymmetry or conformal invariance) that make it easier to derive dualities from superstring theory. Nevertheless, there has been great activity in the field of *AdS/CMT*, which is a contraction of "AdS/CFT" and "Condensed Matter Theory". Much progress has been made by using a so-called "bottom-up" approach, in which an existing correspondence is investigated under the change of certain parameters or the addition of fields. Typically, one uses a simple theory of gravity and then tries to compare the physics of this simple theory to some condensed matter phenomenon. In

[3]note that the gravitational system thus lives in a higher dimensional space than its description without gravity. Therefore, the duality relates two different theories in spacetimes of different dimensionality.

[4]D-branes are massive, dynamical objects in type II superstring theory.

this way, relatively simple gravitational systems may be employed to model interesting condensed matter phenomena. However, since the gravity theory is not a (consistent truncation of) the gravity side of a known duality, the corresponding field theory can not be specified. In order to compare the physics of a condensed matter system to some process in gravity, one would ideally like to find a relatively simple "top-down" model, which nevertheless describes interesting physics. One way of doing this is by starting from the full theory describing the gravitational side of a known duality and "truncating" the theory to a smaller number of fields. This can sometimes be done "consistently", meaning that the solutions of the simple theory are also solutions of the theory with all the fields still present.

A QCP is a natural place to start looking for a gravity dual, since at the QCP the system is effectively scale invariant, and the symmetries in the boundary theory help to constrain the form of the gravity background. Furthermore, QCPs typically occur at strong coupling, which will turn out to be a regime in which the gravity dual should be easier to solve perturbatively. Many examples of systems, both *bottom-up* and *top-down*, have been found to exhibit quantum critical behaviour, signaled by a breakdown of Fermi liquid theory as some external parameter is tuned. In this thesis, we review an example of a holographic duality in which the dual gauge theory undergoes a quantum phase transition. On the gravity side, we will consider Einstein-Maxwell theory with a Chern-Simons term, which is a *consistent truncation* known to describe the gravity dual to an infinite number of supersymmetric field theories. The fact that the field theory duals are known provides in principle the possibility to compare the mechanism causing the quantum phase transition in the gauge theory to a corresponding process in the gravitational dual.

Chapter 2 will contain a short review on the theory of Quantum Phase Transitions. In chapter 3, a review on the holographic principle and one of its realisations, the AdS/CFT correspondence, is given. In chapter 4, the holographic description of systems at finite temperature, charge density and magnetic field is discussed, which is necessary in order to describe processes in condensed matter. An example of a holographic duality in which the system can be tuned towards a quantum critical point is reviewed in chapter 5.

Chapter 2

Quantum Phase Transitions

2.1 Motivation and definition

In thermal equilibrium, the state of a system is the one that minimizes the free energy: $F = U - TS$. At low temperature, the system will be dominated by the low energy states, whereas at high temperature, the states with high entropy dominate. If the dominant states at high and low temperature can not be continuously deformed into each other (for example if the low temperature phase breaks a symmetry which is present in the high temperature phase), then there is a point where the free energy function is non-analytic. In the case that such a nonanalyticity occurs, we speak of a phase transition. We can roughly classify these transitions by the way in which this nonanalyticity occurs: in the case of a discontinuous phase transition, the different phases coexist at the transition point, where their free energies are equal, and the first order derivative of the free energy is discontinuous, exactly at the point where the free energy functions cross. Most phase transitions one encounters in nature are of this kind, as for example the melting of ice. For a continuous phase transition, there is no coexistence of different phases at the transition point, and the system is said to be in a third, "critical" phase. The phases at both sides of the transition point transform continuously into this critical phase as they are tuned to the transition point. An example of a continuous phase transition is the disappearance of spontaneous magnetisation as a ferromagnetic material is heated up to its paramagnetic phase.
When approaching a critical point from the disordered phase, small regions where the material is ordered will begin to form, whose size increases as the system is tuned towards the critical point. In the example of the ferromagnetic transition these are small regions where the spins tend to be aligned. The typical size of these regions defines a distance scale, which diverges at the critical point. The result is that at the critical point, the difference between short- and long-distance behaviour disappears and one can effectively describe the critical phase by a scale invariant theory.

Apart from this large length scale, systems also have short length scales, such as the lattice spacing. However, the description of a system near a critical point turns out to be highly insentive to the precise form of interactions at this small scale. In the limit where the ordered regions become large compared to the lattice spacing, one can neglect the size of the lattice spacing and describe the system in terms of a continuum field theory. The fact that this continuum description of a critical point is insensitive to the microscopic details explains why some magnetic phase transitions may be described by the same theory as nonmagnetic phase transitions.

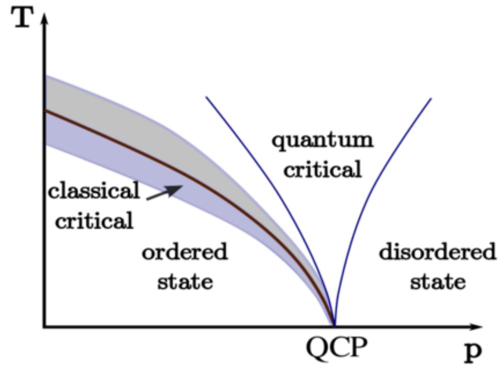


Figure 2.1: QCP in a phase diagram with a line of classical critical points.

In this thesis we will be interested in the holographic description of a *quantum critical point* (QCP), which is a continuous quantum phase transition[1]. By definition, a quantum phase transition is a transition which occurs at zero temperature, as a result of varying some non-thermal control parameter, such as applied magnetic field or pressure. Since classically the entropy at zero temperature has to vanish, a quantum phase transition can not be caused by the competition between energy and entropy, like its finite-temperature counterpart. Rather, it is a result of competition between different terms in the Hamiltonian describing the system.

A common type of phase diagrams with a QCP has a line of finite-temperature critical points where the transition temperature is depressed to zero by varying a coupling constant (see figure 2.1). Around this finite-temperature critical line, the system can be described by a classical field theory, even though the transition temperature may be very low. This is due to the fact that close to a critical point, the length scale above which the behaviour changes qualitatively is very large. Around any nonzero temperature critical point, therefore, one has $kT > \hbar\omega$ where $\hbar\omega$ is some typical energy scale above which the behaviour of the system changes (for example an energy gap). This reasoning clearly breaks down for a quantum phase transition, where the temperature is strictly zero. The behaviour at a QCP is expected to be characterized by competition between low-lying states [67]. This *quantum critical* behaviour is different from typical low energy behaviour, which can be understood in terms of quasiparticles on a ground state. This competition

[1]For a review on quantum phase transitions, see the book by Sachdev [66]

effect tends to break down away from the QCP on the zero temperature line, as an energy gap forms and the system chooses a ground state. However, if the temperature is increased such that this gap may be overcome the interplay between the different energy levels again becomes the dominant behaviour. The finite temperature region of the phase diagram in which this quantum critical behaviour is important is called the *quantum critical regime* (QCR).

Although in real life, it is impossible to actually cool a system down to zero temperature, the existence of a finite-temperature quantum critical regime (QCR) means that the study of a QCP is not merely an academic exercise. On the one hand, the existence of this QCR can serve as an indication that the QCP is present. On the other hand, the fact that the QCP dominates a larger part of the diagram suggests that the quantum critical theory may be a good starting point in understanding the phases that are not correctly described by Fermi liquid theory. An important class of materials where interesting behaviour has been found in the presence of a QCP are heavy fermion metals. These are rare-earth based compounds with quasiparticles behaving like very heavy electrons. There has been evidence of QPTs, which was found to be due to a competition between the interaction of local magnetic moments with conduction electrons, leading to the above heavy Fermi liquid behaviour, and the magnetic interaction between the local moments, favouring a magnetic ground state [24]. When pressure-tuned to a QCP, (some of) these HF metals have been found to be superconducting near the QCP, leading one to believe that some magnetic interaction might serve as the pairing mechanism. [33]. Another interesting possible application of a description of the quantum critical state is in high-temperature superconductivity (HTSC), where the superconducting state survives at such high temperatures, that a non-BCS mechanism is needed to describe the glue by which the bound states are formed. It is speculated that hidden underneath the superconducting dome there is a QPT, which might be relevant for the superconducting regime and other regimes where the system behaves differently from a Fermi liquid. [2]

2.2 RG flow near continuous phase transitions

Phase transitions can often be characterized by an *order parameter*, which is defined as an expectation value that is zero in the disordered phase, and acquires a nonzero value in the ordered phase. In the classic examples of continuous phase transitions, a nonzero value of the order parameter is related to spontaneous symmetry breaking.[2] An example of an order parameter which breaks a symmetry is magnetisation. In a ferromagnetic material, the spin rotation symmetry is spontaneously broken when the system chooses one direction

[2]Continuous phase transitions that are associated with spontaneous symmetry breaking are usually termed “second-order”. Examples of continuous phase transitions which are not associated with spontaneous symmetry breaking are the Kosterlitz-Thouless transition, and the critical endpoint of a discontinuous transition

in which it is magnetized, which we will take to be the z-axis:

$$m = \langle \sum_i S_i^z \rangle.$$

In the disordered phase, the spins point in random directions and the expectation value for the total magnetisation is zero:

$$\langle \sum_i S_i^z \rangle = 0.$$

A measure of the tendency of spins to align is given by the correlation function of the spin operator, which generally has the following form:

$$\langle \vec{S}(\vec{x}) \cdot \vec{S}(0) \rangle \sim \exp(-|\vec{x}|/\xi)$$

This defines a *correlation length* ξ below which the values of the spins are correlated, which is therefore a typical length scale in the system which separates two types of behaviour: short-distance where spins are highly correlated, and long-distance where spins are uncorrelated.[3] Upon reaching the critical point, the correlation length diverges, and this division in short- and long-range behaviour can not be made. This is what scale invariance at continuous phase transitions means.

One can make these ideas exact by applying the ideas of the *renormalization group* (RG) to phase transitions. An RG transformation is like 'zooming out' and looking at the same system from a larger distance, which makes sense, since a phase transition is a change in the macroscopic properties of a system. When describing the system near such a transition, one is not interested in its microscopic details, but rather the effective behaviour of many particles which becomes important at large length scales. The renormalization group is a method of coarse-graining the description of the system, such that the large-length scale behaviour can be obtained.

Starting from a microscopic Hamiltonian describing a system with a short-distance (lattice) cutoff a, a renormalization group (RG) step is as follows:

- Increase the cutoff by a certain factor and integrate out the degrees of freedom below this distance scale (coarse-graining)
- Rescale the distances so that the cutoff is the same as before the coarse-graining
- Rescale the fields so that the kinetic terms have the same form as before the distance rescaling

[3]Length scales below the lattice spacing a are ignored, as one expects the important behaviour for phase transitions to be collective fluctuations, rather than the microscopic interactions at the scale of lattice sites.

If the integration has produced new couplings, these must be added to the original Hamiltonian, and the RG step is iterated, until no new terms appear.
One then compares the Hamiltonian before and after an RG step. The coupling constants in the Hamiltonian typically change after coarse-graining and rescaling. If after an RG step the coupling strength of a term increases, this is called a relevant term, since at longer distances this term will become more important. Conversely, if a coupling constant becomes smaller when looking at larger distance scales, it corresponds to an irrevelant term. A coupling constant which does not change after an RG step is called marginal.[4]

By taking the limit in which the rescaling is infinitesimal, one can obtain a set of differential equations called beta-functions[5] for the coupling constants:

$$\frac{d\{g\}}{dl} = \beta(\{g\}) \tag{2.1}$$

where $\{g\}$ is the collection of coupling constants depending on l, and $l = \ln L$, where L is the length scale of the coarse-graining in the RG step. From this equation one can see that the relevant, irrelevant and marginal terms correspond to positive, negative and zero eigenvalues, respectively.

In general, after an RG transformation, the system will have a smaller correlation length. Exceptions are when the correlation length before the RG step is either zero of infinity. These are called RG fixed points, and they correspond to points in the space of couplings where the system is scale invariant and the beta-function has a zero[6]. If all perturbations around this fixed point are (ir)relevant, then the fixed point is a repulsive (attractive) fixed point. In general, a fixed point will have both relevant and irrelevant directions. Such a fixed point is called a separatrix, and it is these types of fixed points that describe the critical points of a phase transition.

A typical RG flow describing a real-life system starts at a point in the space of couplings which is not a fixed point. We will call this the 'physical point', and the system is described by a theory which has as bare parameters these 'physical' values of the coupling constants

[4]Usually the RG equations can not be solved for the exact Hamiltonian and are solved perturbatively. One generally expects relevant terms to remain relevant, and irrelevant terms to remain irrelevant. Exceptions to the latter rule are termed "dangerously irrelevant". Operators that are marginal to first order and become relevant at higher order are termed "marginally relevant".

[5]Note that this is a *local* equation in energy scale.

[6]A theory may be scale invariant for all couplings, in which case it has a beta function which is identically zero, or it may have some points in the space of all couplings for which beta function has a zero. In the latter case perturbing the system will break the scale invariance and depending on the eigenvalue of the perturbing operator, lead away from the fixed point (relevant) or back to the fixed point (irrelevant). This way, one can define stable and unstable fixed points as being fixed points where all small perturbations lead to an RG flow which brings the system away from the fixed point (unstable), or back towards the fixed point (stable). Typically, there will be both relevant and irrelevant perturbations in the space of all couplings

and as a short-distance cutoff the spacing between the important degrees of freedom (e.g. spins, etc.). This is typically the lattice spacing. As the system is not tuned to a fixed point, an RG transformation will cause the theory to flow to a different theory with smaller correlation length giving the behaviour of the system at slighty longer distances. Following the flow all the way down to zero correlation length, one ends up in a scale invariant theory describing the ultra-long distance behaviour: an *IR fixed point*.
In some cases, one can also extend the flow back to the limit where the lattice cutoff goes to zero, by defining te system to be a low-energy description of a more general renormalizable quantum field theory. The passing to the general theory describing the high-energy limit is called *UV completion*. This general theory will have a *UV fixed point* which governs the ultra-short distance behaviour, from which the RG flow leads towards the physical point, and from there via the original RG flow to the IR fixed point.

The relevant perturbations from the critical theory (temperature, magnetic field, pressure) cause the system to flow to different IR fixed points at both sides of the transition[7]. (The relevant parameters correspond to experimental 'knobs' that are used to tune the system through the phase transition.) The irrelevant directions span the *critical surface*, on which the correlation length is infinite[8]. A phase transition occurs when one passes through the critical surface along a relevant direction which does affect the value of ξ.

2.2.1 Conformal invariance

A critical point is described by an RG fixed point at which the correlation length diverges. This means that the system exhibits scale invariance, but often the theory has more symmetries. For instance, invariance under lattice translations becomes general translation invariance in the scaling limit. Similarly, discrete rotations become continuous rotations, et cetera. In cases where the interactions are short-ranged, the critical point can be described by a conformal field theory (CFT), which is a theory invariant under angle-preserving transformations. A heuristic argument for this statement uses that one can describe a conformal transformation as a rescaling by a factor $b(\vec{x})$ which depends on position: a local rescaling. If the interactions are sufficiently short-ranged, the rescaling will be approximately uniform at the length scale of the interaction. Given that the system is tuned to a fixed point, the couplings will not change under this local rescaling [17].

The relation between the scaling transformation of the time direction and the scaling transformation in the spatial directions that leave the critical theory invariant defines the *dynamical exponent* z:

$$t \to \lambda^z t, \qquad \vec{x} \to \lambda \vec{x}. \tag{2.2}$$

[7]Note that moving away from the fixed point with a relevant perturbation means that the long-distance behaviour will run away from the fixed point, and the correlation length will decrease.

[8]Irrelevant terms are not expected to alter large scale properties, such as the correlation length

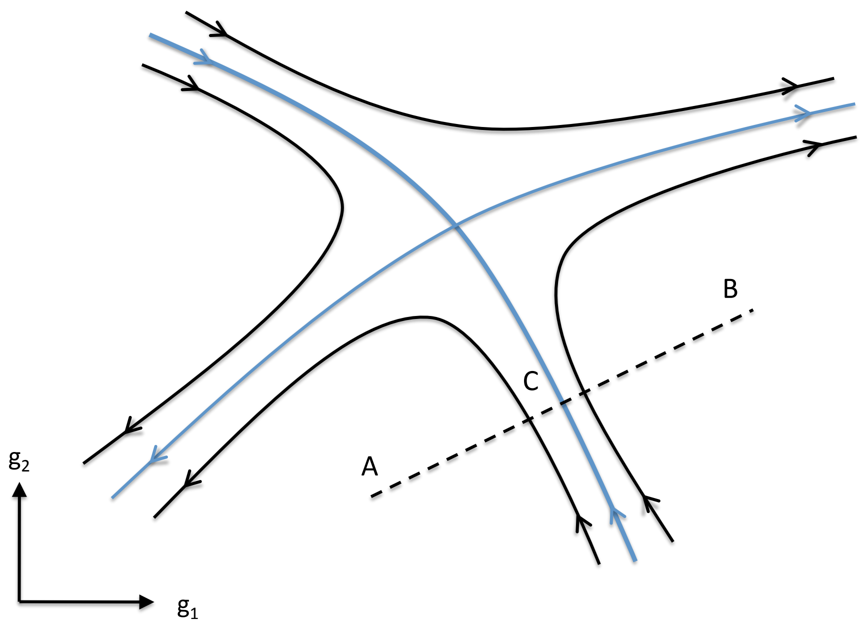


Figure 2.2: Renormalization group flows near a fixed point with both a relevant and an irrelevant direction. A phase transition occurs when one tunes the system through the critical surface along a relevant direction which affects the value of ξ. The dotted line corresponds to the physical values of the coupling constants. The short-distance cutoff on the dotted line is typically the lattice spacing. Unless the system is at a fixed point, RG flows lead away from the physical line, towards a fixed point which gives the long-distance scale description (sometimes called IR fixed point). At points A and B, which do not lie on the critical surface, the correlation length is finite and RG flows will lead away from the critical surface, towards another fixed point which has $\xi = 0$ (not included in the figure). At point C, where the system is tuned through the critical surface, the correlation length is infinite, and the system flows towards the critical point.

2.3 Quantum criticality in metals

Metals close to a magnetic QCP show behaviour that deviates from the Fermi liquid behaviour that characterized the low temperature phase of normal metals. Two hallmark features of Fermi liquid behaviour are the quadratic relation between resistivity and temperature,

$$\rho \sim T^2, \tag{2.3}$$

and the linear temperature dependence of the specific heat,

$$C \sim T. \tag{2.4}$$

Experimentally, a way to see the breakdown of Fermi liquid behaviour as an external magnetic field is tuned close to some *critical value* can often be seen in a divergence of the specific heat coefficient, defined as

$$\gamma \equiv \frac{C}{T}. \tag{2.5}$$

Since $C = T\partial s/\partial T$, with s the entropy density of the system, we can equivalently write $\gamma = s/T$ at low temperatures.

2.3.1 Hertz-Millis theory of quantum criticality

A theory for magnetic quantum phase transitions in metals extending Landau's theory for classical phase transitions was proposed by Hertz [50] and reconsidered and extended by Millis [61]. Hertz pointed out that near a phase transition at $T = 0$ statics and dynamics could not be treated separately, as in the Landau-Ginzburg theory of classical phase transitions. He proposed a generalization of the *Landau-Ginzburg-Wilson functional* (which is the effective theory near a continuous phase transition) in terms of a time-dependent order parameter. The effective action can be obtained by integrating out irrelevant degrees of freedom. If this integration does not lead to an analytic action, the method is invalid[9] This can happen for example in a phase transition described by a magnetic order parameter if additional degrees of freedom (other than the magnetic ones) become critical at the transition [58].

2.3.2 Metamagnetic Quantum Critical Point

A QCP can be the zero-temperature endpoint of a line of finite-temperature critical points, as described in the beginning of this chapter. Here, we shall describe another way in which a zero-temperature critical point can occur, namely by tuning the critical endpoint of a line of first-order metamagnetic transitions to $T = 0$ by varying an extra control parameter. This extra control parameter is usually pressure or chemical doping, and varying this parameter lowers the transition temperature by changing the relative importance of the

[9]Alternatively, one can construct an effective action from general arguments such as global symmetries, but this way one does not know if the theory is correct.

competing terms in the Hamiltonian that are relevant for the quantum phase transition. One example of such a *Quantum Critical Endpoint* (QCEP) occurs when a material which undergoes a *metamagnetic transition* is put under pressure (or tuned otherwise) so that the critical endpoint occurs at zero temperature. Metamagnetism is empirically defined as a nonanalytic increase in magnetisation at a certain value of applied magnetic field H. As the control parameter is a symmetry-breaking field, the transition can not be of the continuous, symmetry-breaking kind[10]. In general, a metamagnetic transition is expected to be a discontinuous transition occurring along some line in the H,T plane, and terminating at some temperature, above which the transition becomes a crossover. This point is called a critical endpoint. By applying pressure, one may be able to reduce the temperature at which the line of discontinuous phase transitions ends, all the way down to T=0, thereby making a so-called *quantum-critical endpoint* (QCEP). A material with such a metamagnetic QCEP can be a very convenient way of studying quantum critical behaviour, since having the magnetic field as a tuning parameter is much easier than applying pressure or changing the doping of a material.

In 2001, strong indications for a metamagnetic QCEP were found in the compound $Sr_3Ru_2O_7$ [36, 79]. $Sr_3Ru_2O_7$ is a layered structure, which for a large magnetic field perpendicular to the layers exhibits a line of first order metamagnetic phase transitions at finite temperature, ending at a finite temperature critical point. By including a component of magnetic field in the plane of the layers, the critical point can be brought to zero temperature. The theoretical treatment [62] of this metamagnetic quantum phase transition is based on Hertz-Millis theory[50, 61]. The order parameter of the transition is the difference in position of the spin-up and spin-down Fermi surfaces, and the *critical fluctuations* are the longitudinal fluctuations of the magnetisation density about its average value at the critical field. The Hertz-Millis theory describing the phase at and above the critical magnetic field corresponds to a dynamical exponent $z = 3$ [62].

As for these metamagnetic QCEPs a description in terms of Hertz-Millis theory is available, they might provide a good check for the holographic treatment of the quantum phase transition, which will be the focus of the rest of this thesis. In fact, the example of a holographic QPT that will be discussed in chapter 5 is of this metamagnetic type, and the results found for the QCP there agree with what one would expect from Hertz-Millis theory applied to a metamagnetic QCEP (albeit a system which has only one spatial dimension).

[10] It has been argued by [13] that non analytic terms would drive all ferromagnetic transitions first order

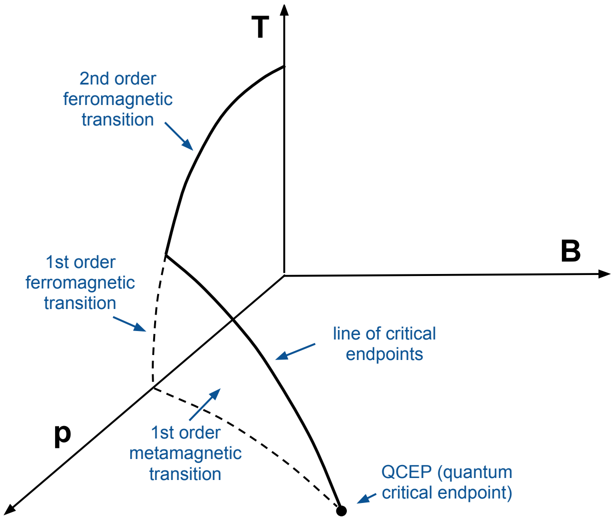


Figure 2.3: Phase diagram with a QCEP, which occurs when a line of critical endpoints (CEPs) is depressed to zero temperature. In the $B = 0$ plane, there is a 2nd order thermal phase transition from the FM phase to a paramagnetic phase. By applying pressure, the critical temperature can be lowered, until at some point the transition becomes 1st order. For a nonzero value of the applied magnetic field B, the second-order transition is no longer there, since the symmetry is broken by the applied magnetic field. However, a *metamagnetic* phase transition can occur, which at sufficiently low temperature is a discontinuous 'jump' in the magnetisation as the applied magnetic field is turned on above some value. Increasing the temperature smooths out the discontinuity in the magnetisation, until at some critical temperature the metamagnetic transition becomes continuous. This point is called a "critical endpoint". At temperatures above the critical temperature, the metamagnetic transition becomes a crossover. One can tune the temperature at which the critical endpoint occurs to zero by changin both the pressure and the magnetic field. The point at which the critical temperature is tuned to zero is called a "quantum critical endpoint" (QCEP).

Chapter 3

AdS/CFT

A possible new tool in studying strongly interacting theories that might describe the region around a Quantum Critical Point is the conjectured duality between certain strongly coupled gauge theories with a large number of "colour charges" N and classical supergravity. Such a duality is called a "gauge/gravity duality", and the various applications are often called "holography". The simplest example is the "AdS/CFT correspondence" [59], which will be reviewed in this chapter.

3.1 Holographic principle

The term "holography" is used because the most important idea on which these holographic dualities rely is the so-called "holographic principle" [73], which will be motivated in this section.

3.1.1 Black hole thermodynamics

According to the no-hair theorem, a classical stationary black hole solution can be characterized by mass, angular momentum and electric charge alone. Nothing can escape from the black hole, so its temperature must be zero. When quantum mechanics is taken into account, the existence of an event horizon puts the background at a finite temperature, as follows: Hawking discovered [49] that black holes radiate via a quantum process in which a vacuum fluctuation creates a virtual pair of particles of which one falls into the black hole and the other escapes to infinity. He showed by a semi-classical computation that a distant observer will detect a thermal spectrum of particles coming from the black hole, at a temperature

$$T_H = \frac{\kappa}{2\pi} \tag{3.1}$$

where κ is the surface gravity at the horizon, which is the gravitational acceleration experienced by an observer at the event horizon, *as measured at infinity*[1]. In other words, the black hole event horizon appears to a distant observer as a hot membrane which emits thermal radiation.[2]
Since the black hole is an object with mass and temperature, the laws of thermodynamics must apply. In 1973, Bardeen, Carter and Hawking found an analogy between the laws of thermodynamics and similar laws in black hole mechanics [11, 10, 48]. After the discovery of Hawking radiation, these "laws of black hole thermodynamics" were found to be more than just an analogy [12, 49].

The "zero-th law of thermodynamics" states that the temperature of a body in thermal equilibrium is uniformly distributed. For a black hole, the equivalent statement is that the surface gravity on the event horizon is constant all over.

The first law of thermodynamics reads

$$dE = TdS. \tag{3.2}$$

Einstein's equations imply an analogous *first law of black hole mechanics* [10]:

$$dM = \frac{\kappa}{8\pi} dA \tag{3.3}$$

where we recognise the Hawking temperature as proportional to the surface gravity κ. The entropy must then be proportional to the area of the event horizon.

The second law of thermodynamics reads:

$$dS \geq 0 \tag{3.4}$$

which was compared [10][3] with the area theorem for black holes:

The area of a black hole event horizon never decreases with time

which we write as:

$$dA \geq 0. \tag{3.5}$$

Bekenstein [12] and Hawking [49] found that

$$S_{BH} = \frac{A}{4}, \tag{3.6}$$

[1]Kiritsis: the surface gravity is the acceleration needed to keep a particle stationary at the horizon (in Killing coordinates).

[2]Hawking's result for the temperature of black hole radiation is consistent with the temperature predicted by the Unruh effect, where an accelerated observer moving in a vacuum will see thermal background radiation, with a temperature proportional to its acceleration.

[3](ignoring radiation)

so the entropy of a black hole is given by a quarter of the area of its horizon in *Planck units* (in full, $S_{BH} = kAc^3/(4G\hbar)$ with G Newton's constant).

When considering systems containing both black holes and ordinary matter outside the event horizon, the second law of thermodynamics holds only for the total entropy of the system, where the entropy of black holes $S_{BH} = A/4$ is included in the balance. It was shown that a *generalized second law of black hole mechanics* (GSL) holds [12]:

$$dS_{total} \geq 0 \tag{3.7}$$

Generalized second law (GSL): *if* $S_{total}^{initial} = S_{matter} + S_{BH}$*, then after interactions have taken place the final state must obey:*

$$S_{\text{total}}^{\text{final}} \geq S_{\text{total}}^{\text{initial}} \tag{3.8}$$

The third law of thermodynamics states that the entropy of a system at absolute zero is a well-defined constant. This is because a system at zero temperature exists in its ground state, so that its entropy is determined only by the degeneracy of the ground state. It means that "it is impossible by any procedure, no matter how idealised, to reduce any system to the absolute zero of temperature in a finite number of operations".

Similarly, the *third law of black hole mechanics* [10] states that it is impossible to reduce the surface gravity of a black hole to zero in a finite number of continuous steps.[4]

Temperature and entropy for charged black hole

The metric for a charged Reissner-Nordstrom black hole is given by:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{3.9}$$

where in our units $c = G = 1$ and the parameters M, Q are to be interpreted respectively as the mass and electrical charge of the body by considering the asymptotic behavior of the metric. For $|Q| > M$ this metric has a naked singularity[5]. If the charged black hole is to be a solution of classical general relativity, a lower bound on the mass in terms of the charge,

$$|Q| \leq M, \tag{3.10}$$

must be obeyed. The Reissner-Nordstrom solution has two horizons, an outer and an inner one. These are defined by

$$\left(1 - \frac{2M}{r_\pm} + \frac{Q^2}{r_\pm^2}\right) = 0 \tag{3.11}$$

[4]Naïvely we would expect that the surface gravity of a non-extremal black hole with $M > |Q|$ would be able to gradually drop to zero by emitting away the electrically neutral matter until $M = |Q|$. This is indeed possible, but only if there is an infinite amount of time available. The reason is that, as the matter is being radiated away, it gets more and more difficult to radiate further since the temperature keeps dropping during the process of radiation.

[5]A *naked singularity* is a timelike singularity with no horizon to cloak it.

where r_+ (r_-) refers to the outer (inner) horizon:

$$r_\pm = M\left[1 \pm \sqrt{1 - \frac{Q^2}{M^2}}\right] \tag{3.12}$$

The metric can be rewritten in terms of the inner and outer horizons:

$$ds^2 = -\frac{(r - r_+)(r - r_-)}{r^2}dt^2 + \frac{r^2 dr^2}{(r - r_+)(r - r_-)} + r^2 d\Omega^2 \tag{3.13}$$

Note that in the extremal limit $M = |Q|$ the inner and outer horizons merge at $r_\pm = M$.

The surface gravity at the outer horizon of a Reissner-Nordstrom black hole is [75]:

$$\kappa = \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2} \tag{3.14}$$

which can also be written as:

$$\kappa_+ = \frac{r_+ - r_-}{2r_+^2} \tag{3.15}$$

For an extremal Reissner-Nordstrom black hole ($r_- = r_+$, or equivalently $M = |Q|$) the surface gravity, and therefore the temperature, vanishes.

3.1.2 Bekenstein bound

From the thermodynamics of black holes, an upper bound on the entropy can be derived for 'matter' in a gravitational theory (this 'matter' need not be a black hole). The clearest way of seeing this is by considering the *Susskind process*, in which a system with spherical symmetry is converted into a black hole by letting a shell of energy collapse onto it. The entropies before and after the collapse are given by

$$S^{\text{initial}}_{\text{total}} = S_{\text{matter}} + S_{\text{shell}}, \tag{3.16}$$

$$S^{\text{final}}_{\text{total}} = S_{\text{BH}} = \frac{A}{4}. \tag{3.17}$$

The generalized second law, applied to this process, then yields a lower bound on the entropy of the matter before it was converted into a black hole (since $S_{\text{shell}} \geq 0$):

$$S_{\text{matter}} \leq \frac{A}{4} \tag{3.18}$$

This bound is called the *spherical entropy bound*. For systems with less symmetry, it is generally very difficult to construct a spacelike holographic bound. For a review on the construction of entropy bounds, see the review by Bousso [14]

Another way to think about the entropy of a system is in terms of its degrees of freedom. If N_{states} denotes the number of microstates of a system, then $S = \ln N_{\text{states}}$. In other words, the Bekenstein-Hawking entropy bound tells us that for a volume of space bounded by area A,

$$N_{\text{states}} \leq e^{A/4} \tag{3.19}$$

As a simple example, one can compare this to the number of microstates of a d-dimensional Ising lattice in a volume V, with (average) lattice spacing a:

$$N_{\text{states}} = 2^{V/a^d} = e^{(\ln 2)V/a^d} \tag{3.20}$$

From this result it is clear that a local field theory has too many degrees of freedom to describe a system where gravity is taken into account, since the degrees of freedom of a local field theory scale with the volume instead of the area. This could be solved by considering a field theory living on the *boundary* of the volume. If the information in a volume of space was somehow encoded by a field theory on the boundary, much like a hologram, its degrees of freedom would scale correctly with system size. This observation led to the formulation of the *holographic principle* [73, 71]:

A gravitational system in a volume of space can be described in terms of a field theory on its boundary (with no more than 1 bit of information per Planck area).

3.1.3 Anti-de Sitter spacetime

AdS space is the vacuum of theories with a negative cosmological constant, and it is the background in which the holographic principle is best understood[6].

AdS metric

One often encounters the following form of the AdS metric in terms of so-called *Poincaré coordinates*[7]

$$ds^2 = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}\left(-dt^2 + dx^i dx_i\right) \tag{3.21}$$

Another way of writing this metric is by redefining the radial coordinate r in terms of a new radial coordinate z:

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + dz^2 + dx^i dx_i\right) \tag{3.22}$$

Now the boundary is located at $z = 0$. This new radial coordinate z[8], which I will call the *Fefferman-Graham coordinate*, measures the radial distance from the boundary of AdS. This will be the radial coordinate used in sections **??** and **??**, as well as in chapter 4. The

[6] It can be shown [70] that any spacelike surface in AdS space will have its entropy bounded by the area.

[7] This is the metric of an incomplete patch of AdS spacetime (the *Poincaré patch*). It is also an approximation to the global AdS metric in the region near the boundary.

[8] not to be confused with the *dynamical exponent*, which uses the same letter

radial coordinate r used in chapter **??** is **not** the same as the coordinate r in (**??**).

From (3.21) it is easy to see that the metric of AdS spacetime is invariant under the following scale transformation:

$$r \to \lambda^{-1} r, \qquad t \to \lambda t, \qquad x^i \to \lambda x^i \tag{3.23}$$

which, if the t, x^i are taken to be the coordinates of the theory on the boundary, seems to imply that the radial coordinate r plays the role of the energy scale of the dual theory. Apart from the above scale transformation, the AdS metric has more isometries. In fact, the group of isometries of $d+1$-dimensional AdS space (or AdS_{d+1}) is given by SO(2,d), which is the same as the group of conformal transformations on $d+1$-dimensional Minkowski spacetime. If the dual field theory is thought of as 'living on the boundary'[9] this implies that the boundary theory is a d-dimensional conformal field theory (CFT).

UV/IR relation

The observation made above that the radial parameter r (or $1/z$) seems to play the role of the energy scale of the dual theory living on the boundary turns out to be true: it was found in [72] that the lare r cutoff that is needed to regularise IR ("infra red" or: "low energy") divergences in the AdS theory is directly proportional to the UV ("ultra violet" or: "high-energy") cutoff needed to regularise the dual CFT. In more general examples of holographic dualities, the relation between the radial parameter in the *bulk* spacetime and the energy scale of the dual field theory on the *boundary* turns out to be less simple. The notion that such a relation exists in a holographic duality is called the *IR-UV connection*

3.2 AdS/CFT correspondence and basic dictionary

The holographic principle is very general, and it does not specify which theory of gravity in the bulk of some spacetime is dual to which theory on the boundary. However, one may improve the situation by looking at examples with a high degree of symmetry. If some field theory is to be equivalent to some theory of gravity, then the symmetries of the field theory should somehow be translated into symmetries of the gravity theory. In particular, the field theory symmetries are translated into isometries of the background spacetime of the gravity theory. A high degree of symmetry therefore restricts the possible gravity background solutions, making the task of finding a correspondence much easier. As was stated in the previous section, in the case of a CFT, the symmetry group corresponds to the group of isometries of AdS spacetime. The example found by Maldacena is indeed called the *AdS/CFT correspondence* [59].

[9] An interesting feature of AdS spacetime is that massless fields can go to the boundary and back in finite proper time [70].

3.2.1 String description of gauge theories

Confining gauge theories
An early hint for where to look for an example of a holographic duality came from the discovery [74] that confining gauge theories at strong coupling have a low energy effective description in terms of a string keeping the fundamental particles together.
Consider for example SU(3) Yang-Mills in 4 dimensions, which is the theory describing the strong interaction between quarks and gluons in quantum chromodynamics (QCD). At high energies (or short distance scales), the quarks are asymptotically free, and are described by perturbative gauge theory. However, at low energies (or long distance scales) the coupling strength becomes large and the perturbative description breaks down. There is, however, a different description in terms of pairs of quarks bound together by a flux tube of gluons. As the coupling strength increases (at lower energy), there will be more "glue" keeping the two quarks together, and also keeping the flux lines together (since the gluons are self-interacting), resulting in something that looks very much like a string. So at lower energy a description in terms of strings becomes better, and one enters a different perturbative regime.

Large N expansion

We can generalize the above theory to N "colours", so we consider SU(N) Yang-Mills theory. The fields can be normalized so that the Lagrangian can schematically be written as

$$\mathcal{L} \sim \frac{1}{g_{YM}^2}\mathrm{Tr}\left((\partial\Phi)^2 + \Phi^2 + \Phi^3 + \ldots\right) \tag{3.24}$$

where the fields $\Phi = \Phi^a_b$ are in the adjoint representation of SU(N).
In this normalization, the weight of a vacuum diagram with E edges, V vertices and F index loops is given by

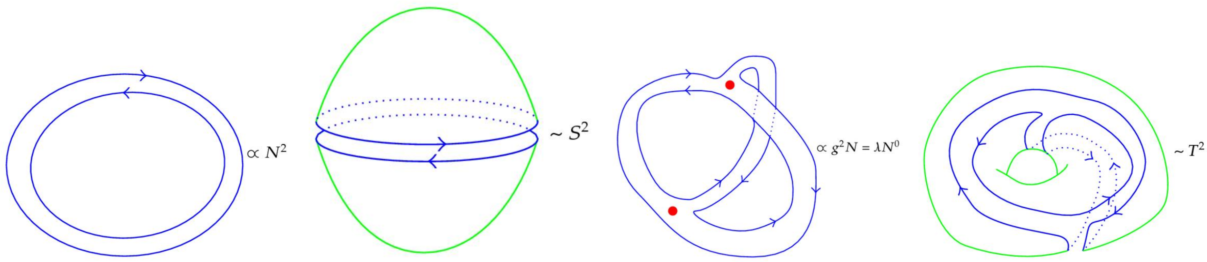
$$(g_{YM}^2)^{E-V}N^F = \lambda^{E-V}N^{2-2g} = (g_{YM}^2)^{2g-2}\lambda^F, \tag{3.25}$$

where the 't Hooft coupling is defined by $\lambda \equiv g_{YM}^2 N$, and g is the number of handles of the diagram, related to the number of edges, vertices and index loops by $2-2g = \chi = E - V + I$[10]. Note that the weight contains a "topological term" which depends only on the genus of the surface on which the diagram can be drawn (given by either N^{2-2g} or $(g_{YM}^2)^{2g-2}$) and not on the number of interactions in the Feynman diagram. The other term does depend on the number of interactions, (either λ^{E-V} or λ^F).

't Hooft found that in the limit $N \to \infty$ and $g_{YM}^2 \to 0$ such that λ remains fixed (the so-called *'t Hooft limit*), the Feynman diagrams are organised in a perturbative expansion in terms of $1/N$. From the first equality in (3.25) This expansion can be regarded as a *topological expansion*, since (see figure 3.1 for examples of a leading-order and a first-order diagram). This expansion is similar to a perturbative string expansion, where the genus

[10]the first equality is a known result from mathematics, stating that for diagrams that can be turned into closed oriented surfaces, the Euler character depends only on the topology of the surface.

of the worldsheet comes with a factor of the string coupling g_s according to g_s^{2g-2}. (note that the string interaction g_s governs the splitting and joining of string endpoints, so if this interaction is weak, worldsheets with genus 0 are most likely).



(a) Planar graph (b) Surface of genus 0 (c) Non-planar graph (d) Surface of genus 1

Figure 3.1: Direct surfaces constructed from a planar and a nonplanar graph.
Images taken from [60]

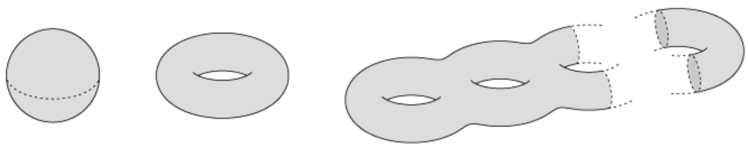


Figure 3.2: Topological expansion of closed string vacuum diagrams in small g_s

Strong/weak coupling

In the gauge theory, one has to take into account not only the Yang-Mills coupling g_{YM}, but also the number of "colours" N, since even if the Yang-Mills coupling is very weak, a very large number of colours means that the weight of a diagram with interaction vertices can become larger due to a colour loop in which one sums over all N colours. From equation (3.25) we can see that if we take the 't Hooft limit, only planar diagrams will survive. Which planar diagrams dominate is then determined by the value of λ. If λ is large the diagrams with many edges (or faces) will become important, meaning the triangulation of the surface will be smoother.

The perturbative string expansion is an expansion in small g_s (lower genus) and small α' (small quantum fluctuations on worldsheet / large radius of curvature in target space?). Similarly, in the gauge theory perturbative expansion there is the number of colours N, and the Yang-Mills coupling g_{YM}^2, although we can also exchange either of these with the 't Hooft coupling $\lambda \equiv g_{YM}^2 N$.

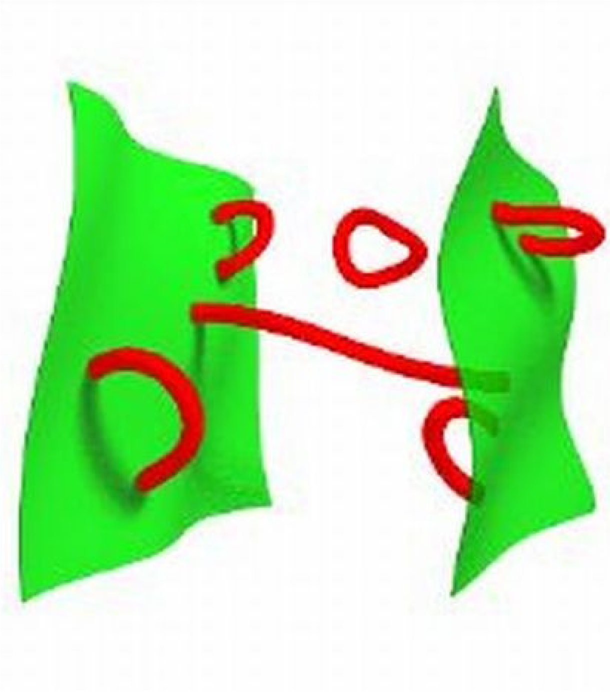
The expansion can be separated into an N-dependent part which only depends on the genus of the surface the diagram triangulates, and a λ-dependent part which goes like

λ^{E-V}, where E is the number of edges of the triangulation and V the number of vertices. If λ is small (meaning the gauge theory is strongly coupled), the diagrams with a large number of edges become more important, and the triangulation of the worldsheet of the string is smooth. This can be related to a weakly-curved target space, or to small α'.

3.2.2 D-branes

A concrete example of a duality between a theory of gravity and a gauge theory was found [59] by comparing two different descriptions of Dp-branes. As will be explained later, these are massive, dynamical objects in type II string theory, extending along p spatial dimensions. Their dynamics are described by open strings which have their endpoints attached to the brane. These endpoints can move along the $p+1$-dimensional worldvolume of the brane, but can not separate from this surface. At the same time, the Dp-branes act as sources for the gravity field of the theory by emission of closed strings, which happens when two open strings collide and join endpoints.

The two different descriptions of Dp-branes are in terms of open strings on their worldvol-



ume and in terms of closed strings moving in the geometry caused by their backreaction on the metric.These different descriptions are valid in different regimes of the coupling "g_sN", which will be introduced below.

Open strings: "D-branes"

A Dirichlet p-brane (or Dp-brane) is a $p+1$ dimensional hyperplane in $9+1$ dimensional space-time where open strings are allowed to end. The endpoints are free to move along the brane's worldvolume, which means free (Neumann) boundary conditions for the $p+1$ longitudinal coordinates. The fact that the endpoints must remain on the brane worldvolume means that the $9-p$ coordinates transverse to the Dp-brane have fixed (Dirichlet) boundary conditions; hence the name "Dirichlet brane."

Strings with both endpoints on the same brane can have an arbitrarily short length, so these have massless states. This is not the case when strings are stretched between two parallel branes which are separated. In that case, the lowest-energy states have a mass proportional to the separation of the branes. However, if the parallel branes are brought closer together so that they coincide, the stretched strings become massless as well.
In order to keep track of which brane the endpoint is attached to, one can assign a non-dynamical label to the endpoints of the string. This is referred to as a *Chan Paton factor*. For N coincident D-branes, each string has two labels $i, j = 1, \ldots, N$, and its massless states transform in the *adjoint* of a U(N) gauge theory[11].
It turns out that the low-energy effective action on a stack of N coincident D3-branes reduces to maximally supersymmetric[12] Yang-Mills theory [78] (in short: "$\mathcal{N} = 4$ SYM"), which is a CFT. In the D-brane action, the string coupling plays the role of the Yang-Mills coupling, and we identify (up to a constant factor):

$$g_{YM}^2 = g_s \tag{3.26}$$

From this we can recognise the 't Hooft coupling $\lambda \equiv g_{YM}^2 N$, which for the Yang-Mills theory on the D-branes is given by $\lambda \sim g_s N$. In the limit $g_s N \ll 1$, the gauge theory is weakly coupled.

So far, we have ignored the interactions with closed strings that live in the 10D background. As mentioned above, a closed string can be formed when open strings on the brane collide and join endpoints. The splitting and joining of string endpoints comes with a factor of the string coupling g_s. When N branes are stacked on top of each other, we may therefore guess that the "strength" of the backreaction of these branes is proportional to $g_s N$.

Closed strings: "black p-branes"

[13]

Consider a generalization of a (charged) black hole solution, where the singularity extends along p spatial dimensions. Just like a Reissner-Nordstrom black hole, when the mass and charge saturate the bound $M \geq |Q|$, we end up with a p-dimensional generalization

[11] One can also consider the stack of branes in its rest frame and ignore the U(1) part, so that the group becomes SU(N).

[12] with $\mathcal{N}$ supersymmetry generators and SU(N) gauge group.

[13] By "black p-brane", "p-brane" or "extremal brane" we will mean the geometry of a p-dimensional black hole. When we consider the object in type II string theory, we will call it a "Dirichlet brane", "Dp-brane", etc. When we use "brane", this may refer to either, depending on the context.

of an extremal black hole. [14] Such an "extremal p-brane" solution appears in type II supergravity, which is the low-energy ($l_s = \sqrt{a'} \to 0$) limit of type II superstring theory. We will assume that the singularity extends along the $t, x^1, \ldots, x^p$ directions, and use spherical coordinates r, Ω_{8-p} for the $(9-p)$ directions perpendicular to the *brane*. The extremal p-brane solution reads:

$$ds^2 = H(r)^{-1/2}(-dt^2 + \sum_{i=1}^{p} dx^i dx^i) + H(r)^{1/2}(dr^2 + r^2 d\Omega_{8-p}^2) \tag{3.27}$$

where the function $H(r)$ is given by

$$H(r) = 1 + \frac{L^{7-p}}{r^{7-p}} \tag{3.28}$$

As alluded to before, it is expected that the black p-brane solution of type II supergravity extends to a solution of the full type II superstring theory[15], with the Dp-branes of the previous section describing their full string dynamics [64].[16]

Such a *model* of an "extremal black brane" can be made e.g. by putting N D3-branes on top of each other. The *supergravity solution* will have the same form as in equation (3.27), with $p = 3$:

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2}(-dt^2 + dx^i dx_i) + \left(1 + \frac{L^4}{r^4}\right)^{1/2}(dr^2 + d\Omega_5^2) \tag{3.29}$$

with the radius now given by:

$$L^4 = 4\pi g_s N l_s^4 \tag{3.30}$$

with a constant dilaton $g_s = e^{\Phi}$. As with the general p-brane solution, the metric becomes flat 10D Minkowski spacetime in the large r limit. In the limit $r \ll 1$, which is near the horizon of the extremal solution, the spherical part of the metric decouples from that of the directions along the brane, and the metric becomes a product geometry

$$ds^2 \approx \frac{r^2}{L^2}(-dt^2 + \sum_{i=1}^{3} dx^i dx^i) + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2 \tag{3.31}$$

which is $AdS_5 \times S^5$. The limit $r \to 0$ is often referred to as the *near-horizon* limit.
Note that the radius of curvature L of the AdS space (and the five-sphere) should be large

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[14] If the mass of this charged black hole were to decrease below the extremal value, there would be a naked singularity. Extremal black holes correspond to zero temperature backgrounds for observers at rest in their geometry.

[15] which may then be subject to α' (l_s) corrections

[16] It was shown in a series of papers starting with [69] that the Bekenstein-Hawking entropy of certain extremal black branes agree with the entropy calculated by counting the states in the gauge theory describing suitable systems of Dirichlet branes. Furthermore, the Hawking radiation rates and absorption cross sections were calculated and successfully reproduced by D-brane models [52].

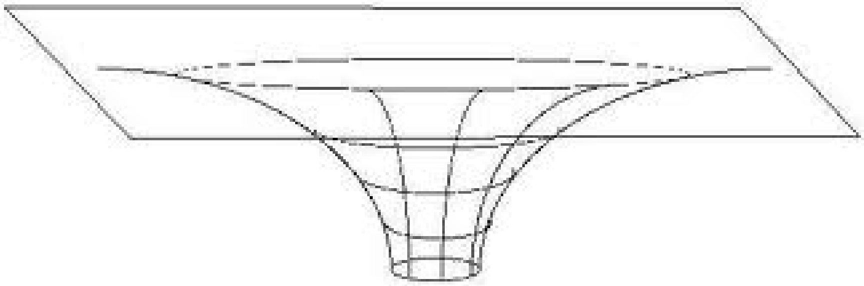

Figure 3.3: Geometry caused by the backreaction of a black p-brane. Close to the branes, the geometry becomes that of a "throat". Far away from the brane, the geometry is just the flat 10D Minkowski spacetime the p-brane was placed in. In the picture, all directions parallel to the brane are not drawn. Also, in the picture there are two transverse directions. In general, if a p-brane is placed in $\mathbb{R}^{1,9}$, there will be $p+1$ parallel directions (including time) and $9-p$ transverse directions.

in order for *stringy corrections* (α') to the supergravity solution to be small. From (3.30) we see that this means that $g_sN \gg 1$. In order for classical supergravity to be a good description, we should also have $g_s \ll 1$, so the regime of validity for the above solution becomes

$$1 \ll g_sN \ll N \tag{3.32}$$

In the following, we will always consider this case of a stack of N coincident D3-branes.

Regimes of validity of the different descriptions

The above black p-branes and D-branes are different simplified pictures of an object in string theory called a Dp-brane. This object is sketched in figure **??**. The process of open strings colliding and forming a closed string that can move away from the brane and backreact on the metric comes with a factor of g_s, which we take to be small. Since each brane contributes this factor, the backreaction of a stack of N branes comes with a factor of g_sN. At the same time, we can see from (3.30) that in the limit $g_sN \ll 1$, the radius of the throat L is small compared to the string length l_s and all stringy probes cannot penetrate and sample the throat region. The open string description therefore is only valid if $g_sN \ll 1$. Classical supergravity, on the other hand, is a good description if $g_sN \gg 1$ (or: $L/\alpha' \gg 1$) and $N \gg 1$ (or: $g_s \ll 1$).

3.2.3 Decoupling limit for N D3-branes

When describing a stack of N D3-branes placed in $\mathbb{R}^{1,9}$, one generally cannot treat the region near the branes separately from the region away from the branes, since in general there are interactions between the branes and the closed strings living in the 10D background. However, in the low-energy limit these interactions vanish, since the low-energy gravity fields that survive this limit in the 10D flat space have too long wavelengths to tunnel into the throat region. In other words, the absorption cross section goes to zero in the low-energy limit. The limit in which the Dp-branes decouple from the 10D spacetime is called the *Maldacena limit* or the *decoupling limit*.

A convenient way to take the decoupling limit is to keep the energies of physical processes fixed while taking the limit $\sqrt{\alpha'} = l_s \to 0$. Away from the branes we have $r > 0$, and we see from the metric (3.29) that in the limit $l_s \to 0$ that the metric becomes everywhere flat for all nonzero values of r. Away from the branes, we therefore are left with free type IIB supergravity in $\mathbb{R}^{1,9}$.

Close to the branes, we have the two perturbative descriptions in terms of open or closed strings. In the open string description, taking the low-energy limit leaves us with $\mathcal{N} = 4$ SYM on the 3+1 dimensional worldvolume of the branes. In the closed string description, the states that survive the decoupling limit are the aforementioned background fluctuations which are decoupled from the throat region, but also closed string states of type IIB string theory that live in the near-horizon region[17]. The reason why full type IIB string theory in the near-horizon limit is part of the low-energy description is because the redshift diverges in the limit $r \to 0$. So we can conclude that the low-energy limit corresponds to the near-horizon limit in the closed string picture.

3.2.4 $\mathbf{AdS_5/CFT_4}$ correspondence

After taking the low-energy limit in both the open string and the closed string description and noting that in both cases we get free supergravity in $\mathbb{R}^{1,9}$ which is decoupled from the branes, Maldacena was led to conjecture that the two *near-brane* parts (as defined in the previous section) must be different descriptions of the same theory, valid in different regimes of the "coupling" $g_s N$.

In its strongest form, the Maldacena conjecture [59] *(for the case of D3-branes)* states that:

Type IIB superstring theory compactified on $AdS_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ SYM theory on $\mathbb{R}^{1,3}$.

This conjecture also goes by the name "AdS/CFT correspondence", and it is a special case of a so-called "gauge/gravity duality". The latter contains all examples where a (decoupling limit can be taken in a gravity system and a) dual description found in terms of a gauge theory on the boundary. Another name by which it is often referred to is the

[17]Due to the gravitational red-shift, an object moving towards $r = 0$ would appear to have lower and lower energy to an observer at $r = \infty$.

"bulk/boundary correspondence", where the theory in the bulk of AdS is said to be dual to a CFT "living on the boundary" of AdS[18].

Maldacena conjecture in the 't Hooft limit

We are interested in the limit in which the gauge theory is "difficult" (strongly coupled), and the gravity theory is "easy" (classical and weakly curved). In the *'t Hooft limit* $N \to \infty$, $\lambda \equiv g_s N =$ constant, we saw before that the diagrams with no string loops became dominant. In this limit, classical superstring theory is a good description.

Weakest statement: $N \to \infty$, $\lambda \equiv g_s N \to \infty$

In the case of a D3-branes, with a constant dilaton $g_s = \exp \Phi$, the 't Hooft coupling is a dimensionless parameter and one can consider the limit in which both $N \to \infty$ and $\lambda \to \infty$. As we saw before, classical supergravity is a good approximation in this limit, and it it this version of the correspondence which will be interesting to us.

3.2.5 Symmetry matching

One basic check of the duality is comparing the symmetries of the dual theories. Recall from section 3.1.3 that the isometry group of AdS_5, which is SO(2,4) (the covering group is SU(2,2)), corresponds to the symmetry group of a CFT on $\mathbb{R}^{1,3}$. From this we concluded that in general, a theory of gravity on AdS_{d+1} should have a dual description in terms of some CFT "on the boundary".

The isometry group of the S^5 part of the geometry is SO(6), which has a covering group SU(4). In the boundary theory, this corresponds to the so-called *R-symmetry group* of $\mathcal{N} = 4$ supersymmetry, which is a symmetry under which supercharges transform into each other.

In both cases, the symmetries are global symmetries of the boundary theory, but appear as gauged symmetries (diffeomorphisms) in the bulk theory. In [44], it is argued that these global symmetries are just the 'large' gauge symmetries of the bulk theory, and one may expect the following general correspondence to hold:

Gauged symmetry of $(d+1)$-dimensional bulk $\leftrightarrow$ Global symmetry of d-dimensional boundary

3.2.6 Kaluza-Klein compactification

The AdS/CFT correspondence is often described as a duality between a $d+1$-dimensional theory of gravity, and a d-dimensional field theory. We saw before, however, that the theory

[18] see section 3.1.3

of gravity is actually ten-dimensional. Five dimensions form the AdS_5 spacetime, while the other five dimensions are compactified into a sphere. We can expand the ten-dimensional supergravity fields into so-called *Kaluza-Klein modes*. In the simplest example of a Kaluza-Klein expansion, one dimension is compactified into a circle. A field can then be expanded in Fourier modes around this circle, thereby replacing the dependence on the compactified dimension with an infinite tower of states labeled by their mass (which take on a discrete set of values). We can separate the Kaluza-Klein modes into "light" and "heavy" fields, which are classified relative to the energy scale we are probing. In the case of compactification on a circle (or an n-torus), all but the lowest-order fields in this expansion are massive and can be ignored when probing large distance scales. When performing a *dimensional reduction*, where the circle (or torus) is shrunk to zero size, these fields become infinitely heavy. For general cases of compactification on some compact manifold $\mathcal{M}^d$ this need not be the case. This does not necessarily mean that all fields have to be kept in order to find a solution. If a *consistent truncation* is found to a subset of the fields, the solutions of the truncated theory can be *lifted* to solutions of the full theory. This will be explained in section 4.1. Consistent truncation does not rely on taking some low-energy limit. Rather, it means that all but a few fields can be consistently set to zero in the equations of motion. Consistency demands that the remaining fields in the truncated theory do not generate fields that were set to zero.

It turns out [53] that type IIB supergravity compactified on S^5 to five dimensions reduces to $\mathcal{N} = 8$ gauged supergravity with SU(4)$\simeq$SO(6) gauge symmetry. Gauged supergravities are supergravity theories with non-abelian gauge fields in the supermultiplet of the graviton.

3.3 GKPW prescription

Having found a theory of gravity and a field theory that are conjectured to be dual, we would like to find the mapping between the observables on both sides. These include the spectrum and correlation functions.

3.3.1 Mapping between fields and operators

In order to look for the precise mapping between fields in gravity and in the dual gauge theory, we will have a look at the fields on both sides of the AdS/CFT correspondence.
First, we note that the gauge theory is a CFT, and therefore has no asymptotic states, so we will consider local operators on the gauge theory side. Second, these operators will need to be gauge-invariant, because the gauge symmetries of the CFT correspond to isometries of the bulk spacetime. One can obtain gauge-invariant combinations of the fundamental fields (which are not gauge-invariant) by taking a product of the fundamental fields, all

evaluated at the same spacetime point, and taking a trace[19]

$$\mathcal{O}^{\alpha\cdots\delta}_{\mu\cdots\sigma}(x) = \mathrm{Tr}\left(\cdots \mathrm{SYM\ fields}(x)\cdots\right) \quad (3.33)$$

In the following we will generally suppress the Lorentz ($\mu\cdots\sigma$) and (supersymmetry) spinor ($\alpha\cdots\delta$) indices.

A hint for the mapping between gravity fields and gauge theory operators came from the previously mentioned comparisons of absorption cross sections. There, the response of the brane systems to external probes coming in from the asymptotic flat region in the D-brane picture was found[20] to agree with that in the p-brane picture [65, 39, 40]

On the D-branes, the external probes correspond to the insertion of certain local gauge-invariant operators, whereas in the p-brane description an external probe perturbs the geometry as it comes in from infinity. The difference with the case considered by Maldacena is that for the absorption calculations the entire geometry was kept, not just the throat region. The AdS/CFT correspondence suggests that if we cut out the throat region and replace it with the D-branes, then the response should be identical. Thus, the insertion of gauge-invariant operators on the D-branes should presumably be identical to the response of the throat geometry to changing the boundary conditions at the edge of the throat.

3.3.2 Correlation functions

Consider a set of gauge-invariant (single-trace) operators $\mathcal{O}_i$. We can deform the CFT by adding source terms for the gauge-invariant operators:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \sum_i J_i(x)\mathcal{O}_i(x) \quad (3.34)$$

so that a generating functional can be written down

$$Z_{\mathrm{CFT}}[J] = \langle e^{-\int \sum_i J_i \mathcal{O}_i}\rangle_{\mathrm{CFT}} \quad (3.35)$$

from which correlation functions can be obtained via

$$\langle \mathcal{O}_{i_1}(x_1)\cdots\mathcal{O}_{i_n}(x_n)\rangle = \frac{\delta}{\delta J_{i_1}(x_1)}\cdots\frac{\delta}{\delta J_{i_n}(x_n)}\ln Z\bigg|_{J=0} \quad (3.36)$$

The coupling of a gravity probe ϕ to the branes can be deduced from the D-brane action, and is of the general form

$$\int d^4x\phi_0(x)\mathcal{O}(x) \quad (3.37)$$

[19] In the large N limit, correlators of multiple-trace operators, denoted e.g. by $\langle\mathcal{O}\mathcal{O}\rangle$ for a double-trace operator, factorise into the single-trace parts: $\langle\mathcal{O}\mathcal{O}\rangle \sim \langle\mathcal{O}\rangle\langle\mathcal{O}\rangle + O(1/N^2)$. The connected diagrams are suppressed by factors of N since the disconnected diagrams have more index loops. This goes by the name *large-N factorization*. The single-trace operators are basically classical objects in the large N limit, and may be expected to be dual to states in gravity (which also becomes classical in this limit).

[20] at low energy

where x denotes the four coordinates on the brane worldvolume, and ϕ_0 denotes the restriction of the *bulk* field ϕ to the brane. Note that ϕ_0 plays the role of the source term J.

The proposal made by GKPW [37, 76] was:

$$Z_{\mathrm{CFT}}[J=\phi_0] = Z_{\mathrm{string}}[\phi \to \phi_0] \tag{3.38}$$

with the generating function of the CFT given by equation (3.35), and on the right-hand side the full string theory partition function, with the boundary condition that the bulk fields $\phi(z,x)$ take on the values $\phi_0(x)$ at the boundary of AdS.

A formula like (3.38) is valid in general, for any field ϕ. Therefore, each field propagating on AdS space is in one-to-one correspondence with an operator in the field theory. We shall therefore take J in the relation (3.38) to mean all sources in the gauge theory.

We will be working in a low-energy limit in which we suppose that we may reliably use classical gravity. This means we will make use of the weakest formulation of the AdS/CFT correspondence which is defined for large N and large λ, and which is dual to a strong-coupling (and planar) limit in the dual field theory. If we use the weakest form of the AdS/CFT correspondence, we can argue that a saddle point to the superstring partition function Z_{string} is given by type IIB supergravity. Thus we can approximate the string partition function $Z_{\mathrm{string}}(\phi \to \phi_0)$ by[21]

$$Z_{\mathrm{string}}(\phi \to \phi_0) \approx \exp(iS_{\mathrm{sugra}}(\bar{\phi} \to \phi_0)), \tag{3.39}$$

where $\bar{\phi}$ denotes the solution of type IIB supergravity with leading asymptotic behavior ϕ_0 near the conformal boundary. In the weakest form, the AdS/CFT correspondence therefore equates

$$\langle e^{i\int d^4x \phi_0 \mathcal{O}} \rangle_{SYM} = \exp(iS_{\mathrm{sugra}}(\bar{\phi} \to \phi_0)), \tag{3.40}$$

The on-shell bulk action, S_{sugra}, acts as the generating functional for correlators involving the operator $\mathcal{O}$. In other words, to compute renormalized correlators of the operator $\mathcal{O}$, we take functional derivatives of S_{sugra} with respect to the source ϕ_0. For example the connected correlator of the operator $\mathcal{O}$, i.e. $\langle \mathcal{O} \rangle_c$, is given by

$$\langle \mathcal{O} \rangle_{\mathrm{c}} = \frac{\delta}{\delta \phi_0} S_{\mathrm{sugra}} \bigg|_{\phi_0 = 0}. \tag{3.41}$$

The GKPW prescription provides us with a method for computing gauge theory correlation functions in gravity, by taking multiple derivatives with respect to the sources,

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle_{\mathrm{CFT}} = \frac{\delta}{\delta \phi_0(x_1)} \cdots \frac{\delta}{\delta \phi_0(x_n)} S_{\mathrm{sugra}} \bigg|_{\phi_0 = 0}. \tag{3.42}$$

[21]we will ignore the fact that there may be more saddle points that are important.

3.3.3 Boundary conditions

Recall (section 3.1.3) that massless particles in AdS can reach spatial infinity in a finite time. Therefore, in order to consistently quantize[22] fields on AdS, one must not only impose initial conditions, but also boundary conditions at infinity [15, 4]. The behaviour of the fields have two linearly independent modes, which behave near the AdS boundary as

$$\phi_\pm \sim \epsilon^{\Delta_\pm} \tag{3.43}$$

where ϵ denotes the (small) distance to the boundary which is located at $z = 0$ in Fefferman-Graham coordinates (introduced in section 3.1.3), and for the scalar field

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + L^2 m^2} \tag{3.44}$$

The mass of a field in AdS (for which the energy is conserved, positive and finite) has to satisfy the *Breitenlohner-Freedman bound*, which for the case of a scalar is given by [15]

$$m^2 > -|m_{\mathrm{BF}}|^2 = -\frac{d^2}{4L^2}. \tag{3.45}$$

So-called "BF-allowed tachyons", with masses in the range $-d^2/4L^2 < m^2 < 0$, surprisingly do not cause an instability of the background.

In the boundary behaviour of the field ϕ (3.43), the mode with the larger value Δ_+ corresponds to a *normalizable mode*, since ϕ_+ vanishes at the boundary, leading to a finite contribution to the partition function[23]. Normalizable modes fluctuate and describe low-energy excitations of the bulk spacetime. The mode with the smaller value $\Delta_- = d - \Delta_+$ in general does not vanish at the boundary, in which case it is called a *non-normalizable mode*. The energy for such a non-normalizable mode is infinite, so it does not contribute to the partition function. Instead, these modes serve as classical, non-fluctuating backgrounds in which normalizable modes propagate [8]. In the GKPW prescription, the boundary values of these non-normalizable modes correspond to sources in the dual gauge theory[24], which deform the CFT. So in short, normalizable modes change the state the system is in, and non-normalizable modes change the theory itself.

We can summarize the asymptotic behaviour of a bulk field ϕ near the boundary in terms of the radial coordinate z[25] as

$$\phi(z, x) \approx \left(\frac{z}{L}\right)^{d-\Delta_+} \phi^{(0)}(x) + \left(\frac{z}{L}\right)^{\Delta_+} \phi^{(1)}(x), \qquad \text{as} \quad z \to 0 \tag{3.46}$$

[22] conserved and positive energy, inner products conserved, no information leaking away at infinity

[23] since normalizable modes are modes which have a finite action in Euclidean AdS, or a finite energy in Lorentzian AdS.

[24] In Euclidean AdS, only the non-normalizable modes are present, and for each given boundary field there is a unique extension into the bulk that is also a solution of gravity [76]. (regularity in the bulk, $z \to \infty$, uniquely fixes ϕ for a given boundary value ϕ_0)

[25] z measures the radial distance from the boundary, see section **??**

where we have used $\phi^{(0)}$ for the coefficient of the leading term, since the non-normalizable modes define the sources in the GKPW prescription [8].

In the presence of a source term $\phi^{(0)}_{\text{reg}}$ (see (3.59)), the expectation value for the dual operator $\mathcal{O}$ becomes

$$\langle \mathcal{O} \rangle_{\phi^{(0)}_{\text{ren}}} = \lim_{\epsilon_0 \to 0} \frac{\delta S_{\text{sub}}[\bar{\phi} \to \phi^{(0)}_{\text{reg}}]}{\delta \phi^{(0)}_{\text{reg}}} \tag{3.47}$$

If the radial evolution of the solution ϕ as a function of the radial parameter z is thought of as a kind of time evolution, we can define the 'canonical momentum' conjugate to ϕ:

$$\Pi_\phi(z) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_z \phi)} \tag{3.48}$$

which can be rewritten as the variation of the action with respect to the boundary value of the field

$$\Pi_\phi(z) = -\frac{\delta S_{\text{reg, on-shell}}}{\delta \phi} \tag{3.49}$$

Then we can evaluate the field momentum in the solution (3.46) and obtain For the scalar field, the result turns out to be [54]

$$\langle \mathcal{O}(x) \rangle = \frac{2\Delta_+ - d}{L} \phi_1(x) \tag{3.50}$$

From the coefficient $\phi^{(1)}$ of the subleading term in (3.46), the state of the system can be read off [9]. If the source term is zero ($\phi^{(0)} = 0$), then $\phi^{(1)}$ is the *vacuum expecation value* of the dual operator.

Note that although in the formula (**??**) for the GKPW prescription only one source ϕ_0 with its dual operator $\mathcal{O}$ is explicitly written, the prescription actually says that for all operators in the boundary theory there should be a corresponding field in the bulk. At the very least, our 'dictionary' should contain an entry for the boundary stress tensor $T^{\mu\nu}$. This should be sourced by the boundary metric, so the deformation of the CFT looks like

$$\int d^d x g_{(0)\mu\nu} T^{\mu\nu} \tag{3.51}$$

where $g_{(0)\mu\nu}$ is the restriction to the boundary of the bulk metric, which behaves near the boundary as [76, 35, 19]:

$$g_{MN} = \frac{L^2}{z^2} g_{(0)\mu\nu} + \dots \qquad \text{as} \quad z \to 0, \tag{3.52}$$

where greek letters $\mu, \nu, \dots$ are used to denote the spacetime coordinates on the boundary, and capital letters $M, N, \dots$ denote the spacetime coordinates in the bulk. We can conclude that in order to describe a local field theory at the boundary, we need a dynamical metric:

$$T^{\mu\nu} \leftrightarrow g_{MN} \tag{3.53}$$

where the symbol "$\leftrightarrow$" means "is dual to".
The expectation value for the boundary stress tensor can be computed from the induced boundary metric, which we set equal to $g^{(0)}_{\mu\nu}$[26].

$$\langle T^{\mu\nu}\rangle = \frac{-2}{\sqrt{-g^{(0)}}}\frac{\delta S_{\text{ren}}}{\delta g^{(0)}\mu\nu}, \tag{3.54}$$

where it is understood that one needs to regularize the expression on the right-hand side according to the procedure outlined in the next section.

In a similar way, we may want to add a massless gauge field A_M $(= A_\mu(z))$ to the bulk theory and find its entry in the dictionary. From

$$\int d^dx A^{(0)}\mu j^\mu \tag{3.55}$$

we conclude that

$$j^\mu \leftrightarrow A_M \tag{3.56}$$

where j is the conserved current of some global symmetry group, which corresponds to the gauge group of A in the bulk (see section 3.2.5). The near-boundary behaviour of the gauge field is given by:

$$A_M = A^{(0)}_\mu + \ldots \qquad \text{as} \quad z \to 0 \tag{3.57}$$

and the expectation value of the boundary current is given by

$$\langle j^\mu\rangle = \frac{1}{\sqrt{-g^{(0)}}}\frac{\delta S_{\text{ren}}}{\delta A^{(0)}_\mu}. \tag{3.58}$$

3.3.4 Renormalization

Generically, both the on-shell bulk action and the CFT generating functional diverge. On the bulk side, the divergences arise from the infinite volume of AdS [72](see the discussion in section 3.1.3), i.e. they are long-distance or infrared (IR) divergences. In the field theory, the divergences are short-distance ultraviolet (UV) divergences. To make the AdS/CFT correspondence meaningful we must regulate and renormalize these divergences. One can obtain a finite value for the action S_{sugra} by subtracting the divergent terms from the on-shell action, which are local[27] and covariant.

The procedure of holographic renormalization is as follows:

1. AdS space is truncated to a finite region, bounded by a new hypersurface at $z = \epsilon_0$. Note that at this point, conformal invariance of the dual field theory is lost because a UV scale is introduced $\Lambda_{UV} = 1/\epsilon_0$ [72].

[26] we define the boundary as the background spacetime on which the dual field theory lives
[27] so as not to change the nonlocal, dynamical, part of the on-shell action [?]

2. The equations of motion are solved with boundary conditions imposed at this new boundary.
   The Dirichlet conditions imposed at this new boundary are related to the conditions imposed at the boundary at infinity via relation

$$\phi(x,z)\big|_{z=\epsilon_0} = \phi^{(0)}(x,\epsilon_0) = \epsilon_0^{d-\Delta_+}\phi^{(0)}_{\text{reg}}(x) \tag{3.59}$$

3. Local terms that diverge in the limit ϵ_0 are subtracted from the action

$$S_{\text{sub}} = S - S_{\text{loc}} \tag{3.60}$$

4. The limit $\epsilon_0 \to 0$ is taken[28] to obtain the renormalized action.

3.3.5 Relevant, marginal and irrelevant operators

We can read off the conformal dimension of a boundary operator dual to a bulk field by looking at the behaviour of the non-normalizable mode (3.59) under a rescaling of z. Since ϕ is dimensionless, we see that $\phi^{(0)}$ has dimensions of $[\text{length}]^{\Delta-d}$ which implies, through the lhs of (3.40), that the associated operator $\mathcal{O}$ has dimension Δ given by $\Delta = d - \Delta_- = \Delta_+$. We can now classify the bulk fields in terms of the conformal dimension of their dual operators. For a scalar field, the relation (3.44) gives the flollowing classification:

- Massless scalars correspond to operators with conformal dimension $\Delta = d$, so these are *marginal.*

- For $m^2 > 0$ the conformal dimension $\Delta > d$, so these correspond to *irrelevant* operators.[29]

- If $-\frac{d^2}{4} < m^2 < 0$, the corresponding operator has $\Delta < d$ and is *relevant*[30].

[28]N.B.: when computing correlation functions, this limit is taken *after* differentiation w.r.t. the sources, which were defined on the regulated boundary.

[29]Since these operators backreact strongly on the boundary, we do not have an asymptotically AdS spacetime anymore.

[30]Recall from section 3.3.3 that $m^2 > -d^2/4L^2$ is required for stability [?]. As the conformal dimension of an operator is given by $\Delta = \Delta_+$, this translates to the requirement that the conformal dimension of an operator be real-valued

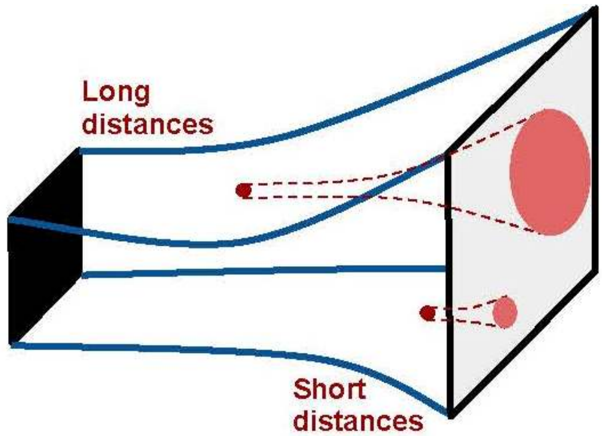

Figure 3.4: For systems with a gravity dual, the AdS/CFT correspondence provides a striking geometric picture for the RG flow and the resulting low energy behaviour. As stated before, the radial direction in the bulk can be associated with the energy scale of the boundary theory [59, 72, 63], and the radial flow in the bulk geometry can be interpreted as the RG flow of the boundary theory [7]. Processes in the interior determine long distance physics (which we call the IR of the dual field theory) while processes near the boundary control the short distance (or UV) physics.
Figure taken from [43].

Chapter 4

Modeling matter

In general, the holographic principle states that a theory of gravity in $d+1$ dimensions should have a description in terms of a field theory in d dimensions. In particular, a CFT in d dimensions (or: CFT_d) should be dual to a theory of gravity on AdS_{d+1}. The standard example which can be derived by taking a decoupling limit of a system of D3-branes gives a holographic description of a CFT_4. However, in realistic systems one will generally want to deform this theory so that the scale invariance (and perhaps other symmetries) is broken. The simplest way of breaking scale invariance is by placing the theory at a finite temperature. One also might like to put the system at a chemical potential and induce a charge density, or turn on a magnetic field. When looking for a holographic description of condensed matter phenomena, there are two ways to do this, which are sometimes called the *top-down* and the *bottom-up* method:

- *Top-down models*, that is models derived directly from string/M-theory constructions (such as the coincident D3-branes discussed in section 3.2.2), might be found in which the gauge theory has some (exotic) features in common with a condensed matter system (such as a quantum phase transition or non-fermi liquid behaviour).
- *Bottom-up models* are usually simple gravitational models that are expected to give a holographic description of some condensed matter phenomenon. One uses general features of the AdS/CFT correspondence to guess what kind of fields are needed in the gravity description, and leaves all other fields out.

In the top-down method, one might try to find a more complicated system of coincident or intersecting D-branes which admits a decoupling limit, and whose D-brane dynamics exhibit the features one is interested in. A very basic example of a top-down construction is the description of a thermal CFT by means of nonextremal D3-branes (to be discussed in section **??**). There has been much work on the holographic description of matter in the fundamental representation by the addition of extra branes. The latter case is often studied by probing the geometry caused by the backreaction of a large stack of Dp-branes by some Dq-branes (where p and q are the number of spatial dimensions of the D-branes, which need not be equal). These probe Dq-branes can intersect the branes, and/or wrap

some of the compact directions. This method neglects the backreaction of the probe branes and the dynamics of the background branes.
It is difficult to find systems in string theory that admit a decoupling limit in which the gauge theory dual has the same behaviour as the condensed matter system one is interested in. Typically, interesting top-down models are very complicated. When such a system has been engineered, the results are generally highly dependent on the model, and one must look for results that are universal[1].

In the bottom-up approach to finding holographic descriptions of condensed matter phenomena, one uses basic knowledge about the holographic duality to construct the gravitational dual to some condensed matter phenomenon. The gravitational theory is simple and contains only the fields necessary for the case at hand. Bottom-up models are a convenient starting point for studying interesting condensed matter phenomena in the gravity dual, as one can get an idea of which bulk interactions are dual to which condensed matter phenomena. Examples are thermalization of a condensed matter system, which has a holographic dual description in terms of energy falling into a black hole, or superconductivity, which corresponds to the condensation of charged scalar fields in the presence of a charged black hole [38, 45, 46], (for a review, see [51]). Studies of charged spinors in the background of a charged black hole have been used to model Fermi surfaces and in some cases provide examples of non-Fermi liquid behaviour [56, 57, 28, 18]

An advantage of the bottom-up method is thus that its results are *universal*, due to the simplicity of the action. (simple action can be a truncation of many different complicated actions. Therefore, a simple action may be a description of many dual field theories...)

The simplicity of the action means it is easy to find solutions with the behaviour one is interested in, but the fact that it is not derived from a string theory construction means that it is generally not clear which dual field theory is described by this background. In fact, for many simple gravity actions it is not known if the solutions these generate are (compactified) solutions of some full higher-dimensional string theory, unless the gravity action is a *consistent truncation* of string theory.

4.1 Consistent truncation

In general, a *truncation* $\mathcal{L}_R$ of a Lagrangian $\mathcal{L}$ describing a system in a certain number of dimensions can be obtained by means of *dimensional reduction* (see section 3.2.6) or by reducing the number of independent fields. Such a truncation is called *consistent* if all solutions of the equations of motion of the truncated theory $\mathcal{L}_R$ are also solutions of the untruncated theory $\mathcal{L}$. Therefore, it must be possible to *uplift* any solution of the truncated theory to a solution of the untruncated theory.

[1] meaning, properties that are generally expected to be present in gravity duals to larger classes of gauge theories

Dimensional reduction should not be confused with *compactification*, in which a lower-dimensional description is found with the fields expanded in terms of the compactified coordinates (the Kaluza-Klein tower of states, introduced in section 3.2.6). It was explained before that although it might be that the Kaluza-Klein modes are all massive and can therefore be ignored in an effective description, the dimensional reduction means that the "heavy" KK-fields are set to zero, and a new theory is defined with only the "light" fields. Such a truncation to a description in terms of only the light fields can only be consistent if the light fields do not source the heavy fields. Recall that for compactification on an n-sphere the dimensional reduction to the massless fields is generally not consistent, and currents built from the massless fields can act as sources for the massive fields that have been set to zero [**?**]. For many cases, the only known lower-dimensional models that are consistent are pure gravity models with no matter. However, consistent truncations to interesting theories can sometimes be found if the theory has a subset of the fields that are invariant under some symmetry.

4.1.1 Einstein-Hilbert action

The simplest consistent truncation to AdS is pure gravity with a negative cosmological constant, which is described by the Einstein-Hilbert action:

$$S_{EH} = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{-g} \left(\mathcal{R} + \frac{d(d-1)}{L^2} \right) \tag{4.1}$$

This is the simplest action we can consider, as it contains the minimal ingredients necessary to describe a CFT holographically: a dynamical metric which is dual to the stress tensor of the CFT, and a negative cosmological constant. The latter ensures that the most symmetric solution is AdS_{d+1}.
The Einstein equations of motion following from this action[2] are:

$$R_{MN} = -\frac{d}{L^2} g_{MN} \tag{4.2}$$

where R_{MN} is the Ricci tensor. The most symmetric solution to these equations of motion is Anti-de Sitter space (AdS) which has metric[3]

$$ds^2 = L^2 \left(\frac{-dt^2 + dx^i dx^i}{z^2} + \frac{dz^2}{z^2} \right). \tag{4.3}$$

[2]if the theory is evaluated on a space with a boundary, one needs to add a so-called "Gibbons-Hawking boundary term", to be discussed below.

[3]in terms of the radial coordinate z measuring the distance from the AdS boundary

Gibbons-Hawking boundary term

Since we are considering gravity on (asymptotically) AdS spacetime and imposing Dirichlet conditions on the boundary[4], the action needs to be supplemented with a boundary term called the *Gibbons-Hawking* term,

$$S_{GH} = -\frac{1}{8\pi G}\int_{\partial(\mathrm{AdS})} d^d x \sqrt{\gamma} K, \tag{4.4}$$

where K is the trace of the extrinsic curvature of the boundary

$$K \equiv \gamma^{\mu\nu}\nabla_\mu n_\nu = \frac{n^z}{2}\gamma^{\mu\nu}\partial_z \gamma_{\mu\nu} \tag{4.5}$$

and n^A is an outward pointing unit normal to the boundary $z = \epsilon$. This boundary term must be added to ensure that the variational problem is well-defined [34]. Without this term, integration by parts in the Einstein-Hilbert term to get the equations of motion produces some boundary terms proportional to variations of derivatives of the metric, which is incompatible with imposing Dirichlet conditions on the metric.

4.1.2 Minimal gauged supergravity in $D = 5$

In [32], it was shown that supersymmetric compactifications on $\mathcal{M}^5$ of type IIB supergravity can be consistently truncated to minimal $D = 5$ gauged supergravity. Thus, any solution of the gauged supergravity can be uplifted on $\mathcal{M}^5$ to obtain an exact solution of type IIB supergravity. In fact, it turns out that minimal gauged supergravity in $D = 5$ is a consistent truncation known to describe all supersymmetric compactifications of type IIB or M-theory to AdS_5 [16],[30, 32].

The bosonic action for minimal gauged supergravity in five dimensions is [41]

$$S = \frac{1}{4\pi G_5}\int \left((\frac{1}{4}\mathcal{R} + \Lambda - \frac{1}{2}F\wedge *F - \frac{2}{3\sqrt{3}}F\wedge F\wedge A \right) \tag{4.6}$$

where $F = dA$ is a U(1) field strength, $\mathcal{R}$ and G_5 are the Ricci scalar and the gravitational constant in five dimensions, and the theory must have a negative cosmological constant Λ.

In the rest of this, we will look for ways of modeling finite temperature and charge density in a d-dimensional field theory. We shall mainly work in the bottom-up method. In chapter 5 we will work with the above action in different notation (which is a consisten truncation to the bosonic part of minimal gauged supergravity in five dimensions).

4.2 Finite temperature

In general, a d-dimensional QFT in equilibrium can be placed at a finite temperature by periodically identifying the Euclidean time, $\tau \sim \tau + \beta$ (with the temperature given by

[4] i.e., keeping the boundary value $g_{(0)\mu\nu}$ of the bulk metric fixed

$\beta = 1/T$ if we set $k_B = 1$). This can be deduced from the partition function in the canonical ensemble:

$$Z = \mathrm{Tr} e^{-\beta H} \tag{4.7}$$

which can be written as a path integral

$$Z = \int [d\phi] e^{-\int_0^\beta d\tau \int d^{d-1}x \mathcal{L}(\phi)} \tag{4.8}$$

with the bosonic (fermionic) fields satisfying periodic (antiperiodic) boundary conditions

$$\phi(x, \tau + \beta) = \pm\phi(x, \tau) \tag{4.9}$$

In section 3.3.3 we identified the induced metric on the boundary $g_{(0)\mu\nu}$ as the field theory metric. We therefore expect the bulk metric extremizing the supergravity action to have a periodic Euclidean time, with a period approaching β at the boundary.

For the case of a d-dimensional CFT at finite temperature, we will use the bottom-up approach to look for the corresponding behaviour of the boundary geometry. Since any field theory can be placed at a finite temperature, all gravity actions with a field theory dual should have a solution corresponding to finite temperature. In our bottom-up approach we might therefore start from the simplest possible gravity action which has AdS spacetime as a solution, and expect it to have also a solution corresponding to the dual CFT at finite temperature. This is the Einstein-Hilbert action with a negative cosmological constant. Indeed, it turns out to have a solution which breaks the scale invariance in the IR while keeping spatial rotational invariance and spacetime translation invariance: the *Schwarzschild AdS solution*[5]

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^i dx^i\right) \tag{4.10}$$

where the so-called *emblackening factor*

$$f(z) = 1 - \left(\frac{z}{z_+}\right)^d \tag{4.11}$$

goes to $f(z) \to 1$ near the AdS boundary $z \to 0$. This spacetime is asymptotically AdS, and is therefore also called an *AdS black hole.* Note that temperature affects the IR, as should be expected.

The Hawking temperature and Bekenstein-Hawking entropy of this black hole correspond to the temperature and entropy of the thermal CFT, respectively [77], with the temperature given by the surface gravity via the relation (3.1). We can also check that the

[5]A finite temperature version of the AdS_5/CFT_4 correspondence of section **??** can be obtained by taking the decoupling limit of non-extremal D3-branes (while keeping the mass above extremality finite). One ends up with a geometry that asymptotically goes to $AdS_5 \times S^5$ at the boundary, but in the interior the AdS_5 part of the geometry contains a black hole, whose geometry is given by the 'Schwarzschild AdS_5' solution. See e.g. [1] for more details.

Euclidean time is periodic, which gives us a convenient way of reading off the temperature [34]. On the gravity side, this period is determined such that the Euclidean metric is regular at the horizon $z = z_+$ (otherwise the solution is not a stationary point of the action). In the case of the Schwarzschild AdS solution (4.10), absence of a *conical singularity* (see [44]) requires that

$$\tau \sim \tau + \frac{4\pi}{|f'(z_+)|} = \tau + \frac{4\pi z_+}{d} \tag{4.12}$$

from which we can read off the temperature of the dual CFT:

$$T = \frac{|f'(z_+)|}{4\pi} = \frac{d}{4\pi z_+} \tag{4.13}$$

The temperature is related to the location of the horizon z_+. Taking the limit $z_+ \to 0$ (zero temperature), we recover the ordinary AdS metric.

It is worth mentioning that since the temperature is the only scale we introduced on the field theory side, the zero temperature limit is not a smooth limit. The zero-temperature geometry is distinct from all nonzero-temperature geometries. All nonzero temperature states are equivalent, as a rescaling of the temperature can always be undone by a rescaling of the coordinates. This also means that once we have characterized the state of the theory at a particular temperature, we know the state of the system at any other temperature and no thermal phase transition can happen. Since we have only one dimensionful scale in our system, the temperature T, the dependence of the free energy, the entropy, or any other thermodynamic quantity is fixed by dimensional analysis.

4.3 Charge density and magnetic field

Another important feature a holographic description of a condensed matter system must have is a description of charged matter. A finite charge density $\rho = \langle J^t \rangle \neq 0$ can be induced by holding the system at a nonzero chemical potential μ. Keeping the chemical potential fixed and letting the charge density fluctuate corresponds to considering the grand canonical ensemble[6]. The partition function in this case is

$$\mathcal{Z} = \mathrm{Tr} e^{-\beta(H-\mu Q)} = \mathrm{Tr} e^{-\beta \int d^d x (\mathcal{H} - \mu J^t)} \tag{4.14}$$

with μ the chemical potential, J^t the charge density operator and $Q = \int d^d x J^t$ is the charge associated with a conserved current J^μ.

When describing charged matter, we may sometimes use an effective condensed matter description in which photons are neglected[7], so the electromagnetic U(1) symmetry is treated as a global symmetry. Recall (section 3.3.3) that a conserved current for a global

[6]One can also consider the canonical ensemble, in which the charge density ρ is held fixed while the chemical potential μ fluctuates.

[7]Electric coupling e is small, and often screened in materials.

U(1) symmetry in the gauge theory is holographically dual to a massless U(1) gauge field in the bulk. In particular, our charge density operator J^t is dual to the time component A_t of a Maxwell field in the bulk. A solution to Maxwell's equations in the bulk behaves near the boundary as

$$A_t(z) = \mu + \langle J^t \rangle z^{d-2} + \dots, \tag{4.15}$$

where the leading behaviour gives the source dual to J^t. Therefore, the chemical potential μ is read off from the leading near-boundary term of the bulk field A_t[8],

$$\mu = \lim_{z \to 0} A_t, \tag{4.16}$$

and the charge density is given by the boundary value of the electric flux,[9]

$$\langle J^t \rangle = \lim_{z \to 0} \frac{\partial \mathcal{L}}{\partial(\partial_z A_t)} = \lim_{z \to 0} F_{zt}. \tag{4.18}$$

Therefore to impose that the quantum field theory is at nonzero density, we must impose that the dual spacetime has an electric flux at infinity. This electric flux must be sourced in the interior of the spacetime. We will encounter the simplest source to the electric flux in the next example: the *Reissner-Nordström-AdS black hole.*

4.3.1 Einstein-Maxwell action

To describe the physics of an electric flux at infinity, the simplest action is obtained by adding a Maxwell term to the Einstein-Hilbert action (4.1).The action reads

$$S_{\text{EM}} = \int d^{d+1}x \sqrt{-g} \left(\frac{1}{2\kappa^2} \mathcal{R} + \frac{d(d-1)}{L^2} - \frac{1}{4e^2} F_{MN} F^{MN} \right) \tag{4.19}$$

Here κ and e are respectively the Newtonian and Maxwell constants. The equations of motion are given by

$$R_{MN} - \frac{\mathcal{R}}{2} g_{MN} - \frac{d(d-1)}{2L^2} g_{MN} = \frac{\kappa^2}{2e^2} \left(2F_{MP} F_N^{\ P} - \frac{1}{2} g_{MN} F_{PQ} F^{PQ} \right) \tag{4.20}$$

$$\nabla_M F^{MN} = 0 \tag{4.21}$$

As one might guess, a solution with electric flux at infinity, preserving the symmetries under rotations and spacetime translations is given by a charged version of the Scharzschild-AdS

[8] with $A_t(z)$ a solution to the bulk Maxwell equations under the assumption that rotations and spacetime translations in the dual field theory are preserved

[9] The precise expression for $\langle J^\mu \rangle$ is

$$\langle J^\mu \rangle = \lim_{\epsilon \to 0} \left(\epsilon^{d-\Delta} \frac{1}{\sqrt{-\gamma}} \frac{\delta S_{\text{sub}}}{\delta A_\mu(\epsilon)} \right), \tag{4.17}$$

where γ is the determinant of the induced metric on the $z = \epsilon$ hypersurface.

solution (4.10). This is the planar Reissner-Nordström-AdS black hole, with metric

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^i dx^i\right) \tag{4.22}$$

The emblackening factor has now changed to

$$f(z) = 1 - \left(1 + \frac{z_+^2\mu^2}{2\gamma^2}\right)\left(\frac{z}{z_+}\right)^d + \frac{z_+^2\mu^2}{2\gamma^2}\left(\frac{z}{z_+}\right)^{2(d-1)}. \tag{4.23}$$

We introduced the dimensionless ratio of the Newtonian and Maxwell couplings

$$\gamma^2 = \frac{(d-1)e^2L^2}{(d-2)\kappa^2} \tag{4.24}$$

The maxwell potential of the solution is

$$A = \mu\left(1 - \left(\frac{z}{z_+}\right)^{d-2}\right)dt, \tag{4.25}$$

where we have required the scalar potential to vanish on the horizon, $A_t(z_+) = 0$, in order for the gauge field to be well-defined at the horizon [55]. The planar Reissner-Nordström-AdS solution is characterized by two scales, the chemical potential $\mu = \lim_{z\to 0} A_t$ and the horizon radius z_+. From the dual field theory perspective, it is more physical to think in terms of the temperature than the horizon radius

$$T = \frac{1}{4\pi z_+}\left(d - \frac{(d-2)z_+^2\mu^2}{\gamma^2}\right) \tag{4.26}$$

So now we do have a dimensionless quantity that we can discuss: the ratio T/μ, and we can continuously take the low-temperature limit $T/\mu \ll 1$ of the solution. If we set $T = 0$, we get

$$\frac{z_+^2\mu^2}{\gamma^2} = \frac{d}{d-2} \tag{4.27}$$

We thereby obtain the extremal Reissner-Nordström-AdS black hole with

$$f(z) = 1 - \frac{2(d-1)}{d-2}\left(\frac{z}{z_+}\right)^d + \frac{d}{d-2}\left(\frac{z}{z_+}\right)^{2(d-1)}. \tag{4.28}$$

The near-horizon extremal geometry, capturing the field theory IR, follows by expanding the solution near $z = z_+$. Setting $z = z_+(1 - z_+/\rho)$, and expanding $f(\rho)$ in powers of z_+/ρ, we see that the redshift factor develops a double zero at the horizon ($\rho \to \infty$)

$$f(\rho) = d(d-1)\left(\frac{z_+}{\rho}\right)^2 + \dots \qquad \text{for } \rho \text{ large} \tag{4.29}$$

taking ρ large and rescaling $\{t, x^i\}$ by dimensionless constants gives the near-horizon metric

$$\begin{aligned} ds^2 &= \frac{L^2}{z_+^2}\left(1-\frac{z_+}{\rho}\right)^{-2}\left(-f(\rho)dt^2+\frac{z_+^4}{\rho^4 f(\rho)}d\rho^2+dx^i dx^i\right) \\ &\approx \frac{L^2}{d(d-1)}\left(\frac{-d^2(d-1)^2dt^2+d\rho^2}{\rho^2}\right)+\frac{L^2}{z_+^2}dx^i dx^i \end{aligned} \tag{4.30}$$

which, under rescaling of t and x^i by dimensionless constants, can be brought to the form

$$ds^2 = \frac{L^2}{d(d-1)}\left(\frac{-d\tilde{t}^2+d\rho^2}{\rho^2}\right)+d\bar{x}^i d\bar{x}^i \tag{4.31}$$

So the near horizon geometry is $\mathrm{AdS}_2\times\mathbb{R}^{d-1}$ (with AdS_2 radius $L/\sqrt{d(d-1)}$), and exhibits an emergent IR scaling invariance of the time and radial parameter ρ. Notice that the spatial coordinates x^i do not scale under this symmetry. This near-horizon AdS_2 geometry suggests that in the *low-frequency limit* the d-dimensional boundary theory at finite charge density is described by some dual CFT_1 [27], to which we refer to as the *IR CFT* of the boundary theory. It is important to emphasize that the conformal symmetry of this IR CFT is *not* related to the microscopic conformal invariance of the higher dimensional theory (the UV theory), which is broken by finite charge density. It apparently emerges as a consequence of collective behaviour of a large number of degrees of freedom.

The horizon area of the extremal geometry is finite, suggesting a finite ground state entropy. This seems unnatural, as in "real-life" condensed matter systems we do not expect to find an exactly degenerate ground state ("Nernst theorem" or: "Third law of thermodynamics"). This problem seems to disappear when various perturbations are included, as low temperature charged AdS black holes are found to be unstable towards a range of processes that discharge the black hole and can lead to spacetimes without black hole horizons. The instabilities include condensation of charged scalar fields [38] and Cooper pairing of charged fermions [42].

One can also consider applying a background magnetic field to the system (at finite density) described by the boundary theory. In the gravity dual, the asymptotic values of the Maxwell field at the AdS boundary give the chemical potential and the external magnetic field

$$\mu = A_t(\mathbf{x}, z\to 0), \qquad B(\mathbf{x}) = F_{xy}(\mathbf{x}, z\to 0). \tag{4.32}$$

The boundary condition at the horizon ("in the IR" of the dual field theory) requires $A_i(\mathbf{x})$ regular and $A_t(\mathbf{x}) = 0$.

4.3.2 Dyonic AdS black hole in AdS_4

For the case $d = 3$ one can find a solution describing such a magnetic field while preserving invariance under rotations and translations: the magnetic brane in AdS_4, or: *dyonic black hole*. In the context of AdS/CMT, it has been applied to the study of 2+1 gauge theories in magnetic field in many papers, for instance [47].

Starting from our Einstein-Maxwell action (4.19), we search for solutions of the same form as the Schwarzschild-AdS (and Reissner-Nordström-AdS) solution, but with a modified emblackening factor. We make an ansatz for the the Maxwell field of the form[10]

$$A = A_t(z)dt + B(z)x_1 dx_2 \tag{4.33}$$

so that the the field strength is of the form

$$F = -\partial_z A_t(z)dt \wedge dz + B(z)dx_1 \wedge dx_2 + \partial_z B(z)x_1 dz \wedge dx_2 \tag{4.34}$$

The metric is again of the form (4.10), but with emblackening factor

$$f(z) = 1 - \left(1 + \frac{z_+^2\mu^2 + z_+^4 B^2}{\gamma^2}\right)\left(\frac{z}{z_+}\right)^3 + \frac{z_+^2\mu^2 + z_+^4 B^2}{\gamma^2}\left(\frac{z}{z_+}\right)^4, \tag{4.35}$$

and the gauge potential is

$$A = \mu\left(1 - \frac{z}{z_+}\right)dt + Bx_1 dx_2. \tag{4.36}$$

corresponding to a field strength

$$F = \frac{\mu}{z_+}dt \wedge dz + B\, dx_1 \wedge dx_2 \tag{4.37}$$

The temperature is given by

$$T = \frac{|f'(z_+)|}{4\pi} = \frac{1}{4\pi z_+}\left(3 - \frac{\mu^2 + z_+^2 B^2}{\gamma^2}\right) \tag{4.38}$$

Since it is of the same form as the electrically charged Reissner-Nordström solution, this dyonic black hole in AdS[11] again has a finite entropy density at extremality. In the next chapter, we will (after the introduction) look at an example of a charged magnetic brane solution which does not have this extremal horizon, and corresponds to a dual "condensed matter system" with no ground state degeneracy.

[10]The second term in this ansatz can be thought of as a magnetic field pointing in the direction perpendicular to the two spatial dimensions of some material which is described by the boundary theory.

[11]in the literature, the terms "AdS black hole", "planar black hole" and sometimes "black brane" are used to indicate that it is a black hole solution which has a planar horizon and is asymptotically AdS near the boundary.

Chapter 5

Holographic Quantum Phase Transitions

There are many examples in the literature of holographic quantum criticality[1]. In particular, quantum critical behaviour has been found [28] by studying the behaviour of bulk fields in the background of the AdS Reissner-Nordström black hole we encountered in section 4.3.1. Recall that this solution to Einstein-Maxwell action is the simplest (bottom-up) model of a finite charge density, and its universality makes it a very attractive candidate. In the charged (or dyonic) Reissner-Nordström solution a mechanism causing a QPT in the bulk is when the mass of some bulk field drops below the Breitenlohner-Freedman bound in the near-horizon AdS_2 region while remaining above the Breitenlohner-Freedman bound of the asymptotic AdS_5 region[2]. The IR region of the bulk can then develop an instability, leading to a new geometry, while leaving the boundary intact. This large effect in the IR for a very small change of a control variable[3] is just what we would expect to see at a continuous phase transition. A less appealing feature of this model is the fact that its zero temperature limit has a finite horizon area, indicating nonzero ground state entropy of the dual field theory[4]. Another disadvantage of using this setup to look for a QCP is that the fermions are explicitly added in the bulk; the fermions can be thought of as gauge singlets (coupled to the large N gauge theory), and the fact that the large N limit makes the gravity theory classical is not exploited (this was emphasized in [29]).

A promising example of a holographic QPT, which will be reviewed in the rest of this

[1] for instance in the probe D-brane systems that were discussed in the previous chapter, see e.g. [26]. For the reasons mentioned in the beginning of the previous chapter, these systems will not be discussed further.

[2] The Breitenlohner-Freedman bound (3.45) depends not only on the AdS radius, but also on the dimensionality of the AdS spacetime.

[3] A control parameter is some variable, such as the chemical potential μ, on which the mass of a bulk field depends.

[4] Since the CFTs dual to the Reissner-Nordström solution typically includes massless bosons, it is expected that these form a condensate at low temperatures, thus describing a superconducting phase which then cloaks the QCP.

chapter, was found [22, 25] by studying gravity solutions that are dual to $d = 3 + 1$ gauge theories at finite charge density and magnetic field. It turns out [31] that a large class of these gauge theories can be described by Einstein-Maxwell-Chern-Simons (EMCS) theory (5.2), which will be introduced in section 5.1. Based on numerical calculations, it was suggested in [21] that the entropy density of finite charge density solutions to the Einstein-Maxwell-Chern-Simons action goes to zero in the zero-temperature limit for large enough Chern-Simons coupling k and applied magnetic field B. This may be understood by noting (see section 5.1) that the effect of the Chern-Simons term is to seemingly allow the Maxwell field to carry some of the electric charge measured at infinity. One might imagine that if the contribution of the Chern-Simons term to the charge density at the boundary is large enough, there might be solutions in which all the charge measured at the boundary originates from the gauge field in the bulk (sometimes termed "fluxes" in the literature), and the interior does not contain a Reissner-Nordström black hole.

In [22], gauge theories whose gravity dual can be described by Einstein-Maxwell-Chern-Simons theory[5] were shown numerically to undergo a continuous quantum phase transition at finite charge density and magnetic field. The phase transition, which shows similarities to the metamagnetic quantum critical endpoint (QCEP) discussed in section 2.3.2, is characterized by nonanalytic behaviour of the magnetisation and specific heat at a finite value of external magnetic field strength. In the large B phase, the entropy density (which gives the specific heat coefficient in the low temperature limit) was found to vanish linearly with temperature, but as the magnetic field was lowered to a critical value, this linear coefficient was found to diverge. In the "Quantum Critical Region" (QCR), the critical exponent was found to change so that the scaling of the entropy with the temperature became $s \sim T^{1/3}$. Later, with $\alpha = 1/3$ for $k \geq 3/4$. In [25] a solution at zero temperature representing the system in its $B \geq B_c$ phase was found, and the low temperature thermodynamics were studied using *matched asymptotic expansions*[6]. Later [23], solutions representing the system at finite temperature for $\hat{B} \geq \hat{B}_c$ and $1/2 < k < 3/4$ were used to deduce the behaviour $s \sim T^{\alpha}$ with $\alpha = (1-k)/k$. An analytic solution for the $B < B_c$ phase has not yet been found, but the behaviour of the entropy density in that region shows that in the limit $T \to 0$ the entropy density goes to a finite value [22, 25].

5.1 Einstein-Maxwell-Chern-Simons action

In five dimensional gravity, one may add a Chern-Simons term of the following form to the Einstein-Maxwell action (4.19),

$$S_{\mathrm{CS}} \sim k \int d^5x F \wedge F \wedge A. \tag{5.1}$$

[5]with Chern-Simons coupling $k > 1/2$

[6]see section 5.4.3

If the Gibbons-Hawking term (see section 4.1.1) and counterterms are included, the full five-dimensional action reads:

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left(\mathcal{R} - \frac{12}{L^2} + F_{MN}F^{MN}\right) + \frac{k}{12\pi G_5}\int F\wedge F\wedge A + S_{GH} + S_{ct}. \quad (5.2)$$

For the so-called *supersymmetric value* $k = 2/\sqrt{3}$ of the Chern-Simons coupling k, this action is a consistent truncation (see section 4.1) to the bosonic part of minimal gauged supergravity in $D = 5$ [31], which describes AdS duals to supersymmetric gauge theories. This makes the results of D'Hoker and Kraus very interesting for two reasons: firstly, as alluded to in the introduction of this chapter, the quantum phase transition found by D'Hoker and Kraus will be present in an infinite class of theories that are dual to the Einstein-Maxwell-Chern-Simons action. Of these theories, $\mathcal{N} = 4$ SYM theory is but one example. Secondly, since the dual gauge theories are completely specified (e.g. $\mathcal{N} = 4$ SYM) one might in principle hope to obtain complementary descriptions of the transition mechanism within both gauge theory and gravity. However, with the advantage of knowing the dual gauge theories comes the disadvantage that now the Maxwell field in the bulk will be the gauge field of some large N group[7]. Since the electromagnetic field is a U(1) field, this large value of N is not realistic.

The Einstein equations are

$$R_{MN} = 4g_{MN} + \frac{1}{3}F^{PQ}F_{PQ}g_{MN} - 2F_{MP}F_N^{\ P} \quad (5.3)$$

while the Maxwell equations become

$$d * F + k\, F \wedge F \quad (5.4)$$

This leads to a new conserved electric charge at the boundary:

$$Q = \int_{S^3} (*F + k\, A \wedge F)\,. \quad (5.5)$$

From this we conclude that in the presence of a Chern-Simons term, it is possible that part of the electric charge is carried by the electric field in the bulk. The larger the value of the Chern-Simons coupling k, the more charge can be carried by the field. This opens up the possibility of looking for solutions in which all the charge density of the boundary theory is sourced by these so-called "fluxes" in the bulk, and there is no Reissner-Nordström black hole with its finite entropy at zero temperature.

[7]recall from the discussion of the AdS/CFT correspondence 3.2 that in order for the gravity dual to be described by classical supergravity, the dual SU(N) gauge theory needs to have large N and large coupling λ.

5.2 Physics of charged particles in a magnetic field

In general, charged particles moving in a strong background magnetic field will undergo a circular motion around the magnetic flux lines due to the Lorentz force. At low temperatures, the effect of quantum mechanics becomes important, and the charged particles can only occupy orbits with discrete energy values, called Landau levels. The Landau levels are degenerate, with the number of electrons per level directly proportional to the strength of the applied magnetic field.
Considering the case of a free[8] $\mathcal{N} = 4$ super Yang-Mills theory in 3+1 dimensions (which is a CFT) in the presence of a magnetic field, we have low energy excitations corresponding to the drift velocity parallel to the field. At low energies the *fermion zero modes* dominate, and the theory flows to a $d = 1 + 1$ CFT (sometimes denoted by CFT_2).

By increasing the number of charged particles (assumed to be fermions), one can build up a Fermi sea. For a very low charge density ρ (compared to the degeneracy of the Landau levels, i.e. compared to the magnetic field strength B), the fermions that are added to the system will be in the lowest Landau level. However, as the density is increased, at some point new behaviour can set in when it will be energetically favourable to start filling up higher Landau levels.

5.3 Magnetic brane solution

On the gravity side of the correspondence, a solution for the simpler case of zero charge density, $\rho = 0$, and finite magnetic field, $B \neq 0$, was found in [20]. The solution corresponds to the point $\hat{B} = \infty, T = 0$ in our phase diagram (figure 5.2), where the dimensionless[9] combination $\hat{B} \equiv B/\rho^{2/3}$ is used instead of two dimensionful quantities B and ρ.

5.3.1 Interpolating solution at $T = 0$

A geometry realizing the RG flow from a UV CFT_4 ($\mathcal{N} = 4$ SYM) to an IR CFT_2 (fermion zero modes) is asymptotically AdS_5, and $\text{AdS}_3 \times \mathbb{R}^2$ in the IR. A solution with these

[8]N.B.: only at *strong* coupling is $\mathcal{N} = 4$ SYM holographically described by the Einstein-Maxwell-Chern-Simons action.

[9]The physical parameters B, ρ and T are dimensionful and can be rescaled by a coordinate transformation that preserves the asymptotic AdS_5 metric. It is thus only dimensionless quantities that are meaningful, and these are defined as

$$\hat{B} \equiv \frac{B}{\rho^{2/3}} \tag{5.6}$$

$$\hat{T} \equiv \frac{T}{B^{1/2}} \tag{5.7}$$

and the entropy density

$$\hat{s} \equiv \frac{s}{B^{3/2}} \tag{5.8}$$

asymptotics was found "almost analytically"[10] in [20].
The dual theory on the boundary was placed in a constant magnetic field B^z in pointing in the z direction in the boundary coordinates,

$$F_{xy} \to \text{const} \qquad \text{approaching boundary} \tag{5.9}$$

which D'Hoker and Kraus described holographically by using the following ansatz:

$$ds^2 = \frac{dr^2}{L(r)^2} + L(r)(-dt^2 + dx_3^2) + e^{2V(r)}(dx_1^2 + dx_2^2) \tag{5.10}$$

$$F = B dx_1 \wedge dx_2 \tag{5.11}$$

5.3.2 Interpolating solution at finite T

Considering the magnetic brane solution at a finite temperature corresponds to placing a black hole in the AdS_3 geometry. Such a black hole in AdS_3 is called a BTZ black hole [5]. An exact solution to the field equations which becomes $AdS_3 \times \mathbb{R}^2$ in the zero temperature limit is BTZ$\times\mathbb{R}^2$. At small[11] finite temperature, the solution therefore interpolates between AdS_5 (large r) and BTZ$\times\mathbb{R}^2$ (small r). Actually, since we now have two dimensionful parameters, B and T, we can find a one-parameter family of solutions in terms of the dimensionless combination $T/\sqrt{B}$. Smooth interpolating solutions were found for all values of $T/\sqrt{B}$.

5.4 Charged magnetic brane solution

When a the boundary theory has a nonzero charge density, the Chern-Simons term (5.1) becomes important, since without it we would have a Reissner-Nordström black brane with finite entropy at zero temperature. Consider a case in which the Chern-Simons coupling k is "large enough". Since the degeneracy of the Landau levels is proportional to the field strength, we expect the value of the charge density at which higher levels (above the lowest Landau level) become important and the behaviour changes to be proportional to the field strength as well.

5.4.1 Interpolating solution at $T = 0$ and $\hat{B} \geq \hat{B}_c$

A solution describing the system at zero temperature in the $B \geq B_c$ phase was found in [25] by assuming that the IR (small r) limit of the solution factorises into some three-dimensional part M_3 (which in the limit of zero charge density becomes AdS_3) and a

[10]Solving the equations of motion with the ansatz given in [20] numerically for $V(r)$, with the conditions that $e^{2V(r)} \to$ const for small r and becomes linear at large r, gives a result which does not depend on any free parameters, since we have only one dimensionful parameter B (coordinates $x_{1,2}$ can be rescaled to undo any change in B). $L(r)$ was found analytically in terms of $V(r)$.

[11]for high temperatures, the black hole horizon will be closer to the boundary, and the black hole can not be regarded as a black hole in AdS_3 anymore.

two-dimensional part: $M_3 \times \mathbb{R}^2$. Assuming translational invariance along the boundary, one can find the general solution of this form:

$$ds^2 = \frac{dr^2}{4B^2r^2} - \left(\alpha_0 r + \frac{q^2}{k(k-\frac{1}{2})} r^{2k}\right) dt^2 + 4Brdtdx_3 + \frac{B}{\sqrt{3}}\left(dx_1^2 + dx_2^2\right) \tag{5.12}$$

$$F = Bdx_1 \wedge dx_2 + qr^{k-1} dr \wedge dt \tag{5.13}$$

The three-dimensional part of this geometry, which we will denote by "WAdS$_3$", is also studied in the context of topologically massive gravity [3] and is known as a "null warped" [3], "Schrodinger" [68, 6] or "pp-wave" solution.

The complete solution for all r, interpolating between WAdS$_3 \times \mathbb{R}^2$ (small r) and AdS$_5$ (large r), was found in [25] by choosing coordinates in which the equations of motion for the functions $L(r)$ and $V(r)$ is the same as for the pure magnetic case so that the results may be transfered, and solving the remaining functions $M(r)$ and $E(r)$ in terms of $V(r)$:

$$ds^2 = \frac{dr^2}{L(r)^2} + M(r)dt^2 + 2L(r)dtdx_3 + e^{2V(r)}\left(dx_1^2 + dx_2^2\right) \tag{5.14}$$

$$F = Bdx_1 \wedge dx_2 + E(r) dr \wedge dt \tag{5.15}$$

The small r part of this solution is given by (5.12).

5.4.2 Quantum Critical Point

For $k < 1/2$ there is no quantum critical point, as the zero temperature entropy density was found numerically in [22] to be nonzero for any value of $\hat{B}$. For $k = 1/2$, the solution (5.12) breaks down and corresponds to a QCP at $\hat{B}_c = \infty$ [25].

The existence of a QCP for the *supersymmetric calue* $k = 2/\sqrt{3}$ was established numerically in [22] by observing nonanalytical behaviour in the specific heat coefficient as the critical magnetic field was approached from the large field side. In the analytic zero-temperature solution (5.12) this critical magnetic field appears for all $k > 1/2$ [25] as the lower bound on $\hat{B}$ for which the solution is a description of our system. This is because in order for the solution to be the zero temperature limit of a finite temperature solution, the function $M(r)$ has to be negative for all values of r. This means that there is a lower bound for the constant α_0 in (5.13) $\alpha_0 \geq 0$. From the relation between α_0 and the boundary parameters ρ, B one can see [?] that this translates into a lower bound on the dimensionless magnetic field to charge density ratio,

$$\hat{B} \geq \hat{B}_c, \tag{5.16}$$

where $\hat{B}_c$ depends on the Chern-Simons coupling k (as might be expected, since the solution exists by virtue of the Chern-Simons term which allows for a finite charge density at the boundary without a charged horizon in the bulk).

The metric of the IR region (small r) at the critical point can be written as

$$ds^2 = \frac{dr^2}{4r^2} - \frac{q^2 r^{2k}}{k(k-\frac{1}{2})}dt^2 + 4brdtdx_3 + dx_1^2 + dx_2^2. \tag{5.17}$$

As shown in [25], the solution at the QCP is invariant under the following scale transformation of the "Schrödinger" part:

$$r \to \lambda^{-1} r, \qquad t \to \lambda^k t, \qquad x_3 \to \lambda^{1-k} x_3. \tag{5.18}$$

which suggests the following value for the dynamical scaling exponent z,

$$z = \frac{k}{1-k}. \tag{5.19}$$

The prediction of the behaviour of the entropy density $\hat{s}$ as a function of $\hat{T}$ in a CFT_2 then would be $\hat{s} \sim \hat{T}^{(1-k)/k}$. As it turns out (see the next section), this prediction is only valid for $1/2 < k < 3/4$. The dynamical critical exponent $z = 3$ for $k > 3/4$ follows from the scaling law $s \sim T^{1/3}$ (again, see section 5.4.3), assuming that the effective IR theory is indeed a 1+1 dimensional CFT.

5.4.3 Low temperature thermodynamics

Analogous to the pure magnetic case, we might look for a finite temperature solution which reduces to (5.12) in the zero temperature limit. For $1/2 < k < 3/4$ we can find such solutions, called *Asymptotically Null Warped* black hole solutions (ANW black holes) [3]. The result for the specific heat coefficient in the Quantum Critical Region is given in the next section.

Since these *ANW black holes* do not exist for general k, we do not have analytic solutions at nonzero temperature for $k > 3/4$. Results for low temperatures were obtained in [25] by matching two expansions which are valid in two different regions. Note that simply expanding around the zero temperature solution does not work, since the perturbative expansion is expected to give bad results in the near-horizon (small r) region, where the zero temperature solution is expected to break down in favour of a black hole solution with the same large r asymptotics.

Near the boundary (large r) it is expected that the effect of a finite temperature is small, so one can expand around the exact $T = 0$ solution (5.14). Near the horizon (small r), the effect of temperature should be significant, so we need a finite temperature solution which is valid for the small r region. We know such a solution for the pure magnetic case: $\mathrm{BTZ}\times\mathbb{R}^2$ with magnetic flux. The three-dimensional BTZ metric can be written as

$$ds^2 = \frac{dr^2}{12r^2 + mnr} - mrdt^2 + 2Brdtdx_3 + ndx_3^2. \tag{5.20}$$

In order to match this onto the finite charge density asymptotically AdS_5 solution, this solution was perturbed with values for E and V at the horizon: $E(0) = q$ and $V(0) = v_0$. In case of small temperatures, there region of r in which both solutions tend to $AdS_3 \times \mathbb{R}^2$ is large. In this region, the parameters of the two asymptotic expansions can be matched.

For the $\hat{B} > \hat{B}_c$ part of the solution this analysis yields the result

$$\hat{s} = \frac{\pi}{6}\left(\frac{\hat{B}^3}{\hat{B}^3 - \hat{B}_c^3}\right)\hat{T} \qquad \hat{T} \to 0, \quad \hat{B} > \hat{B}_c, \tag{5.21}$$

which can be approximated near the QCP as

$$\hat{s} \approx \frac{\pi}{18}\left(\frac{\hat{B}_c}{\hat{B} - \hat{B}_c}\right)\hat{T}, \qquad \hat{T} \to 0, \quad \hat{B} \gtrsim \hat{B}_c. \tag{5.22}$$

Scaling function

A universal scaling function for $\hat{s}/\hat{T}^{1/3}$, given in terms of $(\hat{B} - \hat{B}_c)/\hat{T}^{2/3}$, was determined from the numerical results of [22] and derived in [25] by perturbing around the critical point. The behaviour of the entropy density $\hat{s}$ is given by

$$\hat{s} = \hat{T}^{1/3} f\left(\frac{\hat{B} - \hat{B}_c}{\hat{T}^{2/3}}\right), \tag{5.23}$$

where the scaling function is a solution to the following equation

$$f(x)\left(f(x)^2 + \frac{x}{32k\hat{B}_c^4}\right) = a^3 \tag{5.24}$$

with a some constant which turns out to be proportional to $1/B_c$.

This scaling function was investigated in [25] by perturbing around $\hat{B}_c$ and in small $\hat{T}$. Three qualitatively different behaviours were found in the regions roughly given by $\hat{B} > \hat{B}_c$, $\hat{B} = \hat{B}_c$ and $\hat{B} < \hat{B}_c$.

In the region $\hat{B} > \hat{B}_c$, we can consider the low temperature behaviour (small $\hat{T}$) and take $x \gg 1$. One can make the consistent assumption that $f(x)^3 \ll f(x)x$, so that the $f(x)^3$ term in (5.24) can be ignored, leading to the solution

$$f(x) \sim \frac{32k\hat{B}_c^4 a^3}{x}. \tag{5.25}$$

The Quantum Critical Region (QCR), where $\hat{B} = \hat{B}_c$, has $x = 0$, such that (5.24) becomes simply $f(0) = a$, and we have $\hat{s} \sim \hat{T}^{1/3}$.

For the region $\hat{B} < \hat{B}_c$, we have $x < 0$. If we assume that $f(x) > 0$, then it follows that the two terms on the lhs of (5.24) are separately much larger than the constant on the rhs. Thus, we can approximate (5.24) by

$$f(x)^2 \left(f(x) + \frac{x}{32k\hat{B}_c^4} \right) \approx 0, \tag{5.26}$$

leading to the scaling function for the $\hat{B} < \hat{B}_c$ part of the scaling region:

$$f(x) \sim \frac{\sqrt{-x}}{\sqrt{32k\hat{B}_c^4}}. \tag{5.27}$$

Indeed, in [22] the best fit with the numerical results for the extremal entropy density was found to scale near $\hat{B}_c$ according to:

$$\hat{s} \sim \sqrt{\hat{B} - \hat{B}_c}. \tag{5.28}$$

The results for the scaling function in three different parts of the scaling region close to the QCP (marked by the dotted line in figure 5.1) have been summarized in figure 5.1.

Quantum critical region

The scaling behaviour of $\hat{s}$ at $\hat{B} = \hat{B}_c$ (QCR) given above was determined numerically in [22] and subsequently established by analytic calculations (matched asymptotic expansions) [25], both for $k > 3/4$. For some reason, the behaviour $\hat{s} \sim \hat{T}^{1/3}$, which suggests the dual CFT_2 has dynamical exponent $z = 3$ is not consistent with what one would expect from the form (5.17) of the zero-temperature solution at $\hat{B} = \hat{B}_c$, from which the relation (5.19) was deduced. For the range $1/2 < k < 3/4$, different scaling in the QCR was found numerically [23]. Assembling these data, we have

$$\hat{s} \sim \hat{T}^{\alpha} \qquad \alpha = \begin{cases} \frac{1}{3} & \frac{3}{4} \leq k \\ \frac{1-k}{k} & \frac{1}{2} < k \leq \frac{3}{4} \end{cases} \tag{5.29}$$

In [22] the crossover to the low temperature behaviour of $\hat{B} < \hat{B}_c$ and $\hat{B} > \hat{B}_c$ was shown in the behaviour of the entropy density as the (dimensionless) temperature was decreased to zero. In the first case, where the dimensionless magnetic field was taken slightly below the critical value, the entropy density initially followed the $\hat{s} \sim \hat{T}^{1/3}$ behaviour of the Quantum Critical Region, but at very low temperatures, the entropy density was found to go to a nonzero constant value. For values of the magnetic field slightly above criticality, the crossover was to the linear behaviour of the low temperature $\hat{B} > \hat{B}_c$ region.

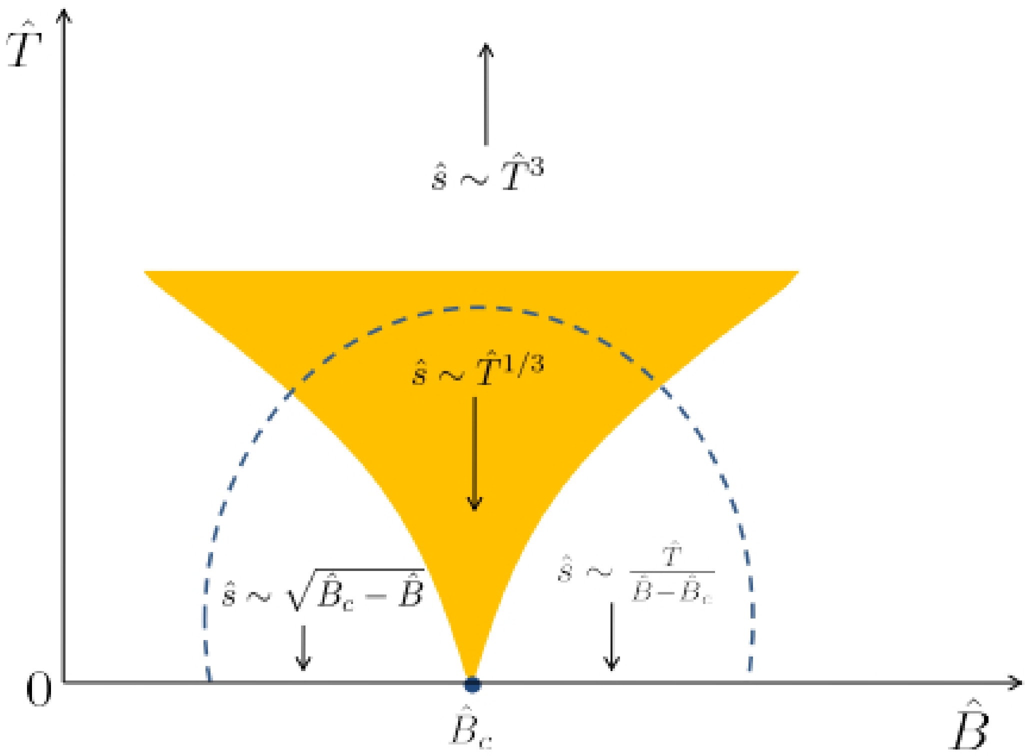

Figure 5.1: Scaling behaviour of the entropy density $\hat{s}$ with temperature $\hat{T}$ in three different regions of the scaling region, marked by the dotted line. These results are valid for $k > 3/4$ (note that for the window. *Image taken from [25].*

5.5 Comparison with a metamagnetic transition

The scaling relation (5.23) is consistent[12] with a QCP with one spatial dimension ($d = 1 + 1$), dynamical critical exponent $z = 3$, and a relevant operator of dimension $\Delta = 2$. As discussed in [22], the critical exponent $z = 3$ also arises in the Hertz-Millis theory of metamagnetic quantum phase transitions, in which a variation in the magnetic field across a nonzero critical value produces nonanalytic behaviour in the specific heat, but no change in symmetry (see section 2.3.2).

5.6 Discussion

It has been established, both numerically [22] and analytically [25] that field theories dual to Einstein-Maxwell theory in $D = 5$ with a Chern-Simons term undergo a Quantum Phase Transition for $k > 1/2$ at a critical magnetic field to charge density ratio. No analytical

[12]Note, however, that the 'real-life' materials in which such metamagnetic quantum phase transitions are found, are in $d = 2 + 1$.

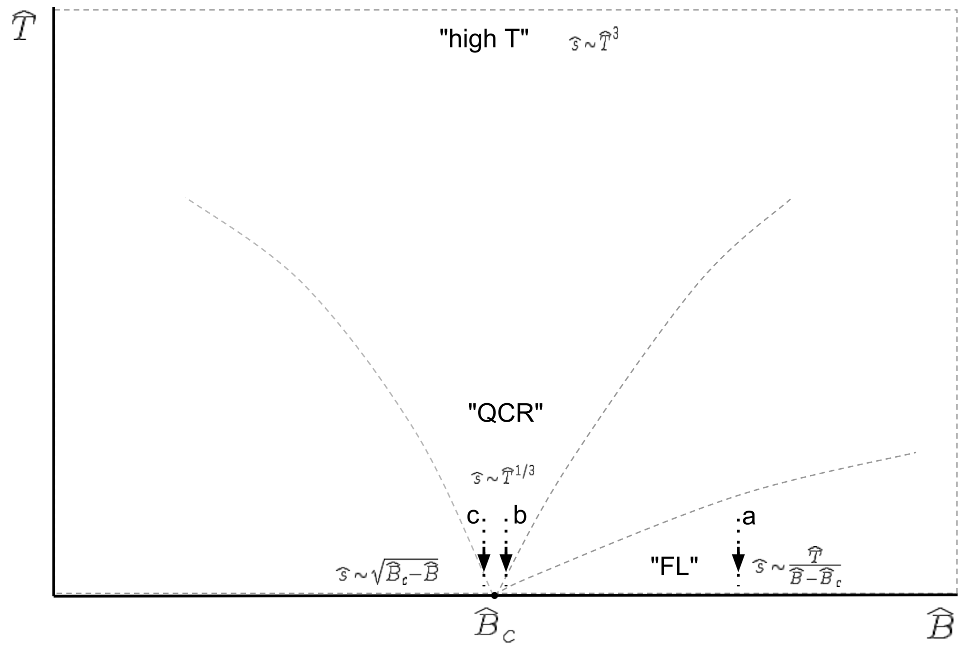

Figure 5.2: The low temperature behaviour of the entropy density was determined numerically. For $\hat{B} > \hat{B}_c$, which is at **a** in the figure, the entropy density vanishes linearly with temperature, which is consistent with Fermi liquid theory. Paths **b** and **c** both start at $\hat{B} \approx \hat{B}_c$ and small finite temperature with the behaviour $\hat{s} \sim \hat{T}^{1/3}$, but for path **b** $\hat{B}$ is slightly above $\hat{B}_c$, and there is a crossover to the linear behaviour $\hat{s} \sim \hat{T}$. Path **c** has $\hat{B}$ slightly below $\hat{B}_c$. Below a certain temperature, the entropy density tends to a constant value $\sqrt{\hat{B}_c - \hat{B}}$ which vanishes in the limit $\hat{B}_c - \hat{B} \to 0$. At large temperatures, the behaviour of the entropy density is the same for all values of $\hat{B}$, as in all cases the geometry is asymptotically AdS_5, leading to a cubic large temperature behaviour of the entropy density $\hat{s} \sim \hat{T}^3$.

solution has been found for the $\hat{B} < \hat{B}_c$ region of the phase diagram, but the finite entropy density at zero temperature suggests that its IR (small r) geometry contains an extremal horizon. As discussed in chapter 4, such a background is expected to be unstable when for instance a charged scalar field condenses, leading to a new geometry which has zero entropy in the $T \to 0$ limit. It would be interesting to have a solution describing this new geometry.

As explained in section 2.2, one typically looks for an order parameter when characterizing a phase transition. Recall that for a phase transition in which the two phases have different symmetry, such as the appearance or disappearance of spontaneous (staggered) magnetisation as a result of applied pressure, the order parameter is the vacuum expectation value for a symmetry-breaking operator. In the phase transition found by D'Hoker and Kraus, the symmetry has already been broken by the magnetic field, so the operator corresponding to the order parameter is not of this type. It is then of interest to find an order parameter for this scenario[13]

[13]it need not be a local operator

Chapter 6

Conclusion and outlook

The mechanism driving the Quantum Phase Transition (QPT) has not been identified, although in the setup reviewed in chapter 5, this might in principle be possible, since the charge carriers of the dual gauge theory are modeled fully holographically in the bulk (as opposed to constructions where charged fields are explicitly added to the bulk theory), and their interaction with the *critical modes* (by which I mean the fields that become massless at the QCP, or the extra states that become accessible in the condensed matter system????) should somehow be encoded in the gravity system. By studying correlation functions in this background, one may hope to see what happens in the gravitational system. If the mechanism causing the instability in the gravity dual were known, one might be able to construct different examples of holographic dualities in which such a QPT takes place.

The holographic methods reviewed in this thesis do have some serious limitations, of which some have been discussed. Among these is the fact that the search for a holographic dual is very difficult unless the dual theories have some amount of supersymmetry, which is (usually) not a feature of condensed matter models. Furthermore, as was mentioned in chapter 5, the gauge theory duals typically need a large number N of "colours" to be able to use classical gravity. However, most models in condensed matter will have $N = 1$, so one must hope to find interesting features that do not depend on N. Still, finding examples of holographic Quantum Critical Points may prove to be a very useful tool in mapping out various possibilities of QPTs, and in constructing toy models for them.

Another important limitation of using holography is that only special types of quantum field theories have been found to be described by a holographic dual. One thus cannot expect that the (strongly coupled) theory describing a QPT necessarily has dual gravity description.

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