String theory and cosmology

cosmology from low-energy string theory and coset models

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Abstract

This thesis is concerned with cosmological models that arise in low-energy string theory and coset models.

The first part of this thesis provides a basic introduction to cosmology and bosonic string theory. After this, the gravitational theory that arises in low-energy string theory is developed and discussed. The cosmological implications of this theory are investigated, with emphasis on the relation to the universe we inhabit. This general account is followed by a detailed review of some known models in the literature and a discussion on the possibility of finding a de Sitter solution.

The second part of this thesis considers coset models, which provide a systematic way to find background fields that are valid solutions in full (as opposed to low-energy) string theory. The construction of such a model using the gauged Wess-Zumino-Witten action is presented. After this, a general way to describe propagation of scalar particles on the coset space is developed. The last chapter focuses on coset models that yield spacetimes with an interesting cosmological interpretation. A well-known example of such a model is the Nappi-Witten universe, which is analyzed in detail using the techniques that were discussed.

Contents

Units, conventions, notation

Throughout this thesis, I use natural units in which $\hbar = c = 1$. The gravitational constant G is kept explicit. Be aware that G also denotes the metric tensor. In particular $\sqrt{G} = \sqrt{\det G_{\mu\nu}}$ is the square-root of its determinant.

For quantities related to curvature, such as the Ricci tensor or curvature scalar, I adopt conventions denoted $+++$ by Misner et.al. [59]. This means in particular that

$$\begin{aligned} R^{\mu}{}_{\nu\alpha\beta} &= \partial_\alpha \Gamma^{\mu}{}_{\nu\beta} - \partial_\beta \Gamma^{\mu}{}_{\nu\alpha} + \Gamma^{\mu}{}_{\rho\alpha}\Gamma^{\rho}{}_{\nu\beta} - \Gamma^{\mu}{}_{\rho\beta}\Gamma^{\rho}{}_{\nu\alpha} \\ R_{\mu\nu} &= R^{\alpha}{}_{\mu\alpha\nu} \end{aligned}$$

For comparison with other conventions, the overview presented on the first page of [59] is excellent. I will sometimes refer to Weinberg [84], who uses $+--$ conventions. This implies that the Ricci tensor and the curvature scalar are defined with an additional minus sign.

To avoid confusion between curvature quantities on the worldsheet and in spacetime, the latter are denoted in boldface. So the worldsheet curvature scalar is denoted as R and the spacetime curvature scalar as $\mathbf{R}$.

Usually, α, β and other letters from the beginning of the Greek alphabet, when used as indices, denote coordinates on the string worlsheet. Starting from μ, ν, the indices denote spacetime coordinates. If a distinction between spacetime and space is made, Greek indices as μ run from 0 to D, whereas Roman indices as i run from 1 to D.

A lot of string theory analysis is on the complex plane, where coordinates z and $\bar{z}$ are used conventionally. Here ∂ is shorthand for ∂_z and $\bar{\partial}$ for $\partial_{\bar{z}}$. For vectors on the complex plane, v^z refers to the z component of the vector v.

1
Introduction

> *"Nature uses only the longest threads to weave her patterns, so each small piece of the fabric reveals the organization of the entire tapestry."*
> – Richard Feynman

It is generally believed that nature comes with four fundamental forces. Two of these are directly visible in our everyday life: electromagnetism and gravity. The first is obviously responsible for electricity and magnetism, but light and radiation are also electromagnetic phenomena. Gravity is the force that keeps the planets in their orbit and makes the apple fall from the tree. On very small scales, these forces are accompanied by two equally fundamental forces known as the weak and strong nuclear force.

Contemporary physics can not describe all four forces in one fundamental framework. The dynamics of the weak, strong and electromagnetic force are described in one unified system known as the *standard model* of particle physics – a fantastic achievement by itself. This is the model that is used to calculate physical quantities in particle accelerator experiments, where highly energetic particles are smashed upon one another to interact and form new particles. The model describes physics at high energies and small length-scales, a world ruled by quantum mechanics.

Gravity is of no importance in this world, as the strength of the force is many orders of magnitude smaller then the strength of the other forces on small length scales. However, gravity 'goes a long way' and contributions of various matter sources generally add up – as opposed to electromagnetism, where positive and negative contributions may cancel. This makes gravity the dominant force on astronomical length scales. The theory that describes this force is Einstein's general theory of relativity.

In their specific regimes, both the standard model and general relativity are well established and verified experimentally. However, the theories are fundamentally very different and cannot be combined. Subjects in which both gravity and quantum mechanics are important, such as black holes or the big bang, are notoriously difficult to describe and clearly indicate the need for a quantum theory of gravity.

A topic related to this issue is the famous cosmological constant, which will be described in more detail in the next chapter. One can think of a cosmological constant as a sort of intrinsic energy of spacetime, though one that may be negative. General relativity allows such a constant in its framework, though it does not make predictions about its value; it's a free parameter. From the standard model perspective, one may think of the cosmological constant as a vacuum energy caused by particles that are created out of the vacuum to annihilate within a very short time. A back-of-the-envelope calculation shows that such a vacuum energy is expected to be of order

$$\rho_V \sim (10^{18}\ \text{GeV})^4$$

If one trusts the standard model up to Plank scale energies[1]. Such a cosmological constant would have enormous implications on cosmology and one may safely conclude that the constant isn't anywhere near this size. For quite some time, the physics community believed that one day it would become clear that (and why) the cosmological constant is exactly zero. However, recent astronomical observations suggest that our universe in fact has a small but non-zero positive cosmological constant of

$$\rho_V \sim (10^{-12}\ \text{GeV})^4$$

The discrepancy between the energy associated with the observed and theoretical cosmological constant is 120 orders of magnitude. Ideally, a true quantum theory of gravity would predict the correct value of the cosmological constant.

The effect of a positive cosmological constant is that it drives the expansion of the universe, thereby steadily diluting the matter content. Eventually the matter contribution to the total energy density of the universe will be negligible and vacuum energy will dominate the universe. The resulting space is known as de Sitter (dS) space, which solves the Einstein field equations with positive cosmological constant.

What about string theory? String theory is a consistent quantum theory that includes a theory of gravity. At low energies (on the string scale of things, $E \ll M_P \sim 10^{19}$GeV where M_P is the Planck mass) this theory is related to general relativity, yet there are fundamental differences. I will not claim that string theory is *the* theory of quantum gravity, but it seems hard to imagine that such a theory, when formulated, will be totally unrelated to string theory.

Most readers will be familiar with string theory or have at least heard something about it – probably something about string theory being this crazy theory that lives in more than four dimensions, namely 26 for bosonic strings and ten for supersymmetric string theories. At first sight, this may look like a disqualification, but from another point of view one may advocate that the very prediction of dimensionality shows the power of the theory. As a comparison, general relativity works in every dimension and the theory has no preferred number of dimensions.

Of course, this does not invalidate the argument that our universe seems to have only four dimensions and not ten or 26. In string theory, it is usually assumed that the remaining dimensions are compactified and form a small internal space that has not yet been discovered due to its smallness. Ideally, the internal space should reproduce the physics of the standard model while the large dimensions constitute the large-scale universe as we observe it. If one day this idea turns out to be true, string theory incorporates both the standard model and general relativity – making it a true theory-of-everything.

1.1 Motivation

In this thesis, I solely consider the large dimensions and investigate how string theory is a theory of gravity that works in these dimensions. I have chosen to focus on two

[1] The reader will find references in the next chapter.

main topics; firstly, cosmology from low-energy string theory and the relation to de Sitter space, associated with a positive cosmological constant. Secondly, coset models that have an interesting cosmological interpretation. Why these?

From the point of view of string theory, or quantum gravity in general, de Sitter space is a notorious subject. To name one issue, it has no globally defined timelike Killing vector which makes it impossible to define a positive conserved energy. Furthermore, it is known that there is no naive de Sitter solution in low-energy string theory. If our universe is indeed headed to become such a space, a better understanding of this space is crucial if string theory is to describe 'the real world'. There is currently a lot of research in the area, making it a prime example of string theory research that is triggered by the experiment.

A striking feature of string theory is the existence of a large-small duality that relates physics at large scales to physics at small scales. This suggests the existence of a minimal length scale. Such a fundamental length scale is unknown in general relativity (or the standard model) and the implications of such a scale for physics at the big bang are enormous. This makes the investigation of singularities in string theory highly relevant. This research is in the regime of high energies, where general relativity may break down and low-energy string theory is not satisfactory. A well-known method to consider dynamical backgrounds (with possible singularities) beyond the low-energy assumption is by using coset models. Knowing the behavior of scalar states on such a background, they can be used to probe the structure of singularities in string theory. Eventually, such knowledge may lead to a better insight into the nature of the big bang.

1.2 Outline

When I started to write this thesis after an initial period of research, I thought about its goal. I wanted to address a number of issues that I had been thinking about and clearly state some insights I developed. On the other hand, I wanted to write something that may serve as a reference to others – and to myself in a possible later stage of research. These considerations have primarily determined the outline and plan of this thesis.

The thesis starts with an introduction to the standard model of cosmology in chapter 2 (not to be confused with the standard model of particle physics). This chapter also briefly discusses the cosmological constant, Brans-Dicke theory and de Sitter space; these topics will be encountered a number of times throughout this thesis in the context of string cosmology. In chapters 3 and 4, conformal field theory and bosonic string theory are introduced. Regarding these theories, I have chosen to highlight issues that are especially relevant for string cosmology, leaving out others. For instance, the subject of interactions in string theory is discussed only very briefly. Chapter 5 introduces the theory of gravity that follows from considering strings in a non-trivial background. The cosmological implications of the theory are discussed; in particular, the possibility of finding a de Sitter solution in this theory. Chapters 6 and 7 consider coset models and discuss models that are interesting from a cosmological point of view, such as the Nappi-Witten universe. A summary and some conclusions are presented in chapter 8, together with suggestions for further research.

The reader may notice that there is one example that appears in a number of sections. It is based on the black hole solution found by Witten, and introduced as such in section 6.4. The associated low-energy spacetime, known as the Milne universe, already appears as a solution in low-energy string theory in section 5.4.2. Propagation of scalars on this background is considered in section 6.6.

2

Theories of cosmology

The first chapter of this thesis is about cosmology. It discusses the cosmological standard model and two extensions to this model: the cosmological constant and the Brans-Dicke scalar field. The theory is well established, and readers that are familiar with cosmology may safely skip it. However, I have chosen to present this review to provide a solid basis for the theory that will follow in the rest of this thesis. In chapters 3 to 5, I will work towards a theory of gravity inspired by string theory and explore some implications of this theory on cosmology. There is a lot of nomenclature and I felt that it would be unwise to refer to cosmological constants, gravitational scalar fields or Brans-Dicke parameters without giving the reader clear definitions and a context. I mostly use the action principle to formulate a theory as it makes comparison between various theories easy.

The outline of this chapter is as follows. In section 2.1, the cosmological standard model is introduced. Section 2.2 discusses the influence of a non-zero cosmological constant and a Brans-Dicke scalar fields on the standard scenario. These extensions will make a comeback in cosmological models based on string theory, starting from chapter 5. Section 2.3 concludes the current chapter with a discussion of the observed universe and a short review on de Sitter space, the maximally symmetric solution to the Einstein equations with a positive cosmological constant.

There are many excellent introductions to cosmology. I usually follow Weinberg [84], though the reader should be aware that Weinberg uses $+--$ sign conventions; see the notation section in the beginning of this thesis. Another nice introduction to cosmology, aimed at an audience of high energy theorists, is the TASI lecture by Carroll [20].

2.1 The cosmological standard model

This section introduces what is often called the standard model of cosmology, which is based on three pillars. The first pillar are the Einstein field equations, which determine the dynamics of spacetime in the presence of energy. Second is the cosmological principle, the hypothesis that space (not spacetime) is isotropic and homogeneous on large scales. Third are equations of state for matter sources in the universe, relating their energy density to the size of the universe. Combining these leads to the Friedmann equations that describe the dynamics of the universe we inhabit. I assume $D = 4$ from the beginning, where D denotes the dimension of spacetime.

2.1.1 The Einstein field equations

Spacetime is described by a gravitational field, or metric, G. Its dynamics may be derived from an action principle. Following Weinberg [84], chapter 12, the total action for some gravitational system with matter splits into a gravitational and a matter part: $S = S_G + S_M$. The purely gravitational part is given by the Einstein-Hilbert action, the matter action is represented by a general Lagrangian density.

$$\begin{aligned} S_G &= \frac{1}{16\pi G}\int d^4x\,\sqrt{G}R \\ S_M &= \int d^4x\,\sqrt{G}\mathcal{L}_M \end{aligned} \tag{2.1}$$

As usual, the demand that the action be stationary under an arbitrary variation leads to an equation of motion. Start by varying the gravitational action with respect to $G_{\mu\nu}$.

$$\begin{aligned} \delta S_G &= \frac{1}{16\pi G}\int d^4x\left(\delta\sqrt{G}\,R + \sqrt{G}\,\delta R\right) \\ &= \frac{1}{16\pi G}\int d^4x\left(\frac{1}{2}\sqrt{G}G^{\mu\nu}\delta G_{\mu\nu}R + \sqrt{G}\,(R_{\mu\nu}\delta G^{\mu\nu} + \delta R_{\mu\nu}G^{\mu\nu})\right) \\ &= \frac{1}{16\pi G}\int d^4x\,\sqrt{G}\left((\frac{1}{2}G^{\mu\nu}R - R^{\mu\nu})\,\delta G_{\mu\nu} + G^{\mu\nu}\,\delta R_{\mu\nu}\right) \end{aligned} \tag{2.2}$$

The last term is a total derivative by the Palatini identity (see [84], section 10.9)

$$\delta R_{\mu\nu} = \nabla_\lambda\,\delta\Gamma^\lambda{}_{\mu\nu} - \nabla_\nu\,\delta\Gamma^\lambda{}_{\mu\lambda} \tag{2.3}$$

Concerning the matter part of the action, the variation of an action with respect to the metric defines the energy-momentum tensor T of the system ([84], 12.2):

$$\delta S_M = \frac{1}{2}\int d^4x\sqrt{G}\,T^{\mu\nu}\,\delta G_{\mu\nu} \tag{2.4}$$

Adding this to the contribution from equation (2.2), one finds the famous Einstein field equations with source, describing the dynamics of a gravitational field G in the presence of energy characterized by an energy-momentum tensor T.

$$R^{\mu\nu} - \frac{1}{2}G^{\mu\nu}R = 8\pi G T^{\mu\nu} \tag{2.5}$$

Contracting both sides of this equation with $G_{\mu\nu}$ implies $R = -8\pi G\,T^\alpha{}_\alpha$ as $D = 4$. This can be used to recast equation (2.5) into the equivalent form

$$R^{\mu\nu} = 8\pi G\,(T^{\mu\nu} - \frac{1}{2}G^{\mu\nu}T^\alpha{}_\alpha) \tag{2.6}$$

As all tensors in the Einstein field equations are symmetric, the field equations are a collection of ten algebraically independent equations. There is however some redundancy. Suppose one has found a solution G of the Einstein equations and performs a coordinate transformation (with four degrees of freedom) $G \to G'$. Then G' will also be solution to the equations – which is just the statement that the field equations are tensor equations. In order to describe a unique solution to the field equations, a coordinate condition has to be specified. This is very similar to choosing a certain gauge in systems with gauge invariance.

From this point of view, it should be no surprise that although the ten equations are algebraically independent, they are actually not functionally independent. This is expressed by the four Bianchi identities

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2}G^{\mu\nu}R\right) = 0 \tag{2.7}$$

From the physical point of view this corresponds to momentum conservation $\nabla_\mu T^{\mu\nu} = 0$ by the field equations. From the geometrical point of view, it follows from symmetries of the Riemann-Christoffel tensor. Taking these identities into account, the Einstein field equations actually consist of only six functionally independent dynamical equations. This leaves four degrees of freedom for the ten unknown elements of G. This freedom connects a given G with all the G'-tensors that can be obtained by coordinate transformations from G.

The above can be summarized by the statement that the Einstein field equations, subject to the Bianchi identities, fix the metric up to general coordinate transformations.

2.1.2 Cosmological principle

The Einstein field equations may be applied to a variety of gravitational systems. Usually, the problem of solving the field equations is simplified by symmetry properties of the particular situation. For instance, the famous Schwarzschild solution that includes black holes assumes radial isotropy and no time-dependence of the metric.

When the Einstein field equations are applied to the universe as a whole, the symmetry argument that is invoked is the *cosmological principle*, the hypothesis that space is homogeneous and isotropic on large scales. This makes physical 'space' a maximally symmetric space which determines its structure to a large extent. Therefore spacetime itself, when sliced into equal-time slices, has a maximally symmetric subspace. From the theory of symmetric spaces (see e.g. Weinberg [84], chapter 13) it is known that the most general metric of such a space is the *Robertson-Walker* metric with line-element (in polar coordinates r, θ, ϕ)

$$ds^2 = -dt^2 + R(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{2.8}$$

$R(t)$ is the cosmic scale factor that describes the dynamical evolution of the universe. There is one additional constant k that can be set to either $-1, 0, 1$ by a suitable choice of units for r. It describes the *spatial* curvature of the spacetime under consideration; for $k = -1$ equal-time hypersurfaces are positively curved, for $k = 0$ these surfaces are flat and in the case $k = 1$ positively curved. These possibilities are usually referred to as open, flat or closed, respectively.

2.1.3 Equations of state

The third simplifying assumption in the cosmological standard model concerns the matter that is present in the universe. Assume that matter (again, on large scales) behaves as a relativistic perfect fluid, meaning that an observer moving along with the fluid sees it as isotropic. Of course this kind of symmetry can only be achieved if the particles that make up the fluid continuously interact, i.e. if the mean free path is small on the observational length scale. On cosmological scales, matter in the universe is indeed observed as isotropic. Mathematically, a perfect fluid is characterized by an energy-momentum tensor of the following form ([84], 2.10)

$$T^{\mu\nu} = (p + \rho)u^\mu u^\nu + pG^{\mu\nu} \tag{2.9}$$

In this expression, p is the pressure of the fluid, ρ its proper energy density and u is the fluid's velocity four-vector. At this point choose comoving coordinates, so that $u = (1, 0, 0, 0)$ and r, θ, ϕ may be taken to be constant. Usually there is some equation of state that relates the pressure p to the energy density ρ of some specific kind of 'fluid'. Many equations of state are as simple as

$$p = w\rho \tag{2.10}$$

Where $w \in \mathbb{R}$ characterizes a specific kind of energy source. In cosmology the most common are matter (also known as dust) and radiation. The former includes all kinds of non-relativistic, non-interacting particles which have negligible pressure and whose energy is mostly rest mass ($w = 0$). The latter applies to all relativistic particles which have $w = 1/3$.

Inserting this generic equation of state into the condition for conservation of energy $\nabla_\mu T^{\mu\nu} = 0$ yields (for $\nu = 0$, $\nu = 1, 2, 3$ is trivially satisfied; see [84], 14.2 and 15.1)

$$\frac{d}{dR}(\rho R^3) = -3pR^2 \tag{2.11}$$

$$\Rightarrow \quad \rho \propto R^{-3(w+1)} \tag{2.12}$$

The time-evolution of the energy density (and subsequently of the pressure) of some source of energy in the universe is thus proportional to the cosmic scale factor to the power $-3(w + 1)$. This is summarized in table 2.1 which for completeness includes a cosmological constant, to be described farther on.

energy source	equation of state	dynamical equation
matter, dust	$p = 0$	$\rho \propto R^{-3}$
radiation	$p = 1/3\rho$	$\rho \propto R^{-4}$
cosmological constant	$p = -\rho$	ρ const

Table 2.1: Dynamics of energy sources in the universe

2.1.4 Friedmann equations

The cards are now on the table. The dynamics of spacetime is described by the Einstein equations, the general form by the cosmological principle and the dynamics of matter sources by their equations of state and the conservation of energy. Now these ingredients can be combined to study the dynamics of the universe. It is important to keep in mind that the predicting power of the theory is limited by the possibility of a phase transition. It is only assumed that the equations of state remain valid at all times.

Plugging the Robertson-Walker metric (2.8) into the Einstein field equations (2.5), gives the two Friedmann equations for a single matter source with pressure p and proper energy density ρ ([84], 15.1)

$$\frac{\dot{R}(t)^2}{R(t)^2} = \frac{8\pi G}{3}\rho(t) - \frac{k}{R(t)^2} \tag{2.13}$$

$$\frac{\ddot{R}(t)}{R(t)} = -\frac{4\pi G}{3}\Big(\rho(t) + 3p(t)\Big) \tag{2.14}$$

It should be noted that the second equation can be derived from the first and (2.11). That is, considering a perfect-fluid energy source with divergenceless energy-momentum tensor, equation (2.13) suffices.

Knowing $\rho(t)$ in terms of $R(t)$, it is possible to solve the above equations for $R(t)$ to find the expansion history and future of the universe. For example, a spatially flat ($k = 0$), matter-dominated universe has a scale factor

$$R(t) \propto t^{2/3} \tag{2.15}$$

As may be verified by inserting this into equations (2.13) and (2.14).

It is customary to introduce the critical energy density as the energy density that is required to have a spatially flat universe. Putting $k = 0$ in equation (2.13), it follows that this energy density is equal to

$$\rho_{crit}(t) = \frac{3}{8\pi H(t)^2}G \tag{2.16}$$

Where $H(t)$ is the Hubble parameter

$$H(t) := \frac{\dot{R}(t)}{R(t)} \tag{2.17}$$

In a realistic model there are of course multiple sources of energy, which is accounted for by expanding $p = \sum_i p_i$ and $\rho = \sum_i \rho_i$. The label i runs over different energy sources with different equations of state. When there are multiple sources, a useful quantity is the energy density parameter Ω_i of a matter source i, defined as

$$\Omega_i(t) = \frac{\rho_i(t)}{\rho_{crit}(t)} \tag{2.18}$$

With these definitions, the first of the Friedmann equations (2.13) for multiple energy sources can be cast into the form

$$\sum_i \Omega_i(t) - 1 = \frac{k}{H(t)^2 R(t)^2} \tag{2.19}$$

And it follows that $k = \text{sign}\left(\sum_i \Omega_i(t) - 1\right)$. If the energy density parameters of all sources exactly add up to one, there is the right amount of energy for the universe to be spatially flat.

For a qualitative discussion on cosmology dominated by matter sources, see section 15.3 of [84]. I will only mention that, for universes dominated by matter and radiation, open

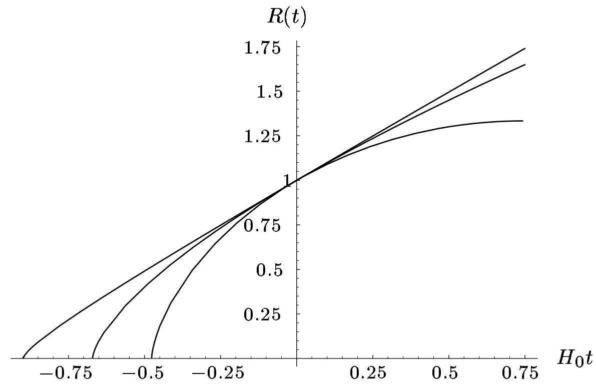

Figure 2.1: Evolution of a matter-dominated universe with, from above to below, $\Omega_M = 0.1, 1.0, 4.0$ ($k = -1, 0, 1$). Note that the time-scale is set by the Hubble parameter at $t = 0$, which has a different value for each model. I have chosen R(0)=1 and set G=1.

($k = -1$) universes will expand forever, while closed ($k = 1$) universes will recollapse. Be aware that this connection is not valid if other energy sources are significant.

In figure 2.1, equation (2.13) was used to calculate the evolution of a universe that consists of matter only. The cosmic scale factor is plotted for various values of Ω_M at $t = 0$. The $\Omega_M = 4.0$ universe will eventually recollapse, the other two increase without limit.

2.2 Extensions to standard cosmology

In the above, the cosmological standard model was described to the extent of finding the Friedmann equations (2.13) and (2.14) and the underlying assumptions were discussed. There are several ways in which this standard model of cosmology may be altered. I will focus on modifications to the first of the three pillars of standard cosmology: the Einstein field equations. As a motivation, the reader should be aware that the field equations (2.5) are not as solidly rooted in general relativity as for instance the equivalence principle (a point of view expressed by Weinberg [84], section 8.3). The exact form of the equations is to some extent guesswork. Therefore modifications such as a cosmological constant or a long-range dynamical scalar field cannot be ruled out from a purely theoretical point of view. The implication of such modifications on the dynamics of the universe is the subject of this section.

2.2.1 A cosmological constant

When Einstein applied his theory of general relativity to the universe as a whole, it was of course known that the relative velocities of celestial objects are much smaller than the speed of light. Therefore Einstein believed that the universe was (quasi-)static. However, the Einstein field equations (2.5) do not allow a static universe ($\dot{R}(t) = 0$) solution. Therefore the original field equations were modified to include the cosmological constant Λ as can be seen in advance in equation (2.21). With $\Lambda > 0$, there is indeed a solution corresponding to a static universe, in which the mass density of 'dust' in the universe is directly related to its geometry. Einstein in particular hoped

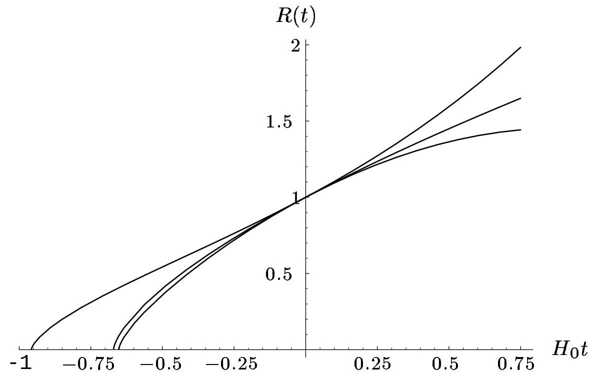

Figure 2.2: Evolution of a universe with matter and a cosmological constant. From above to below, $(\Omega_M, \Omega_\Lambda) = (0.3,\ 0.7)$, $(1.0,\ 0.0)$, $(0.7,\ \text{-}0.7)$. Again, R(0)=1 and G=1.

that it would incorporate Mach's principle, the idea that the mass distribution in the universe defines inertia.

Then de Sitter found another (quasi-)static solution without any matter present, thereby undoing the hope of connecting with Mach's principle. When the redshifts of far-away celestial objects were discovered, it was realized that the universe actually was expanding. Furthermore, these redshifts could be explained by de Sitter's model but not by Einstein's static universe. The idea of a static universe was abandoned and, although mathematically sound, the cosmological constant was discarded. For more details, see Weinberg [85]; a standard text on this issue.

The modified field equations can be derived from the action

$$S_\Lambda = \frac{1}{16\pi G} \int d^4x \sqrt{G}\,(R - 2\Lambda) \tag{2.20}$$

The same analysis as in section 2.1.1 yields the modified Einstein field equations (cf. (2.5)).

$$R^{\mu\nu} - \frac{1}{2} G^{\mu\nu} R + \Lambda G^{\mu\nu} = 8\pi G T^{\mu\nu} \tag{2.21}$$

The left-hand side of this equation is the most general divergenceless symmetric two-tensor that can be constructed from the metric and its first and second derivatives. This makes it a more general starting point than the original field equations without the cosmological constant. As follows from equation (2.21), the effect of the cosmological constant can be taken into account by using the original field equations (2.5) and substituting

$$\begin{aligned} T^{\mu\nu} &\rightarrow T^{\mu\nu} + T^{\mu\nu}_\Lambda \\ T^{\mu\nu}_\Lambda &= -\frac{\Lambda}{8\pi G} G^{\mu\nu} \end{aligned} \tag{2.22}$$

That is, the cosmological constant may be regarded as an omnipresent energy source with energy-momentum tensor T_Λ.

Upon comparison with the general perfect-fluid energy-momentum tensor (2.9) it is clear that one may think of this energy source as a perfect fluid with equation of state $p = -\rho$ ($w = -1$) as anticipated in table (2.1). This means it can be treated in the same manner as 'ordinary' energy sources. In particular, it can easily be included in the Friedmann equations to analyze its effect on the dynamics of the universe. A direct consequence is that spatially closed universes do not necessarily recollapse.

Figure 2.2.1 shows the effect of a non-zero cosmological constant on the evolution of the cosmic scale factor. It is clear that a positive constant speeds up the acceleration, while a negative constant slows it down.

The form of the energy-momentum tensor T_Λ is also the origin of the identification of the cosmological constant with a non-zero *vacuum energy*. Consider the simple case of a single scalar field ϕ with potential energy $V(\phi)$ and action

$$S = \int d^4x\sqrt{G}\left(G^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right) \tag{2.23}$$

It follows that the energy-momentum tensor of this system is given by

$$T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi + \frac{1}{2}G^{\mu\nu}\left(G^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - V(\phi)\right) \tag{2.24}$$

The configuration with the lowest energy is characterized by $\partial_\mu\phi = 0$ and $\phi = \phi_0$ where ϕ_0 is the value of ϕ such that $V(\phi)$ is minimal. There is no a priori reason why $V(\phi_0)$ should be zero. If indeed it is non-zero, the energy-momentum tensor takes the general form

$$T^{\mu\nu}_{vac} = -\rho_{vac}G^{\mu\nu} \tag{2.25}$$

Where $\rho_{vac} = \frac{1}{2}V(\phi_0)$ in the particular model considered here. It follows that, up to a constant

$$T_\Lambda \sim T_{vac} \tag{2.26}$$

A cosmological constant in the sense of (2.21) thus manifests itself exactly as a non-zero vacuum energy.

2.2.2 A fundamental scalar field

In 1961, Brans and Dicke published an article in which they proposed that gravitational effects may be partly caused by a scalar field ϕ (the original article [17] is well readable). Again, this attempt was motivated by the hope of incorporating Mach's principle. They considered the following action, in which ϕ is the scalar field.

$$S_{B-D} = \frac{1}{16\pi}\int d^4x\,\sqrt{G}\,(\phi R + \omega\frac{\partial_\mu\phi\,\partial^\mu\phi}{\phi}) \tag{2.27}$$

Where ω is a dimensionless free parameter. The gravitational constant G has disappeared from the action, its role now being played by the effective 'coupling' $1/\langle\phi\rangle$. Additional matter fields can be included by adding a matter action in the same way as in the Einstein theory: $S = S_\phi + S_M$, where S_M has no dilaton-dependence. This implies that the equations of motion for matter in a gravitational field are not modified – they depend on the metric g and not on the scalar ϕ.[1] However, the equations that

[1] This is the point of view Brans and Dicke take in the original article. The question is actually a little more delicate, as it depends on the frame in which the theory is formulated. This is directly related to the discussion on string and Einstein frames in section 5.2.2.

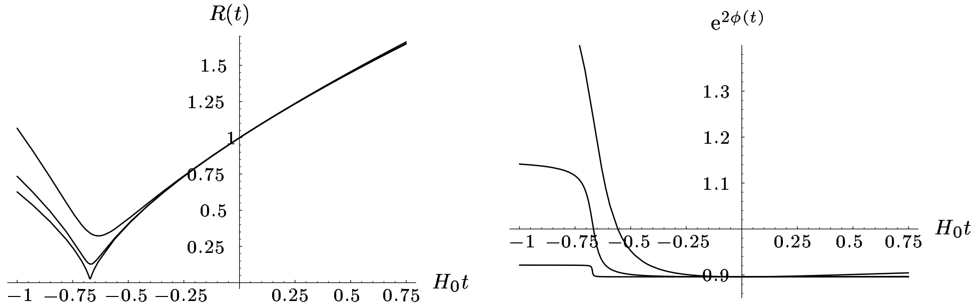

Figure 2.3: Evolution of the scale factor and the dilaton in a spatially flat Brans-Dicke universe with matter only. The free parameter ω is varied; from top to bottom in both figures $\omega = 10, 100, 10000$.

determine the dynamics of the gravitational field g are modified by the presence of the scalar field. They can be derived from the above action in a way similar to the derivation of the original field equations (2.5), the difference being that the Palatini identity cannot be used, so $\delta R^{\mu\nu}$ has to be calculated explicitly. This results in the following equations for the fields g and ϕ ([17] or [84], 7.4):

$$R^{\mu\nu} - \frac{1}{2}G^{\mu\nu}R = \frac{8\pi}{\phi}T^{\mu\nu} + \frac{1}{\phi}\Big(\nabla^\mu\nabla^\nu\phi - G^{\mu\nu}\Box\phi\Big) + \frac{\omega}{\phi^2}\Big(\nabla^\mu\phi\nabla^\nu\phi - \frac{1}{2}G^{\mu\nu}(\nabla\phi)^2\Big) \tag{2.28}$$

$$\Box\phi = \frac{8\pi}{3+2\omega}T^\rho{}_\rho \tag{2.29}$$

In the limit of large $|\omega|$ the second equation implies

$$\phi = \langle\phi\rangle + \mathcal{O}(\omega^{-1}) = 1/G + \mathcal{O}(\omega^{-1}) \tag{2.30}$$

Inserting this in the first equation, it shows that up to $\mathcal{O}(\omega^{-1})$ it is the Einstein field equation (2.5). Hence Brans-Dicke theory becomes standard Einstein gravity in the limit $|\omega| \to \infty$. As mentioned before, ω is a free parameter but in a sensible theory one would expect it to be of order 1.

For various values of ω, the evolution of the scale factor is shown in figure (2.3). As $|\omega|$ gets larger, the dilaton becomes nearly constant and the evolution of the scale factor indeed closely resembles the standard Einstein case. The most striking feature is of course the behavior at early times where there seems to be no singularity for small $|\omega|$.

2.3 Our universe and de Sitter space

The previous sections discussed cosmology from a theoretical point of view. Here I will very briefly comment on observational data related to the cosmological standard model and possible extensions. Starting with the Brans-Dicke scalar field: for a scalar field as introduced in section 2.2.2, observations restrict $|\omega| > 3500$ (see [86], section 3.4). As discussed above, Brans-Dicke theory essentially reduces to Einstein gravity in the limit of large $|\omega|$, so this observation virtually rules out a gravitational scalar field in the sense of equation (2.27). As will be discussed in chapter 5, the scalar field will reappear in a modified form in low-energy string cosmology.

Regarding the energy sources present in our universe (and the related spatial topology), let me just state that there is currently some concordance that our universe is spatially flat and dominated by matter and vacuum energy, with density parameters

$$\Omega_M = 0.3 \qquad \Omega_\Lambda = 0.7 \tag{2.31}$$

The interested reader is referred to a recent review by Sarkar [68], who discusses the collection of (circumstantial) evidence for this claim and provides full references.

The presence of a non-zero positive cosmological constant implies that our universe is a de Sitter space. The following is a short summary to this space; a basic introduction to de Sitter spacetimes can be found in the Les Houches lecture by Spradlin et.al. [73]. The issue of finding a de Sitter space in low-energy string cosmology will be addressed in section 5.4.5 of this thesis.

de Sitter space in n dimensions dS_n is the maximally symmetric solution to the Einstein field equations in n dimensions with positive cosmological constant (cf. eq. (2.21)). An easy way to visualize de Sitter space is a n-dimensional hypersurface in $n+1$ dimensional Minkowski space:

$$-x_0^2 + x_1^2 + \ldots x_n^2 = l^2 \tag{2.32}$$

Where $l > 0$. This embedding makes the $SO(1,n)$ isometry group manifest. Such a manifold solves the Einstein field equations with (see Spradlin et.al. [73])

$$\Lambda = (n-2)(n-1)/(2l^2) \tag{2.33}$$

The maximal symmetry implies that the Riemann tensor takes the form (Weinberg [84], section 13.2)

$$R_{\mu\nu\rho\sigma} = \frac{1}{n(n-1)} R (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) = \frac{2\Lambda}{(n-1)(n-2)} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \tag{2.34}$$

Where the latter equality follows directly from the Einstein equations. There are a number of coordinate systems that cover (part of) de Sitter spacetime. In particular, de Sitter space admits spacelike foliations of the Robertson-Walker form (2.8) for $k = -1, 0, 1$ (see Carroll [20]). The scale factor in such a foliation typically becomes an exponential function of time for late times.

3

Conformal field theory

This chapter presents an introduction to conformal symmetry and conformally invariant quantum field theories (CFTs). It is not meant as an extensive or complete introduction, but is intended to give the reader an overview and a reference for later chapters. The theory in this chapter is developed from general symmetry arguments only, without reference to any particular CFT. It will become clear that conformal symmetry determines the structure of a theory to a large extent.

Sections 3.1 and 3.2 introduce the notion of conformal invariance and conclude with the introduction of the celebrated Virasoro algebra. Section 3.3 continues by providing some basic techniques in field theory that will be used in section 3.4 to discuss basic properties of a CFT focusing on the role of the energy-momentum tensor. Section 3.5 concludes this chapter with a discussion of the Hilbert space of a conformal field theory.

Several good texts on CFT may be found in the literature, such as the excellent self-contained introduction to CFT by Di Francesco et.al. [25]. Where applicable, I have chosen to refer to sections in this book. The introduction to superstrings by Kiritsis [51] also includes a nice chapter on CFT. Both the book [63] and lecture notes [65] on string theory by Polchinski provide an introduction to CFT in the context the string theory. See also the historical article by Belavin et.al. [11], which introduced conformal symmetry in two-dimensional quantum field theories.

3.1 The conformal group

This section introduces conformal symmetry, starting for a system in an arbitrary number of D dimensions. Later the discussion focuses on $D = 2$. As the theory is well established, I have not provided full references in this section. All of the topics addressed here can be found in more detail in Di Francesco et.al. [25], sections 4.1 and 5.1.

The symmetry underlying general relativity is that of general coordinate transformations. It states that true physical laws do not depend on the coordinates in which they are formulated. Mathematically, this leads to the introduction of scalars, vectors and tensors – objects that have a definite transformation law under general coordinate transformations. In particular the metric, as a two-component tensor, transforms under a coordinate transformation $x \to x'$ as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \tag{3.1}$$

The *conformal group* is defined as the subgroup of the group of general coordinate transformation that leaves the metric invariant up to an overall position-dependent scale factor. That is,

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \tag{3.2}$$

Restricting $\Omega(x) = 1$ produces the subgroup of isometries. In flat space, this is the Poincaré group.

For clarity, let's restrict to a Euclidean theory in D dimensions with metric $g = \delta$. Consider an infinitesimal transformation $x \rightarrow x + \epsilon$. To first order, the requirement that the transformation only rescales the metric reads

$$\delta g_{\mu\nu}(x) = -(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \stackrel{!}{=} -f(x) g_{\mu\nu}(x) \tag{3.3}$$

Taking a trace implies $f(x) = 2/D\, \partial_\mu \epsilon^\mu$. This requirement may be reformulated as

$$(\eta_{\mu\nu}\, \partial_\rho \partial^\rho + (D-2)\partial_\mu \partial_\nu)\, (\partial_\sigma \epsilon^\sigma) = 0 \tag{3.4}$$

For $D > 2$, it can be shown from the above equation that all third or higher order derivatives of ϵ vanish, which leaves the following possibilities:

- translations: $\epsilon^\mu = a^\mu$
- rotations: $\epsilon^\mu = \omega^\mu{}_\nu x^\nu$ with $\omega_{\mu\nu} = -\omega_{\nu\mu}$
- scale transformations: $\epsilon^\mu = \lambda\, x^\mu$
- special conformal transformations: $\epsilon^\mu = b^\mu\, x \cdot x - 2x^\mu (b \cdot x)$

It may be verified that coordinate transformations with such ϵ do indeed transform the metric by a factor. The above possibilities add to a total of $(D+2)(D+1)/2$ parameters for a, b, ω and λ. The algebra underlying conformal transformations is isomorphic to $SO(D+1, 1)$.

In $D = 2$ dimensions, which will be of primary interest as the string theory worldsheet is two dimensional, the $(D-2)$ term in equation (3.4) drops out and ϵ may well contain higher order derivatives. In fact, for a Euclidean two dimensional theory, the restrictions on ϵ reduce to

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2 \qquad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1 \tag{3.5}$$

Which is far less restrictive then those given above for the more general case $D > 2$. In fact, there exists an infinite set of transformation parameters ϵ that obey the restriction (3.5). This is particularly clear by introducing complex coordinates $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$. In these coordinates:

$$\partial_z \bar{\epsilon} = 0 \qquad \partial_{\bar{z}} \epsilon = 0 \tag{3.6}$$

So, any transformation of the form $z \rightarrow f(z)$, $\bar{z} \rightarrow \bar{f}(\bar{z})$ is conformal. The functions $f(z)$ and $\bar{f}(\bar{z})$ are understood to be analytic functions depending only on z or $\bar{z}$, respectively. Note that the coordinates z and $\bar{z}$ (or new coordinates $f(z)$ and $\bar{f}(\bar{z})$) are independent but obey the reality condition $z^* = \bar{z}$ to keep two physical independent coordinates.

3.2 The Virasoro algebra

The previous section defined conformal symmetry and introduced complex coordinates. As these coordinates are very convenient, they will be adopted in the following section which is a deeper investigation of conformal symmetry.

The conformal symmetry transformations on z and $\bar{z}$ decouple (up to a reality condition), so it will suffice to consider the z dependence only. The transformations on $\bar{z}$ work out in exactly the same way. Introduce a basis for the transformation parameter ϵ:

$$\epsilon(z) = -\sum_n \epsilon_n z^{n+1} \tag{3.7}$$

The allowed values for n depend on the background under consideration; on the complex plane, regularity at the origin implies $n \geq -1$. For the Riemann sphere, which is the complex plane plus infinity, it can be shown that the only transformations that are valid globally are l_{-1}, l_0 and l_1. In this basis, the generators of conformal symmetry are

$$l_n = -z^{n+1}\partial_z \tag{3.8}$$

That is, a transformation $z \to z - \epsilon_n z^{n+1}$ is generated by l_n. As can easily be verified, these generators satisfy a *Witt* algebra:

$$[l_n, l_m] \quad = \quad (m-n)\, l_{m+n} \tag{3.9}$$

Note that not all of these transformations are well-defined (invertible, single-valued) globally. As mentioned above, on the Riemann sphere only transformations with $n = -1, 0, 1$ are valid. These generators form a subalgebra of the algebra (3.9) that generates the *restricted* conformal group. This group transforms a z coordinate as (see [25], section 5.1.2)

$$z \to \frac{az+b}{cz+d} \qquad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc = 1 \tag{3.10}$$

The parameters a, b, c, d may be regarded as elements of some matrix in $SL(2,\mathbb{C})$, where two matrices that differ only by a minus sign correspond to the same transformation. Hence, the restricted conformal group is isomorphic to $SL(2,\mathbb{C})/\mathbb{Z}_2$.

Completely analogous statements can be made for the $\bar{z}$ sector. The generators are conventionally denoted as $\bar{l}_n$ and commute with the set of generators l_n.

The discussion above is valid for a classical system with conformal symmetry, where the generators l_n act on functions that live on the complex plane. In a quantum field theory, physical states are elements of a Hilbert space, transforming as a representation of the symmetry group. Due to quantum effects, the algebra of these quantum generators may have a central extension term of the form (for a calculation, see Green et.al. [41], section 2.2.2):

$$[L_n, L_m] \quad = \quad (m-n)\, L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{n+m,0} \tag{3.11}$$

This is known as the *Virasoro* algebra with central charge $c \in \mathbb{R}$. For $c = 0$ the symmetry algebra reduces to the classical Witt algebra and c is also known as the conformal anomaly. There is a copy for the generators $\bar{L}_m$ that may a priori have a different central charge $\bar{c}$. Both algebras are decoupled; L_m and $\bar{L}_n$ commute. Note that this quantum algebra reduces to the classical Witt algebra for arbitrary c for the generators L_{-1}, L_0 and L_1.

Apart from the factor $1/12$ in the definition of the central extension term, which is motivated from string theory, the Virasoro algebra was introduced without reference to any explicit theory – just conformal invariance.

3.3 CFT machinery

The previous covers the word 'conformal' in CFT. This section continues with the 'field theory' aspects. It is a collection of some basic techniques encountered in CFT on the complex plane (introduced in section 4.2), an application that will be useful in string theory. The introduction of these techniques may seem a bit disparate, but the next section will put things in perspective by using these tools to analyze the role of the energy momentum tensor in a generic CFT. It is hoped that this section enables the reader to follow CFT applications in string theory in chapter 4, or provides the necessary references.

Of course, the fields are quantum fields and it is assumed that the reader is familiar with quantum field theory. Both path-integral methods and Hilbert space calculations are generally used in CFT calculations. If the reader is unfamiliar with path-integrals: a comprehensive 'short course on path integrals' can be found in Polchinski [63], appendix A. It also discusses the relation between path integrals and the Hilbert space formalism.

Recall that the Hilbert space of a quantum theory is based on canonical quantization. In a Minkowski spacetime, this clearly distinguishes a time coordinate from the spatial coordinates. The analog on the Euclidean complex plane is radial quantization (this is explained in detail in Di Francesco et.al. [25], chapter 6): the theory is quantized along circles of equal radius on the complex plane. The analog of the usual time ordering of products of operators is radial ordering, which will be denoted by $R[\ldots]$

$$R\left[\phi_1(z_1)\phi_2(z_2)\right] = \begin{cases} \phi_1(z_1)\phi_2(z_2) & \text{for } |z_1| > |z_2| \\ \pm\phi_2(z_2)\phi_1(z_1) & \text{for } |z_1| < |z_2| \end{cases} \tag{3.12}$$

Where $\pm$ is -1 only if both ϕ_1 and ϕ_2 are fermionic operators.

In the following sections, the notion of primary conformal fields will be introduced and tools as Ward identities and the operator product expansion. These tools can be used in the path integral formalism to find correlation functions that, in principle, determine the structure of a particular conformal theory. In addition to these tools, one may use the Hilbert space formalism which will be described in section 3.3.3 and used more fully in section 3.5.

3.3.1 Primary fields

In a conformal field theory, fields are classified as either primary, quasi-primary or secondary. A field $\phi(z,\bar{z})$ is primary if under a conformal transformation $z \to f(z)$,

$\bar{z} \to \bar{f}(\bar{z})$ it transforms in the following way[1]

$$\phi(z,\bar{z}) \to \phi'(f(z),\bar{f}(\bar{z})) = \left(\frac{\partial f}{\partial z}\right)^{-h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z}) \tag{3.13}$$

The parameters $(h, \bar{h})$ are called the conformal dimensions of ϕ. A field that transforms according to this transformation law for restricted conformal transformations only is called quasi-primary and all other fields are secondary. The infinitesimal form of the transformation law (3.13) is, for a transformation $z \to z + \epsilon(z)$, $\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$:

$$\delta_{\epsilon,\bar{\epsilon}}\, \phi(z,\bar{z}) = -\left(h\partial\epsilon + \epsilon\partial + \bar{h}\bar{\partial}\bar{\epsilon} + \bar{\epsilon}\bar{\partial}\right) \phi(z,\bar{z}) \tag{3.14}$$

A lot of important fields will depend either on z or $\bar{z}$ but not on both. They are respectively called holomorphic and antiholomorphic and can be expanded in a Laurent series. For a (quasi-)primary field $\phi(z)$ of weight $(h,0)$ the usual definition is

$$\phi(z) = \sum_{n\in\mathbb{Z}} z^{-n-h}\phi_n \qquad \Leftrightarrow \qquad \phi_n = \oint_{C(O)} \frac{dz}{2\pi i} z^{n+h-1}\phi(z) \tag{3.15}$$

Where $C(O)$ denotes a contour integral around the origin. The equivalence of these expressions follows directly from Cauchy's formula. The inclusion of the conformal weight h in these definitions will be justified in section 3.3.3.
Expectation values of scattering processes are usually formulated in terms of an S-matrix. The amplitude of a particular event is then expressed in a correlation function. Consider the unnormalized two-point function

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle = \int [d\phi]\, \phi_1(z_1)\phi_2(z_2)e^{-S[\phi]} \tag{3.16}$$

For $|z_2| > |z_1|$, this corresponds to the amplitude for creating a state associated with the field ϕ_1 at spacetime coordinates z_1 and annihilating a ϕ_2 state at z_2. The physics of the system is encoded in the action $S[\phi]$. Under a symmetry transformation, the action and the integration measure are invariant, so the correlation function transforms as the fields, i.e. covariantly. This condition imposes constraints on the form of correlation functions, known as *Ward identities.* In CFT, it can be shown that both the two- and three-point function are specified up to normalization. The exact form of four-point functions has some freedom that depends on the particular CFT. See [25], sections 2.4.4, 4.3.1 and 4.3.2.

3.3.2 Operator product expansion

When considering products of fields, problems may arise if two or more fields are evaluated at the same point in spacetime, a general problem in quantum theories. Correlation functions are for instance known to give singularities if the positions of two fields coincide. In general, it is possible to express the (radially ordered) product of two fields at positions z_1 and z_2 in terms of other fields. This yields a power series in the distance $(z_1 - z_2)$ from which the singular behavior may be read off. A general formula for some basis of local fields ϕ_k is

$$R[\phi_i(z_1,\bar{z}_1)\phi_j(z_2,\bar{z}_2)\,] = \sum_k c_{ijk}(z_1 - z_2)^{d_{ijk}}(\bar{z}_1 - \bar{z}_2)^{\bar{d}_{ijk}}\phi_k(z_2,\bar{z}_2) \tag{3.17}$$

[1]Following conventions used in Polchinski [63] and Di Francesco et.al. [25]. Another convention that is often used results in a plus sign in the infinitesimal form, as opposed to the minus sign in equation (3.14). See e.g. Kiritsis [51], section 6.2. The interpretation of h and $\bar{h}$ remains unchanged.

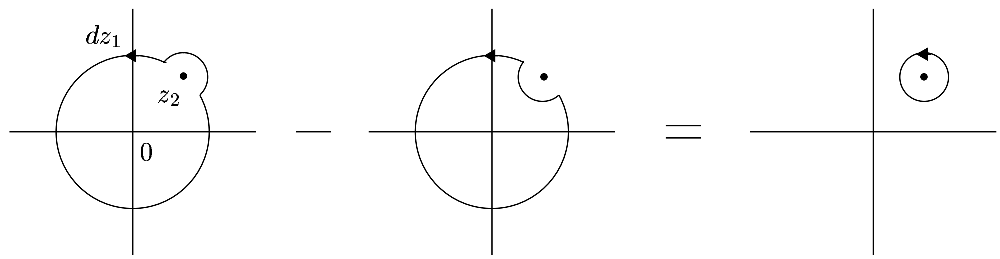

Figure 3.1: Change of contour

This is of course a formal expansion; the coefficients c_{ijk}, d_{ijk} and $\bar{d}_{ijk}$ depend on the particular fields and theory. These expansions are often considered for (anti-)holomorphic fields inside a contour integral, in which case only the singular part is non-zero. Therefore, the regular part is often omitted in such an operator product expansion (OPE).

Now consider a mode expansion such as in equation (3.15). A very useful trick relates the OPE of two fields to the algebra of the modes:

$$\begin{aligned} [\phi_n, \phi'_m] &= \oint_{C(0)} \frac{dz_1}{2\pi i} \oint_{C(0)} \frac{dz_2}{2\pi i} z_1^{n+h-1} z_2^{m+h'-1} [\phi(z_1), \phi'(z_2)] \\ &= \oint_{C(0)} \frac{dz_2}{2\pi i} z_2^{m+h'-1} \oint_{C(z_2)} \frac{dz_1}{2\pi i} z_1^{n+h-1} R\left[\phi(z_1)\phi'(z_2)\right] \end{aligned} \tag{3.18}$$

Which follows by writing out the commutator and changing the contour for the z_1 integration as depicted in figure 3.1. A more detailed description can be found in [25], section 6.1.2.

3.3.3 Hilbert space techniques

In CFT on the complex plane, there is an isomorphism between states in the Hilbert space of the theory and local operators: the operator $\sim$ state correspondence. The mapping is: a path integral on the unit disk with an operator at the origin and a boundary condition at $|z| = 1$ produces a state of the Hilbert space in the Schrödinger representation (see [63], the whole of section 2.8).[2] In particular, the path integral with no field at the origin maps to the ground state (see the argument in [65], section 1.5). I will not describe the isomorphism in more detail, but restrict to the following statement: assuming the existence of a ground state $|0\rangle$, one may associate a state in the Hilbert space to any local operator $\phi(z, \bar{z})$ as

$$|\phi\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle \tag{3.19}$$

Which can be thought of as an asymptotic 'in' state in the complex plane, a point of view expressed by Di Francesco et.al. [25] in section 6.1. Consider the state associated with a conformal field $\phi(z)$ of weight h, using the mode expansion (3.15):

$$|\phi\rangle = \lim_{z,\bar{z}\to 0} \sum_{n\in\mathbb{Z}} z^{-n-h} \phi_n |0\rangle \tag{3.20}$$

[2]From the perspective of string theory, there is a more intuitive way to view this: a path integral on the cylinder with a specified 'in' state at $t \to -\infty$ can be mapped to a path-integral on the complex plane with an operator at $z = 0$. The operator $\sim$ state correspondence assures the equivalence of both perspectives. See [63], section 2.8.

Regularity at $z = 0$ for all modes implies that one should demand

$$\phi_n|0\rangle = 0 \qquad \text{for } n > -h \tag{3.21}$$

So the modes of a field $\phi(z)$ with conformal weight h are annihilation operators for $n > -h$. Of course, the other modes are creators.

Analogous to the 'in' state, one may define a asymptotic 'out' state. Demanding $\langle\phi|_{out} = |\phi\rangle_{in}^{\dagger}$ implies that the Hermitean conjugate of a field ϕ should be defined as (see [25], 6.1.1)

$$\phi(z, \bar{z})^{\dagger} = \bar{z}^{-2h} z^{-2\bar{h}} \phi(\bar{z}^{-1}, z^{-1}) \tag{3.22}$$

Comparing this with a straightforward conjugation of the mode expansion (3.15) implies that the modes of a holomorphic field satisfy

$$\phi_n^{\dagger} = \phi_{-n} \tag{3.23}$$

Which justifies the inclusion of the conformal weight h in the definition of the mode expansion.

The Hilbert space formalism will be used more fully in section 3.5 to find the states in the Hilbert space of a general CFT.

3.3.4 Wick's theorem

Normal ordering was introduced in the beginning of this section as the complex plane analog of time ordering. In field theory, another ordering procedure for quantum fields is creation-annihilation normal ordering (denoted as $: \ldots :$). It prescribes to put all annihilation modes to the right of creation modes in products of fields. The difference of a radial ordered product and a normal ordered product of two fields is defined as the contraction (underbraced), which is a c-number:

$$R\left[\phi_1\phi_2\right] =: \phi_1\phi_2 : + \underbrace{\phi_1\phi_2} \tag{3.24}$$

The definition of a normal ordered product implies that its vacuum expectation value vanishes and hence

$$\underbrace{\phi_1\phi_2} = \langle R\left[\phi_1\phi_2\right]\rangle \tag{3.25}$$

Wick's theorem now states that, for free fields, the radial ordered product of multiple fields is equal to its normal ordered product plus all possible ways of contracting two fields. For the product of three fields, this reads:

$$R\left[\phi_1\phi_2\phi_3\right] =: \phi_1\phi_2\phi_3 : + : \underbrace{\phi_1\phi_2}\phi_3 : + : \phi_1\underbrace{\phi_2\phi_3} : + : \underbrace{\phi_1\phi_2\phi_3} : \tag{3.26}$$

For four or more operators, multiple contractions should be considered. Wick's theorem is discussed by Di Francesco et.al. [25]: a general introduction for free fields in section 2.3.5, and a generalized version for interacting fields in appendix 6.B. See also the related discussion on OPEs and ordering procedures in section 6.5.

3.4 The energy-momentum tensor

The previous section introduced some general techniques that are commonly used in conformal field theories. In the current section, this machinery will be put to use to discuss some general features of a CFT.

3.4.1 The TT OPE

In quantum field theory, symmetry, currents and transformation rules for fields are intimately connected. Noether's theorem states that to every continuous spacetime symmetry corresponds some conserved charge. The space integral of this current yields a conserved charge which on its turn generates the original symmetry transformations.

In an arbitrary number of dimensions, the energy momentum tensor T of a system relates the conserved current to a particular spacetime symmetry transformation:

$$j_\mu(x) = T_{\mu\nu}(x)\epsilon^\nu \tag{3.27}$$

Consider a rigid translation which corresponds to choosing $\epsilon^\mu = a^\mu$ with a constant. Being a current (corresponding to a conserved charge), j is divergenceless:

$$\begin{aligned} \nabla^\mu j_\mu(x) &= 0 \\ &\Rightarrow \; a^\nu\, \nabla^\mu T_{\mu\nu}(x) = 0 \end{aligned} \tag{3.28}$$

Which is the familiar statement that energy and momentum are conserved. Conformal symmetry means that the system is also invariant under a scale transformation, with $\epsilon^\mu = \lambda x^\mu$.

$$\begin{aligned} \nabla^\mu j_\mu(x) &= 0 \\ &\Rightarrow \; \lambda\, \nabla^\mu (T_{\mu\nu}(x)\, x^\nu) = 0 \\ &\Rightarrow \; T^{\mu\nu}(x)\nabla_\mu x_\nu = 0 \end{aligned} \tag{3.29}$$

In the last line, it was used that T is divergenceless. Now, in flat space the covariant derivative reduces to an ordinary partial derivative. Using the fact that T is symmetric in its indices and the identity $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \partial_\rho \epsilon^\rho g_{\mu\nu}$ (cf. equation (3.3)) implies

$$T^\mu{}_\mu = 0 \tag{3.30}$$

Which states that the energy-momentum tensor is traceless classically – a very general feature for conformal field theories. In complex coordinates, the divergence- and tracelessness of the energy momentum tensor imply that it has only two non-zero entries: $T_{zz}(z)$ and $T_{\bar{z}\bar{z}}(\bar{z})$. As these are (anti-)holomorphic functions, they will be denoted

$$T(z) := T_{zz}(z) \qquad \bar{T}(\bar{z}) := T_{\bar{z}\bar{z}}(\bar{z}) \tag{3.31}$$

The conserved current j_μ splits into $j(z) = T(z)\epsilon(z)$ and $\bar{j}(\bar{z}) = \bar{T}(\bar{z})\bar{\epsilon}(z)$

In section 3.2, the generators of conformal transformations were found to be the L_m modes that satisfy the Virasoro algebra (3.11). The conserved current corresponding to these transformations is now expressed in terms of the energy momentum tensor T, which suggests a strong connection. In the following, it will be shown that the modes L_n can be identified as the modes of T, provided that the operator product expansion of T with itself has a form specific to a CFT.

Expand the holomorphic part $T(z)$ of the energy-momentum tensor as follows (the suggestive 2 will be justified farther on but is no loss in generality)

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \qquad \Leftrightarrow \qquad L_n = \oint_{C(0)} \frac{dz}{2\pi i} z^{n+1} T(z) \tag{3.32}$$

The commutator of two modes may be expressed as in equation (3.18):

$$[L_n, L_m] = \oint_{C(0)} \frac{dz_2}{2\pi i} z_2^{m+1} \oint_{C(z_2)} \frac{dz_1}{2\pi i} z_1^{n+1} R\left[T(z_1)T(z_2)\right] \tag{3.33}$$

It can be shown that the righthandside of this equation is equal to the righthandside of the Virasoro algebra (3.11) iff

$$R\left[T(z_1)T(z_2)\right] = \frac{c/2}{(z_1 - z_2)^4} + \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial_2 T(z_2)}{z_1 - z_2} + (\text{regular terms}) \tag{3.34}$$

Where c is again the central charge. This equation should be viewed as the counterpart of the Virasoro algebra and is a very general result for conformal field theories.

3.4.2 Primary fields revisited

In the previous section, the modes of the energy-momentum tensor were identified as the generators of conformal symmetry. Transformations on fields are generated by the associated charge, which is the space integral of the current. In the complex plane, this reads

$$Q_{\epsilon,\bar{\epsilon}} = \oint_{C(0)} \frac{dz}{2\pi i} T(z)\epsilon(z) + \oint_{C(0)} \frac{d\bar{z}}{2\pi i} \bar{T}(\bar{z})\bar{\epsilon}(\bar{z}) \tag{3.35}$$

from which the transformation of fields may be expressed as

$$\begin{aligned} \delta_{\epsilon,\bar{\epsilon}}\, \phi(z_2, \bar{z}_2) &= [Q_{\epsilon,\bar{\epsilon}}, \phi(z_2)] \\ &= \oint_{C(0)} \frac{dz}{2\pi i} \left[T(z)\epsilon(z), \phi(z_2, \bar{z}_2)\right] + (\text{anti-holomorphic}) \\ &= \oint_{C(z_2)} \frac{dz}{2\pi i} \epsilon(z) R\left[T(z)\phi(z_2, \bar{z}_2)\right] + \ldots \end{aligned} \tag{3.36}$$

If $\phi(z)$ is a primary field with conformal dimension $(h, 0)$, then its transformation properties are given in equation (3.14). Matching these expressions implies that the OPE of a primary field with the energy momentum tensor takes the form

$$R\left[T(z_1)\phi(z_2)\right] = \frac{h\phi(z_2)}{(z_1 - z_2)^2} + \frac{\partial_2 \phi(z_2)}{z_1 - z_2} + (\text{regular terms}) \tag{3.37}$$

This OPE may be taken as an alternative definition of a primary field. The antiholomorphic copy is defined in complete analogy.

Comparing the TT OPE (3.34) with this result, it follows that the energy-momentum tensor does not consist of primary fields. Rather, $T(z)$ and $\bar{T}(\bar{z})$ are quasi-primary fields of dimension $(2, 0)$ and $(0, 2)$, respectively. Under an infinitesimal conformal transformation $z \to z + \epsilon$, the holomorphic part $T(z)$ transforms as

$$\delta_\epsilon\, T(z) = -\left(2\partial\epsilon + \epsilon\partial\right) T(z) - \frac{c}{12}\partial^3\epsilon \tag{3.38}$$

Where c is again the central charge. The anti-holomorphic part $\bar{T}(\bar{z})$ transforms in a similar way.

3.5 The Hilbert space

The operator $\sim$ state correspondence assigns a state in the Hilbert space to every field. In particular, there is a state that corresponds to the holomorphic part $T(z)$ of the energy momentum tensor:

$$\lim_{z,\bar{z}\to 0} T(z)|0\rangle = \lim_{z,\bar{z}\to 0} \sum_{n\in\mathbb{Z}} z^{-n-2} L_n|0\rangle \tag{3.39}$$

This state is only well defined if $L_n|0\rangle = 0$ for $n \geq -1$, which in particular implies that the vacuum $|0\rangle$ is invariant under restricted conformal transformations. The state corresponding to $T(z)$ is then $L_{-2}|0\rangle$ and more generally,

$$\partial^n T \sim L_{-2-n}|0\rangle \tag{3.40}$$

Let's denote by $|h\rangle$ the state corresponding to a primary field $\phi(z)$ with conformal dimension $(h, 0)$:

$$|h\rangle \quad := \quad \lim_{z,\bar{z}\to 0} \phi(z)|0\rangle \tag{3.41}$$

From equations (3.32) and (3.37), it is easy to verify that

$$[L_n, \phi(z)] = h(n+1)z^n\phi(z) + z^{n+1}\partial\phi(z) \tag{3.42}$$

Which has direct implications in the Hilbert space:

$$\begin{aligned} L_0\,|h\rangle &= h\,|h\rangle \\ L_n\,|h\rangle &= 0 \qquad \text{for } n > 0 \end{aligned} \tag{3.43}$$

And the commutation relation $[L_0, L_{-n}] = nL_{-n}$ implies that $L_{-n}|h\rangle$ with $n > 0$ is an eigenvalue of L_0 with eigenvalue $h + n$. Applying a combination of negative Virasoro generators leads to a tower of states, each of which has a different eigenvalue under L_0.

This tower of states is known as a *Verma module*. The state $|h\rangle$ is called the heighest weight (HW) state and is in direct correspondence with a primary field. The other states are known as descendants. The L_0 operator takes a prominent role in the construction; this is because it generates the Cartan subalgebra of the Virasoro algebra. In physical theories, it is usually associated with the Hamiltonian.

A general Verma module may contain states of zero or negative norm, where the latter would destroy unitarity. An ongoing line of research called the *conformal bootstrap* is to classify all values for the central charge c and HW parameter h that give rise to a unitary, consistent representation of the Virasoro algebra. For instance, it can be shown that if $0 < c < 1$, only rational values are allowed

$$c = 1 - \frac{6}{n(n+1)} \qquad n = 3, 4, \ldots \tag{3.44}$$

With non-trivial restrictions on h. See e.g. [25], chapter 7.

4

The bosonic string

This chapter introduces bosonic string theory. I have chosen to highlight a number of topics that are especially relevant for string cosmology, thereby paying less attention to other ones. For example, the Polyakov action is described in some detail not only because of its importance in string theory but also because a generalized form will be the starting point of chapter 5. The particle spectrum of (closed, oriented) bosonic string theory is also discussed in detail as these particles will be the building blocks of cosmological models. Interactions in string theory and the Polyakov path integral are only discussed briefly; these topics will be readdressed indirectly in the rest of this thesis.

Sections 4.1 and 4.2 give a short introduction to string theory and the role of the complex plane. They provide a background for a discussion on the Polyakov action, which I present as the defining action of string theory, in section 4.3. Section 4.4 analyzes string theory on the complex plane, thereby using the CFT machinery that was introduced in the previous chapter. The particle spectrum of bosonic string theory is derived in section 4.5 using a procedure known as old covariant quantization. Section 4.6 concludes the chapter with a discussion on interactions, the Polyakov path integral and the Weyl anomaly.

The reader will find a thorough introduction to the subject in the well-known introduction [63, 64] by Polchinski. The lecture notes [65] by the same author form a nice alternative as some topics are addressed from a different point of view. This chapter is largely based on these works, though I have also benefited from the introduction to superstrings by Kiritsis [51], the lectures on string theory by Lüst and Theisen [54] and from the standard text by Green, Schwarz and Witten [41, 42].

4.1 Strings

String theory is about strings: one-dimensional objects that propagate through spacetime. Besides propagation the string can also oscillate. These different oscillation modes are interpreted as different elementary particles, each of which is in possession of its own set of quantum numbers. This is, in a nutshell, the unification of particles and forces that string theory aims to achieve.

A particular string can either be closed or open. In some specific theories both kinds coexist, while other theories consider only closed or open specimen. In this chapter I focus on closed strings as they are most relevant to cosmological scenarios. Most of the analysis can be applied directly to open strings, apart from some conven-

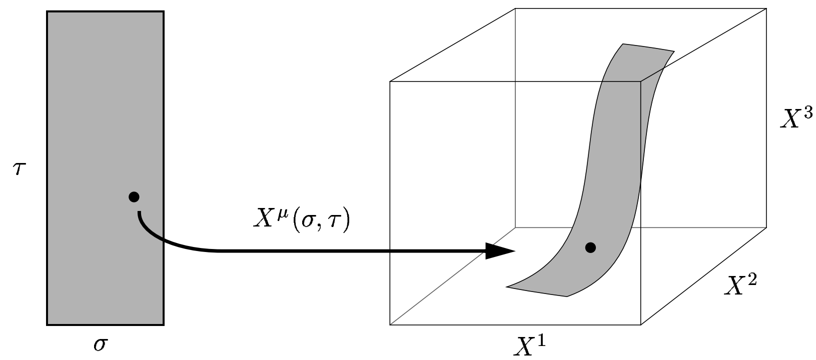

Figure 4.1: The string worldsheet embedded in a target space. X maps the worldsheet to the target space

tional issues. I will consider oriented strings only; it is possible to construct theories that restrict to unoriented strings, which have a definite behavior under worldsheet parity transformations.

As the string propagates in spacetime, it sweeps out a surface called the *worldsheet*. Coordinates on this worldsheet are taken to be $\sigma = \sigma^1$ and $\tau = \sigma^2$. The σ coordinate is interpreted as a space coordinate, while τ is a time coordinate – a distinction that is somewhat arbitrary for Euclidean worldsheets. For open strings, one usually scales $\sigma \in [0, \pi]$ while for closed strings the conventional range is $\sigma \in [0, 2\pi]$ and the endpoints are identified. Throughout this chapter, indices α, β, ... (running from 1 to 2) always label worldsheet coordinates, so derivatives ∂_α are derivatives with respect to worldsheet coordinates.

On this worldsheet there are fields, such as the bosonic field $X(\sigma, \tau)$ that carries an index μ, ν, ... (taking values in $0 \ldots D$). From the worldsheet point of view, the index enumerates a collection of D bosonic fields. From the spacetime or *target space* point of view, $X(\sigma, \tau)$ is a coordinate carrying a Lorentz (vector) index. It gives the position in spacetime of a point (σ, τ) on the worldsheet. For now, the target space is taken to have Minkowski metric η. In string cosmology, this will be relaxed which leads to interesting generalizations of the string theory described hereafter. The distinction between worldsheet and target space can hardly be overemphasized, so have a close look at figure 4.1.

It is possible to include fermionic fields on the worldsheet, which leads to supersymmetric string theories. I have chosen not to discuss these theories, though supersymmetric effective actions will appear in section 5.3.1. The interested reader is referred to any of the standard texts mentioned in the beginning of this chapter.

4.2 The complex plane in string theory

Before starting with the dynamics of classical strings or analyzing its quantum version, it is important to discuss the string worldsheet in some detail.

Following Polchinksi [63], I will consider a Euclidean worldsheet with coordinates σ and τ. For flat metrics, the extension to a Minkowski signature worldsheet is straightforward by analytic continuation $\tau \to i\tau$. If one considers topologically non-trivial (or time-dependent) worldsheets this cannot be done in general, but such backgrounds will not be considered in this chapter. One might object that this choice is inconsistent with the Minkowskian nature of target space; where this is case, analytic continuation on the worldsheet is understood.

As string theory is formulated in terms of a CFT on the worldsheet, complex coordinates are particularly well suited. Introduce coordinates $w = \sigma + i\tau$ and $\bar{w} = \sigma - i\tau$. Derivatives become

$$\partial_w = \frac{1}{2}(\partial_\sigma + i\partial_\tau) \qquad \partial_{\bar{w}} = \frac{1}{2}(\partial_\sigma - i\partial_\tau) \tag{4.1}$$

And the Jacobian of this transformation implies $d^2z = 2d\sigma d\tau$. This is a general coordinate transformation, so vectors and tensors are understood to transform accordingly. In particular, the metric on the Euclidean strip δ transforms to a metric $\hat{h}$:

$$\delta_{\alpha\beta} = \delta^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \to \quad \hat{h}_{\alpha\beta} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \; \hat{h}^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{4.2}$$

A conformal coordinate transformation $w \to z$ maps to the complex plane

$$z = \exp(-iw) \qquad \bar{z} = \exp(i\bar{w}) \tag{4.3}$$

As this is a conformal transformation, the metric transforms only by an irrelevant factor. This coordinate system will prove to be very useful and it will be used almost exclusively in worldsheet calculations. On the complex plane, time (τ) runs radially outward and space (σ) runs in the angular direction. The Levi-Civita tensor ϵ takes the form[1] $\epsilon_{z\bar{z}} = -\epsilon_{\bar{z}z} = i/2$ and $\epsilon^{z\bar{z}} = -\epsilon^{\bar{z}z} = -2i$.

To shorten notation, define $\partial := \partial_z$ and $\bar{\partial} := \partial_{\bar{z}}$. For vectors v in these coordinates, the following shorthand notation is often used: $v := v^z$; $\bar{v} := v^{\bar{z}}$. Finally, I sometimes use the shorthand $z_{12} := z_1 - z_2$.

4.3 The Polyakov action

Now that the background is set out, the dynamics of the classical string will be formulated in terms of an action principle.

4.3.1 An action for classical strings

A general starting point for many introductions to string theory is the action for a classical relativistic point particle. I will not introduce it here, but restrict to the following statement: this action governs the dynamics of a point particle by requiring that its geodesic trajectory in spacetime is minimal.

The dynamics of a classical string now follow from the requirement that the area of the worldsheet is minimal. This statement may be formulated in terms of the *Nambu-Goto* action:

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_A d\sigma d\tau \det\left[\partial_\alpha X^\mu \partial_\beta X_\mu\right]^{1/2} \tag{4.4}$$

[1] Sometimes the antisymmetric tensor is understood to transform as a current, which implies the normalization $\epsilon^{z\bar{z}} = 1$ for the complex plane. The normalization I adopt follows if one lets ϵ transform as a tensor; this complies with Polchinski [63], 3.6. and Di Francesco [25], 5.1.1.

Here A denotes the worldsheet area, and α'^2 is a constant known for historical reasons as the *Regge slope*. It is related to the string tension T as

$$T = \frac{1}{2\pi\alpha'} \tag{4.5}$$

T has dimension energy per length, which is equivalent to energy-squared. Note the Lorentz indices on the X fields; for the time being they just 'go along for the ride', but it should be understood that in general the contraction involves the target space metric η.

The determinant makes the Nambu-Goto action rather difficult to work with, especially for quantization purposes. A useful procedure, which has its analog in the point particle case, is to introduce an auxiliary field h. This leads to the *Polyakov* action

$$S_P = -\frac{1}{4\pi\alpha'}\int_A d\sigma d\tau\, \sqrt{h} h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \tag{4.6}$$

Where $\sqrt{h}$ is shorthand for det $[h_{\alpha\beta}]^{1/2}$. The h field is not dynamical as no derivatives appear in the action. It plays the role of a worldsheet metric and varying the action with respect to h defines the energy momentum tensor (normalized to string theory conventions as in [63], chapter 1):

$$T_{\alpha\beta} = -\frac{1}{\alpha'}\left(\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu\right) \tag{4.7}$$

Contracting both sides of this equation with $h^{\alpha\beta}$ implies that T is traceless (in two dimensions), an important result as may be appreciated from section 3.4. The classical Euler-Lagrange condition is just $T_{\alpha\beta} = 0$, which yields

$$h_{\alpha\beta} \propto \partial_\alpha X^\mu \partial_\beta X_\mu \tag{4.8}$$

Inserting this into the Polyakov action (4.6) leads to the Nambu-Goto action (4.4), which shows the classical equivalence of both actions. Whether this equivalence remains in the quantum theory is in general unclear. I will proceed with the Polyakov action as the defining action for string theory.

4.3.2 Symmetries of the Polyakov action

One may view the Polyakov action as describing the dynamics of a collection of D free scalar fields in a two-dimensional space with metric h. The scalar fields parametrize a D dimensional target space, which is assumed to be Minkowski space in D dimensions. The action has the following symmetries, where I have emphasized the distinction between worldsheet and spacetime symmetries:

- spacetime Poincaré invariance:

$$\begin{aligned} \delta X^\mu &= \omega^\mu{}_\nu X^\nu + a^\mu \qquad (\omega_{\mu\nu} = -\omega_{\nu\mu}) \\ \delta h_{\alpha\beta} &= 0 \end{aligned} \tag{4.9}$$

- worldsheet reparametrization (diffeomorphism) invariance; consider a coordinate transformation $\delta\sigma^a = -\epsilon^a$:

$$\begin{aligned} \delta X^\mu &= \epsilon^\alpha \partial_\alpha X^\mu \\ \delta h_{\alpha\beta} &= \epsilon^\gamma\partial_\gamma h_{\alpha\beta} + \partial_\alpha\epsilon^\gamma h_{\beta\gamma} + \partial_\beta\epsilon^\gamma h_{\alpha\gamma} = \nabla_\alpha\epsilon_\beta + \nabla_\beta\epsilon_\alpha \end{aligned} \tag{4.10}$$

[2] For perspective: $\alpha \sim 1\text{GeV}^{-2}$ [51].

- worldsheet Weyl rescaling invariance:

$$\begin{aligned} \delta X^\mu &= 0 \\ \delta h_{\alpha\beta} &= 2\omega h_{\alpha\beta} \end{aligned} \tag{4.11}$$

The diffeomorphism (diff) invariance corresponds to the divergencelessness of the energy momentum tensor $\nabla_\alpha T^{\alpha\beta} = 0$. The rescaling invariance has no analog in point particle theories and depends crucially on the fact that the string worldsheet is two-dimensional.

Conformal symmetry is a subgroup of the diff $\times$ Weyl symmetry. The action is invariant under a conformal coordinate transformation by its diff invariance. This transformation rescales the metric by an overall factor, which may be undone by a Weyl rescaling. Combining these, the Polyakov action is invariant under the transformation

$$\begin{aligned} \delta X^\mu &= \epsilon^\alpha \partial_\alpha X^\mu \\ \delta h_{\alpha\beta} &= 0 \end{aligned} \tag{4.12}$$

Provided that ϵ denotes a conformal transformation. This symmetry is much smaller than diff $\times$ Weyl but it holds if h is gauge-fixed, which will be done in the next section.

4.4 String theory and CFT

Given the classical string theory action, the quantum theory may be formulated by canonical quantization in a particular coordinate frame. This procedure is described in many introductory textbooks, see e.g. chapter 4 of Kiritsis [51] or chapter 1 of Polchinski [63]. Though insightful, it serves as a first step only as more advanced topics are usually addressed by CFT techniques. In the following, I have chosen a shortcut by identifying bosonic string theory on the complex plane as a conformal field theory, which directly provides a perspective and allows to use the standard CFT machinery as described in the previous chapter.

The rescaling and reparametrization invariance of the Polyakov action can be used to put the metric h in any particular form. A convenient choice is the conformal gauge introduced in the beginning of this chapter: $h = \hat{h}$. This choice of metric implies that the worldsheet is Ricci-flat. As will be seen in section 4.6.3, complications arise when the worldsheet is curved. Be aware that the gauge-fixing by itself is not trivial; the correct way to slice the diff $\times$ Weyl invariance is by coupling a ghost system to the gauge-fixed action for the X field – a procedure that can be found in any introduction on string theory; see e.g. Polchinski [63], section 3.3.

This section is concerned with the Polyakov action in gauge-fixed form of the Polyakov action. This is an action for the bosonic fields X which will be referred to as the matter action. Keep in mind that in the full string theory, there is also a ghost action.

4.4.1 Dynamics of the X-field

The gauge fixing reduces the diff $\times$ Weyl symmetry to conformal symmetry. The metric is completely specified, but be aware that the constraint $T_{\alpha\beta} = 0$ remains valid in its gauge-fixed form. In conformal coordinates, the Polyakov action takes the from

$$S_P = \frac{1}{2\pi\alpha'} \int d^2z\, \partial X^\mu \bar{\partial} X_\mu \tag{4.13}$$

And the equation of motion reads

$$\partial\bar{\partial} X^\mu(z,\bar{z}) = 0 \tag{4.14}$$

Which means that X can be written as a sum of a holomorphic and antiholomorphic function: $X^\mu(z,\bar{z}) = X^\mu_L(z) + X^\mu_R(\bar{z})$. Using the fact that the path integral of a total derivative is zero, the classical equation of motion translates to an operator statement

$$\begin{aligned} 0 &= \int [dX] \frac{\delta}{\delta X_\mu(z,\bar{z})} \exp(-S_P[X]) \ldots \\ &\Rightarrow \langle \partial\bar{\partial} X^\mu(z,\bar{z}) \ldots \rangle = 0 \end{aligned} \tag{4.15}$$

Where the ... denote arbitrary insertions in the path integral, assumed not to be located at the point z. They can be thought of as preparing initial and final states. In the following, an equality such as (4.15) will just be denoted as $\partial\bar{\partial} X^\mu(z,\bar{z}) = 0$, where it is understood that it holds an an operator statement. The next step is to consider the situation where there is another field present that might be coinciding:

$$\begin{aligned} 0 &= \int [dX] \frac{\delta}{\delta X_\mu(z_1,\bar{z_1})} \exp(-S_P[X] X^\nu(z_2,\bar{z}_2) \ldots) = 0 \\ &= \int [dX] \exp(-S_P[X]) \left(\eta^{\mu\nu} \delta^2(z_{12},\bar{z}_{12}) + \frac{1}{\pi\alpha'} \partial\bar{\partial} X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2) \ldots \right) \\ &\Rightarrow \partial\bar{\partial}\, X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2) = -\pi\alpha'\, \eta^{\mu\nu} \delta^2(z_{12},\bar{z}_{12}) \end{aligned} \tag{4.16}$$

Which should be interpreted as an operator equation, stating that a products of two operators obey the equation of motion (4.15) provided they are not located at the same point.

Normal ordering may be defined as to get rid of this proviso. To this end, define the normal ordered product of two X fields as

$$:X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2): \;=\; X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2) + \frac{\alpha'}{2} \eta^{\mu\nu} \ln|z_1 - z_2|^2 \tag{4.17}$$

As $\partial\bar{\partial} \ln|z|^2 = 2\pi\delta^2(z,\bar{z})$ it follows that

$$\partial\bar{\partial} \; :X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2): \;=\; 0 \tag{4.18}$$

And the normal-ordered product of two fields is a solution to the equation of motion. Moreover, it is a harmonic function which is locally the sum of a holomorphic and an anti-holomorphic function. These are non-singular in the limit $z \to z'$, so the OPE of X with itself takes the form

$$X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2) \sim -\frac{\alpha'}{2} \eta^{\mu\nu} \ln|z_1 - z_2|^2 \tag{4.19}$$

The normal ordering prescription (4.17) is in fact set up to give OPEs this simple form. The reader might be more familiar with creation-annihilation normal ordering which makes the calculation of matrix elements of an operator between states easy. In free field theory, these definitions are in fact equivalent; see Di Francesco et.al. [25], section 6.5 or Polchinski [63], section 2.7.

The OPE (4.19) is the basis of many more expansions. For example, taking a derivative with respect to z_1 and one with respect to z_2 yields:

$$\partial_1 X^\mu(z_1)\,\partial_2 X^\nu(z_2) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\frac{1}{(z_1-z_2)^2} \tag{4.20}$$

4.4.2 The energy momentum tensor

The form of energy-momentum tensor T in complex coordinates can be derived directly from (4.7) by substituting $h = \hat{h}$. However, as X denotes a quantum field, their might exist an ordering ambiguity in the classical definition of the energy-momentum tensor T. Therefore it is defined to be normal-ordered:

$$T_{zz} = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu: \qquad T_{\bar{z}\bar{z}} = -\frac{1}{\alpha'} :\bar{\partial} X^\mu \bar{\partial} X_\mu: \qquad T_{\bar{z}z} = T_{z\bar{z}} = 0 \tag{4.21}$$

As T_{zz} and $T_{\bar{z}\bar{z}}$ are (anti-)holomorphic functions, it is convenient to define:

$$T(z) = T_{zz}(z) \qquad \bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}(\bar{z}) \tag{4.22}$$

In conformal coordinates, $T_{z\bar{z}} = T_{\bar{z}z} = 0$ is the statement that T is a traceless tensor (equivalent to the statement $T^\alpha{}_\alpha = 0$ in σ and τ coordinates). This directly implies that the theory is conformally invariant, which can of course be verified by transforming $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$ in (4.13), while keeping $X'(z', \bar{z}') = X(z, \bar{z})$.

Using Wick's theorem, the operator product expansion of T with itself can be calculated directly from its definition and the XX OPE (4.19); see e.g. [63], 2.2.

$$T(z_1)T(z_2) \sim \frac{D/2}{(z_1-z_2)^4} + \frac{2T(z_2)}{(z_1-z_2)^2} + \frac{\partial_2 T(z_2)}{z_1-z_2} \tag{4.23}$$

After comparison with (3.34), this qualifies the matter part of bosonic string theory on the complex plane as a conformal field theory with central charge D, the number of spacetime dimensions of the target space. This makes it apparent that the modes of T generate conformal transformations and satisfy a Virasoro algebra. For completeness, these modes L_m are defined as in equation (3.32):

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \qquad \Leftrightarrow \qquad L_n = \oint_{C(0)} \frac{dz}{2\pi i} z^{n+1} T(z) \tag{4.24}$$

The modes L_n are generators of the Virasoro algebra (cf. section 3.4). In particular, recall from equation (3.7) that L_0 generates a transformation $z \to z + \epsilon z$ which is a holomorphic rescaling. It follows that on the complex plane $L_0 + \bar{L}_0$ generate dilatations, which are associated with time translation. The other real combination $i(L_0 - \bar{L}_0)$ generates rotations, associated with space translation.

Be aware that string theory by itself is not a conformal field theory. String theory is a theory of matter fields in a dynamical background, characterized by diff and Weyl rescaling invariance. Fixing the metric implies that one has to introduce a ghost system that affects the physical spectrum. Furthermore, some backgrounds can only be fixed up to moduli that have to be accounted for in a path integral. The discussion above neglects both moduli and the ghost system and concerns only the matter part of string theory on the complex plane.

4.4.3 Primary fields

The bosonic fields X in this theory are not conformal primary fields. However, ∂X and $\bar{\partial} X$ are primary fields of weight $(1,0)$ and $(0,1)$ respectively. This follows directly from the OPE of these fields with the energy momentum tensor:

$$T(z_1)\partial_2 X(z_2) \sim \frac{\partial_2 X(z_2)}{(z_1-z_2)^2} + \frac{\partial_2(\partial_2 X(z_2))}{z_1 - z_2} \tag{4.25}$$

And similar for the $\bar{T}(\bar{z}_1)\bar{\partial} X(z_2)$ OPE. They are not the only primary fields in bosonic string theory. Consider the composite field

$$V_k(z,\bar{z}) =: \exp\left(ik_\rho X^\rho\left(z,\bar{z}\right)\right) : \tag{4.26}$$

Applying Wick's theorem leads to the OPE:

$$T(z_1)V_k(z_2,\bar{z}_2) \quad = \quad \frac{\alpha' k^2}{4}\frac{V_k(z_2,\bar{z}_2)}{(z_1-z_2)^2} + \frac{\partial_2 V_k(z_2,\bar{z}_2)}{z_1 - z_2} \tag{4.27}$$

The $\bar{T}(\bar{z}_1)V_k(z_2,\bar{z}_2)$ is similar, making $V_k(z,\bar{z})$ a primary field of conformal weight $(\alpha' k^2/4, \alpha' k^2/4)$. This operator will play a role as the tachyon vertex operator in section 4.5.3.

4.4.4 Mode expansions

The previous sections analyzed the conformal field theory aspects from the OPE-point of view. An alternative viewpoint is that of the algebra of modes. In this section, the X field will be expanded in modes; subsequently, the XX OPE can be translated into an algebra statement that is the basis for finding the spectrum of the theory.

As seen in section 4.4.1, the field $X(z,\bar{z})$ splits into $X_L(z)$ and $X_R(\bar{z})$. This implies that $\partial X = \partial X_L$ is a holomorphic function, allowing for a Laurent expansion of the form

$$\partial X^\mu(z) = -\frac{i}{4\pi T}\sum_{n=-\infty}^{\infty}\alpha_n^\mu z^{-n-1} \tag{4.28}$$

The expansion for $\bar{\partial} X$ is completely similar, with independent modes $\tilde{\alpha}$. Integrating these expansions yields

$$X^\mu(z,\bar{z}) = x^\mu - i\frac{\alpha'}{2}p^\mu \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\frac{\alpha_n^\mu}{z^n} + \frac{\tilde{\alpha}_n^\mu}{\bar{z}^n}\right) \tag{4.29}$$

Where $p^\mu = \sqrt{2/\alpha'}\,\alpha_0^\mu = \sqrt{2/\alpha'}\,\tilde{\alpha}_0^\mu$ is identified as a physical momentum. This is a general formula for the motion of a string in its background. The first two terms describe linear motion of the string center of mass; the sum describes its oscillations. Transforming the expansion to σ and τ coordinates makes it clear that it is the general solution to the X equation of motion with periodic boundary conditions. The open string has different boundary conditions which leads to a different expansion. In particular, the modes α and $\tilde{\alpha}$ are identified (see e.g. [51] section 3.3). As X should be real, both x and p are real and the oscillation coefficients α_n obey the reality condition

$$\alpha_n{}^* = \alpha_{-n} \tag{4.30}$$

The XX OPE (4.19) and its antiholomorphic counterpart translate to the following commutation relation for the center of mass coordinates x, p and the oscillation modes α and $\tilde{\alpha}$:

$$
\begin{aligned}
&[x^\mu, p^\nu] = i\eta^{\mu\nu} \\
&[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu} \\
&[\alpha^\mu_m, \tilde{\alpha}^\nu_n] = 0
\end{aligned}
\tag{4.31}
$$

Alternatively, these commutation relations (known as a Heisenberg algebra) could have been derived by imposing canonical commutation relations on X as expanded in (4.29), see [51] sections 3.3 and 4.1.

4.5 The spectrum

To derive the spectrum of bosonic string theory I will follow a somewhat intuitive approach known as old covariant quantization. Other procedures as lightcone quantization or BRST quantization are described in [51], chapter 4 or [63], chapter 4.

Note that, except for the factor m (which may be absorbed by redefining a), the commutation relations (4.31) are the same as in the harmonic oscillator scenario. This suggests that string oscillations, corresponding to particle states, can be quantized analogously. This procedure will be described below; in the last section the result is related to the concept of Verma modules that was introduced in section 3.5.

4.5.1 The procedure

The operators with $m > 0$ are identified as lowering operators and will be called annihilators. Those with $m < 0$ are raising operators and will be referred to as creators. First introduce a ground state, the vacuum, that is by definition annihilated by all annihilators. It carries a momentum k:

$$
\begin{aligned}
\alpha^\mu_{m>0}|0,k\rangle &= 0 \\
p^\mu|0,k\rangle &= k^\mu|0,k\rangle
\end{aligned}
\tag{4.32}
$$

Excited states can be constructed by applying one or several raising operators on the vacuum, leading to a vast amount of particle states. Not all of these states are physical, as a physical configuration is required to satisfy the constraint equation $T(z) = \bar{T}(\bar{z}) = 0$. As the explicit form of T in terms of ∂X is known, its modes L_n can be related to the α_m coefficients by comparing terms in the Laurent expansions (4.24) and (4.29). This gives the following relation:

$$L_m = \sum_{n=-\infty}^{\infty} :\alpha^\mu_{m-n}\alpha_{n\mu}: \tag{4.33}$$

The normal ordering applies only to L_0 which introduces some ambiguity in its definition; this will be of importance later but for now just define L_0 to be normal-ordered:

$$L_0 = \sum_{n=-\infty}^{\infty} :\alpha^\mu_{-n}\alpha_{n\mu}: := \frac{{\alpha_0}^2}{2} + \sum_{n=1}^{\infty} \alpha^\mu_{-n}\alpha_{n\mu} \tag{4.34}$$

The antiholomorphic mode $\tilde{L}_0$ is defined analogously. Translational invariance implies $\tilde{L}_0 = L_0$; recall that $i(L_0 - \bar{L}_0)$ generates rotations in the complex plane, corresponding to translations in the spatial coordinate.

Now these constraints take the form of operator statements with operators L_m acting on states. Naively, these constraints can be translated to operator statements on physical states in the Hilbert space $|\phi\rangle$:

$$L_m|\phi\rangle = 0 \qquad \text{for all } m \neq 0 \tag{4.35}$$

However, the Virasoro algebra would then imply

$$\begin{aligned} 0 &= \langle\phi|\,[L_m, L_{-m}]\,|\phi\rangle = 2m\langle\phi|\,L_0\,|\phi\rangle + \frac{D}{12}m(m^2-1)\langle\phi|\phi\rangle \\ &\Rightarrow \langle\phi|\phi\rangle = 0 \end{aligned} \tag{4.36}$$

And the Hilbert space of the theory is completely trivial. A less restrictive requirement is to impose the operator constraints 'weakly' in the sense that physical states are required to obey:[3]

$$\begin{aligned} 0 &= \langle\phi'\,|L_m|\,\phi\rangle \qquad \text{for all } m \neq 0 \\ &\Rightarrow L_m|\phi\rangle = 0 \qquad \text{for } m > 0 \end{aligned} \tag{4.37}$$

The loosening to $m > 0$ follows from $L_m^\dagger = L_{-m}$. Regarding the L_0 constraint, the definition of this operator (to be normal ordered) is to some extent open for debate. By commutation, other definitions are always equal to the normal-ordered L_0 up to a constant. To keep things general, one introduces an arbitrary constant A in the L_0 constraint[4] so that physical states have to obey

$$(L_0 + A)\,|\phi\rangle = 0. \tag{4.38}$$

4.5.2 The states

Now let's analyze the first mass levels of the closed string. States can be found be applying creators $\alpha_{n<0}$ and $\tilde{\alpha}_{n<0}$ on the vacuum. Physical states are then selected from these by requiring the weak operator constraint $L_{m>0}|\phi\rangle = \tilde{L}_{m>0}|\phi\rangle = 0$. A priori, this procedure might include *spurious* states – states that are orthogonal to physical states in the sense that they can be written as $|\text{spur}\rangle = L_{-n}|\text{any state}\rangle$. States that are both physical and spurious have zero norm and are called *null* states. It can be shown that these states decouple from the Hilbert space in $D = 26$ dimensions, an analysis called the *no ghost* theorem (see e.g. Polchinski [63], section 4.4)

Let's start with the vacuum $|0,k\rangle$. The only non-trivial constraint is from L_0:

$$\begin{aligned} 0 &= (L_0 + A)\,|0,k\rangle = \left(\frac{{\alpha_0}^2}{2} + \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n + A\right)|0,k\rangle = \left(\frac{{\alpha_0}^2}{2} + A\right)|0,k\rangle \\ &\Rightarrow A\,|0,k\rangle = -\frac{{\alpha_0}^2}{2}|0,k\rangle = -\frac{p^2\alpha'}{4}|0,k\rangle \end{aligned} \tag{4.39}$$

Where α_0 was expressed as the string's momentum p according to the expression underneath equation (4.29). The mass $M^2 = -p^2$ of the vacuum state is then $M^2 = 4A/\alpha'$.

[3] In quantizing electrodynamics, this is known as the Gupta-Bleuler approach.

[4] This constant is often absorbed in the definition of L_0. This however violates the Virasoro algebra.

The first excited state can be expressed as $|\epsilon, k\rangle = \epsilon_{\mu\nu}\, \alpha^{\mu}_{-1}\, \tilde{\alpha}^{\nu}_{-1}|0,k\rangle$, where $\epsilon_{\mu\nu}$ is introduced as a polarization vector that will be subject to non-trivial constraints (do not confuse it with the antisymmetric tensor). The constraint $L_0 = \tilde{L}_0$ directly implies that the number of creators α and $\tilde{\alpha}$ should match. Regarding the L_0 constraint, a similar calculation as above shows that the mass of this state is $M^2 = 4(1+A)/\alpha'$. At this level, there is another non-trivial constraint:

$$
\begin{aligned}
0 &= L_1\left(\epsilon_{\mu\nu}\, \alpha^{\mu}_{-1}\, \tilde{\alpha}^{\nu}_{-1}|0,k\rangle\right) = \epsilon_{\mu\nu}\, \tilde{\alpha}^{\nu}_{-1} \sum_{n=-\infty}^{\infty} \alpha^{\rho}_{1-n}\, \alpha_{n\rho}\, \alpha^{\mu}_{-1}|0,k\rangle \\
&= \epsilon_{\mu\nu}\, \tilde{\alpha}^{\nu}_{-1}\alpha^{\mu}_{0}|0,k\rangle = \sqrt{\alpha'/2}\, \epsilon_{\mu\nu}\, \tilde{\alpha}^{\nu}_{-1}\, p^{\mu}|0,k\rangle = \sqrt{\alpha'/2}\, \epsilon_{\mu\nu}\, \tilde{\alpha}^{\nu}_{-1}\, k^{\mu}|0,k\rangle \\
\Rightarrow&\quad \epsilon_{\mu\nu} k^{\mu} = 0
\end{aligned} \tag{4.40}
$$

The second line is derived form the first by pulling annihilators to the right and using the commutation relations (4.31). The constraint states that the holomorphic oscillations are transversely polarized. Of course a similar equation holds for the antiholomorphic oscillations: $\epsilon_{\mu\nu} k^{\nu} = 0$. The other constraints are trivially satisfied but there is some physical redundancy as certain polarizations describe spurious states. This follows from

$$
\begin{aligned}
L_{-1}|0,k\rangle &= \sum_{n=-\infty}^{\infty} \alpha^{\mu}_{-1-n}\, \alpha_{n\mu}\, |0,k\rangle = 2\,\alpha_{-1}\,\alpha_0\,|0,k\rangle = 2\sqrt{\alpha'/2}\, k_{\mu}\, \alpha^{\mu}_{-1}\,|0,k\rangle \\
&\Rightarrow\quad (a_\nu\, k_\mu + b_\mu\, k_\nu)\, \alpha^{\nu}_{-1}\tilde{\alpha}^{\mu}_{-1}|0,k\rangle = \left(L_{-1}\, a_\nu\, \tilde{\alpha}^{\nu}_{-1} + \tilde{L}_{-1}\, b_\nu\, \alpha^{\nu}_{-1}\right)|0,k\rangle
\end{aligned}
$$

So that polarizations $\epsilon_{\mu\nu}$ and $\epsilon_{\mu\nu} + a_\nu\, k_\mu + b_\mu\, k_\nu$ correspond to the same physical state; a and b are understood to be arbitrary vectors with $a_\mu k^\mu = b_\mu k^\mu = 0$.

This process can be continued to arbitrarily high excitations. The L_0 constraint is a mass formula at every level, and more constraint equations apply as the level increases. I will stop at this point as this thesis is mostly concerned with the first excited level.

Taking into account the restrictions on the polarization, it follows that the first excited level has $(D-2)^2$ independent degrees of freedom. As these states are interpreted as particles in the target space (assumed to be flat Minkowski space in D dimensions), they must form representations of the little group of the Lorentz group $SO(D-1,1)$. For massive particles, this is $SO(D-1)$ and for massless particles it is $SO(D-2)$. Now focus on $D=26$; the $24^2 = 576$ physical states can be arranged into a symmetric, antisymmetric and trace part:

$$
\epsilon_{\mu\nu} = \underbrace{\epsilon_{[\mu,\nu]} - (D-2)^{-1}\delta_{\mu\nu}\epsilon_{\rho\rho}}_{G^{\mu\nu}} + \underbrace{\epsilon_{\{\mu,\nu\}}}_{B^{\mu\nu}} + \underbrace{(D-2)^{-1}\delta_{\mu\nu}\,\epsilon_{\rho\rho}}_{\Phi} \tag{4.41}
$$

The states on the right are representations of the little group for massless particles $SO(24)$. The decomposition is $\underline{24} \times \underline{24} = \underline{299} + \underline{276} + \underline{1}$ where $\underline{r}$ denotes an irreducible representation of $SO(24)$ with dimension r. These massless particles are identified as the graviton G, the anti-symmetric tensor B and the dilaton Φ. As the first excited level fits into massless particles, the normal ordering constant should be set to $A=-1$.

The cost of having the massless levels form representations of the massless little group of the target space is that the vacuum state has a negative mass-squared of $-4/\alpha'$, making it a tachyon. Usually, this signals instability of the theory. The tachyon state is an intrinsic drawback of bosonic string theory; in superstring theory there is a procedure to project it out of the spectrum, but I will not discuss this.

The tachyonic vacuum state and the three massless particles will be encountered throughout this thesis. At higher levels of excitation, it can be shown that the physical states can be combined into representations of the massive little group $SO(25)$ (see [54], section 3.3).

4.5.3 Relation to Verma modules and vertex operators

In the above the spectrum was derived by analogy with the well-known harmonic oscillator scenario. This spectrum can be interpreted in terms of Verma modules that were introduced in section 3.5.

As $A = -1$ the restriction $(L_0 + A)\,|\phi\rangle = 0$ translates to $L_0|\phi\rangle = |\phi\rangle$ for physical states ϕ. A similar remark for the $\bar{L}_0$ constraint leads to the conclusion that physical states correspond to highest weight states with $h = \bar{h} = 1$ (cf. equation (3.43)). In this language, spurious states are descendants.

By the operator $\sim$ state correspondence physical states are associated with primary fields of conformal dimension $(1,1)$. These fields are called *vertex operators* and play a crucial role in describing interactions. To every physical state, there corresponds such a vertex operator. For the ground state this is the tachyon vertex operator which is

$$V_{tachyon}(z,\bar{z}) \quad = \quad :\exp\left(ik_\rho X^\rho\left(z,\bar{z}\right)\right): \tag{4.42}$$

As seen in equation (4.27), the tachyon vertex operator has conformal weight

$$(k^2\alpha'/4, k^2\alpha'/4) \tag{4.43}$$

As stated above, the operator is required to have conformal dimension $(1,1)$. This leads to the mass formula $M^2 = -k^2 = -4/\alpha'$.

The vertex operator corresponding to a state at the first excited level $|\epsilon, k\rangle = \epsilon_{\mu\nu}\,\alpha^\mu_{-1}\,\tilde{\alpha}^\nu_{-1}|0,k\rangle$ is given by

$$V_{massless}(z,\bar{z}) \quad = \quad \epsilon_{\mu\nu}\ :\partial X^\mu \bar{\partial} X^\nu \exp\left(ik_\rho X^\rho\left(z,\bar{z}\right)\right): \tag{4.44}$$

It can be shown that such a field is $(1,1)$ primary if $k^2 = 0$ and $\epsilon_{\mu\nu}k^\mu = \epsilon_{\mu\nu}k^\nu = 0$; see e.g. Kititsis [51], chapter 8. Note that these are the vertex operators on the (flat) complex plane. A generalization for curved backgrounds will be presented in section 4.6.2.

4.6 Further developments

In the previous sections, I have introduced string theory and showed some of its structure as a conformal field theory, focusing on the energy momentum tensor and the spectrum. The spectrum is especially important in this thesis as the particles are the building blocks of cosmological scenarios. Furthermore, generalizations of the construction as presented so far will be encountered.

This was all free field theory; in the following I will address interactions and provide a perspective on calculations in string theory. I will be brief as interactions and higher order calculations (though interesting in their own right) are not of primary interest in string cosmology.

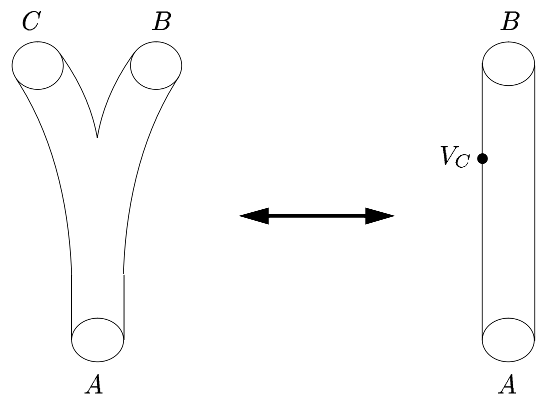

Figure 4.2: A string splitting in two strings

4.6.1 Interactions

In quantum field theory, interactions can be added to a free theory by including terms of higher order in the fields in the Lagrangian density (such as ϕ^3 which leads to a theory of the same name). Using this prescription with the Polyakov action would result in a complicated field theory on the worldsheet – describing non-harmonic oscillation of a single string instead of interactions of two or more strings. String theory is a first-quantized theory whereas the procedure of adding higher order terms to a Lagrangian density is sensible for second-quantized theories. Finding such a second-quantized formulation of string theory – called string field theory – is an area of active research.

Let's take a step back to a more intuitive picture of interacting strings as a generalization of Feynman diagram. For instance, the process of a string A splitting into two string B and C is depicted in figure 4.2. Conformal invariance makes it possible to reshape the worldsheet in a drastic way, as depicted. The state of string C at infinite time is mapped to an operator on the worldsheet that codes for all the relevant information of string C. This is the vertex operator encountered in section 4.5.3. Correlation functions take the general form

$$\langle A \,|\, V_C(z,\bar{z}) \,|\, B \rangle \tag{4.45}$$

Where V_C is the vertex operator corresponding to string C. To give an example, consider strings A,B and C to be tachyons, states with no degrees of freedom except momentum. From section 4.5.3, the vertex operator for a tachyon with momentum k_c is

$$V_C(z,\bar{z}) \quad = \quad : \exp\left(ik_{c\rho}X^{\rho}\left(z,\bar{z}\right)\right) \,: \tag{4.46}$$

Acting on a tachyon state $|0,k_b\rangle$ produces a state

$$|0,k_{b'}\rangle =: \exp\left(ik_{c\rho}X^{\rho}\left(z,\bar{z}\right)\right) \,: |0,k_b\rangle \tag{4.47}$$

Acting with the momentum operator p and using the commutation relations (4.31), it may be verified that $k_b' = k_b - k_c$ which states momentum conservation at the vertex.

The formalism of vertex operators is a general means of calculating tree level interaction processes in string theory.

4.6.2 The Polyakov path integral

Probability amplitudes for physical processes can be calculated in the Hilbert space formalism. However, a much more powerful tool is the path integral which for instance allows calculations beyond one loop. In analogy with point-particle QFT the path integral for the Polyakov action (4.6) is defined as

$$\begin{aligned} Z &= \int [dX][dh] \exp(-S_p[X,h]) \\ S_P &= -\frac{1}{4\pi\alpha'} \int_A d^2x \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \end{aligned} \tag{4.48}$$

The arbitrary x coordinates are used to stress that the metric is not gauge-fixed. This prescription means in particular that integration over all metrics h is assumed. However, due to the diff $\times$ Weyl gauge freedom in the Polyakov action a large number of metrics represent the same physical configuration which leads to a huge overcounting and problems with infinities. The resolution is to fix the gauge such that only one h of each set of gauge-equivalent metrics is counted. This effectively divides the path integral by the gauge volume of diffeomorphism and Weyl rescaling symmetries. The consistent method to do this is by introducing anticommuting Faddeev-Popov ghosts in the path integral. The ghosts constitute an independent CFT with central charge $c_g = -26$. The total central charge of the X CFT and ghost system is then 0 in $D = 26$ dimensions. I will not elaborate on this issue but refer to the literature; see e.g. [63], chapter 3.

The gauge fixing procedure depends on the topology of the worldsheet. Considering closed strings only, the easiest is the sphere where the gauge freedom can be used to completely fix h. On the torus, this is not possible; not all torii are gauge-equivalent and what remains is an integration over a single *Teichmüller* parameter.

Describing interactions in the Poyakov path integral formalism relates to the vertex operators used in the previous section. In the flat space analysis, the operator $\sim$ state correspondence was a general description to find the vertex operator for a physical state. The path integral vertex operator formalism for curved worldsheets is a bit more elaborate. For details the reader is referred to Polchinski [63], section 3.6; I will just describe the recipe. Starting from the known tachyon vertex operator (4.42), an integration over the worldsheet is assumed. This should be no surprise because the worldsheet coordinate dependence has to drop out in the path integral and the integral ensures diff invariance. Furthermore, the closed string coupling constant g_c is inserted. This leads to

$$V_{tachyon} = 2g_c \int_A d^2x \sqrt{h} \exp\left(ik_\rho X^\rho\right) \tag{4.49}$$

At the massive level, the vertex operators for the graviton, anti-symmetric tensor and dilaton state have a different form. The graviton state vertex operator is

$$V_{graviton} = \frac{g_c}{\alpha'} \int_A d^2x \sqrt{h}\, h^{\alpha\beta} s_{\mu\nu}\, \partial_\alpha X^\mu \partial_\beta X^\nu \exp\left(ik_\rho X^\rho\right) \tag{4.50}$$

Where s is a symmetric two-tensor. The α' is included to balance the length dimension of the X fields. In both vertex operators, a regularization procedure is assumed to define (products of) fields in a curved background.

Note that the anti-symmetric state cannot be included just by allowing s to contain an antisymmetric part as this would trivially contract to zero. Rather, the vertex operator resembles the graviton vertex operator but the worldsheet metric h is replaced by the anti-symmetric tensor ϵ. The correct prescription to generalize the flat space dilaton vertex operator is more elaborate and I will postpone this issue until section 5.1.2.

4.6.3 The Weyl anomaly

The symmetries in the Polyakov action are crucial for the consistency of string theory. These symmetries are explicitly present in the classical Polyakov action (4.6). However, in a general quantum theory it is possible that a classical theory breaks down, a process called an *anomaly*. The analysis of string theory as a conformal quantum field theory in sections 4.4 and 4.5 was on the Ricci-flat complex plane. Here conformal symmetry was seen to determine the theory to a large extent. In the Polyakov path integral, integration over metrics is assumed and it is not a priori clear whether the classical symmetries will be respected for all metrics, including including non-flat ones. Though this anomaly is not crucial to the rest of this thesis, I have included the analysis. It uses a lot of the theory introduced before and may serve to put things in perspective.

The evaluation of the Polyakov path integral was discussed in the previous section. It relies on a gauge-fixing procedure for the worldsheet metric h making it imperative that the result does not change when a different gauge is chosen. In other words, if $\hat{h}_1$ and $\hat{h}_2$ are two different fixed metrics,

$$\langle \ldots \rangle_{h=\hat{h}_1} \stackrel{?}{=} \langle \ldots \rangle_{h=\hat{h}_2} \tag{4.51}$$

When actually performing calculations with the path integral, a regularization procedure is often needed. This can usually be done in an explicit diff and Poincaré invariant fashion. An example is *Pauli-Villars* regularization[5] which introduces a massive field. It is easy to verify that adding a mass term to the Polyakov action respects Poincaré and diff invariance, but breaks the Weyl rescaling invariance. This difficulty is generic, so whereas diff and Poincaré invariance are apparent, the Weyl rescaling invariance has to be verified.

Let's return to the definition of the energy momentum tensor T. Starting from a path integral with arbitrary insertions, and using an arbitrary coordinate system x, the path-integral definition of T reads

$$\delta\langle \ldots \rangle = -\frac{1}{4\pi}\int d^2x \sqrt{h(x)}\, \delta h_{\alpha\beta}(x)\, \langle T^{\alpha\beta} \ldots \rangle \tag{4.52}$$

Classically, this coincides with the usual definition of T as used in equation (4.7). For a Weyl transformation $\delta h = 2\omega h$, this takes the form

$$\delta_W \langle \ldots \rangle = -\frac{1}{2\pi}\int d^2x \sqrt{h(x)}\, \omega(x)\, \langle T^{\alpha}{}_{\alpha} \ldots \rangle \tag{4.53}$$

In the quantum theory, Weyl invariance corresponds to the tracelessness of T as an operator statement. Let's assume that, in the quantum theory, T is not traceless. The trace must be diff and Poincaré invariance as the regularization procedure preserves these symmetries. Furthermore, on a flat worldsheet T was found to be traceless; see section 4.4.2. This implies that the trace must be of the form

$$T^{\alpha}{}_{\alpha} = aR \tag{4.54}$$

[5]This regularization procedure is described in [63], appendix A for the harmonic oscillator.

Where R is the curvature scalar of the worldsheet and a is a constant that will be determined in the following. A priori, one might consider a general function depending on R as this is allowed by symmetry, but higher powers of R are forbidden for dimensional reasons ([63], 3.4)

By diff invariance, which is assumed to hold, every metric can be transformed to within a local rescaling of the unit metric. It is therefore no loss in generality to assume the metric in some coordinates x is proportional to δ. Then introduce complex coordinates $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$, such that the metric h takes the form

$$ds^2 = 2h_{z\bar{z}}dzd\bar{z} \tag{4.55}$$

Note that this metric h is not the fixed metric $\hat{h}$ associated with the conformal coordinate system introduced in section 4.2. In particular, it may be position-dependent. In these coordinates, tracelessness of T reads

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{a}{2}h_{z\bar{z}}R \tag{4.56}$$

Now, conservation of energy-momentum (as a result of diff invariance) reads

$$\nabla^z T_{zz} = -\nabla^{\bar{z}} T_{z\bar{z}} = -\frac{a}{2}\nabla^{\bar{z}}\left(h_{z\bar{z}}R\right) = -\frac{a}{2}h_{z\bar{z}}\nabla^{\bar{z}}R = -\frac{a}{2}\,\partial R \tag{4.57}$$

By comparing the behavior of the terms $\nabla^z T_{zz}$ and $-a/2\,\partial R$ under Weyl rescaling, it is possible to find the value of a. In the following, this is done to first order around a flat worldsheet with metric $\hat{h}$.

Being a quasi-primary field in CFT, the energy-momentum tensor transforms according to equation (3.38) under conformal transformations:

$$\delta_\epsilon T(z) = -\left(2\partial\epsilon + \epsilon\partial\right)T(z) - \frac{c}{12}\partial^3\epsilon \tag{4.58}$$

Where c is the central charge of the CFT. As discussed in section 4.3.2, a conformal transformation can be thought of as a diffeomorphism ($\delta z = \epsilon$, $\delta\bar{z} = \bar{\epsilon}$) and Weyl rescaling ($2\omega = \partial\epsilon + \bar{\partial}\bar{\epsilon}$) that combine to leave the metric invariant. In the transformation rule above, the first term in brackets corresponds to tensor transformation under a change of coordinates, corresponding to the diffeomorphism. The last term is identified as the action of a Weyl rescaling on the quasi-primary field $T(z)$. Hence:

$$\begin{aligned} \delta_W T(z) &= -\frac{c}{6}\partial^2\omega \\ &\Rightarrow \delta_W \nabla^z T_{zz} = \nabla^z\left(-\frac{c}{6}\partial^2\omega\right) = -\frac{c}{6}\partial^z\partial^2\omega = -\frac{c}{3}\bar{\partial}\partial^2\omega \end{aligned} \tag{4.59}$$

Which is an expansion to first order around a flat worldsheet.

Under a rescaling of the metric $\delta g = 2\omega g$, the transformation of the Ricci scalar can be deduced from the identity (in $D = 2$ dimensions; see [63], section 3.3)

$$\begin{aligned} \sqrt{g'}R' &= \sqrt{g}(R - 2\nabla^\alpha\nabla_\alpha\omega) \\ &\Rightarrow \delta_W R = -2\nabla^\alpha\nabla_\alpha\omega \end{aligned} \tag{4.60}$$

Which is valid up to first order around a flat metric. This implies

$$\delta_W\left(-a/2\,\partial R\right) = -\frac{a}{2}\,\partial(-2\nabla^\alpha\nabla_\alpha\omega) = a\partial(4\partial\bar{\partial}\omega) = 4a\,\bar{\partial}\partial^2\omega \tag{4.61}$$

Matching the equations (4.59) and (4.61) implies that

$$c = -12a \quad \Rightarrow \quad T^{\alpha}{}_{\alpha} = -\frac{c}{12} R \tag{4.62}$$

In bosonic string theory, the central charge of the X theory is $c = D = 26$. As mentioned in section 4.4, this theory should be coupled to a ghost system that has central charge -26 (cf. section 4.5.2). The total theory then has $c = 0$ which implies Weyl invariance is retained even for curved worldsheets. This provides another motivation for choosing $D = 26$.

5

Low-energy string cosmology

In chapter 2, the reader was introduced to the standard theory of cosmology with the Friedmann equations in a prominent position. Extensions to this scenario as a cosmological constant and a scalar field were discussed. The current chapter introduces a theory of cosmology that is solidly rooted in string theory. Both the cosmological constant and the gravitational scalar field make a reappearance in this theory.

Section 5.1 considers strings that inhabit a non-trivial background, i.e. not 26 dimensional Minkowski space as assumed throughout the analysis of string theory in chapter 4. This leads to an effective action governing the dynamics of the background fields in a low-energy limit. Section 5.2 reinterprets this effective action as a theory of gravity and discusses in general the cosmological implications. Focusing on the universe we inhabit requires more advanced techniques as compactification, supersymmetry breaking and the possibility of a dilaton potential. Section 5.3 gives an overview of these topics and provides references for more detailed study. Sections 5.4 and 5.5 conclude this chapter with an explicit review of two well-known cosmological scenarios in string theory and an outlook on exact solutions.

Two reviews on string cosmology that have been helpful are reference [21] by Copeland et.al. and [57] by Mavromatos.

5.1 Strings in a non-trivial background

How does string theory relate to cosmology? Starting point is the generalization of the Polyakov action in Minkowski target space to arbitrary backgrounds. This describes the dynamics of strings in a coherent background of the massless string states: the graviton (metric), the anti-symmetric tensor and the dilaton. It will be shown that consistency of this theory (diff $\times$ Weyl invariance) imposes constraints on the background. To first order, one of these is that the metric is subject to the vacuum Einstein field equation.

The crux of the construction is that these constraints can be derived from an effective action that is reinterpreted as a string-inspired theory of gravity. The construction is valid for low energies by the limit $\alpha' \ll R_c$, where R_c is the radius of curvature of the spacetime, a typical length scale. This limit corresponds to viewing strings, from a large distance perspective, as point-particles.

In chapter 4, string theory based on the Polyakov action was analyzed. This action describes free bosonic fields X that live on a two-dimensional worldsheet and parametrize a 26 dimensional Euclidean space:

$$S_P = -\frac{1}{4\pi\alpha'}\int_A d^2x\,\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu \tag{5.1}$$

This action was used as a a definition of the classical theory and the path-integral based on the Polyakov action defined the quantum theory. Throughout this analysis, the metric of the target space (implicitly used for contracting the μ indices) was understood to be the Minkowski metric η in $D = 26$ dimensions describing a flat, static spacetime. In this section, the Polyakov action will be generalized to more general backgrounds.

5.1.1 The Gauss-Bonnet term

Besides allowing for non-trivial backgrounds, there is another generalization of the Polyakov action that needs to be mentioned. This results in an additional term known as the *Gauss-Bonnet* term.

As seen in section 4.3.2, the Polyakov action is invariant under worldsheet diff $\times$ Weyl and spacetime Poincaré transformations. However, it is not the most general action with these symmetry properties. Consider the addition

$$S_P \to S_P + \frac{\lambda}{4\pi}\int_A d^2x\,\sqrt{h}R \tag{5.2}$$

Where R is the worldsheet curvature scalar. Under a Weyl rescaling it transforms up to a total derivative. (cf. equation (4.60)) This term is known as the *Gauss-Bonnet* term and is a valid addition to the Polyakov action from a symmetry point of view. For notation, write it as $\lambda\chi$, where χ is a shorthand denoting the integral over the worldsheet area. Then λ is a free parameter, while χ is a topologically invariant quantity known as the Euler number[1]. For a closed surface with h handles, it is equal to $\chi = 2 - 2h$.

As the Gauss-Bonnet term depends only on the worldsheet topology and not on its local properties, it cannot affect the equations of motion. Therefore, it does not influence local string dynamics which is the reason it was not considered in chapter 4. In the path integral however, it gives relative weight to backgrounds with a different topology. This provides an interpretation for the parameter λ: adding a handle in a string diagram changes the weight by a factor $\exp(-2\lambda)$. The extra handle corresponds to the emission and absorption of an additional closed string (cf. diagram 4.2), processes that are characterized by the closed string coupling constant g_c. Hence, identify

$$g_c = \exp(\lambda) \tag{5.3}$$

This suggests that the string coupling constant is a free parameter in string theory. As will become clear in the next section, the situation is in fact a bit more subtle.

5.1.2 Non-linear sigma model

In the previous section, the Polyakov action was generalized by the addition of a new term. In this section, the generalization from a Minkowski to a general target space

[1] A nice account on the Euler number can be found in Green et.al. [41], section 3.4.5.

is considered. This has far-reaching consequences; the generalized action belongs to a broad class of models known as *non-linear sigma models.*

To start with, consider the Polyakov action with the straightforward generalization $\eta \to G(X)$:

$$S = -\frac{1}{4\pi\alpha'}\int_A d^2x\,\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X) \tag{5.4}$$

Where the Gauss-Bonnet term is suppressed for the moment. The crucial point is that the target space metric now depends on the coordinates X. Consider a Taylor expansion for G close to the origin:

$$G_{\mu\nu}(X) \sim G_{\mu\nu}(0) + X^\rho \partial_\rho\, G_{\mu\nu}(0) + \dots \tag{5.5}$$

In the Polyakov action, the first order term is a kinetic term for the X fields. The terms of higher order are interaction terms, with coupling constants given by various derivatives of the metric G. This means that the X fields are no longer free, but instead span some curved manifold. Such a model is historically known as a non-linear sigma model. A more general introduction to such models can be found in the context of coset models in section 6.1.

The replacement of $\eta \to G(X)$ clearly constitutes a generalization of the Polyakov action. Though it may look appealing, it is not obvious that the action with this modification is the correct way to describe string propagation in a general, possibly curved, background. As seen in chapter 4, closed string theory predicts a graviton state in its spectrum and one may question if this is the correct way to account for it. The answer is yes, as the above prescription corresponds (to leading order around flat space) to considering the presence of a coherent background of gravitons. To see this, write the metric as a perturbation around Minkowski space:

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + \chi_{\mu\nu}(X) \tag{5.6}$$

Where χ is small. In this expansion,

$$\begin{aligned} S &= S_P + \frac{1}{4\pi\alpha'}\int_A d^2x\,\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \chi_{\mu\nu}(X) \\ &= S_P + V \end{aligned} \tag{5.7}$$

Comparing this with (4.50), V is identified as the path integral graviton vertex operator in a curved background, with

$$\chi_{\mu\nu}(X) = 4\pi g_c \exp\left(ik_\rho X^\rho\right) s_{\mu\nu} \tag{5.8}$$

Concluding: if χ takes this plane-wave form, the generalization $\eta \to \eta + \chi(X)$ corresponds to adding $S_P \to S_P + V$ with V the graviton vertex operator. The plane-wave ansatz is not essential and can be generalized by taking a superposition. In the Polyakov path integral, various configurations are weighed by

$$\exp\left(-S\right) \;=\; \exp\left(-S_P - V\right) = \exp\left(-S_P\right)\exp\left(-V\right) \tag{5.9}$$

Inserting a single graviton vertex operator V in the path integral represents a single graviton interaction. The exponential corresponds to a coherent background of gravitons. This means that considering a perturbation in the Minkowski metric corresponds to evaluating flat-space string theory in a background of coherent gravitons – which makes perfect sense from the string theory perspective.

Motivated by this success, the other massless background fields will be included: the anti-symmetric tensor field B and the dilaton Φ. As was mentioned in section 4.6.2, this is not straightforward as the dilaton vertex operator takes a non-trivial form. The result is the following non-linear sigma model (see Polchinksi [63], section 3.7):

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int_A d^2x\,\sqrt{h}\left(h^{\alpha\beta}G_{\mu\nu}(X)+i\epsilon^{\alpha\beta}B_{\mu\nu}(X)\right)\partial_\alpha X^\mu\partial_\beta X^\nu \\ &+ \frac{1}{4\pi}\int_A d^2x\,\sqrt{h}R\,\Phi(X) \end{aligned} \tag{5.10}$$

With R the worldsheet curvature scalar. The antisymmetric tensor density ϵ is normalized to $\sqrt{h}\epsilon^{01}=1$ in Euclidean space, in accordance with the discussion in section 4.2; be aware that other definitions are also used in the literature. The form above is chosen as to comply with Polchinski [63] and Tseytlin [78].

Let's focus on the peculiar dilaton term. It resembles the Gauss-Bonnet term (5.2) that may be added to the Polyakov action, but it has a coordinate-dependent coefficient $\Phi(X)$ in place of the fixed parameter λ. The resemblance can be used by expanding $\Phi(X)=\langle\Phi\rangle+\delta\Phi(X)$. From the analysis that led to equation (5.3), the closed string coupling g_c is related to the vacuum expectation value of the dilaton:

$$g_c = \exp\langle\Phi\rangle \tag{5.11}$$

The coordinate-dependent $\delta\Phi(X)$ term is not topologically invariant; in fact, it is not even Weyl rescaling invariant on the classical level. This issue will be addressed at the end of section 5.1.4. It is related to the fact that the dilaton term has no $1/\alpha'$ precoefficient, which is clear on dimensional grounds.

5.1.3 Beta functions and the low-energy effective action

From hereon, the non-linear sigma action (5.10) is understood to be the defining action for string theory in a non-trivial background. This section is concerned with the implications of Weyl rescaling invariance of the non-linear sigma model. It will be shown that this invariance, which is motivated from string theory, leads to a set of constraints on the background fields G, B and Φ. In the literature, one may find quite a few ways to formulate these constraints. I have chosen to follow the notation and conventions of Polchinski [63].

In flat space string theory, it was found that the energy-momentum tensor T is traceless (cf. (4.7)) which assures rescaling invariance. In the analysis on the Weyl anomaly, it was seen that on a curved worldsheet tracelessness of T implies that the total central charge of string theory has to be zero. Given that the ghost system constitutes a CFT with central charge $c=-26$, the X matter CFT is required to have $c=26$. This is the case for 26 bosonic X fields, which span a 26 dimensional target space. In this reasoning, tracelessness of T (rescaling invariance) implies that strings live in a 26 dimensional target space. It must be acknowledged that Weyl rescaling invariance has far-reaching consequences.

Of course a sigma model does not have to be rescaling invariant in general. However, for the sigma model considered here to be interpreted as string theory in a non-trivial background, it should have string theory symmetries – one of which is Weyl rescaling invariance. For this reason it is required that T is traceless. Note that the action is diff invariant by construction.

The trace of T in the quantum theory defined by the action (5.10) is not calculated easily; it involves a regularization procedure such as dimensional regularization. This is described in some detail in Polchinski [63]. I will only posit the result as

$$T^{\alpha}{}_{\alpha} = -\frac{1}{2\alpha'}\left(\beta^G_{\mu\nu} g^{\alpha\beta} + i\beta^B_{\mu\nu}\epsilon^{\alpha\beta}\right)\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{R}{2}\beta^\phi \tag{5.12}$$

The β coefficients encode the scale dependence of the theory and are also known as the renormalization group beta functions. To first order in α', they are equal to (higher order terms can be found in e.g. Tseytlin [76], section 3)

$$\begin{aligned}
\beta^G_{\mu\nu} &= \alpha'\mathbf{R}_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} + \mathcal{O}(\alpha'^2) \\
\beta^B_{\mu\nu} &= \alpha'\nabla^\rho H_{\rho\mu\nu} + \alpha'\nabla^\rho\Phi H_{\rho\mu\nu} + \mathcal{O}(\alpha'^2) \\
\beta^\Phi &= \frac{D-26}{6} - \frac{\alpha'}{2}\nabla^\rho\nabla_\rho\Phi + \alpha'\nabla^\rho\Phi\nabla_\rho\Phi - \frac{\alpha'}{24}H_{\mu\nu\rho}H^{\mu\nu\rho} + \mathcal{O}(\alpha'^2)
\end{aligned} \tag{5.13}$$

Where the boldface $\mathbf{R}$ is the spacetime Ricci tensor (constructed from G), ∇ is the spacetime covariant derivative and H is the field strength of the tensor field B:

$$H_{\mu\nu\lambda} = \partial_{[\mu}B_{\nu\lambda]} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\mu\lambda} + \partial_\lambda B_{\mu\nu} \tag{5.14}$$

The statement that the non-linear sigma model (5.10) is Weyl rescaling invariant at the quantum level now reads

$$\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0 \tag{5.15}$$

So that the lefthandside of each of the equations in (5.13) is identically zero. This is a highly non-trivial constraint on the background fields G, B and Φ. In particular, in the absence of Φ and B background fields, the condition $\beta^G_{\mu\nu} = 0$ implies

$$\mathbf{R}_{\mu\nu} = 0 \tag{5.16}$$

This is a celebrated result in string theory: consistency in the sense of worldsheet rescaling invariance implies that the spacetime graviton field G is subject to the vacuum Einstein field equation! It is apparent where string theory departs from general relativity. If B or Φ are non-trivial, or higher order α' contributions are taken into account, one finds corrections to the Einstein equations.

A trivial solution to the equations (5.15) is string theory in flat Minkowski space without dilaton and antisymmetric tensor field:

$$G_{\mu\nu}(X) = \eta_{\mu\nu} \qquad B_{\mu\nu}(X) = \Phi(X) = 0 \tag{5.17}$$

Which is the theory studied in chapter 4. For this background, β^G and β^B vanish identically, but $\beta^\Phi = 0$ implies that $D = 26$, which is of course a familiar result. If the solution features a dynamical dilaton, the dimensionality of spacetime may be different.

In all three beta functions, Φ only appears differentiated. Shifting it by a constant value therefore defines another valid background. Relating this to the discussion in section 5.1.1, a change in the closed string coupling constant corresponds to considering a different background.

It is possible to construct backgrounds directly from the set of equations (5.15), but a more systematic approach starts by noting that these equations, again to first order in α', can be derived from the following *low-energy effective action*

$$S_{\text{eff}} = \int d^D x \sqrt{G} e^{-2\Phi} \left(\mathbf{R} + 4\nabla^\mu \Phi \nabla_\mu \Phi - \frac{1}{12} H^{\mu\nu\lambda} H_{\mu\nu\lambda} - \frac{2(D-26)}{3\alpha'} \right) \tag{5.18}$$

Note that this is a spacetime action, in an arbitrary number of D dimensions. Its normalization is irrelevant as it is always possible to rescale the action by a constant shift in the dilaton. If the dilaton is constant in spacetime, the exponential is an overall factor and the action reduces to the Einstein-Hilbert action of general relativity, where the B field may be regarded as a source – cf. section 2.1.1. This is in keeping with the vanishing of the beta function β_G as given in (5.13). If Φ is constant, it is the Einstein field equation but a dynamical dilaton yields modifications. The equations of motion that follow from this action coupled to matter sources are explicitly stated in section 5.2.1. At this point, I suffice by claiming that they are equivalent to the vanishing of the beta functions (5.15).

To summarize the above: worldsheet rescaling invariance leads to the vanishing of the beta functions, which is formulated as an action principle in spacetime. The appearance of this action is not some miracle; it is just set up to reproduce (5.15). The remainder of this chapter will be concerned with solutions to the equations of motion of the low-energy effective action. Such a set of fields G, B and Φ is referred to as a background and defines (up to $\mathcal{O}(\alpha')$) a rescaling invariant non-linear sigma model. Before doing so, let's discuss the assumptions and limits that were taken in the construction.

5.1.4 Limits of validity

Effective actions occur in various contexts in quantum field theory. As the name suggests, they do not describe fundamental laws but are usually set up to reproduce results in a specific regime. The low-energy effective action from bosonic string theory is no exception. In the following, the limits of its validity are discussed.

Suppose that the background fields satisfy (5.15) or, equivalently, the equations of motion from the effective action. The non-linear sigma model (5.10) then defines an interacting two-dimensional CFT up to $\mathcal{O}(\alpha')$. The interaction terms are given in the expansion (5.5) for G. The first order term has a spacetime derivative of G, which is roughly proportional to the inverse radius of curvature R_c^{-1}. The dimensionless coupling constant in the series is then $\sqrt{\alpha'} R_c^{-1}$. For the perturbation series to make sense,

$$\alpha' \ll R_c^{-2} \tag{5.19}$$

Meaning that the characteristic length scale of the fields is large compared to the string scale. In other words, one is looking at strings from a large-scale perspective in which the internal structure of the string plays no role. This perspective also justifies the exclusion of massive states; on large scales one considers large wavelengths and low energies. This was implicitly assumed earlier as the vertex operators used to construct the sigma model (5.10) were those corresponding to the massless states G, B and Φ only. The tachyon state vertex operators was also neglected. The inclusion of this operator is considered in e.g. Polchinski [63], section 6.6. See also reference [13] by Bilal.

The beta functions are explicitly considered up to first order in α', which corresponds to considering interaction terms up to the second derivative of G in the non-linear sigma model. It is possible to include higher-order terms which results in additional higher-order derivatives in the effective action. Which terms to include is a matter of choice in perturbation theory. The approximation to consider only terms of first order in α' is better if the gradients of the fields are small.

Usually, one is interested in a string theory with a weak coupling constant so that the effects of higher order loops can be addressed in a perturbation expansion. Because of the relation (5.11) between this constant and the dilaton vev, it is desirable that the dilaton does not grow arbitrarily large.

The issue that was raised in the end of section 5.1.2 can be readdressed in the language of the α' perturbation expansion. It was noted that the dilaton term by itself is not rescaling invariant at the classical level. However, in the full sigma model (5.10) the term is of higher order in α' compared to the first term with G and B. Therefore its classical transformation under a rescaling should be compared with quantum corrections from the G and B fields. Classically, Weyl rescaling invariance is then apparent and in the quantum theory rescaling invariance leads to the vanishing of the beta functions.

5.2 A theory of cosmology: basics

In the previous section, bosonic string theory in a general background was developed to the point of finding consistency equations for allowed background fields. The main result is that bosonic string theory may be formulated in a non-trivial background if the background fields G, B and Φ obey the equations of motion derived from the spacetime action

$$\begin{aligned} S_{\text{eff}} &= \int d^D x \sqrt{G} e^{-2\Phi} \left(\mathbf{R} + 4\nabla^\mu \Phi \nabla_\mu \Phi - \frac{1}{12} H^{\mu\nu\lambda} H_{\mu\nu\lambda} - V \right) \\ V &= \frac{2(D-26)}{3\alpha'} \end{aligned} \tag{5.20}$$

The reason to introduce the symbol V is that this term will be modified in various scenarios as will be discussed farther on. Using the symbolic notation allows for a more general discussion.

Starting here, the effective action will be regarded in a different light. In the previous section, the action was introduced as a trick to find valid backgrounds for string theory. In this section and the following, the effective action – with a specific implementation of V – is considered as the definition of a 'string-inspired' theory of gravity. This gravitational theory will be referred to as the low-energy string theory of gravity and should be regarded as a string theory extension of Einstein gravity, i.e. general relativity.

The most striking difference is of course the existence of a B and Φ field that are on equal footing with the metric G. In particular, the dilaton Φ plays the role of Newton's constant yet one that is coordinate dependent. It closely resembles the Brans-Dicke scalar field, as may be appreciated by comparing equation (5.20) with (2.27).

If it is a constant, the theory reduces to general relativity where B may be regarded as a source term, or it may assumed to be absent. In the general case, these fields imply modifications to Einstein gravity.

This section discusses general aspects of this theory of gravitation. Section 5.2.1 derives the equations of motion for the fields G, B and Φ that follow from the action (5.20) coupled to matter sources. This is followed by a discussion on frames in section 5.2.2. This section concludes by presenting a large-small duality that is typical to string cosmology but unknown in general relativity in section 5.2.3.

5.2.1 Matter, symmetry and equations of motion

The low-energy effective action governs the dynamics of the background fields G, B and Φ in the absence of other fields and should be regarded as the counterpart of the Einstein equations in vacuum.

What are the implications on cosmology? Recall from section 2.1 that the Einstein field equations constitute one of three basic pillars of the standard model of cosmology, the other two being the dynamical behavior of mass and the symmetry properties of spacetime by the cosmological principle. In order to study the impact of replacing Einstein gravity by the string theory of gravity on cosmological scenarios, one has to appreciate the behavior of matter and the symmetry properties of spacetime. Ideally, these would follow from string theory but usually one has to make some assumptions, the act of which defines a specific string-motivated theory of cosmology.

In section 5.4, these assumptions will be discussed for the specific models considered there. As for the symmetry of spacetime, it is usually assumed that the metric is maximally symmetric of the Robertson-Walker type or strictly diagonal with time-dependent spatial entries. For such an ansatz, it is natural to assume that both the dilaton Φ and the antisymmetric field B have no spatial dependence. To keep things general, comments regarding the nature of 'stringy' matter are postponed until section 5.4. However, it is assumed that the action is a functional of G and Φ only:

$$S_M = \int d^D x \sqrt{G} \mathcal{L}_M \tag{5.21}$$

The equations of motion from the general bosonic string cosmology action appear throughout the literature in a variety of forms; with or without B-field, dilaton potential or matter. For the sake of definiteness, I will state the equations of motion for the effective action with general $V = V(\Phi)$, including a B field and matter. Varying the action with respect to G, B and Φ, respectively, results in the following set of equations

$$\begin{aligned}
\mathcal{S}_G &= \mathbf{R}_{\mu\nu} + \frac{1}{2} G_{\mu\nu} \left(-\mathbf{R} + V + 4\nabla^\mu \Phi \nabla_\mu \Phi - 4\nabla^\mu \nabla_\mu \Phi + \frac{1}{12} H^2 \right) \\
&+ 2\nabla_\mu \nabla_\nu \Phi - \frac{3}{12} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} \\
0 &= \nabla_\mu \left(e^{-2\Phi} H^{\mu\nu\rho} \right)
\end{aligned}$$

$$\mathcal{S}_\Phi \quad = \quad \mathbf{R} - V - \frac{1}{12}H^2 + 4\nabla^\mu\nabla_\mu\Phi - 4\nabla^\mu\Phi\nabla_\mu\Phi + \frac{1}{2}\frac{\partial V}{\partial\Phi} \tag{5.22}$$

Where $\mathcal{S}_G$ and $\mathcal{S}_\Phi$ are source terms given by (recall the assumption that the matter action is independent of B)

$$\mathcal{S}_G := \frac{e^{2\Phi}}{\sqrt{G}}\frac{\delta S_M}{\delta G^{\mu\nu}} = \frac{e^{2\Phi}}{2}T_{\mu\nu} \qquad \mathcal{S}_\Phi := \frac{-e^{2\Phi}}{2\sqrt{G}}\frac{\delta S_M}{\delta\Phi} \tag{5.23}$$

Where T is the usual energy-momentum tensor of the matter sources. An explicit calculation, which also discusses the matter sources in some detail, can be found in Gasperini [31], appendix C.[2]

Without the source terms and restricting to V as given in eq. (5.20), these equations are equivalent to the vanishing of the beta functions as expressed in equation (5.13). For example, the vanishing of β^G follows from inserting the third of the equations (5.22) into the first. The first equation can be rewritten in the particularly easy form

$$\mathbf{R}_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{3}{12}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} = 0 \tag{5.24}$$

The addition of matter sources must be regarded from the point of view of the effective action being a theory of gravity.

The H field is subject to the field equations that follow from the action as expressed the second equation of (5.22). Furthermore, it obeys a closure condition

$$\partial_{[\alpha}H_{\mu\nu\rho]} = 0 \tag{5.25}$$

which follows directly from its definition in terms of B, equation (5.14). In $D = 4$, it is customary to introduce a pseudo-scalar axion field σ as (see e.g. Lidsey et.al. [53], section 3.3)

$$H^{\mu\nu\rho} = e^{2\Phi}\epsilon^{\mu\nu\rho\alpha}\nabla_\alpha\sigma \tag{5.26}$$

This automatically solves the field equation (5.22) for H. The closure condition, needed to interpret H as a field strength, is non-trivial:

$$2\nabla_\mu\sigma\nabla^\mu\sigma + \nabla_\mu\nabla^\mu\sigma = 0 \tag{5.27}$$

In fact, the closure equation takes the form of a field equation for the dual field $\bar{H}_{\alpha\beta\gamma} = e^{-2\Phi}\epsilon_{\mu\nu\rho\alpha\beta\gamma}/n!\, H^{\mu\nu\rho}$. This duality transformation interchanges the field and closure equations. As long as both equations are satisfied, the dynamics of a solution is equivalent to that of its dual.

5.2.2 String and Einstein frame

The reader will have noted the overall factor of $\sqrt{G}e^{-2\Phi}$ in the effective action; this is typical in the *string frame* form of the action. By a redefinition of G, it is possible to cast the action into a form more familiar from general relativity, known as the *Einstein*

[2] The derivation of the first equation is bit tricky, as one needs to explicitly calculate the variation of the Ricci tensor. In general relativity (i.e. without the dilaton prefactor), this is a total derivative by the Palatini identity.

frame. This section comments on this redefinition.

Using primes to indicate quantities in the Einstein frame, consider rescaling

$$G'_{\mu\nu} = e^{-2p\Phi} G_{\mu\nu} \qquad p := \frac{2}{D-2} \tag{5.28}$$

This is a spacetime rescaling under which the effective action is not invariant. The various terms in the string frame action (5.20) are related to quantities in the Einstein frame as

$$\begin{aligned}
\sqrt{G}e^{-2\Phi} &= e^{(Dp-2)\Phi}\sqrt{G'} = e^{2p\Phi}\sqrt{G'} \\
\mathbf{R} &= e^{-2p\Phi}\left(\mathbf{R}' - \frac{4(D-1)}{D-2}\left(\nabla'^{\mu}\nabla_{\mu}\Phi - \nabla'^{\mu}\Phi\nabla_{\mu}\Phi\right)\right) \\
\nabla^{\mu}\Phi\nabla_{\mu}\Phi &= G^{\mu\nu}\nabla_{\mu}\Phi\nabla_{\nu}\Phi = e^{-2p\Phi}G'^{\mu\nu}\nabla_{\mu}\Phi\nabla_{\nu}\Phi = e^{-2p\Phi}\nabla'^{\mu}\Phi\nabla_{\mu}\Phi \\
H^{\mu\nu\lambda}H_{\mu\nu\lambda} &= e^{-6p\Phi}H'^{\mu\nu\lambda}H_{\mu\nu\lambda}
\end{aligned} \tag{5.29}$$

The primes are used to indicate that the index is raised with the rescaled metric G'. The relation between the curvature scalars $\mathbf{R}$ and $\mathbf{R}'$ follows from a straightforward computation[3], the result may be found in Polchinski [63], section 3.7. Inserting the equations (5.29) in the string frame action changes its form to

$$S_{\text{eff}} = \int d^D x\sqrt{G'}\left(\mathbf{R}' - \frac{4}{D-2}\nabla'^{\mu}\Phi\nabla_{\mu}\Phi - \frac{1}{12}e^{-4p\Phi}H'^{\mu\nu\lambda}H_{\mu\nu\lambda} - Ve^{2p\Phi}\right) \tag{5.30}$$

The reader will recognize the canonical Einstein-Hilbert term $\sqrt{G'}\mathbf{R}'$ from general relativity. The rescaling was set up to make this term appear without the dilaton prefactor. Nevertheless, the departure from Einstein's theory of gravity is also apparent by the other terms in this frame. The equations of motion that follow from this action, the Einstein-frame analogs of equations (5.22), can be found in Copeland et. al. [21].

The actions in the string and in the Einstein frame provide complementary views on the same physics. If the dilaton is constant, the rescaling is trivial and the form of both actions is virtually identical. If, however, the dilaton is dynamical, the actions are quite different in form and a solution for the background fields may be qualitatively different in both views. For example, a static metric in the string frame may be expanding in the Einstein frame. In the string frame, solutions often take a simpler form and it seems a more natural point of view from the perspective of a physical string. On the other hand, for comparison to cosmological scenarios in general relativity or to astronomical data, the Einstein frame may be more convenient. I will usually use the perspective of the string frame.

Their is a fundamental question behind this choice: does an 'observer' get data in the string or in the Einstein frame? Of course all experimental input comes from particles that couple to the gravitational system, so the real question is whether or not particles in general couple to the dilaton. A related observation is that the effective action for open string theory has a somewhat different dilaton coupling (see e.g. [21], equation (2.11)). Even in the string frame, the dilaton does not couple to the antisymmetric tensor field.

[3] Be aware that the Palatini identity, which relates the transformation of the Ricci tensor to the transformation of the affine connection, holds to first order only.

This implies that the 'natural' frame depends on us being made of open or closed strings – or perhaps point particles after all. In the literature, one encounters both matter sources coupled to the dilaton and matter sources that are not. As mentioned in the beginning of section 5.2.1, I will only discuss the nature of matter sources for some explicit examples so I leave the discussion at this. Besides, there is experimental evidence for the dilaton being constant in the present universe so that both frames are identical. This will be discussed in more detail in section 5.3.3.

5.2.3 Large-small duality

An important concept in string theory are dualities, relationships between seemingly different theories. This section is devoted to a large-small duality that relates string theory on 'small' and 'large' backgrounds. This large-small duality is unknown in general relativity, yet an intrinsic property of string theory modifications to relativity. It is investigated in detail by Veneziano [82]; see also more general comments in Tseytlin [78]. It plays a fundamental role in so-called pre-big-bang scenarios, see e.g. the introduction by Gasperini [31]. See also the review by Giveon et.al. [38].

Large-small duality is related to T-duality, which (in an elementary form) states that closed string theory compactified on a circle with radius R is dual to closed string theory compactified on a circle with radius α'/R. It is understood that winding modes in one theory correspond to momentum modes in the other theory and vice versa. This particular example of T-duality is described in Polchinski [63], section 8.3. From a more general point of view, such a duality relates processes at a large scale to processes at a small scale, thereby strongly suggesting the existence of a minimum length scale equal to the string length. In a cosmological setting, this leads to quite fundamental questions regarding the big bang.

The following example is meant to illustrate that the existence of the dilaton is crucial to this duality. Consider a term in the string theory effective action of the kind

$$S_V = \int d^D x \sqrt{G} e^{-2\Phi}\, V \tag{5.31}$$

The V is understood not to scale with the metric G. It could be the dimensionality term $-2(D-26)/(3\alpha')$ in the string theory effective action, or some dilaton potential term but let's keep it general. Now consider the large-small transformation $G \to G^{-1}$. The action is invariant if, at the same time, Φ is transformed according to:

$$\begin{aligned} G &\to G^{-1} \\ \Phi &\to \Phi - \frac{1}{2}\mathrm{Tr}[\ln G] \end{aligned} \tag{5.32}$$

Where ln is understood to be inverse of matrix exponentiation. These two transformations exactly balance each other and S_V is invariant under the large-small duality. This should be contrasted with general relativity. Assuming Φ is fixed, S_V will transform non-trivially by the transformation of $\sqrt{G}$ and there is no way to compensate for this. With a specific transformation law, the dilaton may accommodate large-small duality – a specific feature of string theory.

The large-small transformation (5.32) is an example of a larger class of duality transformations on the background fields G, B and Φ, which will be discussed in section

5.4.4. Such a transformation relates two different backgrounds to each other, so it is not a symmetry in the usual sense of the word. Just as the large-scale duality described above they provide insight into the nature of various solutions. With a duality transformation, it is possible to trade properties of the graviton field G for properties of the anti-symmetric tensorfield B or the dilaton Φ. In particular, two Gs with different curvature may in fact be dual if their accompanying B and Φ fields are matched properly. This strongly suggests that strings 'feel' geometry in a different way then point-particles do.

5.3 String cosmology and our universe

The previous section introduced the reader to a theory of cosmology inspired by string-theory. The current section introduces more advanced concepts that aim to relate this general theory to the universe we inhabit.

A fairly obvious question of this kind is how string cosmology can admit a four-dimensional universe as a solution. The standard answer to this question uses the mechanism of compactification, which will be the subject of section 5.3.2. Before doing so, supersymmetric extensions[4] to the effective action from bosonic string theory will be the subject of section 5.3.1. Section 5.3.3 describes experimental constraints from astrophysical observations on the dynamics of the dilaton and discusses mechanisms that may affect these dynamics.

5.3.1 Supersymmetry

The effective action discussed so far, being motivated from bosonic string theory, includes only the massless states from bosonic string theory. This section is concerned with the effective actions of supersymmetric string theories.

Low-energy effective actions similar to the bosonic action (5.20) can be constructed for the massless fields of the various superstring theories. The bosonic part of these actions is universal and equal to the bosonic string theory effective action. Without addressing this in detail, it should be mentioned that the construction of these actions is not a straightforward generalization of the procedure of the low-energy effective action of bosonic string theory as discussed in section 5.1.2. In particular, the 'vertex operator' approach that was described is much more difficult in superstring theory. However, it should be clear that it is possible to construct an effective action that reproduces scattering amplitudes calculated in the low-energy limit of the full theory. This task is made easier by supersymmetry; in fact, the various superstring effective actions are largely determined just by their particle content and supersymmetry. As suggested by symmetry, the low-energy actions of ten-dimensional superstrings are intimately related (by dimensional reduction) to supergravity theories in eleven dimensions. For

[4]Recall the physical motivation for supersymmetry: there is a huge difference in the standard model energy scale 10^2 GeV and the Planck mass of 10^{19} GeV, which is a natural scale for gravity. Higher order gravitational corrections tend to take standard model masses to the Planck scale. The avoidance of this effect requires extreme fine-tuning, a situation known as the naturalness problem. Supersymmetry provides a resolution as fermionic contributions exactly cancel bosonic ones. In addition, SUSY predicts a unification of gauge couplings near the Planck scale. As supersymmetry has not yet been experimentally observed, it is of course assumed that the symmetry is broken around the 10^3 GeV scale. See e.g. the introductory article by Schwarz and Seiberg [69] or the TASI lecture by Kane [50].

a nice overview, see Lidsey et.al. [53] and references therein. A more pedagogical account is presented by Kiritsis [51], chapter 10. Apart from these relations, there exist various relations between the effective actions, see e.g. [53], section 5.2.

To present one example, consider the bosonic sector of type IIA superstring theory. I will be very general and the reader is referred to e.g. Polchinski [64], chapter 10 for a proper introduction The bosonic NS-NS sector is identical to that of bosonic string theory, so that the physical states split into the dilaton Φ, a symmetric two-tensor G and an anti-symmetric two-tensor B. In $D = 10$, these are the **1**, **35** and **28** representation of the little group of the Lorentz group.

The R-R sector provides two other bosonic fields, a vector field A ($\mathbf{8_v}$) and an antisymmetric three-tensor C ($\mathbf{56_t}$). Their field strengths will be denoted as F and G; they are defined in the same way as H is defined as the field strength of B in equation (5.14). These bosonic fields make up the following effective action (see e.g. [53], chapter 2)

$$\begin{aligned} S_{\mathrm{IIA}} &= \int d^{10}x\sqrt{G}\left(e^{-2\Phi}(R + 4(\nabla\Phi)^2 - \frac{1}{12}H^2) - \frac{1}{4}F^2 - \frac{1}{48}\left(G + A \wedge H\right)^2\right) \\ &+ \frac{1}{2}\int B \wedge G \wedge G \end{aligned} \tag{5.33}$$

The theory is assumed to live in $D = 10$ dimensions, making R the ten-dimensional spacetime curvature scalar. I will not go into detail as this example is just intended to give a general idea of the structure of a superstring effective action. The other superstring theories have similar effective actions with of course different fields. The IIB theory is a bit problematic as it is not possible to derive the selfduality of $\mathbf{35}_+$ directly from an action principle.

In a supersymmetric theory, the fermionic fields of some background can consistently be put to zero as required by rotational (Lorentz) invariance. Consider a solution to the equations of motion that follow from the effective action with both bosonic and fermionic fields. Under supersymmetry transformations by a constant spinorial parameter, the variation of the fermionic fields is generally zero. The variation of the bosonic fields is proportional to the fermionic fields. Assuming these fermionic fields are zero makes the bosonic fields invariant under supersymmetry transformations. This means that a set of purely bosonic fields may be a supersymmetric solution to the equations following from a supersymmetric effective action.

5.3.2 Compactification

From the cosmological point of view, the five superstring effective actions define scalar-tensor theories of gravity with various antisymmetric tensor fields living in ten dimensions. To find a solution resembling our universe, the obvious first step is to find a way to reduce the number of large (non-compact) dimensions. Of course, this argument applies also to the bosonic string. The following section gives a description of the *Kaluza-Klein* compactification procedure. For a more rigorous discussion, the reader is referred to e.g. Kiritsis [51], chapter 12 or Lidsey et.al. [53], chapter 3. As a perspective: compactification is not the only way to account for our four dimensions. An alternative is to assume that our universe is restricted to a 3-brane that inhabits a higher-dimensional spacetime, as proposed by Randall and Sundrum; see e.g. [66].

The most common compactification procedure is Kaluza-Klein dimensional reduction. Focusing on $D = 10$, this assumes that six dimensions are curled up and form an internal compact space. From a geometrical point of view, this corresponds to writing a ten-dimensional manifold $\mathcal{M}$ as a product space $\mathcal{M} = \mathcal{J} \times \mathcal{K}$, $\mathcal{J}$ being a four-dimensional non-compact manifold and $\mathcal{K}$ a six-dimensional compact manifold that represents the internal space. Coordinates (x, y) on $\mathcal{M}$ split into a set x that parametrizes $\mathcal{J}$ and a set y for $\mathcal{K}$. Matter fields are understood to be independent of the internal manifold coordinates y, which implies that $\mathcal{K}$ is Ricci-flat (see [53]). The compactification results in scalar moduli fields that live on $\mathcal{J}$. They parametrize the internal space, in the sense that a different value of such a field corresponds to a different internal space.

A somewhat more abstract description is in terms of CFTs. A solution to the full – up to all orders of α' – Weyl invariance equations defines a CFT on the string worldsheet. Such a solution can usually be associated with a ten-dimensional manifold. This is however not always possible, which makes the CFT approach the more general one. The analog of assuming a geometrical product space is then to decompose the full CFT into the tensor product of some internal compact CFT and a non-compact CFT with target spaces of the appropriate dimensions. In many cosmological scenarios, the existence of such an internal CFT with appropriate central charge and symmetry properties is taken for granted and the internal dimensions are merely spectators.

For supersymmetric theories, compactification on a background will in general break supersymmetry. If however $\mathcal{K}$ is of a special form, some or all of the supersymmetry remains. For this, it is essential that the manifold allows a covariantly constant spinor that is defined on the whole manifold ([51], section 12.2). Without the notion of such a spinor, supersymmetry transformations cannot be defined properly as there is no valid spinorial transformation parameter. The torus has the maximum number of covariantly constant spinors ([53], section 3.1), so it preserves the most supersymmetry. Other manifolds that allow at least one constant spinor are for instance *Calabi-Yau* manifolds. If the compact space is such that some supersymmetry remains, the fermionic fields can be put to zero without breaking supersymmetry. This was discussed more generally in the previous section.

The above argument is valid if the field strength $H = 0$ (see e.g. the historical article by Candelas et.al. [19]). If one allows fluxes (non-zero field strengths) for various background fields, some interesting new options arise. See e.g. [16, 56, 72].

5.3.3 A dilaton potential

At first sight, the existence of a scalar field and an anti-symmetric tensor field that accompany the usual gravitational field may seem quite a strong prediction of string theory. The existence of these fields is crucial in a typical string theory duality (discussed in section 5.2.3) that has no analog in general relativity. However, this point of view is a bit misleading. String theory as such does not (yet?) yield definite predictions about the dynamical behavior of B or Φ – there are various modifications to the effective action that may change the scenario.

Recall that the vacuum expectation value (vev) of the dilaton controls both the string coupling g_s (cf. equation (5.11)) and the gravitational coupling in analogy with Newton's constant (cf. equation (5.20)). Non-perturbatively, the absence of a dilaton potential implies that there is no natural, stable dilaton vev. Besides this theoretical flaw, there are observational constraints on the possible existence of a massless gravitational scalar field.

This observational constraint stems from an older modification to Einstein gravity known as *Brans-Dicke* theory; see section 2.2.2. In that theory, it is proposed that a long-range scalar field interacts with the Einstein gravitational field, so that the dynamics of G change. Matter fields are understood to be coupled only to G, as in general relativity (this is required if one respects the equivalence principle). Brans-Dicke theory features a free parameter ω; the string frame action resembles the Brans-Dicke theory for $\omega = -1$, while the limit $|\omega| \to \infty$ takes Brans-Dicke theory to general relativity. Astrophysical observations lead to the constraint $|\omega| > 3500$ ([86], section 3.4), which makes the existence of such a scalar field highly unlikely.

It should be noted that the dilaton is not quite the Brans-Dicke scalar field as it does couple to the antisymmetric tensor field. This does not change the conclusion that a dilaton with dynamics as given in the effective action is ruled out by observational data as noted by Tseytlin [76]. However, there are higher order or non-perturbative effects (in the α' expansion) that may give rise to a dilaton potential. Effectively, the dilaton then gets a mass and a finite range which changes the whole story. In our present universe, the dilaton is assumed to be frozen at the minimum of its potential but it may have played a dynamical role in the early universe.

Such a potential has enormous implications for cosmological scenarios based on the effective action. In reference [29], Ellis et.al. provide a 'cookbook recipe' to find a potential $V(\Phi)$ that will result in some desired behavior of the universe's scale factor $a(t)$. The particular analysis is based on a Bianchi I type universe with a perfect fluid matter source; see also Gasperini [31].[5] For such a scenario, virtually any behavior of the scale factor can be incorporated by choosing a suitable dilaton potential.

Hopefully, the reader will feel a little uncomfortable with the idea of inserting a specific potential in the effective action without physical motivation. After all, the effective action as constructed from perturbative string theory has no dilaton potential at all – see equation (5.20). So what potentials are allowed from a string theory point of view? At present, this question cannot be answered in general; there are various models, inspired by string theory, that give rise to a dilaton potential and other models are being developed. In the following, I will discuss two general effects that lead to a dilaton potential. This discussion will be very relevant in section 5.4.5, where the possibility of de Sitter solutions in string cosmology is investigated.

First of all, consider one-loop string amplitudes. Such a loop amplitude represents the probability that two strings are created out of the vacuum and immediately rejoin. In a gravitational setting, a non-zero chance contributes to the vacuum energy density. This is the same as a non-zero cosmological constant, which corresponds to a constant shift in V. In supersymmetric theories, contributions from bosons and fermions cancel, resulting in a cosmological constant that is identically zero. In a more realistic theory, where supersymmetry is broken at some energy scale, the cosmological constant is related to that energy scale. An example of such a scenario with explicit calculations can be found in Rohm [67]; a more general account is given by Polchinski [63], section 7.3.

[5] To be explicit: $\Phi = \Phi(t)$ and $p_i/\rho = \text{const}$,

$$G_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -a_i(t)^2\delta_{ij} \end{pmatrix} \qquad T_\mu{}^\nu = \begin{pmatrix} \rho(t) & 0 \\ 0 & -p_i(t)^2\delta_i{}^j \end{pmatrix}$$

The antisymmetric tensor field is put to zero, and it is assumed that matter represented by T as given above is coupled minimally to the dilaton in S_M.

In supersymmetric theories, the low-energy behavior is described in terms of a four-dimensional $\mathcal{N} = 1$ supergravity field theory. These theories may be described by a Kähler potential and a superpotential. In turn, these potentials control the dilaton dynamics. Hence non-perturbative effects on these potentials may influence the dilaton potential. An example of a mechanism which such an effect is gaugino condensation (giving it a non-trivial vev), see e.g. Barreiro et.al. [4] and references therein. This mechanism leads generally to potentials that are a combination of exponentials and polynomials in Φ. An investigation as to which potentials are allowed in such a scenario can be found in Brustein and Steinhardt [18]. See also comments in Tseytlin [76].

5.4 Cosmological solutions

This section is concerned with explicit solutions to the equations of motion of string cosmology. The first section discusses some literature to provide the reader with references for further investigations and to put things into perspective. I have chosen to highlight two interesting and well-known models from the literature in sections 5.4.2 and 5.4.3. Section 5.4.4 discusses $O(d,d)$ duality which is an important tool to relate two given solutions to each other, or to find new solutions from old ones. The last section is concerned with the efforts undertaken in the literature to find de Sitter solutions in low-energy string theory.

5.4.1 A guide to the literature

The aim of this section is to provide the reader with references to a number of useful articles on low-energy string cosmology. It has no pretension of being complete, which would be an enterprise beyond the scope of this thesis. Instead, I have chosen to discuss a number of articles that have proven to be useful in my research on low-energy string cosmology. The reader will find extensive references in these articles.

An important article from the early days of string cosmology is by Myers [61]. It introduces the linear dilaton background that will be discussed in section 5.4.2 and analyzes string theory on this background using the light-cone gauge on the world-sheet. Another well-known paper from that era is by Mueller [60], also to be discussed in section 5.4.2. It presents a nice introduction to string cosmology and introduces the 'rolling radii' class of solutions to the equations of bosonic string cosmology. These solutions are analyzed in more detail for $D < 26$, $D = 26$ and $D > 26$. Both these articles focus on bosonic string cosmology (though Myers discusses the superstring generalization) in the absence of a B field, dilaton potential and matter. The linear dilaton background assumes a Minkowski metric $G = \eta$ and a dilaton that is a linear function of any coordinate. Mueller uses an an ansatz that the metric consists of diagonal, time-dependent entries; the dilaton is time-dependent.

Section 5.4.3 discusses an article [74] by Tseytlin featuring a maximally symmetric spacetime metric and a time-dependent dilaton. It is one in a series of articles on the subject of low-energy string cosmology by Tseytlin. Another well-known article in this series is reference [80] by Tseytlin and Vafa. It discusses the general form of the equations of motion of bosonic string cosmology, with attention to large-small duality and the role of the dilaton. The metric is assumed to be diagonal with time-dependent spatial entries, the dilaton depends only on time and the B field is absent. Matter is incorporated as a string gas in thermal equilibrium for which a notion of temperature and entropy is discussed. Assuming the matter action is reparametrization invariant (the discussion is similar to that presented in section 5.4.3) leads to a general set of equations for the background fields $G(t)$ and $\Phi(t)$ in the presence of a string gas source characterized by a density ρ and pressure p. Cosmological scenarios in the critical dimension are discussed with attention to the role of winding modes in compact spatial dimensions. The article concludes with a discussion of non-critical strings, thereby focusing on relations between Mueller-type backgrounds (as discussed in section 5.4.2) and exact backgrounds from WZW models, a topic that will be discussed in more detail in section 5.5.

The article [75] by Tseytlin is closely related to the Tseytlin-Vafa article and uses the same ansatz for G and Φ; $B = 0$. It discusses matter sources in more detail, making a distinction between classical and quantum matter sources.

The last article by Tseytlin I want to mention is [76]. It starts with a discussion on the relation between the string cosmology dilaton and the Brans-Dicke scalar field. Based on the experimental bounds known from Brans-Dicke theory, the possible dynamical behavior of the dilaton is investigated; in particular, it includes a discussion on the dilaton potential that may arise by the mechanism of gaugino condensation. Some cosmological solutions for a non-trivial dilaton potential and scalar moduli field are discussed. The article also comments on various generalizations to WZW models.

Another useful series of articles on string cosmology is by Antoniadis et.al. The article [1] investigates a cosmological scenario for time-dependent dilaton and axion field. The metric is assumed to be of the Robertson-Walker form and the setup allows for a non-zero charge deficit δc that plays the role of V. It is found that, without matter contributions or higher order terms in the α' expansion, the only asymptotic solutions are a static Einstein universe or a Milne universe. Lifting these solutions to exact CFTs implies discretization of the central charge deficit. The article concludes with the observation that string loop effects may result in a non-zero dilaton potential allowing for a de Sitter solution; this issue will be readdressed in section 5.4.5.

Reference [2] reviews some elements of string cosmology and the construction of cosmological solutions. The linear dilaton and Robertson-Walker solution are discussed and the associated CFTs are analyzed. In the last part of the article, the discussion is extended to include superstrings and interactions.

All of the above articles show a very limited interest in the antisymmetric tensor filed B – it is usually put to zero. Copeland et.al. focus on the role that B may play in cosmological scenarios in references [21, 22]. The latter article also considers a modulus field that lives in four dimensions as a consequence of compactification.

5.4.2 The rolling radii solution

The first model I will discuss is based on the much referred to 'rolling radii' solution found by Mueller [60]. A review of various solutions based on the model is given by Craps et.al. in reference [23]. It features bosonic fields only, and the anti-symmetric tensorfield $B = 0$.

The action that serves as the starting point of the model is related to the general effective action (5.20) by putting

$$B = 0 \qquad V = 2\Lambda \tag{5.34}$$

In the Mueller article [60], the physical dilaton Φ is half of the physical dilaton used here. Note that Λ plays the role of a cosmological constant in this theory of gravity; to see this, compare equations (5.20) and (2.20).

Now assume that the metric has only diagonal entries, all of which depend only on time. That is,

$$G_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & e^{2\alpha_i(t)}\,\delta_{ij} \end{pmatrix} \qquad \Phi = \Phi(t) \qquad B = 0 \tag{5.35}$$

Where μ and other Greek indices take values from 0 to $N = D - 1$; Roman indices as i range from 1 to N. For notational simplicity, it is convenient to introduce a rescaled dilaton as

$$\varphi = 2\Phi - \sum_{i=1}^{N} \alpha_i \tag{5.36}$$

Similar rescalings will prove to be convenient in many other scenarios. Evaluating the various terms in the action for the specific ansatz results in an integral over t only; there is no spatial dependence by the ansatz and the volume of space is just an irrelevant overall factor. For details on this calculation see appendix A.1; here the result is posited as

$$S = \int dt \sqrt{-G_{00}}\, e^{-\varphi} \left(\sum_{i=1}^{N} G^{00} \dot{\alpha}_i^2 - G^{00} \dot{\varphi}^2 + 2\Lambda \right) \tag{5.37}$$

The equations of motion derived from this action can be written as

$$\begin{aligned} \ddot{\alpha}_i &= \dot{\alpha}_i \dot{\varphi} \qquad \Leftrightarrow \qquad \dot{\alpha}_i = c_i e^{\varphi} \\ \ddot{\varphi} &= \dot{\varphi}^2 - 2\Lambda = \sum_{i=1}^{N} \dot{\alpha}_i^2 \end{aligned} \tag{5.38}$$

Where c_i is an arbitrary integration constant. Alternatively, these solutions can be derived by evaluating the various terms in the set of equations (5.22) for this ansatz.

General solutions (c_i, φ) to the equations (5.38) for the rolling radii ansatz are discussed in reference [23], section 2.2. Making an assumption for the set of c_i simplifies the problem considerably. Various assumptions are studied in the literature; I will present two interesting examples in the following.

The linear dilaton

A nearly trivial solution is found by setting $c_i = 0$ for all i. The α_i factors are constant and may be scaled to 1. The metric reduces to the Minkowski metric but the dilaton is non-trivial:

$$G = \eta \qquad B = 0 \qquad \Phi = \pm\sqrt{\Lambda/2}\, t \tag{5.39}$$

It may be verified that these background fields are not only solutions up to order α', but in fact imply the vanishing of the beta functions to all orders of α' – it is an exact solution and as such defines a full conformal field theory. The vanishing of β^{Φ} implies

$$D = 26 + 3\alpha'\Lambda \tag{5.40}$$

Making this an example of bosonic strings living in a spacetime with dimension different then 26. The background is closely related to the solution (5.17) which is just string theory in flat 26-dimensional Minkowski spacetime. This resemblance is particularly clear by inserting the fields (5.39) in the general non-linear sigma action (5.10):

$$S = \frac{1}{4\pi\alpha'}\int_A d^2x\,\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \pm \frac{\sqrt{\Lambda/2}}{4\pi}\int_A d^2x\,\sqrt{h}R\,X^0 \tag{5.41}$$

The reader will recognize the Polyakov action with an additional term. The conformal field theory that is defined by this action is known as the (timelike) linear dilaton CFT. The theory is closely related to string theory as formulated in section 4.4. The X fields are still free, but the energy-momentum tensor has an additional term (cf. equation (4.21)). In conformal coordinates,

$$T(z) = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu: \pm\sqrt{\Lambda/2}\,\partial\partial X^0 \tag{5.42}$$

A similar expression holds for $\bar{T}(\bar{z})$. From the TT OPE, it is easily seen that the CFT has central charge

$$c = D - 3\alpha'\Lambda \tag{5.43}$$

In $D = 26 + 3\alpha'\Lambda$ dimensions, this results in $c = 26$, which exactly balances the $c = -26$ ghost CFT that is associated with diff $\times$ Weyl invariance.

A more detailed account on the linear dilaton CFT can be found in Polchinski [63], sections 2.5 and 3.7. See also the article [61] by Myers.

Generalized Milne

A different assumption leads to a model with the geometry of the two-dimensional Milne universe. Take

$$c_1 = 0\,, \qquad c_i = 0 \text{ for } i \neq 1 \tag{5.44}$$

Introducing $Q := \sqrt{\Lambda/2}$, this corresponds to

$$\begin{aligned} \alpha_1(t) &= \frac{1}{2}\ln\left(Q^{-2}\tanh^2(Qt)\right) \\ \Phi(t) &= -\ln\left(\cosh(Qt)\right) \end{aligned} \tag{5.45}$$

So that the geometry of the full spacetime is flat, static space in $D-2$ dimensions times a dynamical spacetime with line-element

$$ds^2 = -dt^2 + Q^{-2}\tanh(Qt)^2\,dx_1{}^2 \tag{5.46}$$

These field constitute a a valid string theory background up to order α' but it is not an exact solution like the linear dilaton background. However, as will be discussed in section 5.5, this background may be identified as the leading order of an exact solution that is found using a gauged WZW model.

5.4.3 The maximally symmetric solution

In this section, I discuss an interesting cosmological solution proposed by Tseytlin in reference [74]. It assumes a Friedmann-Robertson-Walker type universe with $k = -1, 0, 1$ in $3+1$ dimensions. The connection to well-known solutions of cosmology based on general relativity makes it a particularly interesting model. Generic matter is included in the general analysis, but excluded in the specific example that will be presented in the end of this section. The solution includes bosonic fields only.

Relating the notation (5.20) used in this thesis to the article [74], the general term V takes the form

$$\begin{aligned} V &= V(\Phi) - c \\ c &= -\frac{2}{3\alpha'}(D_{\text{eff}} - D_{\text{crit}}) \end{aligned} \tag{5.47}$$

Where $V(\Phi)$ is an unspecified dilaton potential and both D_{eff} and D_{crit} depend on the particular string theory that underlies the effective action – for superstrings $D_{\text{eff}} = \frac{3}{2}D$, $D_{\text{crit}} = 15$; for the bosonic string $D_{\text{eff}} = D$, $D_{\text{crit}} = 26$. In the following, c will be considered to be an arbitrary parameter, but keep in mind that it is motivated from string theory. From the effective action (5.20), it is clear that $-c$ plays the role of a cosmological constant in the theory of gravity, which will be of importance in the end of this section.

The B field is left out of the effective action, but may be present in the matter Lagrangian.

The ansatz used to solve the equations (5.22) that follow from the effective action is

$$G_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & e^{2\lambda(t)}\tilde{G}_{ij} \end{pmatrix} \qquad \Phi = \Phi(t) \qquad B = 0 \tag{5.48}$$

Here $\tilde{G}$ is the metric of a maximally symmetric $N = D-1$ dimensional subspace, with parameter $k = -1, 0, 1$. The zero coordinate is interpreted as a time coordinate, the other coordinates span the maximally symmetric physical space. The ansatz is basically the assumption that the spatial universe is isotropic and homogeneous, hence maximally symmetric. This makes it natural to assume that the dilaton also has no spatial dependence. As noted above, $B = 0$ here but it may be included later on in the matter action.

The maximal symmetry allows to study the spatial background without further specifications. The analysis, including a number of results that may be useful in calculations, is presented in some detail in appendix A.2. It is convenient to introduce the rescaled dilaton φ (the original Φ will be referred to as the physical dilaton)

$$\varphi := 2\Phi - N\lambda \tag{5.49}$$

Using the notation of the original paper for compatibility, also introduce

$$S'_M[G_{00}, \lambda, \varphi] := S_M[G_{00}, \lambda, \Phi] \qquad V'(\varphi, \lambda) := V(\Phi) \tag{5.50}$$

Inserting the ansatz in the general effective action (keeping G_{00}) results in an integral over t as all spatial dependence is trivial.[6] The volume of space is some constant which

[6] In the original article, the dilaton potential term and the curvature term (including k) are not included in the action. It can easily be verified from the general effective action and the ansatz that these terms should be present.

may be neglected.

$$\begin{aligned} S &= \int dt\sqrt{-G_{00}}\left(c - G^{00}N\dot{\lambda}^2 + G^{00}\dot{\varphi}^2 - V'(\lambda,\varphi) + N(N-1)ke^{-2\lambda}\right) \\ &- S'_M[G_{00},\lambda,\varphi] \end{aligned} \tag{5.51}$$

The equations of motion for the maximally symmetric ansatz can be derived from this action. Of course, they can also be found by inserting the ansatz (with $G_{00} = -1$) in the general equations of motion (5.22).

$$\begin{aligned} c - N\dot{\lambda} + \dot{\varphi}^2 &= 2U \\ \ddot{\lambda} - \dot{\varphi}\dot{\lambda} &= W_1 \\ \ddot{\varphi} - N\dot{\lambda}^2 &= W_2 \end{aligned} \tag{5.52}$$

The dot denotes a derivative with respect to time t; there is no spatial dependence by the assumption of maximal symmetry. In the above formulae the matter, potential and curvature terms are included in U, W_1 and W_2:

$$\begin{aligned} U &:= -\frac{1}{2}kN(N-1)e^{-2\lambda} + \frac{1}{2}V - e^{\varphi}\frac{\delta S'_M}{\delta G_{00}} \\ W_1 &:= -k(N-1)e^{-2\lambda} - \frac{1}{2}\frac{\partial V}{\partial\varphi} - \frac{1}{2N}e^{\varphi}\frac{\delta S'_M}{\delta\lambda} \\ W_2 &:= \frac{1}{2}\frac{\partial V}{\partial\varphi} - e^{\varphi}\left(\frac{\delta S'_M}{\delta G_{00}} - \frac{1}{2}\frac{\delta S'_M}{\delta\varphi}\right) \end{aligned} \tag{5.53}$$

Where G_{00} is understood to be equal to -1 after the variation.

From general principles in this action formulation, the behavior of the matter sources can be further specified. The argument is as follows: under a reparametrization of t, the factor G_{00} – being an einbein – transforms as

$$\int \sqrt{-G'_{00}}\,dt' = \int \sqrt{-G_{00}}\,dt \tag{5.54}$$

Now suppose the matter action depends on G_{00} only through this prefactor:

$$S'_M[G_{00},\lambda,\varphi] = \int dt\sqrt{-G_{00}}\,\mathcal{L}_M(\lambda,\varphi) \tag{5.55}$$

It is then invariant under reparametrizations of time by the einbein transformation law above. From this, it immediately follows that

$$\frac{\delta S'_M}{\delta G_{00}} = \frac{\partial\sqrt{-G_{00}}}{\partial G_{00}}\mathcal{L}_M = -\frac{1}{2}\frac{1}{\sqrt{-G_{00}}}\mathcal{L}_M \to -\frac{1}{2}\mathcal{L}_M \tag{5.56}$$

Where the arrow denotes putting $G_{00} = -1$, which is always assumed after the variation. To complete the argument, consider the dependence of the dilaton potential on λ and φ.

$$\begin{aligned} 0 &= \frac{dV(\phi)}{d\lambda} = \frac{\partial V'(\varphi,\lambda)}{\partial\lambda} + \frac{\partial V'(\varphi,\lambda)}{\partial\varphi}\frac{\partial\varphi}{\partial\lambda} = \frac{\partial V'(\varphi,\lambda)}{\partial\lambda} - N\frac{\partial V'(\varphi,\lambda)}{\partial\varphi} \\ &\Rightarrow \frac{\partial V'(\varphi,\lambda)}{\partial\lambda} = N\frac{\partial V'(\varphi,\lambda)}{\partial\varphi} \end{aligned} \tag{5.57}$$

Combining the equations (5.56) and (5.57) then implies

$$\frac{\partial U}{\partial \lambda} = -NW_1 \qquad \frac{\partial U}{\partial \varphi} = W_2 \tag{5.58}$$

So all matter, curvature and dilaton potential contributions to the system are included in the single function $U(\varphi, \lambda)$. The article [74] by Tseytlin discusses a mechanical interpretation of the equations of motion for the maximally symmetric ansatz in terms of this single function U. It also includes a discussion on the behavior of solutions with the B field as a matter source for some choices of U. I will discuss another example presented in the article, which is used to investigate the possibility of a de Sitter solution in this context.

Assume that matter sources and dilaton potential are absent. The equations of motion (5.52), (5.53) reduce to[7]

$$\begin{aligned} c - N\dot{\lambda}^2 + \dot{\varphi}^2 &= -N(N-1)ke^{-2\lambda} \\ \ddot{\varphi} - N\dot{\lambda}^2 &= 0 \end{aligned} \tag{5.59}$$

In general relativity, there exists a Robertson-Walker type de Sitter solution to the Einstein equations with a positive cosmological constant. The spatial curvature of this solution is characterized by $k = -1, 0, 1$. Let's attempt to reproduce this solution by assuming a constant dilaton. Be aware that $-c$ plays the role of a cosmological constant, so it is required that $c < 0$. It can easily be verified that

$$\begin{aligned} e^{\lambda} &= H^{-1}\cosh(Ht) \qquad H^2 = -\frac{c}{N(N-1)} \\ \Phi &= \text{const} \qquad \Rightarrow \qquad \dot{\varphi} = -N\dot{\lambda} \end{aligned} \tag{5.60}$$

Solves the first of the two equations for $k = 1$; solutions for $k = -1$ or $k = 0$ can be found in the article [74]. However, this ansatz does not solve the second of the equations; in fact, this equation implies that the dilaton cannot be constant, thus introducing a deformation in the de Sitter solution. This also applies to the $k = 0, 1$ solutions and leads to the conclusion that the de Sitter solution of general relativity is not allowed in cosmology based on low-energy string cosmology.

5.4.4 $O(D, D)$ duality

In section 5.2.3, scale factor duality – a typical string theory feature – was discussed. This duality is in fact part of a larger duality group, known as $O(d, d)$ duality. The duality relates two different sets of backgrounds fields that do not depend on d dimensions. I will comment on this duality for the cosmological models discussed before, assuming that the fields only depend on cosmic time t and not on spatial coordinates, implying $d = D - 1$. For a more general account, see e.g. Giveon and Rocek [39]. Another application of this duality is in the context of compactification: if a dimensions are compactified, $O(a, a)$ duality relates different backgrounds on the uncompactified space. See e.g. Lidsey et.al. [53], section 4.3.

The general effective action from bosonic string theory, including some antisymmetric tensorfield B, can be reformulated in terms of $2d \times 2d$ matrices $\mathcal{M}$, $\mathcal{E}$ and a shifted

[7] In [74], there are three equations but they are not functionally independent; the derivative of the first equation can be expressed in a linear combination of the second and third equation.

dilaton $\tilde{\Phi}$ as

$$S = \int dt \, e^{-2\tilde{\Phi}} \left(4\,(\partial_t \tilde{\Phi})^2 - V + \frac{1}{8}\mathrm{Tr}\,[\dot{\mathcal{M}}\,\mathcal{E}\,\dot{\mathcal{M}}\,\mathcal{E}] \right) \tag{5.61}$$

Where the overdot denotes a derivative with respect to time t. The matrix $\mathcal{M}$ consists of the spatial parts of G and B, which will be denoted as g and b.

$$\mathcal{M} = \begin{pmatrix} -g^{-1} & -g^{-1}b \\ bg^{-1} & g - bg^{-1}b \end{pmatrix} \qquad \mathcal{E} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{5.62}$$

And the shifted dilaton absorbs the volume of space in the sense that

$$e^{2\tilde{\Phi}} = e^{2\Phi}\sqrt{\tilde{g}} \tag{5.63}$$

The equivalence of (5.20) and (5.61) can be shown by a lengthy but straightforward calculation. It can be found in Meissner et.al. [58], the article that introduced this duality. As an alternative, it is presented in detail in Gasperini [31], appendix B.

The action is manifestly invariant under a transformation with some matrix $\mathcal{N} \in O(d,d)$

$$\begin{aligned} \mathcal{M} &\rightarrow \mathcal{N}\,\mathcal{M}\,\mathcal{N}^T \\ \tilde{\Phi} &\rightarrow \tilde{\Phi} \end{aligned} \tag{5.64}$$

Which is easily seen as elements of $O(d,d)$ preserve the off-diagonal form

$$\mathcal{N} \in O(d,d) \Rightarrow \mathcal{N}^T \mathcal{E} \mathcal{N} = \mathcal{E} \tag{5.65}$$

This duality can be extended to models that include matter sources, provided that these sources obey some transformation rule under the duality transformation. The perfect fluid assumption is not invariant under such a transformation, suggesting a viscosity of some sort. See e.g. Gasperini [31] or Gasperini and Veneziano [32].

The scale-factor duality that was discussed in section 5.2.3, is a specific example of an $O(d,d)$ duality. The anti-symmetric field is understood to be absent: $b = 0$. Now take the transformation matrix

$$\mathcal{N} = \mathcal{E} \tag{5.66}$$

Which interchanges $g \leftrightarrow g^{-1}$. Assuming $G_{00} = -1$ extends the duality to the full spacetime metric G. Invariance of the rescaled dilaton $\tilde{\Phi}$ is equivalent to the transformation rule (5.32) for the physical dilaton Φ.

5.4.5 A de Sitter solution?

Recently, de Sitter spacetimes (see the introduction in section 2.3) have received considerable interest in string theory. This is motivated from cosmology, where de Sitter space plays a role in inflationary scenarios. Apart from a de Sitter phase of the early universe, recent astronomical data suggest that our universe is currently expanding at an increasing rate, a signature of de Sitter spacetime; for a review, see Sarkar [68]. Formulating a quantum theory of gravity on such a spacetime is a notoriously difficult subject and it is hoped that string theory may provide a theoretical framework to address these problems; see e.g. the overview article by Witten [89] or the recent article by Balasubramanian et al. [3]. The possibility of a de Sitter solution in various supergravity theories, which are closely related to the low-energy limit of superstring theories, is discussed in detail in the review by Hull [48].

There is a large number of articles that present de Sitter solutions in string-inspired theories of gravity such as scalar-tensor gravity. Usually, the behavior of matter sources and the form of the dilaton potential – or free parameters such as the Brans-Dicke parameter ω – are fine-tuned to yield such a solution. Though interesting in their own right, I have chosen to restrict to constructions that are more directly related to string theory. In this section, I present some references to the literature where de Sitter backgrounds were considered in low-energy string theory, including a short discussion of a no-go theorem for de Sitter compactifications in string theory.

Following the work of Boulware and Deser [14, 15], Bento and Bertolami [12] discuss in general the possibility of a de Sitter solution in string theories of gravity. The analysis includes higher order curvature terms, but no dilaton potential or explicit cosmological constant. The basic conclusion is that, without a dilaton, de Sitter solutions exists but are generally unstable – a small perturbation in the metric always grows large and changes the structure of spacetime. The inclusion of a dilaton leads to more constraints on the background fields which remove the de Sitter solution.

A totally different approach that I want to mention is described by De Vega and Sánchez in [24]. The analysis is based on a gravitational theory with zero cosmological constant coupled to a gas of string sources. The conclusion in section VI.B. is that a Robertson-Walker type gravitational background with de Sitter-like dynamical behavior requires the dilaton to have a constant imaginary part $\mathrm{Im}[\Phi] = \pi/2$ (in conventions used here) making it a kind of anti-gravity coupling. From a string theory point of view, an imaginary dilaton is clearly an undesirable feature as it would for instance result in a complex string coupling.

The introduction of a cosmological constant, either explicit or in disguise, seems unavoidable to find a de Sitter solution. This leads to two questions. First, can string theory motivate the inclusion of a positive cosmological constant in the effective action? Secondly, what (if any) backgrounds solve the equations of motion including such a constant?

The latter question was already addressed in previous sections of this chapter, where two examples of cosmological backgrounds were presented that may a priori be valid for a positive cosmological constant. Recall the cosmological solutions found by Tseytlin in [74], which were discussed in section 5.4.3. The parameters of the model that was probed for a de Sitter solution can be summarized as

$$V = -c = 2\Lambda \qquad B = 0 \qquad \Phi = \mathrm{const} \tag{5.67}$$

And G is of the Robertson-Walker type as can be seen in equation (5.48). It was found that the de Sitter solution of general relativity cannot be lifted to the string effective action theory of gravity with this ansatz for the additional fields B and Φ. This was also noted by Tseytlin in [76].

On the other hand, in section 5.4.2 a class of solutions was described for

$$V = 2\Lambda \qquad B = 0 \qquad \Phi = \Phi(t) \tag{5.68}$$

And G diagonal with time-dependent entries as expressed in equation (5.35). These solutions typically have a large dilaton gradient, which to some extent invalidates the first order α' approximation. With a time-dependent dilaton, the interpretation of Λ as a cosmological constant is debatable. In the Einstein frame, $V = 2\Lambda$ couples to the dilaton which results in an 'effective' time-dependent cosmological constant $\exp[4/(D-2)\,\Phi(t)]\,\Lambda$. To see this, compare the action effective action in the string frame (5.20) to the same action in the Einstein frame (5.30).

The above discusses the (non)existence of a valid solution to the equations of motion that follow from the string theory effective action with a positive cosmological constant. But how, from a string theory point of view, can such a constant appear in the theory of gravity in the first place?

This question is strongly related to the question which dilaton potentials are to be allowed as was discussed in the end of section 5.3.3. String loop effects may lead to a constant shift in V, which is equivalent to introducing a cosmological constant. In reference [1], Antoniadis et.al. hinted at the possibility of finding a de Sitter solution with constant dilaton using this effect. A similar observation was made by Mavromatos in the review [57], chapter 3.2. The effect of string loops on higher-order solutions is investigated by Bento and Bertolami [12]. They conclude that, if string loop corrections are taken into account, the negative conclusion described above (no de Sitter solutions if the dilaton is included) may be avoided.

Recall from section 2.1.3 that a true cosmological constant (a constant non-zero V) may be regarded as an energy source with equation of state $p = -\rho$. Non-constant dilaton potentials may result in exotic scaling behavior of the dilaton, characterized by some equation of state $p = w\rho$ with $-1 < w < -1/3$. In contrast with an inert cosmological constant, w can be dynamical. These models are known as *quintessence* models, see e.g. Hellerman et.al. [44] or Fischler et.al. [30].

Another way to introduce a cosmological constant in the gravity action is by assuming that strings do not live in a target space of the critical dimension. This effect is directly clear from the bosonic action (5.20). Choosing a supercritical value $D = 26 + \Delta D > 26$ leads to a positive cosmological constant of

$$\Lambda = \frac{\Delta D}{2\alpha'} \tag{5.69}$$

This connection was already mentioned by Mueller [60], the article that is the basis of the solutions described in section 5.4.2. It is discussed in some more detail by Tseytlin and Vafa in [80].

A no-go theorem

The crux of the scenarios discussed above is to find a string theory motivation to include a cosmological constant in the effective action. A natural question is whether this can be done by a specific compactification procedure of the (super)string effective action?

A systematical approach to answer this question is given by Maldacena and Nuñez in [55] and results in a no-go theorem. The authors consider a non-singular warped compactification of the low-energy superstring effection actions:

$$ds_D^2 = \Omega(y)^2\left(\eta_{\mu\nu}dx^\mu dx^\nu + \hat{g}_{mn}dy^m dy^n\right) \tag{5.70}$$

The total D-dimensional spacetime splits into a d dimensional cosmological spacetime with coordinates x and indices μ, ν and an internal manifold with coordinates y and indices m, n. The warp factor $\Omega(y)$ depends on the internal space only. The metric of the 'large' spacetime η is understood to be either Minkowski or de Sitter.

The authors derive general properties for the massless superstring fields in such a background. Plugging these in the D-dimensional Einstein equations for η Minkowski or de Sitter results in inconsistencies. In the analysis, it is assumed that the action has no $\mathcal{O}(\alpha^2)$ terms and that the dilaton potential is strictly non-negative $V(\Phi) \geq 0$[8]. Furthermore, it is assumed that the massless fields have positive kinetic terms and that the effective Newton's constant remains finite. The article leads to the conclusion that, under these assumptions, de Sitter nor Minkowski spacetime can arise from the compactification of superstring effective actions with the general ansatz (5.70). A similar conclusion is drawn by Hari Dass in [43].

Maldacena and Nuñez point out that their negative conclusion can be avoided by including $\mathcal{O}(\alpha^2)$ corrections or starting from a modified theory that has a positive cosmological constant before compactification. A nice example of a model that avoids the no-go criterion is presented in reference [72] by Silverstein, including a comment on how the no-go theorem is circumvented. A review and generalization of this model is given by Maloney et.al. [56].

5.5 Extension to exact solutions

The whole of chapter 5 so far was devoted to the low-energy limit (order α') of string theory on non-trivial backgrounds. This section discusses possible relations between such low-energy solutions and exact (up to all orders in α') solutions. Such an exact solution defines a worldsheet CFT, for which powerful tools are available to analyze the theory. The discussion is rather brief and serves only to provide a perspective.

The backgrounds that were considered in the previous sections solve the equations of string cosmology (5.22) to first order in the α' expansion. From the string theory perspective, such a background defines a theory that is conformally invariant on the worldsheet up to first order in α'. Incidentally, such a solution may by itself be conformally invariant up to all orders in α'. An example of such a background is the linear dilaton CFT, which was discussed in section 5.4.2.

In general, one is not so lucky but a more common possibility is that the low-energy solution can be identified as the leading order of an exact solution. That is, there exists a CFT whose leading term in an α' expansion is the CFT associated with a low-energy solution. The formulation of such an exact CFT as a two-dimensional field theory that lives on a worldsheet is highly non-trivial. A well-known method to do this is by using gauged Wess-Zumino-Witten (WZW) models, which will be studied in more detail in chapter 6. As was already hinted at in section 5.4.2, this is possible for the Milne background found there. This example is pursued further in sections 6.4 and 6.6.

An interesting possibility to find exact solutions is by using duality. As was discussed in sections 5.2.3 and 5.4.4, there exist duality relations between low-energy ($\mathcal{O}(\alpha')$) solutions: using $O(d,d)$ transformations, one could acquire new solutions from old ones. In fact, this duality transformation extends to higher order in α' and is an exact symmetry. Therefore, exact solutions may also be generated by duality transformations. See Gershon [34] and references therein.

[8]For V as defined in equation (5.20). In the article, V is assumed to have the same sign as a cosmological constant, which requires it to be non-positive.

There are other methods to lift a low-energy solution to an exact one (e.g. F-models) but I will not comment on these; see e.g. the account on exact solutions by Tseytlin [79].

6

Coset models

This chapter concerns a well-studied approach to construct non-trivial, possibly time-dependent backgrounds for string theory. Starting point is the non-linear sigma action which, from a geometrical point of view, defines geodesic motion on a group manifold. Introduction of a topological term defines a *Wess-Zumino-Witten* (WZW) theory which is conformally invariant at the quantum level. Restricting to a coset of the original symmetry group yields a conformal field theory that has a natural interpretation as string theory on a non-trivial background. The exact construction of the coset allows for controlling the background and investigating the properties of strings that live on it.

In sections 1 and 2, I choose a physicists' approach by starting with the gauged WZW action to define dynamics on a coset space. I introduce the WZW action and comment on its interpretation as a conformal field theory. Section 3 introduces the gauged WZW action and discusses how the coset construction leads to a theory of strings in a background. This coset construction is illustrated in section 4 by discussing Witten's black hole construction. In sections 5 and 6, propagation of a scalar particle on a background originating from a coset construction is discussed and applied to the black hole solution. The analysis is more mathematical and uses a group theory approach to cosets.

The reader may refer to Di Francesco et.al. [25] for a more detailed account on WZW models and the relation to conformal field theory (the first two sections of this chapter). Another good introduction to this subject is chapter 3 of reference [49] by Kaku. Other references will be given in the corresponding sections.

6.1 The Wess-Zumino-Witten action

Consider a theory with matrix-valued fields g that live on a two-dimensional worldsheet A. To be more specific, choose $g(x_1, x_2) \in G$, where G is the defining representation of a certain group. The *non-linear sigma model* is defined by the action (see Di Francesco et. al. [25], section 15.1.1)

$$S_{nl\sigma}[g] \quad = \quad \frac{1}{4a^2}\int_A d^2x \,\mathrm{Tr}\left[\partial_i g^{-1}\partial^i g\right] \tag{6.1}$$

Indices are raised or lowered with the metric on the worldsheet. The metric is of course the origin of possible non-linearity as it may depend on the worldsheet coordinates. In a unitary representation, $g^{-1} = g^\dagger$ and the action is real. When written out in components, one may regard the action as an economic way of writing down the action for a collection of free scalars. From the more geometrical point of view, the above action defines the dynamics of an elements g on the group manifold of G. The non-linear sigma action (6.1) has global $G_L \times G_R$ symmetry, being invariant under the symmetry operation

$$g \to g' = m\, g\, n \qquad m, n \in G \tag{6.2}$$

Although the non-linear sigma model is explicitly conformally invariant at the classical level, the quantum theory is not. Details on this anomaly and the construction of a resolution can be found in Di Francesco et.al. [25], section 15.1. In the following section, I will address the issue by looking at the currents associated with the non-linear sigma model. For now, the claim is that the action can be modified to gain conformal symmetry at the quantum level by including a topological term. This *Wess-Zumino* term integrates over a three-dimensional space V, chosen such that $A = \partial V$. Introduce $\hat{g}$ as the three-dimensional continuation of g on V such that $\hat{g}(x_1, x_2, x_3) \in G$, and $\hat{g} = g$ on the surface of V.

$$\Gamma[g] = \frac{ik}{24\pi} \int_V d^3x\, \epsilon^{ijk}\, \mathrm{Tr}\left[\hat{g}^{-1}\partial_i \hat{g}\, \hat{g}^{-1}\partial_j \hat{g}\, \hat{g}^{-1}\partial_k \hat{g}\right] \tag{6.3}$$

The variation of $\Gamma[g]$ with respect to g can be written as a total derivative in three dimensions, making it a two-dimensional functional. If G is a compact group, the parameter k needs to be integer for topological reasons. In the next section, it will be argumented that if k is related to a by $a^2 = \pm 4\pi/k$ the resulting theory is conformally invariant at the quantum level. Explicit examples on how to calculate the contribution of the Wess-Zumino term for groups $SL(2, \mathbb{R})$ and $SU(2)$ can be found in appendix D.

Assuming $a^2 = 4\pi/k$ and assing the topological term to the action (6.1) defines the *Wess-Zumino-Witten* (WZW) action:

$$\begin{aligned} S_{WZW}[g] &= \frac{k}{16\pi} \int_A d^2x\, \mathrm{Tr}\left[\partial_i g^{-1} \partial^i g\right] - \Gamma[g] \\ &= -\frac{k}{16\pi} \int_A d^2x\, \mathrm{Tr}\left[g^{-1}\partial_i g\, g^{-1}\partial^i g\right] - \Gamma[g] \end{aligned} \tag{6.4}$$

The two forms above are easily seen to be equivalent (note the minus-sign); I have stated them both as either form appears in the literature frequently.

6.2 WZW models as conformal field theories

The previous section introduced the Wess-Zumino-Witten action as an extension of the non-linear sigma model. This section investigates the symmetry properties by analyzing the conserved currents. It will be argumented that the currents constitute an affine Lie algebra that includes the Virasoro algebra – which establishes conformal invariance. The following section follows Di Francesco et.al. [25] by discussing the WZW model from a rather abstract CFT point-of-view. For an introduction with more emphasis on string theory, see Bars and Nemeschansky [5] or Gepner and Witten [33].

6.2.1 Classical currents

Already at the classical level, it can be argumented that the non-linear sigma model is not the kind of CFT encountered before. Recall from section 3.4.1 that on the complex plane, the currents of a CFT split naturally into a holomorphic and an antiholomorphic part. For the WZW action (6.4) it can be shown that varying the action with respect to g yields the following equation of motion (see [25], 15.1.1):

$$\partial^\mu(g^{-1}\partial_\mu g) + \frac{ia^2k}{4\pi}\epsilon_{\mu\nu}\partial^\mu(g^{-1}\partial^\nu g) = 0 \tag{6.5}$$

Putting $k \to 0$ reduces the WZW theory to the non-linear sigma model (6.1) as the topological term vanishes. In this case, it is natural to define a current J as

$$J_\mu = (g^{-1}\partial_\mu g) \tag{6.6}$$

In conformal coordinates on the complex plane, the vector J consists of $J_z = g^{-1}\partial_z g$ and $J_{\bar{z}} = g^{-1}\partial_{\bar{z}} g$. However, $\partial^\mu J_\mu = 0$ in these coordinates reads

$$\partial_z J_{\bar{z}} + \partial_{\bar{z}} J_z = 0$$

And the holomorphic and antiholomorphic current are not conserved separately, which signals a lack of conformal invariance at the quantum level. For the WZW action (6.4) both a and k are a priori arbitrary parameters. Of course their overall scaling is irrelevant classically, but there is some freedom to choose a relation between them. Use this to choose $a^2 = 4\pi/k$, so that (6.5) reduces to

$$\partial^\mu(g^{-1}\partial_\mu g) + i\epsilon_{\mu\nu}\partial^\mu(g^{-1}\partial^\nu g) = 0 \tag{6.7}$$

Evaluating this equality in complex coordinates[1] implies

$$\partial_z(g^{-1}\partial_{\bar{z}} g) = 0 \tag{6.8}$$

Which identifies $J_{\bar{z}} = (g^{-1}\partial_{\bar{z}} g)$ as an antiholomorphic current, depending only on $\bar{z}$ – note that it is conserved independently of any holomorphic current. To find this corresponding holomorphic current J_z, use the equality

$$\partial_z(g^{-1}\partial_{\bar{z}} g) = g^{-1}\partial_{\bar{z}}(\partial_z g g^{-1})g \tag{6.9}$$

Which is easily verified by writing out both sides of the equation. Now conservation of $J_{\bar{z}}$ implies the vanishing of both sides of the above equation. In turn, this corresponds to conservation of $\partial_z g g^{-1}$. This identifies the corresponding holomorphic current J_z as

$$J_z = \partial_z g g^{-1} \tag{6.10}$$

Note the difference with the holomorphic current from the non-linear sigma model. Alternatively, one could have chosen to relate a and k by $a^2 = -4\pi/k$. This results in the conservation of dual currents $J_z = (g^{-1}\partial_z g)$ and $J_{\bar{z}} = \partial_{\bar{z}} g g^{-1}$. Other relations between a and k do not lead to separately conserved (anti-)holomorphic currents.

To summarize: in the WZW theory, the (anti-)holomorphic currents $J_z(z)$ and $J_{\bar{z}}(\bar{z})$ are conserved separately. This indicates that the global $G_L \times G_R$ symmetry of the non-linear sigma model is extended to a semi-local $G(z)_L \times G(\bar{z})_R$ symmetry for the WZW model (see [25], 15.1.2).

[1] Recall from section 4.2 that on the complex plane, ϵ is normalized to $\epsilon_{z\bar{z}} = i/2$. Furthermore, the non-vanishing components of the metric are $g^{z\bar{z}} = g^{\bar{z}z} = 2$ and $g_{z\bar{z}} = g_{\bar{z}z} = 1/2$.

6.2.2 Current and Virasoro algebra

For further analysis, it is customary to introduce rescaled currents

$$J(z) := kJ_z(z) \qquad J(\bar{z}) := kJ_{\bar{z}}(\bar{z}) \tag{6.11}$$

These currents take values in the Lie algebra of G, so it is useful to expand them in a basis of generators τ_i, where $i = 1, \ldots, \dim G$.

$$J(z) = \sum_a J^a(z)\tau_a \tag{6.12}$$

A standard CFT calculation ([25], 15.1.3) shows that the currents J^a obey the OPE

$$J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + i f^{ab}{}_c \frac{J^c(w)}{z-w} \tag{6.13}$$

Where $f^{ab}{}_c$ are the structure constants of G.[2] Performing a Laurent expansion introduces the modes J^a_n of the currents J^a as

$$J^a(z) = \sum_n J^a_n z^{-n-1} \tag{6.14}$$

The mode algebra corresponding to the OPE (6.13) is

$$[J^a_n, J^b_m] = \sum_c i f^{ab}_c J^c_{n+m} + kn\, \delta^{ab}\delta_{n,-m} \tag{6.15}$$

In the literature, this algebra is known is a level k *current algebra* or *Kac-Moody algebra*. Note the central extension term proportional to k. There is some similarity with the central extension in the Virasoro algebra characterized by the central charge c in CFT; cf. equations (3.11) and (3.34).

For the zero-modes J^a_0, this term vanishes so they constitute a Lie algebra. The anti-holomorphic current $\bar{J}$ can be expanded analogously as

$$\bar{J} = \sum_a \bar{J}^a(z)\tau_a = \sum_a\sum_n \bar{J}^a_n \tau_a \bar{z}^{-n-1} \tag{6.16}$$

The modes $\bar{J}^a_n$ form another, independent copy of the affine algebra. In the following, I will focus on the holomorphic copy; the analysis extends to the anti-holomorphic symmetry.

For an affine algebra as introduced above, it is in general possible to combine modes J^a_n of the affine current to form new modes L_n that satisfy a Virasoro algebra. This statement implies that $G(z)_L \times G(\bar{z})_R$ invariance includes conformal invariance! From the CFT perspective, L_n is one of the modes of a traceless energy momentum tensor T. There are currents associated with conformal invariance, but these are not the only currents: the J^a modes that are still present are interpreted as primary fields with conformal dimension $(1, 0)$. This can be made more precise by defining the energy

[2] The δ^{ab} stems from the inner product of two elements of the Lie algebra of G. It is assumed that the group is simple, and the basis is chosen such that this inner product is proportional to δ, with normalization such that the long roots have length-squared 2. See comments in [64], 11.4 and 11.5; [51], 6.10.

momentum tensor T in terms of the currents, a procedure known as the *Sugawara* construction:

$$T(z) = \frac{1}{2(k+g)} \sum_a J^a(z) J^a(z) \tag{6.17}$$

Where g is the dual Coxeter number for the group G. This form is derived in detail in [25], section 15.2. The modes L_n of T constitute a Virasoro algebra with central charge

$$c = \frac{k \dim G}{k+g} \tag{6.18}$$

To complete the argument, it can also be shown that, with this energy-momentum tensor, the current J^a is indeed a conformal primary field of weight $(1, 0)$ ([25], 15.2).

6.2.3 The coset construction

The Wess-Zumino-Witten model as described in the previous section is a theory for elements g of a group G. The construction may be generalized by considering tensor products of different groups, that is $G = G_1 \times G_2 \times \ldots$. It is easily seen that the WZW action of such a product group is the sum of the WZW actions of the constituent groups. Another possibility is to consider a *coset* theory G/H where elements on the orbit of some subgroup H of the original symmetry group G are identified. These coset theories are especially relevant in string theory as their classical limit can be interpreted as a theory of strings in a non-trivial background; this will be described farther on.

If T_G is the energy-momentum tensor (by the Sugawara construction) of the G WZW model, and T_H is that for H, then the energy-momentum tensor for the coset theory is given by $T_{G/H} = T_G - T_H$, and the central charge is

$$c_{G/H} = c_G - c_H \tag{6.19}$$

The construction is known as the GKO construction, after the authors of [40]. It is also described in [25], section 18.1.

6.3 Gauging the WZW action

The commonly used procedure to analyze a coset theory starts from the WZW action for elements of the group G. The restriction to the coset space is done by parametrizing the elements g of G and choosing a gauge that includes only one element of H. To do this properly, one first has to gauge the theory. This section outlines and comments the gauging procedure. In the following section, an example will be presented where it is explicitly shown how the model relates to string theory in a background. It is assumed that G is semi-simple; the extension to non-semi-simple groups is not straightforward as formulating a WZW theory in not trivial. See the work by Sfetsos [70, 71].

Due to the peculiar $\Gamma[g]$ term in the action only subgroups that obey the following condition of anomaly cancellation can be gauged (see Witten [88]):

$$\mathrm{Tr}\,[T_{a,L} T_{b,L}] = \mathrm{Tr}\,[T_{a,R} T_{b,R}] \tag{6.20}$$

where T_a $(a = 1, 2)$ generate the action of H on g in the sense that

$$\delta g = \epsilon^a \left(T_{a,L}\, g + g\, T_{a,R} \right) \tag{6.21}$$

Assuming H is such a subgroup, one can gauge the vector[3] symmetry $g \to ugu^{-1}$ where $u(z, \bar{z}) \in H$ is a matrix-valued field. If H is abelian, it is also possible to gauge the axial symmetry $g \to ugu$. This choice represents a T-duality. As a CFT, the theory is self-dual but the target space interpretation can be very different; see e.g. Dijkgraaf et.al. [26].

The explicit form of the vector gauged action can be constructed by a trick that I will present in some detail.[4] In conformal coordinates, the WZW action is (with a shorthand notation for the Wess-Zumino term)

$$S_{WZW}[g] = \frac{k}{4\pi} \int_A d^2z \, \mathrm{Tr} \left[\partial g^{-1} \, \bar{\partial} g \right] + \frac{ik}{12\pi} \int_V d^3z \, \mathrm{Tr} \left[g^{-1} dg \right]^3 \tag{6.22}$$

Where the integration measure $d^2z = d^2x$ to match the normalization from section 6.1 with explicit examples discussed farther on and with appendix D.

Now consider the following action, where an auxiliary field $h \in H$ is introduced:

$$S_{gWZW}[g; h] := S_{WZW}[hgh^\dagger] - S_{WZW}[hh^\dagger] \tag{6.23}$$

It is manifestly invariant under

$$g \to ugu^{-1} \qquad h \to hu^{-1} \qquad h^\dagger \to uh^\dagger \tag{6.24}$$

Now one may apply a useful relation, known as the Polyakov-Wiegmann formula (see e.g. [77], which describes the same procedure)

$$S_{WZW}[gh] = S_{WZW}[g] + S_{WZW}[h] - \frac{k}{2\pi} \int_A d^2z \, \mathrm{Tr} \left[g^{-1} \partial g \, \bar{\partial} h h^{-1} \right] \tag{6.25}$$

Using this, expand the action (6.23):

$$\begin{aligned} S_{gWZW}[g; A_i] &= S_{WZW}[g] \\ &+ \frac{k}{2\pi} \int_A d^2z \, \mathrm{Tr} \left[-A \bar{\partial} g g^{-1} + \underline{\bar{A} g^{-1} \partial g} + \underline{g^{-1} A g \bar{A}} - A \bar{A} \right] \end{aligned} \tag{6.26}$$

Where gauge fields $A := h^{-1} \partial h$ and $\bar{A} := h^\dagger \bar{\partial} (h^\dagger)^{-1}$ that are elements of the algebra of H are introduced. By construction, the gauged action is invariant under the local transformation

$$g \to ugu^{-1} \qquad A \to u(A + \partial)u^{-1} \qquad \bar{A} \to u(\bar{A} + \bar{\partial})u^{-1} \tag{6.27}$$

The term $S_{WZW}[hh^\dagger]$ in the action (6.23) was necessary to avoid non-local $S_{WZW}[h]$ terms in the resulting action (6.26).

For abelian subgroups H, it is also possible to gauge the axial symmetry. A similar construction as above results in an action that resembles (6.26), but has additional

[3] The nomenclature is a little obscure; I will refer to this symmetry as a vector symmetry but in the literature there is some ambiguity. Note that this procedure differs from the more standard left or right coset construction that is based on symmetry transformations $g \to ug$ or $g \to gu$, respectively.

[4] A nice alternative motivation for this form of the gauged WZW action can be found in Ginsparg and Quevedo [37], section 2.1

minus-signs for the underlined terms. To see this, start with $S_{WZW}[hg\tilde{h}] - S[h\tilde{h}^{-1}]$, where $\tilde{h} = (h^{-1})^*$. It is invariant under $\tilde{h} \to u^{-1}\tilde{h}$, $h \to u^{-1}h$ as H is abelian. This axially gauged action is invariant under the combined transformation

$$g \to ugu \qquad A \to u(A + \partial)u^{-1} \qquad \bar{A} \to u(\bar{A} + \bar{\partial})u^{-1} \tag{6.28}$$

Having an action that is locally invariant under the transformations induced by the subgroup H, it is possible to fix the gauge and reduce the symmetry of the model. To do so, first introduce coordinates on (a patch of) the coset space G/H. Usually, there are several coordinate patches and it can be difficult to find global coordinates that cover the full coset space.

As the gauge fields appear at most quadratically, they may be integrated out. This is done be finding their equations of motion from the action, solving these and reinserting the dynamical behavior into the action. This leaves an integral over a two-dimensional worldsheet for the $D = \dim(G) - \dim(H)$ scalars that parametrize the coset.

The crucial point for string cosmology is that this action may be identified with the standard action for strings in a non-trivial massless background (cf. equation (5.10), in conformal coordinates)

$$S = \frac{1}{2\pi\alpha'} \int_A d^2z\, (G_{\mu\nu} + B_{\mu\nu})\, \partial x^\mu \bar{\partial} x^\nu + \frac{1}{8\pi} \int_A d^2z R\, \Phi \tag{6.29}$$

The coset model thus yields an D-dimensional target space for string theory. The metric field G and the antisymmetric background field B can usually be read off directly from the action after integrating out the gauge fields. The dilaton Φ emerges from a non-trivial determinant when actually performing this integration, but can also be reconstructed by considering its low-energy string equation of motion. The reader will find comments on the relation between isometries for the background fields and symmetries of the gauged WZW model in the following section, in the context of an explicit example.

The correspondence implies $k = 1/\alpha'$, so the low-energy limit $\alpha' \to 0$ (discussed in section 5.1.4) corresponds to $k \to \infty$. As k is usually related to the central charge of the underlying CFT (via equation (6.18)), it is not obvious that this limit can be taken. If not so, higher order α' corrections should in principle be considered.

In references [6, 10] Bars and Sfetsos argue that in fact the gauged WZW model as introduced above is only valid in the classical regime. They suggest that quantum corrections to the background fields should be calculated using a Hamiltonian method [6], or using an effective quantum action [10], which is a modified version of the (gauged) WZW model. For the ungauged WZW action, the modification is an overall renormalization $k \to k - g$, with g the Coxeter number of the group G. For the gauged action the modification is more difficult and includes non-local terms. In the classical limit $k \to \infty$, which will be of primary interest in the following sections, this quantum theory has the same classical limit as the gauged WZW action introduced above. This is also discussed by Tseytlin [77].

6.4 Example: Witten's black hole

To illustrate the coset construction described in the beginning of this chapter, I will present and discuss a famous example introduced by Witten [87] based on the coset

$SL(2,\mathbb{R})/\mathbb{R}$. Introducing the gauged WZW action for the model and integrating out the gauge fields results in a non-linear sigma model from which a black-hole geometry can be read off. I consider the Lorentzian black hole but use a different coordinate patch as in [87].

Consider a theory in which the matrices in the WZW action are elements of the non-compact group $SL(2,\mathbb{R})$. To be explicit, take the following matrices as a basis[5] of the Lie algebra:

$$\tau_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{6.30}$$

In the following, the non-compact subgroup H generated by τ_2, is going to be gauged in an axial sense. That is, the action is modified in such a way that it becomes invariant under the local transformation $g \to ugu$ with $u = i\epsilon(z,\bar{z})\tau_2$. Written explicitly, g transforms as

$$g \to \epsilon(z,\bar{z}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g + \epsilon(z,\bar{z})\, g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.31}$$

The gauged action can be taken directly from the general equation (6.26) from the previous section – with the minus-signs as this is an axial symmetry. Express the gauge fields as

$$A = -ia(z,\bar{z})\tau_2 \qquad \bar{A} = -i\bar{a}(z,\bar{z})\tau_2$$

where $a(z,\bar{z})$ and $\bar{a}(z,\bar{z})$ are independent fields that are c-numbers in matrix space. Now fix the gauge by introducing coordinates (x,t) on the coset space G/H as

$$g(x,t) = \begin{pmatrix} \cosh t & e^{-x}\sinh t \\ e^{x}\sinh t & \cosh t \end{pmatrix} \tag{6.32}$$

These coordinates do not cover the whole coset space. By using this specific parametrization one only considers a region of the total space, a situation often encountered in this kind of models.

The gauged action now contains the physical fields x, t as well as the gauge fields $A, \bar{A}$ and reads

$$\begin{aligned} S &= \frac{k}{2\pi}\int_A d^2z \left(-\partial t\bar{\partial}t + \sinh^2 t\,\partial x\bar{\partial}x\right) + 0 \\ &+ \frac{k}{2\pi}\int_A d^2z \left(-2a\sinh^2 t\bar{\partial}x + 2\bar{a}\sinh^2 t\partial x - 4a\bar{a}\cosh^2 t\right) \end{aligned} \tag{6.33}$$

Where 0 denotes the vanishing of the Wess-Zumino term for these coordinates (cf. [87], equation (33) or appendix D) in this coordinate patch. As the action is at most quadratic in the gauge fields, it is easy to derive their equations of motion and solve them. Consequently, these fields can be integrated out by inserting their dynamics back in the action. For the example studied here, this results in the following action:

$$S = \frac{k}{2\pi}\int_A d^2z \,\left(-\partial t\bar{\partial}t + \tanh^2 t\,\partial x\bar{\partial}x\right) \tag{6.34}$$

[5] I use the convention that group elements are generated by $e^{i\alpha_i\tau_i}$, with parameters α_i and generators τ_i. For a short summary on $SU(2)$, $SU(1,1)$ and $SL(2,\mathbb{R})$, see appendix A.

Comparing this with the general action for strings in a background (6.29), suggests that the model presented here can be interpreted as string theory in a two-dimensional spacetime with metric

$$ds^2 = \frac{k}{2\pi}\left(-dt^2 + \tanh^2 t\, dx^2\right) \tag{6.35}$$

And no B field. This metric has a spacetime interpretation as a region of the Lorentzian signature black hole. A closer analysis shows that the dilaton should be included (compare to [87], equation (12)):

$$\Phi = -\ln\cosh t \tag{6.36}$$

So that the following equation holds (cf. (5.22))

$$\mathbf{R}_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi = 0 \tag{6.37}$$

6.4.1 Isometries

The symmetries of the background fields are closely related to the global symmetries of the original WZW action. The gauged action has, apart from the local symmetry which it possesses by construction, the original global $SL(2,\mathbb{R})$ symmetry. Therefore, transforming $g \to ugu$ with $u \in SL(2,\mathbb{R})$ in the gauged WZW action is just a reformulation of the theory. If one applies this transformation of g to the gauge fixing equation (6.32), the string action that follows from the construction necessarily has the same form. For the background fields, that are read off from this action, this is an isometry. For example, choosing $u = i\tau_1$, the transformation corresponds to $x \to -x$ which is clearly a symmetry of the metric and the dilaton. This argument applies to abelian groups G; for non-abelian groups the global symmetry is lost in the gauging procedure (see [37], sections 2.2 and 4).

This should be contrasted with the possibility to gauge a different subgroup. In that case, the resulting spacetime can (and generally will) be very different. In the same article that introduces the example described above, the $SL(2,\mathbb{R})$ symmetry is also divided by the compact group generated by τ_3. The resulting string action has an interpretation as a part of the Euclidean black hole.

6.5 Scalar propagation on a coset

The coset construction is a tool to find non-trivial string backgrounds. Having found such a background, it may be interesting to study the behavior of scalar states on these backgrounds. For example, the nature of possible singularities can be probed in such a way, a subject that will be readdressed in section 7.1.3. Consider a scalar X of mass m minimally coupled to the metric and dilaton:

$$S_X = \int d^D x\sqrt{G}e^{-2\Phi}\left(-G^{\mu\nu}\partial_\mu X\partial_\nu X - m^2X^2\right) \tag{6.38}$$

The equation of motion for X, which will be referred to as the wave equation, is

$$\left(\nabla_\mu\nabla^\mu - m^2\right)X - 2\nabla_\mu\Phi\nabla^\mu X = 0 \tag{6.39}$$

Which is the usual wave equation for a scalar particle if Φ is constant. It is assumed that X is small enough not to influence the background fields (no backreaction). In a generic background this equation can be highly complex. Of course, one can attempt to straightforward solve it, but this bypasses the symmetries that are known to be present from the coset construction. Using the theory of group representations, these symmetries can be used to find solutions to the wave equation. In the following, this method will be decribed in more detail and applied to the background considered in the previous section. It is based on the work of Vilenkin and Klimyk [83], a thorough introduction in the subject. For a more general account on group theoretical aspects of cosets, see Gilmore [36], especially sections 6.II, 9.IV and 9.V.

For definiteness, I restrict to the coset model from section 6.4: G/H with $G = SL(2, \mathbb{R})$ and $H = U(1)$. The generalization to other groups and subgroups is straightforward. The basic strategy is to find an infinite dimensional representation of elements of G on the space of functions. The freedom one has to explicitly construct such a representation is invested to diagonalize the action of the subgroup H that is gauged. That is, choose a basis in function space in such a way that the action of the subgroup is trivial. Once the action of the subgroup on this basis is established, the physical space G can be reduced to G/H by considering only states that are invariant under the action of the subgroup. What follows is a more explicit discussion of this construction.

Adopting the notation of [83], a general matrix $g \in SL(2, \mathbb{R})$ can be expressed as

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \alpha\delta - \beta\gamma = 1 \qquad \alpha, \beta, \gamma, \delta \in R \tag{6.40}$$

The aim of the following is to construct the *principal continuous representation*, which is an infinite dimensional representation of $SL(2, \mathbb{R})$ acting on functions, labeled by $\chi = \{\tau, \epsilon\}$. For clarity, I will denote the representation of a group element by $\hat{T}_\chi[g]$ and use g strictly to refer to the defining representation, i.e. two dimensional matrices. Let $f(x)$ be a function on the Hilbert space of square-integrable functions $L_2(\mathbb{R})$. This space has the usual inner product

$$\langle f_1, f_2 \rangle = \int_{-\infty}^{\infty} dx\, f_1(x) \overline{f_2(x)} \tag{6.41}$$

Now define the action of an element g of the abstract group on a function f:

$$\left(\hat{T}_\chi[g] \cdot f \right)(x) = |\beta x + \delta|^{2\tau} \operatorname{sign}^{2\epsilon}(\beta x + \delta)\, f\left(\frac{\alpha x + \gamma}{\beta x + \delta} \right) \tag{6.42}$$

It is easily verified that

$$\hat{T}_\chi[g_1] \cdot \hat{T}_\chi[g_2] \cdot f = \hat{T}_\chi[g_1\, g_2] \cdot f \tag{6.43}$$

Which assures this is a valid representation. Unitarity with regard to the above inner product reads

$$\langle f_1, f_2 \rangle = \langle \hat{T}_\chi[f_1]\, ,\, \hat{T}_\chi[f_2] \rangle \tag{6.44}$$

Writing this out implies $\tau = is - 1/2$, where $s \in R$. The parameter ϵ takes values in $\{0, 1/2\}$. As will become clear, its role is rather limited but the aboved introduction is formally correct. In the following, the analysis is extended to the space of generalized functions.

In the space of functions one is free to choose a basis. It is very convenient to choose a set of functions that are eigenfunctions under the action of a subgroup of $SL(2,\mathbb{R})$. Continuing the example from the previous section, consider the *hyperbolical* subgroup generated by τ_2.

$$e^{it\tau_2} = e^{-t\sigma_3} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \tag{6.45}$$

For such a one-parameter subgroup, the associated generator $\hat{A}$ in the space of functions follows readily from the operator equation

$$\hat{A} = -i\frac{d}{dt}\hat{T}_\chi\left[e^{-t\sigma_3}\right]\Big|_{t=0} = 2i(\tau - x\partial_x) \tag{6.46}$$

Now choose as a basis functions $|f_m\rangle$ with $m \in \mathbb{R}$ that are eigenfunctions of the operator $\hat{A}$. The basis consists of two distinct sets of functions (see [83], 7.1.3):

$$\begin{aligned} |f_m\rangle &= \{x_+^{\tau+im}, x_-^{\tau+im}\} \\ \hat{A}\cdot|f_m\rangle &= 2m\,|f_m\rangle \\ \hat{T}_\chi[e^{it\tau_2}]\cdot|f_m\rangle &= e^{2imt}\,|f_m\rangle \end{aligned} \tag{6.47}$$

Here x_+ and x_- are generalized functions defined to be equal to x for $x \geq 0$, $x \leq 0$, respectively. The explicit form of the functions will be needed later, at this point it is possible to continue more formally.

The usefulness of this construction follows when considering the matrix element

$$(f_m, f_{m'}) := \langle f_m|\,\hat{T}_\chi[g]\,|f_{m'}\rangle \tag{6.48}$$

and apply the symmetry transformation $g \to e^{t\sigma_3}\,g\,e^{t'\sigma_3}$:

$$\langle f_m|\,\hat{T}_\chi[e^{t\sigma_3}\,g\,e^{t'\sigma_3}]\,|f_{m'}\rangle = e^{-2i(tm+t'm')}\langle f_m|\,\hat{T}_\chi[g]\,|f_{m'}\rangle \tag{6.49}$$

Which leads to conclude that the matrix element is an eigenfunction of this transformation; the $g \to e^{t\sigma_3}\,g\,e^{t'\sigma_3}$ symmetry on the group manifold is diagonalized.

Now consider the Casimir operator if the group, i.e. the operator that commutes with all generators. The same reasoning as above leads to the conclusion that $(f_m, f_{m'})$ is an eigenfunction of the Casimir operator on the group manifold. As this operator is the generator for free propagation, these eigenfunctions are naturally associated with free scalar particle states on the group manifold. By construction, the behavior of these states under the transformation by the gauged subgroup is straightforward. This means one can study particle propagation and interactions on the smooth $SL(2,\mathbb{R})$ manifold and apply the results to the coset space by restricting to states invariant under the transformation.

The discussion will be limited to the principal continuous representation; there are different representations, such as the principal discrete representation that may lead to different but equally valid scalar states. See Vilenlik and Klimyk [83].

6.6 Example: propagation on the black hole solution

To make the construction explicit, let's revisit the coset model introduced in section 6.4. Introducing $Q = \sqrt{k/\pi}$ and rescaling $t \to Qt$ the metric (6.35) and dilaton (6.36) take the form

$$\begin{aligned} ds^2 &= -dt^2 + Q^{-2}\tanh^2 Qt\, dx^2 \\ \Phi(t) &= -\ln\cosh Qt \end{aligned} \tag{6.50}$$

I will outline the procedure described in [23] that solves the wave equation (6.39) for a scalar T in exactly this background. To simplify things a little bit, consider plane waves with momentum p

$$T(x,t) = \tilde{T}(t)e^{ipx} \tag{6.51}$$

The wave equation for a such a scalar with mass M in the background reads[6]

$$Q^2\left(p^2\frac{\cosh^2 Qt}{\sinh^2 Qt} + M^2\right)\tilde{T}(t) + Q\left(\frac{1+2\sinh^2 Qt}{\sinh Qt\cosh Qt}\right)\frac{d}{dt}\tilde{T}(t) + \frac{d^2}{dt^2}\tilde{T}(t) \tag{6.52}$$

This equation could be solved directly in terms of hypergeometric functions but, as stated in the beginning of this section, it is much more elegant (and instructive) to use the representation method.

With the knowledge that the background originated from a coset of $SL(2,\mathbb{R})$, it is natural to consider matrix elements of the representation of this group as introduced above:

$$(f_m, f_{m'}) := \langle f_m|\,\hat{T}_\chi[g]\,|f_{m'}\rangle \tag{6.53}$$

The claim was that this expression solves the equation of motion for a scalar particle on $SL(2,\mathbb{R})$. This will not be shown explicitly as the coset space is of primary interest. Restricting to the coset implies that the matrix element should be invariant under the transformation $g \to e^{t\sigma_3}\, g\, e^{t\sigma_3}$. From equation (6.49) it is clear that this means restricting the basisfunctions by imposing $m = -m'$.

Then next step is to fix the gauge in the subgroup by choosing coordinates on G/H. Parametrize g similar to equation (6.32)

$$\begin{aligned} g(x,t) &= \begin{pmatrix} \cosh Qt & e^{-x}\sinh Qt \\ e^{x}\sinh Qt & \cosh Qt \end{pmatrix} \\ &= \begin{pmatrix} e^{-x/2} & 0 \\ 0 & e^{x/2} \end{pmatrix}\begin{pmatrix} \cosh Qt & \sinh Qt \\ \sinh Qt & \cosh Qt \end{pmatrix}\begin{pmatrix} e^{x/2} & 0 \\ 0 & e^{-x/2} \end{pmatrix} \\ &= e^{-x/2\,\sigma_3}\, g_I(Qt)\, e^{x/2\,\sigma_3} \end{aligned} \tag{6.54}$$

Defining the matrix g_I in the last line. With the choice of basisfunctions the matrix element reduces to

$$(f_m, f_{m'}) := e^{ix(m-m')}\langle f_m|\,\hat{T}_\chi[g_I(Qt)]\,|f_{m'}\rangle \tag{6.55}$$

Upon identifying $p = 2m = -2m'$ it shows that the plane-wave ansatz was correct. Up to this point the calculations were simplified by a smart choice of basisfunctions. The difficult part is to calculate the t-dependence of the solution as the basisfunctions are not eigenfunctions of the associated generator. This will not be done here; the

[6]This is equivalent to equation (4.6) in [23].

reader is referred to [83], section 7.2.1 where region h corresponds to g_I. I just state as a result that the functions $\langle f_m | \hat{T}_\chi[g_I(Qt)] \, | f_{m'} \rangle$ fall into two independent sets:

$$\begin{array}{ll} f_{++}(Qt; m, m', \tau) & \text{and} \\ f_{--}(Qt; m, m', \tau) & \end{array}$$

corresponding to the two sets of basisfunctions. The functions $f_{\pm\pm}$ can be found in appendix B.2. After matching the parameters $2s = \sqrt{M^2 + p^2 - 1}$, it can be verified that

$$T(t, x) = (f_m, f_{m'}) = e^{ix(m-m')} f_{\pm\pm}(Qt; m, m', s) \tag{6.56}$$

is indeed a solution to the wave equation (6.52).

7

Cosmological models from cosets

The previous chapter introduced the reader to gauged WZW models and their relation to string theory. The two-dimensional black hole that was discussed is a well-known toy model for more complex cosmological scenarios. This chapter is concerned with 'realistic' cosmological scenarios that originate from a coset construction – realistic in the sense that the model has a target space of four dimensions, with a single time coordinate.

This chapter splits into three sections. Section 7.1 discusses in some more detail the '$3+1$' requirement mentioned in the previous paragraph. It also discusses how spacetime singularities, interesting from a cosmological point of view, typically arise in gauged WZW models. Section 7.2 presents some cosets that fulfill the requirements and were studied in the literature. Finally, section 7.3 focuses on one of these models that was presented by Nappi and Witten. The model is reviewed in some detail and propagation of scalar states is discussed in analogy with section 6.5.

7.1 General features

The section discusses some general properties that a coset should have to allow for a cosmological interpretation. A number of models in the literature are discussed in the next section. Besides physical requirements, these cosets (as any coset) must obey a consistency condition in the sense of anomaly cancellation; see equation (6.20).

Recall from equation (6.18), that the level k of the current algebra is related to the central charge of a WZW model. For a coset theory G/H, the central charge is related to the level of the current algebras associated with G and H as expressed in equations (6.18) and (6.19). The string theory interpretation of a coset model thus requires matching k to get an appropriate value for the central charge. For pure bosonic string theory, this implies the condition condition $c = 26$. More generally, it is possible to consider a theory that splits into an internal CFT for the compact dimensions and an external CFT for the 'large' dimensions, as discussed in section 5.3.2. This can be used to reduce c to e.g. $c = 4$ (flat Minkowski space in $D = 4$) or any other appropriate value. Because of this possibilities, I will not consider specific conditions on k. However, keep in mind that at the end of the day the value of c should be justified.

7.1.1 Dimensionality

Let's first consider the possibilities to construct a coset G/H with a four dimensional target space. The manifold G has dimension dim G; gauging the subgroup H reduces the number of degrees of freedom by dim H, so the first constraint is obviously

$$\text{dim } G - \text{dim } H = 4 \tag{7.1}$$

Of course, G may be a direct product of lower-dimensional groups. In this way, the $SL(2,\mathbb{R})$ model or its analytic continuation $SU(2)$ can be used as building blocks to generate higher-dimensional backgrounds. The resulting groups are typically semi-simple.

A large class of cosets can be constructed by taking G to be a direct products of $SU(n)$s and $SL(n,\mathbb{R})$s. The dimension of G follows from dim $SU(n)$ = dim $SL(n,\mathbb{R}) = n^2 - 1$. To end up with a four dimensional space, one chooses H to consist of lower-dimensional subgroups; if the resulting coset has dimension greater than four, the difference can be cancelled by including additional $U(1)$s or $\mathbb{R}$s in H. However, as n gets large, the gap between dim $SU(n)$ and dim $SU(n-1)$ grows as $2n$ and one needs an increasing number of $U(1)$s or $\mathbb{R}$s. As there is a limit to the number of commuting $U(1)$s and $\mathbb{R}$s in a group, this process stops at some value of n.[1]

7.1.2 Signature

The next obvious requirement is that of a single time coordinate. In the theory of Lie groups, it is well known that one may define a metric on the space spanned by elements of the algebra of a group. This metric is known as the Cartan-Killing metric; for an excellent introduction, see Gilmore [36], section 7.III. The signature of this metric is directly related to the compactness of generators of the group. Gilmore uses conventions in which the Cartan-Killing metric associated with a compact group is negative definite. For reasons that will become clear below, I will not be very precise about an overall minus sign but use the fact that the metric associated with a group with n compact and m non-compact generators has signature (n, m). Restricting to a coset G/H where H has n' (m') (non)compact generators leads to a metric on the coset with signature $(n - n', m - m')$.

The gauged WZW formulation of a coset theory differs in two ways from the standard coset construction as presented in e.g. Gilmore [36]. Firstly, it is based on gauging the vector symmetry $g \to ugu^{-1}$ or the axial symmetry $g \to ugu$ rather then a left symmetry $g \to ug$ or right symmetry $g \to gu$. Secondly, the Wess-Zumino term adds an element of torsion to the background, which is interpreted as a non-zero B field. However, the signature of the background metric that originates from the gauged WZW model corresponding to a coset model is the same as the more standard coset metric (Ginsparg and Quevedo, [37], 2.3). To identify cosets with signature $(1, 3)$, one may use standard literature on cosets, such as table 9.7 of [36]. This table is the basis of table 7.1.2, which shows the number of (non)compact generators for some groups that will be used farther on.

A last subtlety is the appearance of the level of the current algebra k in front of the WZW action (6.4). Replacing $k \to -k$ yields an overall minus-sign in the background fields (cf. (6.35)). From the CFT perspective, this defines a dual theory as was

[1] I would like to acknowledge Jan de Boer for this argument.

discussed in section 6.2.1. For the purpose of finding a cosmological background this makes the overall sign of the metric irrelevant as it is always possible to consider the dual theory after putting $k \to -k$. Again, at the end of the day, k should be justified in the sense that is related to the Virasoro central charge of the underlying CFT.

Group G	compact generators G_c	non-compact generators G_{nc}
$SO(n)$	$n(n-1)/2$	0
$SO(m,n)$	$n(n-1)/2 + m(m-1)/2$	$n\,m$
$SO(n;\mathbb{C})$	$n(n-1)/2$	$n(n-1)/2$
$SL(n;\mathbb{R})$	$n(n-1)/2$	$n(n+1)/2 - 1$
$SU(n)$	$n^2 - 1$	0
$U(1)$	1	0
$\mathbb{R}$	0	1

Table 7.1: Number of compact, noncompact generators for some well-known groups.

7.1.3 Singularities

It is hoped that string theory may provide a framework to investigate the nature of spacetime singularities. This hope is partly motivated from the abstract notion of large-small duality (see section 5.2.3), which leads to quite fundamental questions about the big bang singularity. Propagation of strings in the presence of singularities is for example discussed by Craps et.al. in reference [23]. See also the work of Horowitz and Steif [46, 47].

Spacetime singularities are known to emerge in coset models. Typically, these singularities occur at fixed points of the gauge transformation symmetry operation $g \to ugu^{-1}$ or $g \to ugu$. This implies that there is some residual, discrete symmetry left after fixing the gauge on the coset space. Though the curvature may be finite at these points in spacetime, other physical quantities blow up and the singularity is called an orbifold singularity. An example of this is presented for the Nappi-Witten universe in section 7.3.2. Here I will discuss a general argument that shows how a fixed point of the symmetry transformation can result in a singularity in the resulting metric.

Consider the vector gauged WZW action as stated in equation (6.26). Expand the gauge fields $A, \bar{A}$ in term of generators τ_i of H

$$A = A^i \tau_i \qquad \bar{A} = \bar{A}^i \tau_i \tag{7.2}$$

Integrating out the fields A^i and $\bar{A}^i$ results in (see Ginsparg and Quevedo [37], section 2.2)

$$\begin{aligned} S &= S_{WZW}[g] - \frac{k}{\pi} \int d^2z \mathrm{Tr}[\tau_i g^{-1} \partial g] \, \mathrm{Tr}[\tau_j \bar{\partial} g g^{-1}] \, \Lambda_{ij}^{-1} \\ \Lambda_{ij} &:= \mathrm{Tr}[\tau_i \tau_j - \tau_i g \tau_j g^{-1}] \end{aligned} \tag{7.3}$$

Now consider the transformation $g \to ugu^{-1}$ for infinitesimal $u = \mathbb{1} + \epsilon^i \tau_i$:

$$g \to (\mathbb{1} + \epsilon^i \tau_i) g (\mathbb{1} - \epsilon^i \tau_i) = g + \epsilon^i (\tau_i g - g \tau_i) \tag{7.4}$$

So g is left invariant by the transformation if $\tau_i g = g \tau_i$ – it is a fixed point. Multiplying this equality with $g^{-1} \tau_j$ and taking a trace implies that $\Lambda_{ij} = 0$. The action, from which the background fields are read off, blows up resulting in a singular metric.

7.2 Models in the literature

In the previous section, the requirements for an interesting coset construction from a cosmological point of view were stated: the target space is supposed to be four-dimensional with a single timelike direction. This section presents some references to the literature where exactly such cosets were considered. The result is presented in table 7.2, which has no pretension of being complete but presents some models encountered in the literature.

In most references, $U(1)$ and $\mathbb{R}$ are used interchangeably to denote a one-parameter subgroup. Of course, they are closely related as $U(1)$ elements are generated by e^{it}, where $t \in \mathbb{R}$. The important difference is that $U(1)$ is compact, whereas $\mathbb{R}$ is non-compact. As discussed in the previous section, this is of importance in determining the metric on a group, so I will be rather strict and use $U(1)$ to denote a compact subgroup and $\mathbb{R}$ to denote a non-compact subgroup.

Coset (G/H)	G_c	G_{nc}	H_c	H_{nc}	sign.	refs.
$\frac{SO(3,2)}{SO(3,1)}$	4	6	3	3	(1,3)	[5, 37]
$\frac{SO(2,2)}{SO(1,1)\times SO(2)}$	2	4	1	1	(1,3)	[37]
$\frac{SO(4,1)}{SO(3,1)}$	6	4	3	3	(3,1)	[8, 37]
$\frac{SO(3;\mathbb{C})}{SO(3)} \times U(1)$	4	3	3	0	(1,3)	[37]
$\frac{SO(3,1)}{SO(3)} \times U(1)$	4	3	3	0	(1,3)	[37]
$\frac{SO(2,2)}{SO(2,1)} \times \mathbb{R}$	2	5	1	2	(1,3)	[37]
$\frac{SL(2,\mathbb{R})}{SO(1,1)} \times \mathbb{R} \times \mathbb{R}$	1	4	0	1	(1,3)	[37]
$\frac{SO(2,\mathbb{R})}{U(1)} \times U(1) \times \mathbb{R}$	2	3	1	0	(1,3)	[37]
$\frac{SL(2,\mathbb{R})\times SO(1,1)^2}{SO(1,1)}$	1	4	0	1	(1,3)	[37, 52]
$\frac{SL(2,\mathbb{R})\times SU(2))}{\mathbb{R}\times U(1)}$	4	2	1	1	(3,1)	[8, 35, 62]
$\frac{SL(2,\mathbb{R})\times SL(2,\mathbb{R})}{\mathbb{R}\times U(1)}$	2	4	1	1	(1,3)	[8, 45]
$\frac{SO(3,1)}{SO(2,1)} \times U(1))$	4	3	1	2	(3,1)	[7]
$\frac{SL(2,\mathbb{R})\times U(1)^2}{\mathbb{R}}$	3	2	0	1	(3,1)	[35]
$\frac{SL(2,\mathbb{R})\times SU(2)}{SO(1,1)\times U(1)}$	4	1	1	1	(3,1)	[52]
$\frac{SL(2,\mathbb{R})\times SL(2,\mathbb{R})}{SO(1,1)\times U(1)}$	2	4	1	1	(1,3)	[52]

Table 7.2: Some cosets with $3+1$ dimensional metric. It is understood that all possible embeddings of the subgroup $H \subset G$ should be considered. The references are discussed in this section.

In reference [37], Ginsparg and Quevedo provide a list of coset spaces with a single timelike coordinate in table 3. They claim that the list is the most general if G is a product of simple non-compact groups, $U(1)$s and $\mathbb{R}$s. Cosets based on non-semisimple groups are discussed in a qualitative way, but explicit examples are not presented.[2]

The cosets mentioned in this article constitute the first part of table 7.2. Note that most of these associate compact generators with timelike directions ($k > 0$), but there is one exception for which it is assumed that $k < 0$.

[2] The extension to non-semi-simple groups is not straightforward as formulating a WZW theory in not trivial. See the work by Sfetsos [70, 71].

The article explicitly discusses the D dimensional coset theory based on the coset $SL(2,\mathbb{R}) \times SO(1,1)^D/SO(1,1)$ for general D. In $D = 4$, the coset has a $3+1$ dimensional metric which makes it interesting for cosmology. The coset based on a vector gauging procedure is that of a two dimensional black hole with two flat dimensions; the axially gauged coset leads to a spacetime that is the product of a three dimensional black string and an extra dimension. Both constructions yield a non-zero B field.

Bars and Sfetsos have published a number of articles that discuss the construction of coset models with interesting cosmological interpretation. In reference [8], they consider the anti-de Sitter coset $SO(3,2)/SO(3,1)$. It is shown that the coset results in a spacetime with singularities. The possibility of a coset $SO(4,1)/SO(3,1)$ for negative k is mentioned in a footnote.

The article [9], by the same authors, discusses two cosets that were not mentioned in [37], see table 7.2. The first coset is also considered by Nappi and Witten [62] and will be discussed in detail in section 7.3. The geometry following the second coset can be derived from the first by analytic continuation.

Another article [7] by Bars and Sfetsos starts with a list of cosets (arbitrary dimensions) that yield a single time coordinate based on simple non-compact groups. This list can also be found in [37], table 1. The article focuses on the three dimensional cosets $SO(2,2)/SO(2,1)$ and $SO(3,1)/SO(2,1)$. In section 7, it is argumented that these models have a cosmological interpretation by adjoining an additional $U(1)$ or $\mathbb{R}$. The first possibility was already included in table 7.2.

Gershon [35] considers a coset similar to the one considered by Nappi and Witten. The parametrization is different which results in a spacetime interpretation of a four-dimensional Lorentzian black hole. Another coset $[SL(2,\mathbb{R}) \times U(1)^2]/\mathbb{R}$ is discussed and interpreted as a black membrane in four dimensions.

In reference [52], Kounnas and Lüst consider the $[SL(2,\mathbb{R}) \times SO(1,1)^2]/SO(1,1)$ coset model that was also investigated in detail in [37] for an arbitrary dimension. The authors discuss the relation between c and k for this coset and find that the classical limit $k \to \infty$ can be considered by taking $c = 4$, requiring an internal CFT with $c = 22$ for the full string theory. In a region of the full space, they find that the metric at late times describe an isotropic, linearly expanding Robertson-Walker type universe. At early times, the model is anisotropic and there is no big bang singularity. Two other cosets are mentioned briefly (see table 7.2).

Reference [45] by Horava completes this list. The article presents a class of solutions based on the gauging of the direct product of two $SL(2,\mathbb{R})$s. The gauging procedure is discussed in detail, as is the anomaly cancellation condition. The exact gauging procedure depends on a model-defining parameter $\alpha \in \mathbb{R}$ that introduces a 'distortion'; for $\alpha = 0$ the geometry is the product of a Euclidean and a Minkowskian $2D$ black hole.

7.3 The Nappi-Witten universe

In this section, I discuss a model introduced by Nappi and Witten [62]. It is a well-known coset model that yields cosmologically interesting background fields. The resulting spacetime can be interpreted as a universe that starts from a big bang and recollapses with a big crunch. The WZW action is based on the direct product $SU(2) \times SL(2,\mathbb{R})$. It has a free model-defining parameter α that determines the gauging of the subgroup.

In this section, I will comment on the construction as presented in the article [62] and discuss two different coordinate patches on the coset space. Regarding the free parameter, I will focus on the behavior of the coset space at $\alpha = 0$ and $\alpha = \pi/2$ as these will prove to be interesting 'limits'. In analogy with the discussion in sections 6.5 and 6.6, propagation of a scalar state on the background is analyzed using group representation techniques.

Recently, there has been quite some interest in the model. Elitzur et.al. [27] consider a model based on the NW scenario for $\alpha = 0$ that describes a series of expanding and contracting universes and investigate particle propagation on it. Where applicable, I will relate some findings for $\alpha = 0$ to this article. In the follow-up article [28] they expand the model by an extra $U(1)$ group and consider a very general action of the subgroup that covers general values of α.

7.3.1 The NW subgroup and the gauged action

In the following, take $g_1 \in SL(2,\mathbb{R})$ and $g_2 \in SU(2)$. The Nappi-Witten (NW) model is defined by gauging a two-dimensional subgroup $H \subset SL(2,\mathbb{R}) \times SU(2)$ that acts on group elements g_1, g_2 as

$$\begin{aligned} g_1 &\rightarrow g_1' = e^{\epsilon\sigma_3} g_1 e^{\bar{\epsilon}\cos\alpha\sigma_3 + \epsilon\sin\alpha\sigma_3} \\ g_2 &\rightarrow g_2' = e^{i\bar{\epsilon}\sigma_2} g_2 e^{i\epsilon\cos\alpha\sigma_2 - i\bar{\epsilon}\sin\alpha\sigma_2} \end{aligned} \tag{7.5}$$

In these transformations, α is a free model-defining parameter that introduces 'mixing' in the model. For $\alpha = \pi/2$, H is a direct product of two subgroups and the transformation (7.5) is just a combination of an axial transformation on g_1 with parameter ϵ and a vector one with parameter $\bar{\epsilon}$ on g_2. Setting $\alpha = 0$, the transformation on each group depends on both parameters ϵ and $\bar{\epsilon}$. Recall the condition for anomaly cancellation (6.20):

$$\mathrm{Tr}\left[T_{a,L}T_{b,L}\right] = \mathrm{Tr}\left[T_{a,R}T_{b,R}\right] \tag{7.6}$$

For the product group, the trace operator is defined as $\mathrm{Tr} = k_1\mathrm{Tr}_1 + k_2\mathrm{Tr}_2$ with Tr_1 the trace in the defining representation of $SL(2,\mathbb{R})$ and k_1 the level of its WZW action; likewise for Tr_2, k_2. For H defined by (7.5) this implies $k_1 = -k_2$. The central charge of the WZW model for elements of G is (cf. (6.18))

$$c = \frac{3k_1}{k_1+2} + \frac{3k_2}{k_2+2} \tag{7.7}$$

As H is isomorphic to $\mathbb{R} \times U(1)$, the gauging procedure reduces the central charge by 2. Demanding that the central charge of the coset model is 4 (just like flat Minkowski space) implies $k_1 = k_2 + 4$. This is inconsistent with the condition from anomaly cancellation and in principle implies that it is not possible to construct a $c = 4$ coset CFT with these G and H. However, the main interest is in the classical regime $k_1, k_2 \rightarrow \infty$, where the discrepancy disappears. This was also noted in [62].

Introducing gauge fields A and B corresponding to ϵ and $\bar{\epsilon}$, the gauged action takes the form[3]

$$S_{NW} = \frac{k_1}{4\pi}\int_A d^2z\,\mathrm{Tr}\left[\partial g_1^{-1}\bar{\partial}g_1\right] + \frac{ik_1}{24\pi}\int_V d^3z\,\mathrm{Tr}\left[g_1^{-1}dg_1\right]^3 \tag{7.8}$$

[3]Regarding the notation, A here corresponds to A_z in [62]; $\bar{A} \sim A_{\bar{z}}$, $B \sim \widetilde{A_z}$, $\bar{B} \sim \widetilde{A_{\bar{z}}}$. Furthermore, (k', k) in [62] is (k_1, k_2) here.

$$
\begin{aligned}
&+ \frac{k_1}{2\pi}\int_A d^2z\,\mathrm{Tr}\left[-A\sigma_3\bar\partial g_1 g_1^{-1} - \sin\alpha\,\bar A\sigma_3 g_1^{-1}\partial g_1 - \sin\alpha\,A\sigma_3 g_1\bar A\sigma_3 g_1^{-1}\right]\\
&+ \frac{k_1}{2\pi}\int_A d^2z\,\mathrm{Tr}\left[-\cos\alpha\,\underline{\bar B\sigma_3 g_1^{-1}\partial g_1} - \cos\alpha\,\underline{A\sigma_3 g_1\bar B\sigma_3 g_1^{-1}} - A\sigma_3\bar A\sigma_3\right]\\
&+ \frac{k_2}{4\pi}\int_A d^2z\,\mathrm{Tr}\left[\partial g_2^{-1}\,\bar\partial g_2\right] + \frac{ik_2}{24\pi}\int_V d^3z\,\mathrm{Tr}\left[g_2^{-1}dg_2\right]^3\\
&+ \frac{k_2}{2\pi}\int_A d^2z\,\mathrm{Tr}\left[-Bi\sigma_2\bar\partial g_2 g_2^{-1} + \sin\alpha\,\bar B i\sigma_2 g_2^{-1}\partial g_2 + \sin\alpha B i\sigma_2 g_2\bar B i\sigma_2 g_2^{-1}\right]\\
&+ \frac{k_2}{2\pi}\int_A d^2z\,\mathrm{Tr}\left[-\cos\alpha\,\underline{\bar A i\sigma_2 g_2^{-1}\partial g_2} - \cos\alpha\,\underline{Bi\sigma_2 g_2\bar A i\sigma_2 g_2^{-1}} - Bi\sigma_2\bar B i\sigma_2\right]
\end{aligned}
$$

As discussed above, it is understood that $k_1 = -k_2$. The action is exactly equal to equation (8) of [62], yet written a bit more explicit.

Most terms are left unsimplified to avoid obscuring the structure. Note how the gauge fields mix in the underlined terms, all of which have a factor $\cos\alpha$. As this already suggests, for $\alpha = \pi/2$ the mixing terms drop out and the action reduces to the sum of an axially gauged WZW action for g_1 and a vector gauged action for g_2. Putting $\alpha = 0$ and identifying both gauge fields, it reduces to the sum of two axially gauged WZW actions. This is consistent with the specific action of the subgroup as specified in equation (7.5).

7.3.2 Coordinates and the spacetime background

Before integrating out the gauge fields, define coordinates on the coset space. A general element of the covering space $SU(2)\times SL(2,\mathbb{R})$ can be parametrized as (for $SL(2,\mathbb{R})$, see e.g. Elitzur et.al. [27], section 2.2) :

$$
\begin{aligned}
g_1(a,\psi,b;\epsilon_1,\epsilon_2,\delta) &= e^{a\sigma_3}(-1)^{\epsilon_1}(i\sigma_2)^{\epsilon_2}g_\delta(\psi)e^{b\sigma_3}\\
g_2(\gamma,s,\beta) &= e^{i\gamma\sigma_2}e^{is\sigma_3}e^{i\beta\sigma_2}
\end{aligned}
\tag{7.9}
$$

The notation is chosen to comply with the original article [62], as is the parametrization of $SU(2)$ elements which is different from that described in appendix B.1. The action (7.5) of the subgroup H now takes the convenient form

$$
\begin{aligned}
a &\to a+\epsilon\\
b &\to b+\bar\epsilon\cos\alpha+\epsilon\sin\alpha\\
\gamma &\to \gamma+\bar\epsilon\\
\beta &\to \beta+\epsilon\cos\alpha-\bar\epsilon\sin\alpha\\
\psi,s,\epsilon_1,\epsilon_2,\delta &\to \psi,s,\epsilon_1,\epsilon_2,\delta
\end{aligned}
\tag{7.10}
$$

This makes it particularly clear how the gauge orbit is parametrized by ϵ and $\bar\epsilon$. Restricting to the coset space can be thought of as picking only one point out of every set of points that can be reached by the subgroup transformation (7.10). In this context, this process is usually referred to as choosing a gauge. The resulting coordinates parametrize a certain patch of the coset space.

There are a number of ways to fix the gauge. In this section, I will follow the construction in [62], which imposes the gauge fixing conditions on the $SL(2,\mathbb{R})$ sector as $a = b = 0$. Other possibilities are discussed in section 7.3.3. Following [62], parametrize g_1 by

$$
g_1(\psi) = e^{i\psi\sigma_2}
\tag{7.11}
$$

This fixes the gauge and limits the discussion to one region in $SL(2,\mathbb{R})$. This is the region denoted as II in [27], corresponding to $\epsilon_1 = \epsilon_2 = 0$; $\delta = II$ in equation (7.9). In particular, this choice implies that the Wess-Zumino term vanishes for g_1 vanishes, as may be verified from appendix D. For g_2, it takes the form

$$\Gamma[g_2] = \frac{ik_2}{4\pi}\int_V d^2x\, \epsilon^{jk}(\partial_j\beta\partial_k\gamma)\cos 2s \tag{7.12}$$

In conformal coordinates, this is the $\cos 2s$ term in equation (13) of [62]. Having parametrized g_1 and g_2, the machinery as described in section 6.4 can be applied to the model. The gauge fields in the action (7.8) can be integrated out. This yields the string action from which the spacetime metric G and anti-symmetric tensorfield B can be read off. Without copying the analysis in [62], I quote the resulting G in coordinates $\rho = \gamma + \beta$ and $\lambda = \gamma - \beta$:

$$\begin{aligned}
G_{\psi\psi} &\propto -1 \\
G_{ss} &\propto 1 \\
G_{\rho\rho} &\propto \frac{2\cos^2 s\cos^2\psi(1+\sin\alpha)}{1-\cos 2\psi\cos 2s+\sin\alpha(\cos 2\psi-\cos 2s)} \\
G_{\lambda\lambda} &\propto \frac{2\sin^2 s\sin^2\psi(1-\sin\alpha)}{1-\cos 2\psi\cos 2s+\sin\alpha(\cos 2\psi-\cos 2s)}
\end{aligned} \tag{7.13}$$

The proportionality constant is $k_2/2\pi$. The antisymmetric tensorfield B is equal to

$$B_{\rho\lambda} \propto \frac{(\cos 2\psi-\cos 2s)+\sin\alpha(1-\cos 2s\cos 2\psi)}{1-\cos 2\psi\cos 2s+\sin\alpha(\cos 2\psi-\cos 2s)} \tag{7.14}$$

And of course $B_{\lambda\rho} = -B_{\rho\lambda}$. The dilaton follows either from the gauge field integration or by imposing the equations for the background fields in low-energy string theory. It is equal to [62]

$$\phi = -\frac{1}{2}\ln(1-\cos 2\psi\cos 2s)+\sin\alpha(\cos 2\psi-\cos 2s) \tag{7.15}$$

These fields will be referred to as the NW background. As none of the components of the metric depends on either λ or ρ, it is clear that the metric has two isometries.

For generic values of α, the background (7.13) is perfectly well-behaved. For example, putting $\alpha = 0$ reduces G to

$$\begin{aligned}
G_{\psi\psi} &\propto -1 \\
G_{ss} &\propto 1 \\
G_{\rho\rho} &\propto -\frac{2\cos^2 s\cos^2\psi}{\cos 2s\cos 2\psi-1} \\
G_{\lambda\lambda} &\propto -\frac{2\sin^2 s\sin^2\psi}{\cos 2s\cos 2\psi-1}
\end{aligned} \tag{7.16}$$

However, at $\alpha = \pi/2$, the λ coordinate decouples from the metric as $G_{\lambda\lambda} = 0$. The metric degenerates and does no longer properly describe the coset space.

Spacetime singularities

A number of features of the spacetime defined by these fields is easily understood for arbitrary α. As $G_{\rho\rho} \geq 0$ and $G_{\lambda\lambda} \geq 0$, the space has signature $(-{+}{+}{+})$ and it is natural to interpret ψ as the time coordinate. At $\psi = 0$ there is the singularity $G_{\lambda\lambda} = 0$ while at $\psi = \pi/2$ one has $G_{\rho\rho} = 0$. This suggests the interpretation as a universe that starts to expand from a collapsed state at $\psi = 0$ to recollapse at $\psi = \pi/2$. The nature of the singularities is not obvious. At $\psi = 0$, the curvature scalar R is

$$R|_{\psi=0} = -\frac{2\cos^2 s(-7+3\sin\alpha)}{\sin^2 s(1+\sin\alpha)} \tag{7.17}$$

So for $s = 0$ and arbitrary α there is a curvature singularity. For other values of s the singularity is of a different nature: it is an orbifold singularity. In constructing backgrounds from gauged WZW models, a general phenomenon is that fixed points of the subgroup that is gauged lead to spacetime singularities.

In the Nappi-Witten model, the transformation (7.5) has fixed points, i.e. there is a discrete subgroup of H that leaves $g_1(\psi)$ and $g_2(s,\rho,\lambda)$ invariant for special values of ψ. Consider $\alpha = 0$, $s \neq 0$, for which values the curvature scalar is clearly finite, yet there is a singularity for special values of ψ. The action of the subgroup (7.10) becomes

$$\begin{aligned} g_1 &\to g_1' = e^{\epsilon\sigma_3} g_1 e^{\bar{\epsilon}\sigma_3} \\ g_2 &\to g_2' = e^{i\bar{\epsilon}\sigma_2} g_2 e^{i\epsilon\sigma_2} \end{aligned} \tag{7.18}$$

As $\exp(2\pi n\, i\sigma_2) = \mathbb{1}$ for every $n \in \mathbb{Z}$, g_2 is invariant under the $\mathbb{Z}\times\mathbb{Z}$ subgroup of H defined by $\epsilon = 2\pi n$, $\bar{\epsilon} = 2\pi m$. At $\psi = 0$, the group element g_1 is invariant under this transformation if $n = -m$. At the time of recollapse $\psi = \pi/2$, g_1 is invariant for $n = m$.

For general values of α the situation is more subtle, but similar [62]. The spacetime singularities at both $\psi = 0$ and $\psi = \pi/2$ correspond to the invariance of g_1 and g_2 under a discrete subgroup of H isomorphic to $\mathbb{Z}$.

7.3.3 Other patches

Before considering different gauge choices, let's recollect the construction in the previous section. The Nappi-Witten scenario stems from a WZW model based on $G = SL(2,\mathbb{R})\times SU(2)$ with a gauged subgroup $H \subset G$ that acts on elements of G according to equation (7.5). For definiteness, I will restrict to the region $(\epsilon_1 = \epsilon_2 = 0;\ \delta = II)$ of $SL(2,\mathbb{R})$ that was also considered in the above, so that

$$\begin{aligned} g_1(a,\psi,b) &= e^{a\sigma_3} e^{i\psi\sigma_2} e^{b\sigma_3} \\ g_2(\gamma,s,\beta) &= e^{i\gamma\sigma_2} e^{is\sigma_3} e^{i\beta\sigma_2} \end{aligned} \tag{7.19}$$

The spacetime background (7.13) that was explored in the previous sections originates from the gauged action by the particular coordinate patch (7.11). In the language of equation (7.10) this choice of coordinates is equivalent to fixing the gauge as

$$a = b = 0 \tag{7.20}$$

In this section, I comment on other choices of coordinates. For some choices, there is a discrete subgroup of H that preserves the gauge condition which implies that one has to make identifications. I will focus on the behavior at $\alpha = 0$ or $\alpha = \pi/2$, as every patch becomes invalid at one of these choices.

Alternative one: $\gamma = \beta = 0$

Following Elitzur et.al. [27] consider the alternative gauge choice[4]

$$\gamma = \beta = 0 \tag{7.21}$$

Which slices through the $SU(2)$ sector rather than through the $SL(2,\mathbb{R})$. This particular choice is not unique in the sense that there are different choices for γ and β that lead to the same g_2. This important subtlety will be addressed farther on, but first the resulting spacetime is discussed. Again, it will prove to be useful to introduce coordinates $\rho = a + b$ and $\lambda = a - b$.

In the parametrization (7.19) with $\gamma = \beta = 0$, the Wess-Zumino term vanishes for g_2. For g_1, it takes the form (see appendix D)

$$\Gamma[g_1] = \frac{ik_1}{2\pi} \int_V d^2x \, \epsilon^{jk} (\partial_j \lambda \partial_k \psi) \sin\psi \cos\psi \ln\left(e^{\rho} \cos\psi\right) \tag{7.22}$$

The background fields that arises can be found for arbitrary α in appendix C, equations (C.2) and (C.4). In the case $\alpha = 0$, the resulting metric G is exactly the same as found in the Nappi-Witten patch, as given in equation (7.16).[5] The antisymmetric field B is different though, as can be seen by comparing equation (7.14) from the Nappi-Witten patch with (C.4) for the patch currently considered.

For $\alpha = \pi/2$, the metric degenerates. As follows from (C.2), $G_{\lambda\lambda} = 0$ which implies this coordinate patch is no longer valid.

As mentioned above, the gauge choice (7.21) is not unique. Recall from equation (7.19) that

$$g_2(\gamma, s, \beta) \quad = \quad e^{i\gamma\sigma_2} e^{is\sigma_3} e^{i\beta\sigma_2} \tag{7.23}$$

As can easily be verified, g_2 is invariant under a transformation

$$\gamma \to \gamma + (n_1 + n_2)\pi \qquad \beta \to \beta + (n_1 - n_2)\pi \tag{7.24}$$

Where n_1 and n_2 are understood to be integers and this particular form will be convenient farther on. This means that both γ and β can only be defined modulo π and the gauge fixing condition (7.21) is only valid up to a shift with π. This implies that there exists a discrete subgroup of H that leaves the gauge condition invariant. Comparison with equation (7.10) shows that the subgroup is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ as

$$\begin{aligned} \bar{\epsilon} \quad &= \quad (n_1 + n_2)\,\pi \qquad \epsilon\cos\alpha - \bar{\epsilon}\sin\alpha = (n_1 - n_2)\,\pi \qquad \Rightarrow \\ \epsilon \quad &= \quad \frac{\pi}{\cos\alpha}(n_1(\sin\alpha + 1) + n_2(\sin\alpha - 1)) \end{aligned} \tag{7.25}$$

[4]This article only considers $\alpha = 0$. Furthermore, the authors consider the action of the subgroup that acts as $(g_1, g_2) \to (e^{\epsilon\sigma_3} g_1 e^{\bar{\epsilon}\sigma_3}, e^{i\bar{\epsilon}\sigma_3} g_2 e^{i\epsilon\sigma_3})$. In the $SU(2)$ sector, this amounts to gauging the subgroup generated by σ_3 rather then σ_2. The parametrization of $SU(2)$ is also different so that this subgroups acts conveniently. I will stick to the original subgroup but do consider the gauge choice discussed in this article. The results are very similar as the reader may appreciate by comparing this section with [27].

[5]This is the same geometry as given in equation (3.7) of [27]. To see this, associate coordinates $(\theta, \theta', \lambda_- \lambda_+)$ used in that article with (ψ, s, λ, ρ) used here. Note that the region in $SL(2,\mathbb{R})$ specified by (7.19) corresponds to region II so that λ_- and λ_+ are understood to be interchanged in (3.7).

Where the last step is only valid for $\alpha \neq \pi/2$. So the condition (7.21) fixes the gauge up to transformations with ϵ, $\bar{\epsilon}$ given in (7.25). This means that any two elements of $SL(2,\mathbb{R}) \times SU(2)$ that are connected by such a transformation should be identified. In the $SL(2,\mathbb{R})$ sector this leads to non-trivial modifications. Keeping things fully general in that sector – not restricting to the region ($\epsilon_1 = \epsilon_2 = 0$; $\delta = II$) – it follows that an arbitrary matrix g_1 transforms under (7.5) with transformation parameters (7.25) as

$$\begin{aligned} g_1^{(1,1)} &\rightarrow g_1^{(1,1)} \exp\left(2\pi n_1 \frac{1+\sin\alpha}{\cos\alpha}\right) \\ g_1^{(1,2)} &\rightarrow g_1^{(1,2)} \exp\left(-2\pi n_2 \frac{1-\sin\alpha}{\cos\alpha}\right) \\ g_1^{(2,1)} &\rightarrow g_1^{(2,1)} \exp\left(2\pi n_2 \frac{1-\sin\alpha}{\cos\alpha}\right) \\ g_1^{(2,2)} &\rightarrow g_1^{(2,2)} \exp\left(-2\pi n_1 \frac{1+\sin\alpha}{\cos\alpha}\right) \end{aligned} \tag{7.26}$$

Which reduces to equation (3.5) of Elitzur et.al. [27] for $\alpha = 0$. The resulting spacetime is described in more detail in [27]; it consists of a series of closed universes with 'whiskers' attached at the big crunch - big bang singularity. For generic non-zero values of α, things change only quantitatively. The identifications imply that one should consider a fundamental domain for the coordinates that come from $SL(2,\mathbb{R})$. In terms of a general element g_1, such a domain is

$$\begin{aligned} 1 &\geq \left|\frac{g_1^{(1,2)}}{g_1^{(2,1)}}\right| > \exp\left(-4\pi\frac{1-\sin\alpha}{\cos\alpha}\right) \\ 1 &\geq \left|\frac{g_1^{(1,1)}}{g_1^{(2,2)}}\right| > \exp\left(-4\pi\frac{1+\sin\alpha}{\cos\alpha}\right) \end{aligned} \tag{7.27}$$

This reduces to equation (3.6) of [27] for $\alpha = 0$. If α is non-zero the fundamental domains get larger or smaller. For the region ($\epsilon_1 = \epsilon_2 = 0$; $\delta = II$) corresponding to g_1 from (7.19), the fundamental domain reads

$$\begin{aligned} 0 &\geq \lambda > -2\pi\frac{1-\sin\alpha}{\cos\alpha} \\ 0 &\geq \rho > -2\pi\frac{1+\sin\alpha}{\cos\alpha} \end{aligned} \tag{7.28}$$

So that the 'size' of the compact spatial directions ρ and λ change if $\alpha \neq 0$.

This description, and equation (7.26) in particular, breaks down for $\alpha = \pi/2$. To analyze this situation, start with the original transformations (7.10). For $\alpha = \pi/2$, it follows that a transformation with $\bar{\epsilon} = n_1\pi$ preserves the gauge condition without any restrictions on ϵ. With these parameters, g_1 transforms as

$$\begin{aligned} g_1^{(1,1)} &\rightarrow g_1^{(1,1)} \exp(2\epsilon) \\ g_1^{(2,2)} &\rightarrow g_1^{(2,2)} \exp(-2\epsilon) \end{aligned} \tag{7.29}$$

Which would imply that a and b have to be fixed in order to restrict to a fundamental domain in $SL(2,\mathbb{R})$. This proves the problematic behavior at $\alpha = \pi/2$.

Alternative two: $a = \beta = 0$

It is of course interesting to investigate this further. Is the problematic behavior for $\alpha = \pi/2$ a problem specific to the coordinate patches discussed so far or is it a general feature of the coset itself? The following section introduces a third gauge choice that has no pathologies for $\alpha = \pi/2$ which shows that the coset is in fact well defined.

The setup is very similar to what was discussed above. Consider the parametrization of $SL(2,\mathbb{R}) \times SU(2)$ as given in equation (7.19). Fix the gauge as

$$a = \beta = 0 \tag{7.30}$$

Which means that I apply one gauge fixing condition on both the $SL(2,\mathbb{R})$ and the $SU(2)$ sector.

Let's look at the background fields that emerge from this model. From the discussion in appendix D, it follows that the Wess-Zumino term for g_2 vanishes. For g_1, it takes the form

$$S_{WZ}[g_1] = \frac{k_1}{2\pi} \int d^2z \ln\left(e^b \cos\psi\right) \sin 2\psi \left(\partial\psi\bar{\partial}b - \partial b\bar{\partial}\psi\right) \tag{7.31}$$

Inserting this, plus the coordinates, in the gauged action (7.8) and integrating out the gauge fields yields an action from which the low-energy metric G and anti-symmetric tensor field B can be read off. The general result can be found in appendix C. The reader may verify that there is no decoupling of coordinates for $\alpha = \pi/2$; in fact the metric takes the form

$$\begin{aligned} G_{\psi\psi} &\propto -1 \\ G_{ss} &\propto 1 \\ G_{bb} &\propto \tan^2\psi \\ G_{\gamma\gamma} &\propto \tan^{-2} s \end{aligned} \tag{7.32}$$

Where the proportionality factor is again $k_2/2\pi$. This shows that there is a coordinate patch in which the coset is perfectly well-defined for $\alpha = \pi/2$. Taking $\alpha = 0$, the metric (C.5) again degenerates. With this choice of coordinates it is not immediately clear, but one may verify that G is no longer linearly independent for this value of α. This is particularly clear in coordinates $\gamma + b$, $\gamma - b$.

Just as in the previous analysis, the gauge choice $a = \beta = 0$ is not unique. The choice $a = 0$ uniquely specifies a group element g_1 – a result of the non-compactness of the subgroup generated by $\exp(a\sigma_3)$. However, a general g_2 is left invariant by the transformation

$$\beta \to \beta + 2\pi n_1 \tag{7.33}$$

Note the factor 2 relative to (7.24). Again, this leads to the conclusion that there exists a discrete subgroup of H that preserves the gauge choice. From equation (7.10),

$$\epsilon = 0 \qquad \bar{\epsilon} = \frac{2\pi n_1}{\sin\alpha} \tag{7.34}$$

This transformation acts on the other coordinates as

$$\begin{aligned} b &\to b + 2\pi n_1 \frac{\cos\alpha}{\sin\alpha} \\ \gamma &\to \gamma + \frac{2\pi n_1}{\sin\alpha} \end{aligned} \tag{7.35}$$

These identifications are perfectly well-defined for $\alpha = \pi/2$, for which value there are no identifications on b. The situation for $\alpha = 0$ is similar to the case $\alpha = \pi/2$ in the previous patch. From (7.10), it follows that any transformation with $\epsilon = 0$ and arbitrary $\bar{\epsilon}$ preserves the gauge condition which invalidates the description.

Three patches, one space

In this section, I considered three coordinate patches on the Nappi-Witten coset, corresponding to the choices $a = b = 0$, $\gamma = \beta = 0$ and $a = \beta = 0$. The resulting background fields can be found in equations (7.13), (C.2) and (C.5), respectively. It was found that the metric of the first two patches degenerates at $\alpha = \pi/2$ but is well-defined at $\alpha = 0$. The situation is completely opposite in the last patch; it behaves properly at $\alpha = \pi/2$ but degenerates at $\alpha = 0$.

Do the coordinate patches describe the same space where they overlap? All three geometries emerge from the Nappi-Witten coset with the subgroup defined by equation (7.5) so they are all of signature $(-{+}{+}{+})$. They all have two isometries, a necessary condition to be identical. The fact that all metrics only depend on ψ and s suggest a more detailed way to compare them: consider a point on the six-dimensional space with coordinates

$$\begin{array}{ccc} a = 0 & \psi = \psi & b = 0 \\ \gamma = 0 & s = s & \beta = 0 \end{array} \tag{7.36}$$

Each of the patches considered includes this particular point. A straightforward calculation shows that, for $0 < \alpha < \pi/2$, the curvature scalar for each of the patches is identical and equal to

$$R = -\frac{(22 + 6\cos 2\alpha)(\cos 4\psi - \cos 4s) + 32\sin\alpha(\cos 2\psi \sin^2 2s + \cos 2s \sin^2 2\psi)}{4(1 + \sin\alpha\cos 2\psi - \cos 2s(\cos 2\psi + \sin\alpha))^2}$$

Which reduces to (C.3) for $\alpha = 0$, but this is only valid in the first two patches. For $\alpha = \pi/2$, only the geometry of the last patch is well defined. For this value of α, the scalar curvature indeed reduces to (C.6).

7.3.4 Construction of scalar states

The previous section discussed a number of coordinate patches on the Nappi-Witten coset space. In this section, I will restrict to the original patch that was defined by equation (7.11). The resulting space has geometry as given in (7.13) and antisymmetric tensorfield (7.14). These fields will be referred to as the Nappi-Witten background in this section, that discusses propagation of scalar states on that background. The discussion is similar to that of section 6.5.

Scalar states on the NW background must obey the wave equation (6.39) for the background. For a general scalar field $T = T(\psi, s, \lambda, \rho)$ this equation is both lengthy and untransparent so it will not be stated here. However, assuming that the scalar field is constant in two spatial dimensions and making the ansatz $T = T(\psi, s)$, the equation reduces to

$$\begin{aligned} m^2 T(\psi, s) &= \left(\tan^{-1} s - \tan s\right) \frac{\partial}{\partial s} T(\psi, s) + \left(\tan\psi - \tan^{-1}\psi\right) \frac{\partial}{\partial \psi} T(\psi, s) \\ &+ \frac{\partial^2}{\partial s^2} T(\psi, s) - \frac{\partial^2}{\partial \psi^2} T(\psi, s) \end{aligned} \tag{7.37}$$

Which is independent of α, a feature special to the simplified situation considered here. The general wave equation for an arbitrary four-dimensional scalar field does depend on α.

In order to find solutions of the scalar wave equation on the NW background, the technique of group representations will be applied. Using the parametrization introduced in the previous section in equations (7.9) and (7.11), the general solution $T(\psi, s, \lambda, \rho)$ of the wave equation on $SL(2,\mathbb{R}) \times SU(2)$ splits into

$$\begin{aligned} T(\psi, s, \lambda, \rho) &= T_1(\psi)\, T_2(s, \lambda, \rho) = \langle f_m |\, \hat{T}[g_1]\, | f_{m'} \rangle \, \langle \tilde{f}_n |\, \hat{T}[g_2]\, | \tilde{f}_{n'} \rangle \\ &= (f_m, f_{m'})\, (\tilde{f}_n, \tilde{f}_{n'}) \end{aligned} \tag{7.38}$$

The bases $|f_m\rangle$ and $|\tilde{f}_n\rangle$ can of course be chosen differently if this is convenient. The bases can be constructed to diagonalize the transformation (7.10) in the sense that

$$\langle f_m |\, \hat{T}[g_1']\, | f_{m'} \rangle = e^{-2i(m+m'\sin\alpha)\epsilon - 2i(m'\cos\alpha)\bar{\epsilon}} \, \langle f_m |\, \hat{T}[g_1]\, | f_{m'} \rangle \tag{7.39}$$

$$\langle \tilde{f}_n |\, \hat{T}[g_2']\, | \tilde{f}_{n'} \rangle = e^{-2i(n'\cos\alpha)\epsilon - 2i(n-n'\sin\alpha)\bar{\epsilon}} \, \langle \tilde{f}_n |\, \hat{T}[g_2]\, | \tilde{f}_{n'} \rangle \tag{7.40}$$

Invariance of $T(\psi, s, \lambda, \rho)$ under the transformation (in order to consider the coset space) then implies

$$\begin{aligned} m + m'\sin\alpha + n'\cos\alpha &= 0 \\ m'\cos\alpha + n - n'\sin\alpha &= 0 \end{aligned} \tag{7.41}$$

For $\alpha = \pi/2$ the equations separate into one restriction on the functions $|f_m\rangle$ and one on the functions $|\tilde{f}_n\rangle$ while for different values of α the restrictions mix. Choosing $\alpha = 0$ the mixing is maximal: m relates to n' and m' relates to n. In the following, the matrix elements $(f_m, f_{m'})$ will be evaluated first and then $(\tilde{f}_n, \tilde{f}_{n'})$.

The $SL(2,\mathbb{R})$ sector

The basis of functions $|f_m\rangle$ should be chosen such that it diagonalizes the transformation (7.5)

$$g_1 \to e^{\epsilon\sigma_3}\, g_1\, e^{\bar{\epsilon}\cos\alpha\sigma_3 + \epsilon\sin\alpha\sigma_3} \tag{7.42}$$

The reader will recognize the hyperbolic subgroup $e^{t\sigma_3}$ introduced in equation (6.45). Applying the results of the analysis in that section shows that the basis $\{x_+^{\tau_1+im}, x_-^{\tau_1+im}\}$ leads to the correct transformation behavior as expressed in equation (7.39). Not the whole analysis that follows can be taken over as the gauge condition used here is different from the one used there; an other region of $SL(2,\mathbb{R})$ is considered.

As before, the evaluation of $(f_m, f_{m'}) = \langle f_m |\, \hat{T}_{\chi_1}[e^{i\psi\sigma_2}]\, | f_{m'} \rangle$ is taken from Vilenkin and Klimyk [83]. The reader may refer to section 7.2.1 where region u corresponds to the region chosen here[6]. The matrix elements are given by the functions $g_{++}(\psi; m, m', \tau)$ and $g_{--}(\psi; m, m', \tau)$, which can be found in appendix B.2.

The $SU(2)$ sector

Up to this point, all matrix elements originated from representing elements of $SL(2,\mathbb{R})$ by the principal continuous representation series. A similar construction can be set up for $SU(2)$ – a well-studied subject in quantum mechanics so I will be brief. Consider an

[6] See also [27] where this is region II.

system of arbitrary spin with the three spin generators J_1, J_2, J_3. As these operators do not commute among themselves it is convenient to choose a basis that diagonalizes one operator, say J_3. Let $|\phi_m\rangle$ be such a basis with eigenvalues

$$J_3 \cdot |\phi_n\rangle = n\,|\phi_n\rangle \tag{7.43}$$

As is well known, n takes integer or half-integer values ranging from $-j$ to j, where $j(j+1)$ is the eigenvalue corresponding to the Casimir operator J^2. This choice of basis implies the triviality of the matrix element

$$\langle\phi_n|e^{J_3 t}|\phi_{n'}\rangle = \delta_{nn'}\,e^{nt} \tag{7.44}$$

Explicit calculations are needed to evaluate matrix elements that involve J_1 and J_2, just as in the $SL(2,\mathbb{R})$ analysis.

These rather formal statements do not specify the representation of states and operators. For example, one can think of a spin-1/2 system, and take $J_i = \sigma_i/2$ as operators in a two-dimensional vector space. On the other hand, one can think of the basis states as functions in function space and the spin operators as some representation of $SU(2)$ acting on these functions. This point of view provides the link with the analysis on representations of $SL(2,\mathbb{R})$. In the following, this connection is stressed by using the notation $J_i = \hat{T}_l[\sigma_i]$, where l labels different representations.

Opposed to every textbook example, it is the subgroup generated by J_2 that is to be diagonalized. Suppose that the basis $|\phi_n\rangle$ is defined as above, while the basis $|\tilde{f}_n\rangle$ is a set of eigenvectors of J_2:

$$J_2 \cdot |\tilde{f}_n\rangle = \hat{T}_l[\sigma_2] \cdot |\tilde{f}_n\rangle = n\,|\tilde{f}_n\rangle \tag{7.45}$$

Using the coordinates γ, s and ρ that parametrize the $SU(2)$ sector in the NW analysis, the matrix elements may be computed as[7]

$$\begin{aligned}(\tilde{f}_n, \tilde{f}_{n'}) &= \langle\tilde{f}_n|\,\hat{T}_l[e^{i\gamma\sigma_2}e^{is\sigma_3}e^{i\beta\sigma_2}]\,|\tilde{f}_{n'}\rangle \\ &= \langle\phi_n|\,\hat{T}_l[e^{i\gamma\sigma_3}e^{-is\sigma_2}e^{i\beta\sigma_3}]\,|\phi_{n'}\rangle \\ &= e^{-2in\gamma}e^{-2in'\beta}\,\langle\phi_n|\,\hat{T}_l[e^{-is\sigma_2}]\,|\phi_{n'}\rangle\end{aligned} \tag{7.47}$$

The last matrix element is given by the Wigner D-function $d^l_{-n,-n'}(2s)$ stated explicitly in appendix B.2. For further reference, see e.g. [81], which uses a parametrization in Euler angles[8]

$$\hat{D}^{1/2}(-2\gamma, 2s, -2\beta) = e^{i\gamma\sigma_3}e^{-is\sigma_2}e^{i\beta\sigma_3} \tag{7.48}$$

It is easily verified that $(\tilde{f}_n, \tilde{f}_{n'})$ as defined above has the correct transformation behavior (7.40) under the action of the NW subgroup (7.10).

[7] In the second line, I use a trick that may need some comment. Denote $J_i = \hat{T}_l[\sigma_i]$. From equations (7.43) and (7.45), the exponential factors follow directly. Consider a unitary operator U that acts as $\phi_n = U\tilde{f}_n$ which is just a change of basis. As J_2 and J_3 have the same eigenvalues, it can be shown that $J_2 = U^{-1}J_3U$. Use this to find

$$\langle\phi_n|\,J_2\,|\phi_{n'}\rangle = \langle\phi_n|\,U^{-1}J_3U\,|\phi_{n'}\rangle = \langle\tilde{f}_n|\,U^{-2}J_3U^2\,|\tilde{f}_{n'}\rangle \tag{7.46}$$

Which may be explicitly considered in the defining representation to find eq. (7.47).

[8] Though the definition of Euler angles differs from that adopted in appendix B.1; $\hat{D}^{1/2}(\alpha_1,\alpha_2,\alpha_3) = u(-\alpha_1 - \pi/2, -\alpha_2, -\alpha_3 + \pi/2)$ where u is defined as in equation (B.2)

This completes the construction of scalar states on the Nappi-Witten background. Summarizing, the following set of scalar states was found (in coordinates ψ, s, λ and ρ):

$$T(\psi, s, \lambda, \rho) = e^{-i(\rho+\lambda)n} e^{-i(\rho-\lambda)n'} \, g_{\pm\pm}(\psi; m, m', \tau) \, d^l_{-n,-n'}(2s) \tag{7.49}$$

The functions g_{--}, g_{++} and $d^l_{n,n'}$ are given explicitly in appendix B.2. There are α-dependent restrictions on (n, n', m, m') as expressed in eq. (7.41). From the $SU(2)$ sector there is the additional constraint that both n and n' are (half-)integer valued and bounded by $|n|, |n'| \leq l$.

This clarifies the fact that for the ansatz $T(\psi, s)$ the wave equation was independent of α. The ansatz corresponds to setting $n = n' = 0$ in the general solution. Except for the special value $\alpha = \pi/2$ this implies $m = m' = 0$ for generic α and restrictions on the states are no longer α-dependent.

8

Conclusion

String theory is a huge subject. I hope that this thesis serves the reader as a useful resource for one aspect of string theory: how string theory is a theory of gravity and what implications this may have on cosmology. In this conclusion, I will summarize the main line of argumentation and emphasize some findings.

A first introduction to string theory usually starts by considering classical strings that vibrate and propagate in D-dimensional Minkowski spacetime, sweeping out an area known as the worldsheet. The dynamical behavior can be formulated by the Polyakov action which is then used to analyze the quantum theory (chapters 3 and 4):

$$S_P = -\frac{1}{4\pi\alpha'}\int_A d\sigma d\tau\,\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu\,\eta_{\mu\nu}$$

From the string worldsheet point of view, the Polyakov action defines the dynamics of a set of scalar fields X that live on the two-dimensional worldsheet. The X fields are coupled covariantly to the worldsheet. They are contracted by the spacetime Minkowksi metric η which just enumerates the fields. From the worldsheet perspective, this action defines a free field theory for each X^μ so that this set spans Minkowski space in D dimensions.

From a cosmological point of view, it is highly interesting to consider strings in a more general background. It can be shown that a naive generalization of the Polyakov action by substituting $\eta \to G(X)$ is the correct way to account for a coherent background of graviton states (section 5.1.2). From the worldsheet perspective, this generalization means that the X fields are no longer free but have interaction terms with coupling constants related to derivatives of G so that they span some manifold (section 5.1.2). Such a model is historically known as a non-linear sigma model. It may be generalized to include the other background fields of closed bosonic string theory: the dilaton Φ and anti-symmetric tensorfield B (section 5.1.2).

String theory in Minkowski space is characterized by reparametrization and rescaling invariance (section 4.3.2). Rescaling invariance is a typical feature of conformal field theories (cf. chapter 3), which explains their relevance to string theory. If the non-linear sigma model is to describe strings in a non-trivial background, it is reasonable to demand that these symmetries are present. This imposes highly non-trivial constraints on the set of allowed background fields G, B and Φ (section 5.1.3). In turn, these constraints can be derived from a low-energy effective action – which defines the bosonic string theory of gravity (section 5.2). This is valid for low energies and large distances, where a string reduces to a point particle (more precise: up to first order in an α' expansion; section 5.1.4).

The theory reduces to general relativity if the dilaton is constant. From comparison with Brans-Dicke theory (cf. section 2.2.2) we know that the dilaton cannot play a dynamical role in our present universe. In string theory, this means that there should be some dilaton potential so that the dilaton is now settled near its minimum. Such a potential has huge implications on the dynamical evolution of the universe (section 5.3.3). A currently interesting question is whether the de Sitter (dS) solution of general relativity (which solves the Einstein field equations with a positive cosmological constant; cf. section 2.3) is also allowed in string gravity. The example of section 5.4.3 shows that assuming a constant dilaton is inconsistent. There are solutions for a time-dependent dilaton but one may question the exact interpretation of a cosmological constant in that case (section 5.4.2). The construction of a dS solution is not trivial and there are no-go theorems as discussed in section 5.4.5.

All of the above applies to low-energy string theory; the background fields solve the constraint equation from rescaling invariance only up to $\mathcal{O}(\alpha')$ (section 5.5). For more exotic phenomena as black holes or the big bang (where energies are around the Planck scale), one needs to consider backgrounds that are valid beyond this approximation – defining a fully rescaling and reparametrization invariant non-linear sigma model. One procedure to find such backgrounds is by using a coset construction (chapter 6). One may start from a non-linear sigma model for some group G and restrict to a coset G/H by gauging the associated Wess-Zumino-Witten action (sections 6.1 to 6.3). Integrating out the gauge fields makes contact with the low-energy theory of gravity – but it is now possible to associate an exact solution with a low-energy solution (section 6.4).

This makes it particularly convenient to study the propagation of scalar states on a background that is acquired from a coset construction (section 6.5 and the explicit example in section 6.6). The scalars can be used as probes to investigate the spacetime structure.

In a cosmological setting, such research may provide new insight into the nature of spacetime singularities such as the big bang. Some general requirements for a coset to yield a background that is interesting from a cosmological perspective (such as a $3+1$ metric) can be formulated (section 7.1. A number of such cosets have already been studied in the literature, and the reader will find a (non-complete) list in section 7.2.

A prominent example of such a model is the Nappi-Witten (NW) universe [62], which is investigated in detail (section 7.3). The model has a free parameter α that determines how the subgroup is gauged. There has been recent interest in this model, such as in the work of Elitzur et.al. [27] who consider a different patch as the original authors and restrict to $\alpha = 0$. This article is discussed and some results are generalized to non-zero α (section 7.3.3). Another coordinate patch on the NW coset is found. Opposed to the patches of [27, 62] it is well-behaved for $\alpha = \pi/2$ (section 7.3.3). Finally, scalar propagation on the NW background is investigated for general α (section 7.3.4).

8.1 Suggestions for further research

The (non-)possibility of finding a de Sitter solution in low-energy string theory is quite an interesting subject. Let me stress that the discussion on section 5.4.5 is not conclusive – a more thorough investigation of the subject would be very welcome. Such a study may cover a more detailed analysis of allowed dilaton potentials, or consider higher-dimensional objects as branes or orientifolds (see e.g. Silverstein [72] and Maloney et.al. [56]).

Another interesting topic is the possible propagation of scalar states through spacetime singularities. Using the theory described in chapters 6 and 7, one could investigate the nature of a singularity that arises in some coset background. An example of such a study, with quite detailed calculations, can be found in Elitzur et.al. [27] for the NW coset. It would be very interesting to compare these results with singularities in other backgrounds.

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A

Formulae for cosmological scenarios

A.1 The rolling radii solution

This appendix presents some geometrical quantities for the metric ansatz in the rolling radii scenario (5.35). The G_{00} term is kept explicit here, as it may be useful for variation purposes. It is understood that $G_{00} = -1$ after the variation. A straightforward calculation shows that the non-vanishing components of the connection are

$$\begin{aligned}
\Gamma^0{}_{00} &= \frac{\dot{G}_{00}}{2G_{00}} \\
\Gamma^i{}_{0i} &= \dot{\alpha}_i \qquad \text{(no sum)} \\
\Gamma^0{}_{ii} &= \Gamma^i{}_{i0} = -\frac{\dot{\alpha}_i}{G_{00}} e^{2\alpha_i} \qquad \text{(no sum)}
\end{aligned} \tag{A.1}$$

This implies the following non-vanishing entries of the Ricci tensor:

$$\begin{aligned}
\mathbf{R}_{00} &= -\sum_{i=1}^{N} (\ddot{\alpha}_i + \dot{\alpha}_i^2) + \frac{\dot{G}_{00}}{2G_{00}} \sum_{i=1}^{N} \dot{\alpha}_i \\
\mathbf{R}_{ij} &= -\frac{e^{2\alpha_i}}{G_{00}} \left(\ddot{\alpha}_i + \dot{\alpha}_i \sum_{k=1}^{N} \dot{\alpha}_k - \frac{\dot{G}_{00}}{2G_{00}} \dot{\alpha}_i \right) \delta_{ij} \qquad \text{(no sum)}
\end{aligned} \tag{A.2}$$

Various terms that are encountered in reducing the general low-energy effective action (5.20) to the specific form (5.37) are

$$\begin{aligned}
\nabla_\mu \nabla_\nu \Phi &= \begin{pmatrix} \ddot{\Phi} - \frac{1}{2} \frac{\dot{G}_{00}}{G_{00}} \dot{\Phi} & 0 \\ 0 & \frac{1}{G_{00}} e^{2\alpha_i} \dot{\alpha}_i \end{pmatrix} \\
\nabla_\mu \nabla^\mu \Phi &= -\ddot{\Phi} + \frac{1}{2} \frac{\dot{G}_{00}}{G_{00}} \dot{\Phi} + \frac{N}{G_{00}} \\
\nabla_\mu \Phi \nabla^\mu \Phi &= \frac{1}{G_{00}} \dot{\Phi}^2 \\
\mathbf{R} &= -G^{00} \left(\sum_{i=1}^{N} (2\ddot{\alpha}_i + \dot{\alpha}_i^2) + \sum_{i=1}^{N} \sum_{j=1}^{N} \dot{\alpha}_i \dot{\alpha}_j \right) - \frac{d}{dt}(G^{00}) \sum_{i=1}^{N} \dot{\alpha}_i
\end{aligned} \tag{A.3}$$

Putting $G_{00} = -1$, this is equal to the result in Mueller [60], section 3.

A.2 The maximally symmetric solution

In this appendix, the reader will find some results in deriving the equations of motion (5.52), (5.53) from the ansatz (5.48). The action (5.51), which is an integral over t, can be derived from the general effective action by calculating various geometrical quantities and keeping G_{00} explicit.

Please be aware of the notation as this may cause some confusion. As usual, Greek indices run from 0 to $D-1$ and Roman indices from 1 to $D-1$. Quantities with a tilde are evaluated in the $N = D-1$ dimensional subspace, those without a tilde in the D dimensional spacetime. For instance, $\tilde{\mathbf{R}}_{ab}$ is the (a,b) component of the Ricci tensor of the maximally symmetric subspace. In contrast to this, $\mathbf{R}_{ab}$ is the (a,b) component of the D-dimensional Ricci tensor, which in general is a different quantity. The connection for the geometry (5.48) is related to the connection of the maximally symmetric subspace as:

$$\begin{aligned}
\Gamma^0{}_{00} &= \frac{\dot{G}_{00}}{2G_{00}} \\
\Gamma^i{}_{0i} &= \dot{\lambda} \qquad \text{(no sum)} \\
\Gamma^0{}_{ij} &= \Gamma^0{}_{ji} = -\frac{\dot{\lambda}e^{2\lambda}}{G_{00}}\tilde{G}_{ij} \\
\Gamma^i{}_{jk} &= \tilde{\Gamma}^i{}_{jk}
\end{aligned} \tag{A.4}$$

Now the fact that the 'space' subspace is maximally symmetric can be used. For an introduction to maximally symmetric (sub)spaces, see e.g. Weinberg [84], chapter 13.[1] The Ricci tensor in a maximally symmetric space is

$$\tilde{\mathbf{R}}_{ij} = k(N-1)\tilde{G}_{ij} \tag{A.5}$$

Using this, it can be shown that the D-dimensional Ricci tensor for the ansatz has non-zero entries

$$\begin{aligned}
\mathbf{R}_{00} &= -N(\ddot{\lambda} + \dot{\lambda}^2) + \frac{N}{2}\frac{\dot{G}_{00}}{G_{00}}\dot{\lambda} \\
\mathbf{R}_{ij} &= \left(k(N-1) - \frac{1}{G_{00}}e^{2\lambda}(\ddot{\lambda} + N\dot{\lambda}^2) + \frac{1}{2}\frac{\dot{G}_{00}}{(G_{00})^2}e^{2\lambda}\dot{\lambda}\right)\tilde{G}_{ij}
\end{aligned} \tag{A.6}$$

The G_{00} is kept explicit as it may be needed for variational purposes. After the variation, $G_{00} = -1$. Other terms in the equations of motion are found by working out the connection coefficients of this geometry, resulting in

$$\begin{aligned}
\nabla_\mu\nabla_\nu\Phi &= \begin{pmatrix} \ddot{\Phi} - \frac{1}{2}\frac{\dot{G}_{00}}{G_{00}}\dot{\Phi} & 0 \\ 0 & \frac{1}{G_{00}}e^{2\lambda}\dot{\lambda}\,\dot{\Phi}\tilde{G}_{ij} \end{pmatrix} \\
\nabla_\mu\nabla^\mu\Phi &= \frac{\ddot{\Phi}}{G_{00}} + \frac{N}{G_{00}}\dot{\lambda}\dot{\Phi} - \frac{1}{2}\frac{\dot{G}_{00}}{(G_{00})^2}\dot{\Phi} \\
\nabla_\mu\Phi\nabla^\mu\Phi &= \frac{1}{G_{00}}\dot{\Phi}^2
\end{aligned}$$

[1] Be aware that Weinberg uses $(+--)$ sign conventions, as opposed to the $(+++)$ convention adopted throughout this thesis. This leads in particular to a different sign for the Ricci tensor and curvature scalar. See the 'notation' section in the beginning of this thesis.

$$\mathbf{R} \quad = \quad \frac{-2N}{G_{00}}\ddot{\lambda} + \frac{-N(N+1)}{G_{00}}\dot{\lambda}^2 + N(N-1)ke^{-2\lambda} + \frac{\dot{G}_{00}}{(G_{00})^2}N\dot{\lambda} \quad (A.7)$$

These quantities can be inserted in the general equations of motion of string cosmology (5.22) with $G_{00} = -1$. After putting $B = 0$ this leads to (5.52), (5.53) with a bit of linear algebra. Recall the introduction of the rescaled dilaton $\varphi = 2\Phi - N\lambda$.

B

Some group theory and special functions

B.1 $SU(2)$, $SU(1,1)$, $SL(2,\mathbb{R})$

The Pauli σ matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.1}$$

They form the usual basis of $SU(2)$. A commonly used parametrization of nearly all group elements of $SU(2)$ is in terms of the real Euler angles ϕ, θ, ψ.

$$u(\phi, \theta, \psi) = \begin{pmatrix} \cos\theta/2\, e^{i(\phi+\psi)/2} & i\sin\theta/2\, e^{i(\phi-\psi)/2} \\ i\sin\theta/2\, e^{i(\psi-\phi)/2} & \cos\theta/2\, e^{-i(\phi+\psi)/2} \end{pmatrix} \tag{B.2}$$

Which has the convenient factorization $u(\phi,\theta,\psi) = u(\phi,0,0)\, u(0,\theta,0)\, u(0,0,\psi)$. With an obvious reparametrization, a generic element can be written in real coordinates α_i $(i = 1,2,3)$ as

$$u(2\alpha_1 - \pi/2, 2\alpha_2, 2\alpha_3 + \pi/2) = e^{i\alpha_1\sigma_3} e^{i\alpha_2\sigma_2} e^{i\alpha_3\sigma_3} \tag{B.3}$$

The group $SU(1,1)$ can be regarded as the analytic continuation of $SU(2)$. Complexifying ϕ, θ, ψ – or, equivalently, the α_i's – leads to the group $SL(2,\mathbb{C})$. The group $SU(1,1)$ is then extracted by requiring θ to be purely imaginary; starting from $SU(2)$ this corresponds to continuing $\theta \to -it$. The parametrization in Euler angles ϕ, t, ψ is

$$g(\phi, t, \psi) = \begin{pmatrix} \cosh t/2\, e^{i(\phi+\psi)/2} & \sinh t/2\, e^{i(\phi-\psi)/2} \\ \sinh t/2\, e^{i(\psi-\phi)/2} & \cosh t/2\, e^{-i(\phi+\psi)/2} \end{pmatrix} \tag{B.4}$$

Equivalently, a group element can be expressed in real coordinates β_i as

$$e^{i\beta_1\sigma_3} e^{\beta_2\sigma_2} e^{i\beta_3\sigma_3} \tag{B.5}$$

A natural choice of basis in $SU(1,1)$ is such that the subgroup corresponding to a generator ρ_i is associated by analytic continuation with the subgroup generated by σ_i in $SU(2)$. This basis is

$$\rho_1 = -i\sigma_1 \qquad \rho_2 = -i\sigma_2 \qquad \rho_3 = \sigma_3 \tag{B.6}$$

The non-compact group $SL(2,\mathbb{R})$ is isomorphic to $SU(1,1)$:

$$g \in SL(2,\mathbb{R}) \leftrightarrow t^{-1}gt \in SU(1,1) \qquad t = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{B.7}$$

Which suggests the basis ($\tau_i \leftrightarrow \rho_i$)

$$\tau_1 = -i\sigma_1 \qquad \tau_2 = i\sigma_3 \qquad \tau_3 = \sigma_2 \tag{B.8}$$

All entries are purely imaginary as expected. A frequently encountered parametrization of $SL(2,\mathbb{R})$, which covers elements with all entries non-zero, is as follows (see e.g. reference [27] or [83])

$$g(a,t,b;\epsilon_1,\epsilon_2,\delta) = e^{a\sigma_3}(-1)^{\epsilon_1}(i\sigma_2)^{\epsilon_2} g_\delta(t) e^{b\sigma_3} \tag{B.9}$$

Where the discrete labels ϵ_1 and ϵ_2 are either 0 or 1 and δ labels the following matrices

$$\begin{aligned} g_I(t) &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \qquad -\pi/2 < t < \pi/2 \\ g_{II}(t) &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \qquad -\infty < t < \infty \end{aligned} \tag{B.10}$$

As a possible help, here are some exponentiated matrices often encountered in the analysis:

$$e^{t\sigma_2} = \begin{pmatrix} \cosh t & -i\sinh t \\ i\sinh t & \cosh t \end{pmatrix} \quad e^{it\sigma_2} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad e^{t\sigma_3} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

B.2 Explicit form of matrix elements

B.2.1 f_{++} and f_{--}

In the space of generalized functions, choose the basis $|f_m\rangle := \{x_+^{im+\tau}, x_-^{im+\tau}\}$ where $m \in \mathbb{R}$. Consider the following region of $SL(2,\mathbb{R})$ acting on these functions

$$g_I(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \tag{B.11}$$

Matrix elements $\langle f_m | \hat{T}_{\tau,\epsilon}[g_I(t)] \, | f_{m'} \rangle$ are given by $f_{++}(t;m,m',\tau)$ or $f_{--}(t;m,m',\tau)$ where

$$\begin{aligned} f_{++}(t) &= \frac{1}{2\pi i} \cosh(t)^{-i(m+m')} \sinh(t)^{2\tau+i(m+m')} \\ &\times B(-\tau - im, -\tau + im) \\ &\times {}_2F_1(-\tau - im, -\tau - im', -2\tau, -\sinh^{-2} t) \\ f_{--}(t) &= \frac{1}{2\pi i} \cosh(t)^{-i(m+m')} \sinh(t)^{-2-2\tau+i(m+m')} \\ &\times B(1+\tau+im', 1+\tau-im') \\ &\times {}_2F_1(1+\tau-im, 1+\tau-im', 2\tau+2, -\sinh^{-2} t) \end{aligned} \tag{B.12}$$

In these formulae B is the Euler beta function and ${}_2F_1$ is a hypergeometrical function. See Vilenkin and Klimyk [83], section 7.2.1 where region h corresponds to g_I.

B.2.2 g_{++} and g_{--}

Now consider the same basisfunctions $|f_m\rangle$ with a different region

$$e^{i\psi\sigma_2} = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \tag{B.13}$$

The corresponding matrix elements $\langle f_m|\,\hat{T}_{\tau,\epsilon}[e^{i\psi\sigma_2}]\,|f_{m'}\rangle$ are equal to either $g_{++}(t;m,m',\tau)$ or $g_{--}(t;m,m',\tau)$, defined as:

$$\begin{aligned}
g_{++}(t) &= \frac{1}{2\pi i}(\cos t)^{-i(m+m')}(\sin t)^{2\tau+i(m+m')} \\
&\times\; B(1+\tau-im',-\tau-im) \\
&\times\; {}_2F_1(-\tau-im,-\tau-im',1-i(m+m'),-\tan^{-2}t) \\
g_{--}(t) &= \frac{1}{2\pi i}(\cos t)^{i(m+m')}(\sin t)^{-2-2\tau-i(m+m')} \\
&\times\; B(1+\tau+im',-\tau+im) \\
&\times\; {}_2F_1(1+\tau+im,1+\tau+im',1+i(m+m'),-\tan^{-2}t)
\end{aligned} \tag{B.14}$$

See [83], section 7.2.1 where region u corresponds to $e^{i\psi\sigma_2}$.

B.2.3 Wigner D-function $d^l_{mm'}$

The Wigner D-function $d^l_{mm'}$ can be expressed in Jacobi polynomials:

$$\begin{aligned}
d^l_{mm'}(t) &= \xi^{mm'}\left(\frac{(l-m)!(l+m)!}{(l-m')!(l+m')!}\right)^{1/2}(\cos t/2)^{|m+m'|}(\sin t/2)^{|m-m'|} \\
&\times\; P^{(|m-m'|,|m+m'|)}_{l-(|m+m'|+|m-m'|)/2}(\cos t)
\end{aligned} \tag{B.15}$$

Where $P^{(a,b)}_n(x)$ denotes the nth Jacobi polynomial in x for parameters a and b. The factor $\xi^{mm'}$ is equal to 1 for $m' \geq m$ and to $(-1)^{m'-m}$ for $m' < m$.

Various other forms can be found in reference [81], chapter 4. These matrix elements are also given in [83], section 6.8. The region considered there is $u(0,t,0)$ which is equal to $e^{i\pi/4\,\sigma_3}e^{it/2\,\sigma_2}e^{-i\pi/4\,\sigma_3}$. This leads to a phase factor of $i^{m-m'}$ relative to scalar states in the region $\hat{D}^{1/2}(0,t,0) = e^{-it/2\,\sigma_2}$ considered in the text.

C

Background fields related to the Nappi-Witten universe

This appendix explicitly states the background fields found for different coordinate patches on the Nappi-Witten coset. Recall from (7.19) that I parametrize the six-dimensional manifold $SL(2,\mathbb{R}) \times SU(2)$ as

$$\begin{aligned} g_1(a,\psi,b) &= e^{a\sigma_3} e^{i\psi\sigma_2} e^{b\sigma_3} \\ g_2(\gamma,s,\beta) &= e^{i\gamma\sigma_2} e^{is\sigma_3} e^{i\beta\sigma_2} \end{aligned} \tag{C.1}$$

Which does not cover the whole manifold. Nappi and Witten choose a coordinate patch by setting $a = b = 0$ [62].

C.1 Alternative one: $\gamma = \beta = 0$

The geometry is conveniently expressed in coordinates $\rho = a + b$ and $\lambda = a - b$:

$$\begin{aligned} G_{\psi\psi} &\propto -1 \\ G_{ss} &\propto 1 \\ G_{\rho\rho} &\propto -\frac{2\cos^2 s\cos^2\psi(1-\sin\alpha)}{-1+\cos 2\psi\cos 2s+\sin\alpha(\cos 2s-\cos 2\psi)} \\ G_{\lambda\lambda} &\propto -\frac{2\sin^2 s\sin^2\psi(1+\sin\alpha)}{-1+\cos 2\psi\cos 2s+\sin\alpha(\cos 2s-\cos 2\psi)} \end{aligned} \tag{C.2}$$

The proportionality factor is $k_2/2\pi$. Note that it has signature $(-{+}{+}{+})$ for $k > 0$. The geometry is well-behaved for $\alpha = 0$; in particular, the curvature scalar takes the form

$$R|_{\alpha=0} = \frac{7\,(\cos 4s - \cos 4\psi)}{(\cos 2s\cos 2\psi - 1)^2} \tag{C.3}$$

For $\alpha = \pi/2$, $G_{\rho\rho} = 0$ which means the metric degenerates and this is not a valid description of the coset geometry at that point.
The antisymmetric tensorfield B takes the form

$$\begin{aligned} B_{\psi\lambda} &\propto -\ln\left(e^{\rho}\cos\psi\right)\sin 2\psi \\ B_{\rho\lambda} &\propto \frac{2\cos^2\psi\cos^2 s(\sin\alpha-\cos 2s)}{-1+\cos 2\psi\cos 2s+\sin\alpha(\cos 2s-\cos 2\psi)} \end{aligned} \tag{C.4}$$

C.2 Alternative two: $a = \beta = 0$

Sticking to the original coordinates, the geometry can be expressed as

$$
\begin{aligned}
G_{\psi\psi} &\propto -1 \\
G_{ss} &\propto 1 \\
G_{bb} &\propto -\frac{1+\cos 2\psi\cos 2s - \sin\alpha(\cos 2s + \cos 2\psi)}{-1+\cos 2\psi\cos 2s + \sin\alpha(\cos 2s - \cos 2\psi)} \\
G_{\gamma\gamma} &\propto -\frac{1+\cos 2\psi\cos 2s + \sin\alpha(\cos 2s + \cos 2\psi)}{-1+\cos 2\psi\cos 2s + \sin\alpha(\cos 2s - \cos 2\psi)} \\
G_{\gamma b} &\propto \frac{(1+\cos 2s\cos 2\psi)\cos\alpha}{-1+\cos 2\psi\cos 2s + \sin\alpha(\cos 2s - \cos 2\psi)}
\end{aligned}
\tag{C.5}
$$

The proportionality factor is $k_2/2\pi$ and the signature is again $(-{+}{+}{+})$ for $k > 0$. Opposed to the previous example, this geometry is well defined for $\alpha = \pi/2$. The curvature scalar is

$$
R|_{\alpha=\pi/2} = -\frac{2(\cos 2s + \cos 2\psi)}{\sin^2 s \cos^2 \psi}
\tag{C.6}
$$

The antisymmetric tensorfield B is found to be

$$
\begin{aligned}
B_{\psi b} &\propto 2\sin 2\psi \ln\left(e^b \cos\psi\right) \\
B_{b\gamma} &\propto \frac{(1-\cos 2s\cos 2\psi)\cos\alpha}{-1+\cos 2\psi\cos 2s + \sin\alpha(\cos 2s - \cos 2\psi)}
\end{aligned}
\tag{C.7}
$$

D

The Wess-Zumino term

This appendix discusses the topological Wess-Zumino term that arises in WZW models. More specific, it presents a general formula for the WZ term for WZW models based on $SL(2,\mathbb{R})$.

Recall from equation (6.3) the definition

$$\Gamma[\hat{g}] = \frac{ik}{24\pi}\int_V d^3x\,\epsilon^{ijk}\,\mathrm{Tr}\left[\hat{g}^{-1}\partial_i\hat{g}\,\hat{g}^{-1}\partial_j\hat{g}\,\hat{g}^{-1}\partial_k\hat{g}\right] \tag{D.1}$$

Where $\hat{g}(x_1,x_2,x_3)$ is valued in G and lives on a three-dimensional space V whose boundary corresponds to the string worldsheet. This term is easily evaluated if one can find some 2×2 matrix D such that

$$\epsilon^{ijk}\,\mathrm{Tr}\left[\hat{g}^{-1}\partial_i\hat{g}\,\hat{g}^{-1}\partial_j\hat{g}\,\hat{g}^{-1}\partial_k\hat{g}\right] = \epsilon^{ijk}\partial_i D_{jk} \tag{D.2}$$

Now the Wess-Zumino term can be evaluated as follows (cf. Di Francesco et.al. [25], 15.1.2)

$$\begin{aligned}\Gamma[\hat{g}] &= \frac{ik}{24\pi}\int_V d^3x\,\epsilon^{ijk}\,\partial_i D_{jk} = \frac{ik}{24\pi}\int_V d^3x\,\epsilon^{jki}\,\partial_i D_{jk}\\ &= \frac{ik}{24\pi}\int_V d^2x\,\epsilon^{jk}D_{jk}\end{aligned} \tag{D.3}$$

For $G = SL(2,\mathbb{R})$ this can be done in generality. Express an element g_1 of this group as

$$g_1 = \begin{pmatrix} a & u \\ -v & b \end{pmatrix} \tag{D.4}$$

Where a, b, u and v are real, depend on x_1, x_2 and x_3 and obey $ab+uv=1$. The reader may verify that equation (D.2) holds for

$$D_{jk} = -6(\partial_j u\,\partial_k v)\,\ln a \tag{D.5}$$

So that Γ takes the form

$$\Gamma[g_1] \;=\; -\frac{ik}{4\pi}\int_V d^2x\,\epsilon^{jk}(\partial_j u\,\partial_k v)\,\ln a \tag{D.6}$$

In conformal coordinates, $x_1 = z, x_2 = \bar{z}$, the term Γ takes the form

$$\begin{aligned}\Gamma[g_1] &= -\frac{ik}{4\pi}\int_V d^2z\,\left(\epsilon^{z\bar{z}}(\partial u\bar{\partial}v) + \epsilon^{\bar{z}z}(\bar{\partial}u\partial v)\right)\ln a \\ &= -\frac{k}{2\pi}\int_V d^2z\,(\partial u\bar{\partial}v - \bar{\partial}u\partial v)\,\ln a \end{aligned} \tag{D.7}$$

Recall from section 4.2 that $\epsilon^{z\bar{z}} = -\epsilon^{\bar{z}z} = -2i$. I have implicitly used $d^2x = d^2z$ here. The result (D.7) is in keeping with equation (31) of reference [87], the article by Witten on which section 6.4 is based. Note that, in particular, the Wess-Zumino term vanishes for any element of $SL(2,\mathbb{R})$ with $u \propto v$.

For $SU(2)$, I have not found such a general formula. However, consider elements $g \in SU(2)$ that can be parametrized by

$$g_2 = e^{ia\sigma_2}e^{ib\sigma_3}e^{ic\sigma_2} \tag{D.8}$$

In this parametrization, (D.2) holds for

$$D_{jk} = -6\,(\partial_j c\,\partial_k a)\,\cos 2b \tag{D.9}$$

So that

$$\Gamma[g_2] = -\frac{ik}{4\pi}\int_V d^2x\,\epsilon^{jk}(\partial_j c\,\partial_k a)\,\cos 2b \tag{D.10}$$

Which in conformal coordinates reads

$$\Gamma[g_2] = -\frac{k}{2\pi}\int_V d^2x\,(\partial c\,\bar{\partial}a - \bar{\partial}c\,\partial a)\,\cos 2b \tag{D.11}$$

This is consistent with equation (13) of [62] by Nappi and Witten. The Wess-Zumino term in that equation follows from the above equation by identifying $b = s$, $a = (\rho+\lambda)/2$ and $c = (\rho-\lambda)/2$.

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