New ways of supersymmetry breaking

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To my parents,
Athanasios and Maria

Abstract

The starting point for this thesis is gauge theory in four-dimensional flat spacetime. After a short introduction to the relevant ideas and methods we introduce supersymmetry and a very convenient way to represent it, namely the notion of superspace. After that we proceed to the construction of theories which are simultaneously Poincaré-invariant, supersymmetric and gauge-invariant.

However, supersymmetry is obviously not present in nature and, thus, its breaking is necessary. In that direction we analyze spontaneous supersymmetry breaking, introduce various criteria regarding it, and explain their implications. Furthermore, we explore the very promising perspective offered by the possibility of coexistence of both supersymmetric and nonsupersymmetric ground states in the same theory. For that we analyze a special case of the supersymmetric extension of quantum chromodynamics and, indeed, we find that it experiences supersymmetry breaking in metastable vacua.

In the remainder we try to motivate the string and M-theory realizations of the above ideas. In order to achieve that we use string-theory brane configurations to obtain the classical dynamics and their corresponding lifts to M-theory to get the full quantum dynamics.

Preface

Although a very appealing idea, supersymmetry is not a part of nature. As of today numerous experiments have not seen any sign of supersymmetry and, hence, if we want to retain its power we need to find a phenomenologically acceptable way to break it. Now, there are mainly two approaches regarding supersymmetry breaking: Either it is explicit, or spontaneous. In the former case the theory not only contains supersymmetric terms but, also, terms (one or more) which break supersymmetry. In the latter case, which is also the subject of this thesis, although the theory has supersymmetry, its vacuum does not.

In the first chapter I give a very short reminder of the ideas and methods of ordinary gauge theory in four-dimensional flat spacetime. I describe how supersymmetry extends the Poincaré algebra and introduce the notion of superspace in order to be able to use classical methods to represent supersymmetry. After that I construct Lagrangians which have three very important properties: Poincaré invariance, supersymmetry and gauge invariance. In that way one incorporates supersymmetry in the edifice of quantum field theory and makes explicit that supersymmetry is indeed a beautiful extension of the established ideas of the Standard Model.

In the second chapter I delve into the spontaneous breaking of supersymmetry, motivated by the fact that supersymmetry is nowhere around us and, therefore, its breaking is necessary. I introduce various ways to check whether supersymmetry is broken and extensively analyze a particular example up to one-loop order. I give an explicit proof of the Goldstino theorem and argue that if supersymmetry is spontaneously broken at tree-level, then the breaking persists to all orders in perturbation theory. Likewise, if supersymmetry is unbroken at tree-level, then it is not broken at any order in perturbation theory.

In the third chapter I introduce the supersymmetric extension of quantum chromodynamics (SQCD) and show that the dynamics of this theory depends heavily on the number of colors and flavors. I extensively analyze the vacuum structure of SQCD both at tree-level and in the quantum theory, and present a very important duality for a particular range of the number of flavors. Finally, I use this duality to find that massive SQCD has both supersymmetric and nonsupersymmetric vacua. In particular, although at tree-level the theory breaks supersymmetry, there is a nonperturbative (dynamical) mechanism which, without spoiling the nonsupersymmetric vacua, brings supersymmetric vacua in the quantum theory. That is an example where mechanisms beyond perturbation theory result in a wide variety of new and unexpected phenomena.

In the fourth chapter I try to motivate the form the above phenomena take in string theory realizations. I introduce the idea of branes in superstring theories and use brane configurations of type-IIA superstring theory to obtain supersymmetric field theories at low energy. However, the brane configurations give us the classical dynamics in the low-energy effective theory; in order to explore the full quantum dynamics one has to lift the type-IIA brane configurations to M-theory. I extensively analyze the relevant interpretation and find the shape of the so-called M-theory curve.

The fifth chapter contains a particular string and M-theory realization of the phenomenon of dynamical supersymmetry breaking in metastable vacua. I describe how massive SQCD arises in the low-energy limit of type-IIA brane configurations and try to see whether the dynamical supersymmetry breaking it experiences can be seen in M-theory. Here I stumble upon a rather disappointing result; M-theory considerations indicate that the low-energy theory does not host both supersymmetric and nonsupersymmetric vacua. On the contrary, one can see that these two sets of vacua belong to different theories. This situation is unresolved as of this writing.

With this thesis I aim to finish my master's on theoretical physics at the university of Amsterdam. I want to express my gratitude to my adviser, Jan de Boer, with whom it was a pleasure to discuss about physics and beyond it. I feel very privileged I had the opportunity to come in contact with his way of thinking. I would also like to thank Kyriakos Papadodimas for explaining me many difficult concepts in the clearest way possible.

Last, but certainly not least, I would like to thank my family for always being there for me. Many thanks go to my roommates, Dimitris and Menelaos, for offering me a nice time at home. I wish them all the best in their professional and personal endeavors. I would also like to thank Idse, Mark and Michele for very useful discussions on and outside physics.

Andreas Stergiou,
Amsterdam, July 2007

CONTENTS

Supersymmetry

The Standard Model of particle physics is one of the most elegant, profound and important achievements of modern theoretical physics. It is the theory that describes the fundamental particles and three of the four interactions they experience, namely the electromagnetic, weak and strong interactions. Its predictions have been stringently tested in a series of experiments in the past, and it was proved to be the most successful theory in our possession. However, the Standard Model is not a final theory. It fails to incorporate the omnipresent gravitational interaction, it predicts zero mass for the neutrinos, particles that actually have nonzero mass and, in general, it looks rather artificial, since it needs a lot of experimental input in order to operate. As of today, the consensus is that the Standard Model, within its limits of applicability, is a theory that adequately describes our universe. Moreover, beyond those limits, new physics is expected to appear and give more concrete answers.

The search for this new physics led to several new ideas. One of those was supersymmetry, a symmetry that relates bosons and fermions, the two categories of elementary particles observed in the universe. Supersymmetry enabled physicists to shed light on several problems the Standard Model failed to handle. For example, supersymmetry achieves the so-called unification of the couplings. More specifically, the running of the three couplings of the Stan-

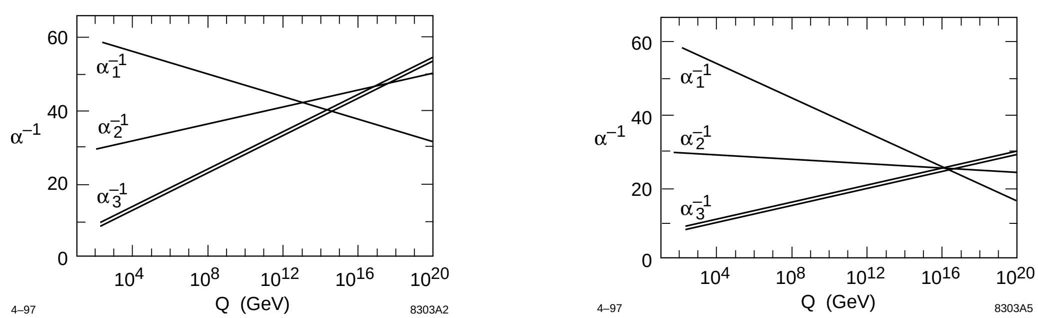


Fig. 1: Running of the couplings in the Standard Model (left) and the Minimal Supersymmetric Standard Model (right)

dard Model is such that the corresponding "fine-structure constants" do not meet at a specific point at any energy scale. However, when supersymmetry is taken into account the situation is amended, that is the three "fine-structure constants" all have the same value at some high energy scale (see Fig. 1) [1]. Additionally, unbroken supersymmetry predicts exactly zero cosmological constant, and it is strongly believed that some hints about the smallness of the cosmological constant will emerge from supersymmetry. Furthermore, supersymmetry suppresses the ubiquitous infinite corrections of quantum mechanics in classical quantities and, finally, there is a particular supersymmetric model, namely the Minimal Supersymmetric Standard Model, which almost perfectly reduces to the Standard Model, an indispensable attribute of any theory we might construct.

However, our universe is definitely not supersymmetric since we do not observe the particles supersymmetry predicts. In addition, we do not yet know if nature utilizes supersymmetry at high energies, even though there are several reasons to believe so. But the beautiful solutions supersymmetry offers are too attractive to be dismissed at the first difficulty, and the Large Hadron Collider (LHC) at CERN is expected to give a definitive answer within five years. Since this answer is strongly believed to be positive, this gives a rather strong impetus to the idea of supersymmetry.

Therefore, it appears as the best possibility that we retain supersymmetry up to some point, and then find a physical mechanism that will explain the transition from the supersymmetric to the nonsupersymmetric regime. As of today, the proposed solutions to this problem are far from satisfactory and intense research is conducted in order to unravel the mystery. One of the most promising directions the research has taken involves the breaking of supersymmetry in the framework of string theory. This is a very appealing prospect since string theory automatically incorporates gravity, thus being a theory which offers unification of the interactions straight from the beginning.

String theory

After string theory was found inadequate to describe the strong interactions among quarks and gluons, it was realized that quantum mechanics and gravity could be amalgamated by replacing point particles by strings. The idea of this replacement per se sounds so naive that it might be hard to believe that it is truly fundamental. But this very idea is perhaps as basic as introducing complex numbers in mathematics. The orbit of a particle in spacetime is one-dimensional—a line (Fig. 2(a)). On the other hand, the orbit of a string in spacetime is two-dimensional—a surface (Fig. 2(b)). Therefore, we are tempted to say that physics without strings is roughly analogous to mathematics without complex numbers.

Although an area of intense research, the construction of consistent string theories was delayed by the complex problem of respecting all physical requirements a theory has to respect. A vast effort towards this direction resulted in five different, yet perfectly consistent, theories, bearing the strange names type-I, type-IIA, type-IIB, heterotic $\mathsf{E8}\times\mathsf{E8}$ and heterotic $\mathsf{SO}(32)$. Nevertheless, this plethora of theories posed an embarrassing problem for string theory; it was highly unlikely that the long-sought unification would eventually be achieved in five different ways. If one of these ways describes our universe, then what do the other four describe?

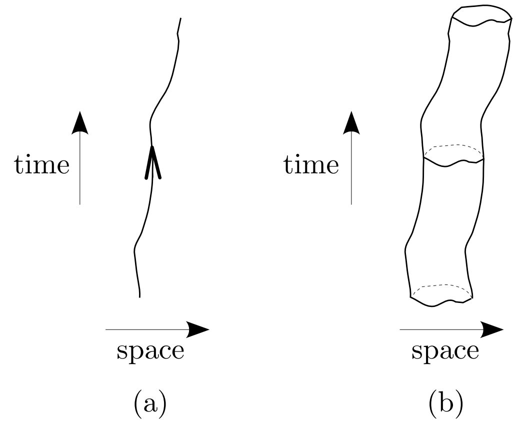


Fig. 2: The spacetime orbit of a point particle (a) or a string (b) is a manifold of real dimension one or two respectively

Various attempts to solve the aforementioned problem failed, until in 1995 Edward Witten derived a great result: All five different string theories along with another interesting theory, the eleven-dimensional supergravity, are limits of a single, underlying theory (Fig. 3). The new theory was baptized M-theory.

M-theory is the only auspicious candidate for the missing unification of the forces of nature. As of today, though, its intricate nature has deprived physicists from the insights that would lead to the understanding of its meaning and predictions. Therefore, further developments in high energy physics (and, of course, in physics in general) rely heavily on the deep understanding of this challenging theory. This fact declares the imperative need for research in M-theory. In particular, since our world is not supersymmetric, it is required that a theory contain nonsupersymmetric vacua. Therefore, the discovery of phenomenologically acceptable nonsupersymmetric vacua in M-theory would signal a giant leap forward.

Research question

The basic research question pertains to the identification of nonsupersymmetric vacua. This is an important and long quest both in field-theory and in string/M-theory considerations. In particular, an alluring possibility is the one where supersymmetric and nonsupersymmetric vacua coexist in the same theory. In this case the nonsupersymmetric vacua are metastable.

Although beyond the scope of this thesis, there is a very important phenomenological question which stems from the answer to the question we address. If field theory and/or string/M-theory admit metastable nonsupersymmetric vacua, then which are the models that most accurately resemble the observed universe? Can those vacua be as long-lived as required so that our universe be perturbatively-away from them? Evidently, it is only the answer to these questions that will judge whether we can achieve with the aid of field theory and/or string/M-theory what we set out to achieve in the first place.

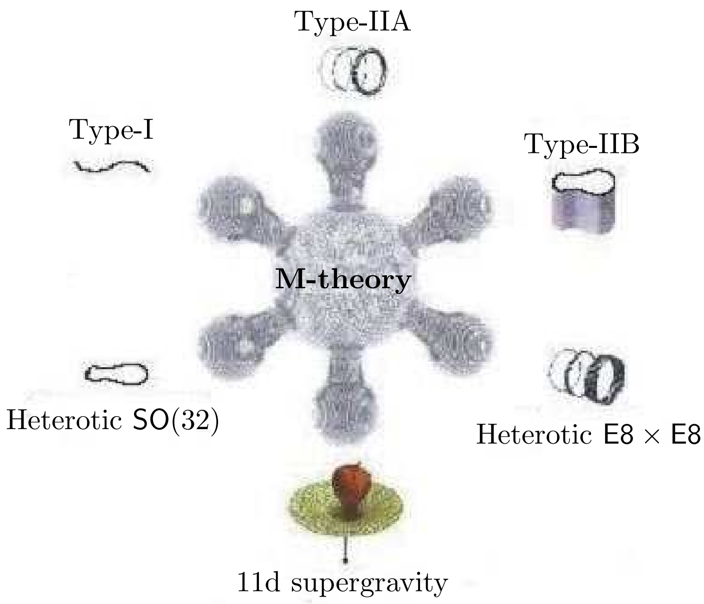


Fig. 3: M-theory and its various limits

Relevance for science

At the moment M-theory is the most promising candidate for the unification of all interactions, the problem that occupied Albert Einstein himself until the very end of his life. After many years of intensive research, the verisimilitude of string/M-theory results in its ever increasing acceptance. The lack of incontrovertible evidence for the factuality of string/M-theory further bolsters up the research activity, thus indicating that something important is under way.

In fact, the aspired start of LHC at CERN in early 2008 will test both supersymmetry and the quality and potential of our research efforts. As the expectation that supersymmetry will be verified in LHC is universal, a positive result will indicate that the current approach is on the right path and needs to be elaborated further. In addition, the implications and prospects of string theory extend to astronomy and cosmology, in that future detailed astronomical observations may offer hints in favor of string theory. More specifically, the identification of objects called cosmic strings could possibly provide us with some pieces of evidence for the correctness of string theory. Despite the limited reference to the relevant issues, I hope it is by now evident that this is an absolutely key epoch for research in string/M-theory.

Generalities

Throughout this thesis we work in units where

$$\hbar = c = 1$$

The metric we use is the mostly-minus metric

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Unless otherwise stated Einstein's summation convention is at work, i.e. in any expression if an index appears both in an upper and a lower position, then it is assumed to be summed over the values it takes.

All the spinors we will be using are two-component Weyl spinors which anticommute. Left-handed spinors carry an undotted index from the beginning of the greek alphabet. Undotted indices are raised and lowered with the antisymmetric symbol $\epsilon_{\alpha\beta}$, $\alpha, \beta = 1, 2$,

$$\theta^\alpha = \epsilon^{\alpha\beta}\theta_\beta \;\text{ and }\; \theta_\beta = \theta^\alpha \epsilon_{\alpha\beta}$$

for which we choose the convention $\epsilon^{12} = -\epsilon_{12} = 1$. For example,

$$\theta^1 = \epsilon^{1\beta}\theta_\beta = \epsilon^{12}\theta_2 = \theta_2$$

and

$$\theta^2 = \epsilon^{2\beta}\theta_\beta = \epsilon^{21}\theta_1 = -\theta_1$$

The contraction of two left-handed spinors, ψ^α and χ^α, acquires a minus sign if we change the position of the indices:

$$\psi^\alpha\chi_\alpha \equiv \epsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\epsilon^{\beta\alpha}\psi_\beta\chi_\alpha = +\epsilon^{\beta\alpha}\chi_\alpha\psi_\beta = +\chi^\beta\psi^\beta = -\psi_\beta\chi^\beta$$

Right-handed Weyl spinors carry dotted indices from the beginning of the greek alphabet. Dotted indices are raised and lowered with the antisymmetric symbol $\epsilon_{\dot\alpha\dot\beta}$, $\dot\alpha, \dot\beta = \dot1, \dot2$, with the convention $\epsilon^{\dot1\dot2} = -\epsilon_{\dot1\dot2} = 1$. If we have two right-handed spinors, $\bar\psi^{\dot\alpha}$ and $\bar\chi^{\dot\alpha}$, then

$$\bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} \equiv \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}\bar\chi^{\dot\alpha} = -\epsilon_{\dot\beta\dot\alpha}\bar\psi^{\dot\beta}\bar\chi^{\dot\alpha} = +\epsilon_{\dot\beta\dot\alpha}\bar\chi^{\dot\alpha}\bar\psi^{\dot\beta} = +\bar\chi_{\dot\beta}\bar\psi^{\dot\beta} = -\bar\psi^{\dot\beta}\bar\chi_{\dot\beta}$$

The massless Dirac equation in terms of Weyl spinors splits into

$$i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\psi_\alpha = 0$$

for left-handed, and

$$i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\psi}^{\dot{\alpha}} = 0$$

for right-handed spinors. The 2×2 identity matrix is often denoted by σ^0, and σ^i, $i = 1, 2, 3$, are the usual Pauli matrices:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Therefore, the components of σ^μ are $(\mathbb{1}_2, \sigma^1, \sigma^2, \sigma^3)$. One can also see that the components of $\bar{\sigma}^\mu$ are $(\mathbb{1}_2, -\sigma^1, -\sigma^2, -\sigma^3)$.

Finally, note that any expansion in the anticommuting superspace coordinates terminates in order θ^2 and $\bar{\theta}^2$, since

$$\theta^2 \equiv \epsilon^{\alpha\beta}\theta_\beta\theta_\alpha = \epsilon^{12}\theta_2\theta_1 + \epsilon^{21}\theta_1\theta_2 = 2\theta_2\theta_1 = -2\theta_1\theta_2 = 2\theta^2\theta^1 = -2\theta^1\theta^2$$

and similarly for $\bar{\theta}^2$. Note here that θ^2 is used for both θ squared and the second component of θ.

Part I

Supersymmetric Field Theories

CHAPTER 1

Supersymmetric gauge theories

Contents

The aim of this chapter is to introduce ordinary gauge theory as well as gauge theory in superspace. Although a few ideas are described, I assume some familiarity of the reader with the supersymmetry algebra and the representations of the "group" it generates.

1.1 Ordinary gauge theories

1.1.1 The Abelian case

Ordinary gauge theories constitute the cornerstone of modern theoretical physics. The power of these theories is manifested through the assumption that Lagrangian densities have to remain invariant under gauge transformations. Once we impose that constraint it is easy to see that terms which describe interactions come into play.

Before we attempt to put the above discussion in mathematical language, let us point out the difference between Lagrangians and Lagrangian densities. Suppose we have a classical field theory described by a Lagrangian, L, which depends on some classical fields, $\phi_i(x)$, and their derivatives:

$$L = L(\phi_i(x), \partial_\mu \phi_i(x))$$

Then, the action is given as the integral of the Lagrangian over time from t_1 to t_2:

$$S = \int_{t_1}^{t_2} dt\, L(\phi_i(x), \partial_\mu \phi_i(x))$$

Obviously, in the above equation time is distinguished from the other coordinates of spacetime. Since we intend to rely heavily on Lorentz invariance that is certainly an undesirable feature and, hence, we should find a fundamental quantity which has to be integrated over all four spacetime coordinates in order to give the action. This function is the Lagrangian density, $\mathscr{L}$, and it is given by the formula

$$S = \int_{t_1}^{t_2} dt\, L(\phi_i(x), \partial_\mu \phi_i(x)) = \int_{\mathcal{R}} d^4x\, \mathscr{L}(\phi_i(x), \partial_\mu \phi_i(x))$$

where $\mathcal{R}$ indicates the spacetime region over which we have to evaluate the integral. From now on we will never use Lagrangians and, hence, I will refer to Lagrangian densities simply as Lagrangians.

With the above clarification in mind let us consider the specific example of the Dirac Lagrangian:

$$\mathscr{L}_{\text{Dirac}} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{1.1.1}$$

where $\bar{\psi} = \psi^\dagger \gamma^0$ and $\partial\!\!\!/ = \gamma^\mu \partial_\mu$.[1] It is easy to see that under the global transformation

$$\psi \to e^{iq\alpha}\psi \quad \text{and} \quad \bar{\psi} \to e^{-iq\alpha}\bar{\psi} \tag{1.1.2}$$

where q and α are real numbers (α is the parameter of the transformation), the Lagrangian (1.1.1) remains invariant. Therefore, Noether's theorem guarantees the existence of a conserved current density J^μ. A straightforward calculation gives

$$J^\mu = q\bar{\psi}\gamma^\mu\psi \tag{1.1.3}$$

from which we can verify using the Dirac equation that

$$\partial_\mu J^\mu = 0$$

With the identification of q as the electric charge, (1.1.3) is recognizable at once as the electromagnetic current density of the spinor field.

Now suppose that the global transformation (1.1.2) becomes local, i.e. the parameter α is substituted with a real function $\alpha(x)$ over spacetime. The complication we encounter now stems from the Leibniz rule of differentiation:

$$\partial\!\!\!/\psi \to e^{iq\alpha(x)}(\partial\!\!\!/\alpha(x))\psi + e^{iq\alpha(x)}\partial\!\!\!/\psi$$

[1] Note that we suppress the x-dependence of the spinor field ψ.

Gauge invariance is lost and we should do something to reinstate it. The key observation is that if we perform the so called minimal substitution:

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$$

and we demand that the vector (or gauge) field A_μ transform according to

$$A_\mu \to A_\mu - \partial_\mu \alpha(x) \tag{1.1.4}$$

then we recover invariance under gauge transformations.[2] Indeed, the Lagrangian

$$\begin{aligned}\mathscr{L}'_{\text{Dirac}} &= \bar{\psi}(i\not{D} - m)\psi \\ &= \bar{\psi}(i\not{\partial} - m)\psi - qA_\mu\bar{\psi}\gamma^\mu\psi \\ &= \mathscr{L}_{\text{Dirac}} - A_\mu J^\mu \end{aligned}\tag{1.1.5}$$

where J^μ is given by (1.1.3), is gauge-invariant as it can be easily checked.

If $q = e$ where e is the absolute value of the electron charge, then the additional term in (1.1.5) is the well known vector-spinor interaction of quantum electrodynamics (QED) with gauge coupling e. If we now add the kinetic term for the vector field,

$$\mathscr{L}_{\text{EM}} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

which is manifestly invariant under local gauge transformations (since partial derivatives commute), we get the Lagrangian of QED:

$$\mathscr{L}_{\text{QED}} = \bar{\psi}(i\not{\partial} - m)\psi - eA_\mu\bar{\psi}\gamma^\mu\psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{1.1.6}$$

The gauge principle is therefore seen to introduce the so-called minimal coupling of the electromagnetic field to the electron field in a natural fashion. We say that QED is an Abelian gauge theory with gauge group $\mathsf{U}(1)$—its gauge group is the group $\mathsf{U}(1)$ because we assumed gauge invariance under the transformation

$$\psi \to e^{ie\alpha(x)}\psi \;\text{ and }\; \bar{\psi} \to e^{-ie\alpha(x)}\bar{\psi}$$

and, of course, $e^{ie\alpha(x)}$ is an element of $\mathsf{U}(1)$ expressed in its most general form, and it is Abelian because the group $\mathsf{U}(1)$ is Abelian.

1.1.2 The non-Abelian case

The non-Abelian generalization of the above construction was introduced by Yang and Mills in 1954 [2]. At that time quarks were not known and so it was naturally assumed that the Dirac Lagrangian can be used for protons and neutrons,

$$\mathscr{L} = \bar{p}(i\not{\partial} - m)p + \bar{n}(i\not{\partial} - m)n$$

[2] In modern terminology we would say that with the above procedure we gauge the global symmetry.

where in the absence of electromagnetism protons and neutrons were assumed to have the same mass. If we introduce the composite spinor

$$\psi = \begin{pmatrix} p \\ n \end{pmatrix}$$

then we can write the Lagrangian more compactly:

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{1.1.7}$$

Evidently, the Lagrangian (1.1.7) is invariant under global isospin rotations,

$$\psi \to \exp\left(\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\alpha}}{2}\right)\psi$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ describes the 2×2 Pauli isospin matrices,

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is an arbitrary constant (3-vector) parameter of the transformation with $\alpha_i^* = \alpha_i, i = 1, 2, 3$. The conserved current density is

$$\boldsymbol{J}^\mu = \bar{\psi}\gamma^\mu \frac{\boldsymbol{\tau}}{2}\psi$$

which is the isospin current density.

The idea of Yang and Mills was to gauge the $\mathsf{SU}(2)$ global symmetry and explore the physical consequences of this procedure. Let's follow their reasoning. A gauge transformation of the field $\psi(x)$,

$$\psi(x) \to \psi'(x) = G(x)\psi(x) \tag{1.1.8}$$

where

$$G(x) = \exp\left(\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\alpha}(x)}{2}\right)$$

results in the transformation

$$\partial_\mu\psi \to G(\partial_\mu\psi) + (\partial_\mu G)\psi$$

for the gradient. Now we perform the minimal substitution

$$\partial_\mu \to D_\mu = \partial_\mu + igB_\mu \tag{1.1.9}$$

B_μ is the 2×2 matrix defined by

$$B_\mu = \tfrac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{b}_\mu = \tfrac{1}{2}\tau^a b_\mu^a = \tfrac{1}{2}\begin{pmatrix} b_\mu^3 & b_\mu^1 - ib_\mu^2 \\ b_\mu^1 + ib_\mu^2 & -b_\mu^3 \end{pmatrix}$$

where the three gauge fields are $\boldsymbol{b}_\mu = (b^1_\mu, b^2_\mu, b^3_\mu)$ and the isospin index a runs from 1 to 3.[3] Then, we can prove that if B_μ transforms as

$$B_\mu \to B'_\mu = G B_\mu G^{-1} + \frac{i}{g}(\partial_\mu G) G^{-1}$$

under the gauge transformation (1.1.8), then the Lagrangian

$$\begin{aligned}\mathscr{L}' &= \bar{\psi}(i\not{D} - m)\psi \\ &= \mathscr{L} - g\bar{\psi}\gamma^\mu B_\mu \psi \\ &= \mathscr{L} - \frac{g}{2}\boldsymbol{b}_\mu \cdot \bar{\psi}\gamma^\mu \boldsymbol{\tau}\psi\end{aligned}$$

is indeed gauge-invariant. Again, the "price" we pay in order to maintain gauge invariance is an interaction term.

What is missing, at this point, in order to arrive to the Yang–Mills Lagrangian is the term that describes the propagation of the gauge fields $\boldsymbol{b}_\mu$. This term, of course, has to be gauge-invariant on its own. In analogy with electromagnetism we should first find a field-strength tensor,

$$F_{\mu\nu} = \tfrac{1}{2}\boldsymbol{F}_{\mu\nu} \cdot \boldsymbol{\tau} = \tfrac{1}{2} F_{a\mu\nu}\tau^a$$

from which to construct the gauge-invariant kinetic term

$$-\tfrac{1}{4}\boldsymbol{F}_{\mu\nu} \cdot \boldsymbol{F}^{\mu\nu} = -\tfrac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$

The equality in the last equation follows from the identity

$$\mathrm{tr}(\tau^a \tau^b) = 2\delta^{ab}$$

of the Pauli matrices. Our problem, therefore, is to construct a field-strength tensor that is gauge-covariant, i.e. transforms according to

$$F_{\mu\nu} \to F'_{\mu\nu} = G F_{\mu\nu} G^{-1} \tag{1.1.10}$$

Again in analogy with electromagnetism we choose

$$F_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu]$$

which, upon insertion of D_μ from equation (1.1.9), takes the form

$$F_{\mu\nu} = \partial_\nu B_\mu - \partial_\mu B_\nu + ig[B_\nu, B_\mu]$$

which can be verified to satisfy (1.1.10). Thus, we arrive to the Yang–Mills Lagrangian

$$\mathscr{L}_{\mathrm{YM}} = \bar{\psi}(i\not{D} - m)\psi - \tfrac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \tag{1.1.11}$$

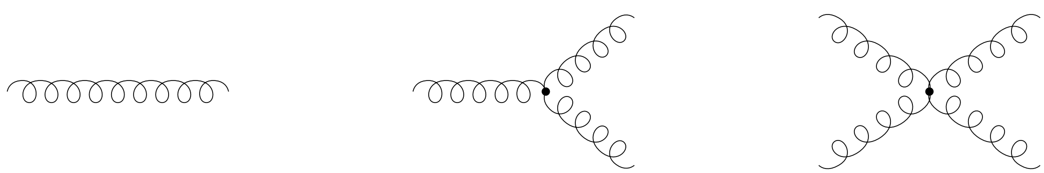
[3] $\boldsymbol{b}_\mu$ is an isovector.

Remember that the central feature for Yang and Mills was the isospin symmetry. The emerging $\mathsf{SU}(2)$ gauge bosons of their theory were identified with the pions. While it is now understood that isospin symmetry is not a true gauge symmetry, this initial confusion was historically important for the development of the overall ideas of gauge invariance.

It is now time to consider the differences between QED and Yang–Mills theory. Sourceless QED is a free (noninteracting) field theory. Only bilinear combinations of the photon gauge field A_μ appear in the QED Lagrangian (1.1.6) and, thus, the only Feynman rule to be found is that of the photon propagator (Fig. 1.1).

Fig. 1.1: The photon propagator of QED

On the other hand, Yang–Mills theory (1.1.11) has a much richer structure. First of all, even in the absence of fermion sources there will be interactions due to the nonlinear term in $F_{\mu\nu}$. The non-Abelian character of the gauge group introduces trilinear and quadrilinear terms in $F_{\mu\nu}F^{\mu\nu}$, besides the usual bilinear terms. Hence, in addition to the gauge field propagator the theory contains the three- and four-gauge-boson vertices (Fig. 1.2).



(a) (b) (c)

Fig. 1.2: Gauge boson propagator (a) and self-interactions in Yang–Mills theory—3-gauge-boson vertex (b) and 4-gauge-boson vertex (c)

These additional interactions and further characteristics of $\mathsf{SU}(2)$ Yang–Mills theory proved to be ideal to describe the weak interactions of leptons. Therefore, although originally misunderstood, the $\mathsf{SU}(2)$ Yang–Mills theory holds a remarkable position in modern theoretical physics. Furthermore, the formulation of an $\mathsf{SU}(3)$ Yang–Mills theory, namely quantum chromodynamics (QCD), for the description of strong interactions, strongly indicated that Yang–Mills theories are indeed the correct mathematical way to describe particles and their interactions.

1.2 Supersymmetry algebras

A truly remarkable result concerning quantum field theories is the famous Coleman–Mandula theorem [3]. Suppose that the symmetry group of the S-matrix of a theory contains the Poincaré group, generated by the $(\frac{1}{2},\frac{1}{2})$ generators P_μ of translations and the $(1,0)\oplus(0,1)$

generators $M_{\mu\nu}$ of proper Lorentz transformations, and a unitary internal symmetry group, generated by the $(0,0)$ generators T_A of various internal symmetries.[4] Then, the Coleman–Mandula theorem asserts, under mild assumptions, that the symmetry group of the S-matrix is isomorphic to a direct product of the Poincaré group and the internal symmetry group.

With that result at hand it seems impossible to ever find some symmetry group of the S-matrix larger than what ordinary quantum field theory takes into consideration in its various models. However, if we relax one of the assumptions of the Coleman–Mandula theorem, namely that we work in a Lie algebra,[5] then additional operators can be utilized as well. The symmetry group of the S-matrix is then enlarged; we can define a larger (graded) algebra which contains the bosonic generators P_μ, $M_{\mu\nu}$ and T_A, as well as some new generators, Q, and is realized by anticommutators between those new generators and commutators in any other case. Since the new generators obey anticommutation relations it is clear that, naively, we can think of them as carrying representations of the Lorentz group with spin-$\frac{1}{2}$, spin-$\frac{3}{2}$, etc. However, the Haag–Łopuszański–Sohnius theorem [5] states in part that the new generators can only belong to spinor representations of the Lorentz group. Therefore, the new generators, Q, are fermionic and enlarge the usual Poincaré algebra to the so-called super-Poincaré or supersymmetry algebra. The "group" generated by the super-Poincaré algebra is called the super-Poincaré group.

Theories with this enlarged symmetry group are called supersymmetric. Global supersymmetry transformations are generated by the fermionic quantum operators Q, called supercharges, which change fermionic states into bosonic states and vice-versa:

$$Q|\text{fermion}\rangle \propto |\text{boson}\rangle \quad \text{and} \quad Q|\text{boson}\rangle \propto |\text{fermion}\rangle$$

As we saw the supercharges carry spinor representations of the Lorentz group. Of course, we want to work with irreducible representations of the Lorentz group and we immediately remember that in four dimensions (one time and three space dimensions) a Dirac spinor, Q_D, has eight real components and does not furnish an irreducible spinor representation of the Lorentz group. However, a Majorana spinor, Q_M, satisfies the condition

$$Q_M = Q_M^c$$

where $Q_M^c = C\bar{Q}_M^T$ with C the charge conjugation matrix, has four independent real components and indeed furnishes an irreducible spinor representation of the Lorentz group of real dimension four. Likewise, we can consider Weyl (two-complex-component) spinors, Q^α, $\alpha = 1, 2$. These have four real components and, just like Majorana spinors, furnish an irreducible spinor representation of the Lorentz group of real dimension four. This is denoted as the $(\frac{1}{2}, 0)$ spinor representation and referred to as the left-handed Weyl spinor representation of the Lorentz group. Now, if we have a Weyl spinor, then we can find its complex conjugate,

[4] Here, $(\frac{1}{2}, \frac{1}{2})$ denotes the vector representation of the Lorentz group, $(1,0) \oplus (0,1)$ the representation of the Lorentz group carried by parity-invariant 2-form fields like, for example, the electromagnetic field strength $F_{\mu\nu}$, and $(0,0)$ the scalar representation of the Lorentz group. For a general account of irreducible representations of the Lorentz group the reader is referred to [4, Section 5.6]

[5] We work instead in a graded Lie algebra, a Lie algebra which contains anticommutators as well as commutators.

$\bar{Q}^{\dot{\alpha}}$, $\dot{\alpha} = \dot{1}, \dot{2}$, where the bar denotes complex conjugation and dotted indices label the complex conjugate representation of the one that is labeled by undotted indices. $\bar{Q}^{\dot{\alpha}}$ transforms under $(0, \frac{1}{2})$, that is under the right-handed Weyl spinor representation of the Lorentz group. Evidently, quantities with dotted indices are left-handed Weyl spinors and quantities with undotted indices right-handed Weyl spinors. If we arrange them in a spinor as

$$Q = \begin{pmatrix} Q^{\alpha} \\ \bar{Q}^{\dot{\alpha}} \end{pmatrix}$$

we obtain a spinor which transforms in $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, i.e. the Dirac spinor representation of the Lorentz group.

At this point we are in position to ask an obvious question: Is there any kind of limitation in the number of supersymmetry transformations we can have in a field theory or supergravity[6] model? The answer is yes and the limits originate in the requirement that the supercharges should act on multiplets of physical states and that the underlying theory should be either general relativity or an ordinary renormalizable field theory in four spacetime dimensions. As is well known renormalizable field theories in four-dimensional spacetime cannot accommodate spins larger than 1. In addition, gravity considerations are limited to spins not higher than 2.

Now, in order to find the exact limits we have to use the supersymmetry algebra and the action of the raising operators on massless one-particle states.[7] It is an attribute of the supersymmetry algebra that half of the supercharges have to be represented by zero when acting on massless one-particle states. From the remaining supercharges half are lowering operators and, thus, we can safely exclude them from the discussion that follows. In addition, possible central charges do not change the helicity of massless one-particle states and, thus, it is sufficient to work only with the $\mathcal{N}$ nonzero raising operators. But those anticommute and, hence, applying n of them to a one-particle state of minimum helicity λ_{min} and four-momentum p^{μ} gives $\mathcal{N}!/n!(\mathcal{N}-n)!$ states with the same momentum and helicity $\lambda_{\text{min}} + n/2$. The maximum value of n that gives a nonzero state is $n = \mathcal{N}$ and, hence, the maximum helicity in a supermultiplet is given by

$$\lambda_{\text{max}} = \lambda_{\text{min}} + \mathcal{N}/2$$

Thus, if we want to exclude helicities λ with $|\lambda| > 2$ then our theory can contain up to thirty-two supercharges, while in pure field theory considerations, where we exclude helicities with $|\lambda| > 1$, we can have up to sixteen supercharges.

A useful quantity with which we label supersymmetry algebras is the number $\mathcal{N}$ defined above, which is also given by dividing the total number of supercharges by the dimension of the irreducible spinor representation of the Lorentz group in the dimensions we are working in. In four dimensions the real dimension of the irreducible spinor representation of the Lorentz group is four (Weyl or Majorana) and, hence, in field theory considerations we can have up to

[6] Supergravity is the supersymmetric extension of general relativity.

[7] At energy scales large enough to allow us to neglect the effects of supersymmetry breaking we can treat the known particles as well as their superpartners as massless. That is why we are particularly interested in supermultiplets of massless particles.

$\mathcal{N} = 4$ supersymmetry. Evidently, if we consider supergravity we can work with up to $\mathcal{N} = 8$ supersymmetry. The second definition of $\mathcal{N}$ is extremely useful when we work in theories which are not four-dimensional. Then, we know the total number of supercharges, since that does not depend on the dimensionality of spacetime, and we can find $\mathcal{N}$ by knowing the (real) dimension of the irreducible spinor representation of the Lorentz group in the specific dimensions.

Going back to four dimensions one can see that the case $\mathcal{N} = 3$, although theoretically allowed, results in a multiplet structure which is automatically the same with that of an $\mathcal{N} = 4$ supersymmetry. In this thesis we will limit ourselves to the study of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry.

1.3 Superspace formulation of $\mathcal{N} = 1$ supersymmetry

Field theories with $\mathcal{N} = 1$ supersymmetry can be conveniently described by using the notion of superspace [6]. In superspace, in addition to the ordinary coordinates, x^μ, there is also a set of anticommuting coordinates, labeled by Grassmann numbers θ^α and $\bar{\theta}^{\dot{\alpha}}$, $\alpha, \dot{\alpha} = 1, 2$, such that

$$\{\theta^\alpha, \theta^\beta\} = \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \{\theta^\alpha, \bar{\theta}^{\dot{\beta}}\} = 0$$

Derivatives with respect to the anticommuting coordinates are defined as

$$\left\{\frac{\partial}{\partial\theta^\alpha}, \theta^\beta\right\} = \delta_\alpha{}^\beta, \quad \left\{\frac{\partial}{\partial\theta^\alpha}, \bar{\theta}^{\dot{\alpha}}\right\} = \left\{\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, \theta^\alpha\right\} = 0 \text{ and } \left\{\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, \bar{\theta}^{\dot{\beta}}\right\} = \delta_{\dot{\alpha}}{}^{\dot{\beta}}$$

where $\delta_\alpha{}^\beta$ (and $\delta_{\dot{\alpha}}{}^{\dot{\beta}}$ of course) is the Kronecker delta, and they obey the anticommutation relations

$$\left\{\frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\theta^\beta}\right\} = \left\{\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, \frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}}\right\} = \left\{\frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}}\right\} = 0$$

Integration in superspace is defined by means of the Berezin integral:

$$\int d\theta\, \theta = 1 \text{ and } \int d\theta\, 1 = 0$$

As we observe integration and differentiation with respect to anticommuting coordinates are equivalent operations.

The goal of the superspace formulation is to provide a classical description of the action of supersymmetry on fields, just as in the description of the action of the Poincaré transformations. In other words, in order to exponentiate the supersymmetry generators we need the anticommuting coordinates in order to form well-defined exponentials:

$$e^{\theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}}$$

Now, consider functions of the superspace variables,

$$\Phi = \Phi(x, \theta, \bar{\theta})$$

The supersymmetry generators are represented by differential operators,

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \quad\text{and}\quad \bar Q_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \tag{1.3.1}$$

where σ^0 is the unity matrix and σ^i, $i = 1, 2, 3$, are the Pauli matrices:

$$\sigma^0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sigma^1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma^2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad\text{and}\quad \sigma^3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

The θs have mass dimension $-1/2$ and it can be checked that the supercharges in the representation (1.3.1) indeed satisfy the $\mathcal{N} = 1$ supersymmetry algebra:[8]

$$\{Q_\alpha, Q_\beta\} = \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0 \quad\text{and}\quad \{Q_\alpha, \bar Q_{\dot\alpha}\} = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \tag{1.3.2}$$

One can think of the Qs as generating infinitesimal transformations in superspace with (anticommuting) parameter ϵ^α (and $\bar\epsilon^{\dot\alpha}$). Then, finite transformations can be constructed by exponentiating the Qs and since the θs are anticommuting the θ-expansion of these exponentials will contain only a finite number of terms. With these additional transformations the Poincaré group is enhanced to the super-Poincaré group. The result is expressed as

$$e^{\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha}}\Phi(x^\mu, \theta_\alpha, \bar\theta_{\dot\alpha}) = \Phi(x^\mu - i\epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}, \theta^\alpha + \epsilon^\alpha, \bar\theta^{\dot\alpha} + \bar\epsilon^{\dot\alpha})$$

and from this it is easy to understand that supersymmetry is to be thought of as a spacetime symmetry.

Expanding Φ in powers of θ one would get a finite number of terms. However, the representation of the super-Poincaré group obtained by such an expansion is reducible and, thus, not a satisfactory one. An irreducible representation of the super-Poincaré group can be constructed by introducing the superspace-covariant derivatives, D_α and $\bar D_{\dot\alpha}$. These are objects which anticommute with the supersymmetry generators and, thus, are useful for writing down invariant expressions. In the representation we are working in they are given by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \quad\text{and}\quad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \tag{1.3.3}$$

and they satisfy the wrong-sign supersymmetry algebra (cf. (1.3.2)):

$$\{D_\alpha, D_\beta\} = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0 \quad\text{and}\quad \{D_\alpha, \bar D_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$$

The fact that the Ds anticommute with the Qs makes the condition

$$\bar D_{\dot\alpha}\Phi = 0$$

invariant under supersymmetry transformations. Superfields that satisfy this condition are called chiral superfields and they carry an irreducible representation of the super-Poincaré group. Their construction is rather easy: Setting

$$y^\mu = x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} \tag{1.3.4}$$

[8] Here we show only the anticommutation relations among supercharges.

it is straightforward to verify that

$$\Phi(y) = \phi(y) + \sqrt{2}\theta^\alpha \psi_\alpha(y) + \theta^2 F(y)$$

is a chiral superfield. Expanding in θ we get

$$\begin{aligned}\Phi(x,\theta,\bar\theta) = \phi(x) &+ i\theta^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\phi(x) + \frac{1}{4}\theta^2\bar\theta^2\partial^2\phi(x) \\ &+ \sqrt{2}\theta^\alpha\psi_\alpha(x) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi^\alpha(x)\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} + \theta^2 F(x)\end{aligned} \tag{1.3.5}$$

The components of the chiral superfield (1.3.5) are a complex scalar field (ϕ), a spinor field (ψ_α) and an auxiliary field (F). The transformation laws for these can be worked out by starting with

$$\delta\Phi = (\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha})\Phi$$

The result is

$$\delta\phi = \sqrt{2}\epsilon^\alpha\psi_\alpha, \;\; \delta\psi_\alpha = \sqrt{2}\epsilon_\alpha F + \sqrt{2}i\sigma^\mu_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}\partial_\mu\phi \;\text{ and }\; \delta F = i\sqrt{2}\bar\epsilon_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\partial_\mu\psi_\alpha \tag{1.3.6}$$

and it makes evident the fact that supersymmetry maps bosons to fermions and vice versa.

However, we are not done yet since, normally, any spectrum of states that is derived from a Lorentz-invariant field theory will exhibit CPT-symmetry. This implies that for each state with helicity λ there should exist a parity-reflected state with helicity $-\lambda$. So far this property is absent in our spectrum and, hence, in order to render our theory truly Lorentz-invariant we have to add the CPT-conjugate multiplet. Consequently, we find that the $\mathcal{N}=1$ chiral multiplet contains one state with helicity $-1/2$, two states with helicity 0 and one state with helicity $+1/2$.

Another irreducible representation of the super-Poincaré group, that is vector superfields, satisfy the condition

$$V = V^\dagger \tag{1.3.7}$$

which is invariant under supersymmetry transformations. The θ-expansion of V yields

$$\begin{aligned}V(x,\theta,\bar\theta) = C(x) &+ i\theta^\alpha\chi_\alpha(x) - i\bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha}(x) \\ &+ \frac{i}{2}\theta^2(M(x)+iN(x)) - \frac{i}{2}\bar\theta^2(M(x)-iN(x)) \\ &- \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}A_\mu(x) + i\theta^2\bar\theta_{\dot\alpha}\left(\bar\lambda^{\dot\alpha}(x) + \frac{i}{2}\bar\sigma^{\mu\dot\alpha\beta}\partial_\mu\chi_\beta(x)\right) \\ &- i\bar\theta^2\theta^\alpha\left(\lambda_\alpha(x) + \frac{i}{2}\sigma^\mu_{\alpha\dot\beta}\partial_\mu\bar\chi^{\dot\beta}(x)\right) \\ &+ \frac{1}{2}\theta^2\bar\theta^2\left(D(x) + \frac{1}{2}\partial^2 C(x)\right)\end{aligned} \tag{1.3.8}$$

The scalar fields C, D, M, N and the vector field A_μ must all be real for (1.3.8) to satisfy (1.3.7). Moreover, we have two spinor fields (χ_α and λ_α) in the supermultiplet. Evidently, the name of the entire supermultiplet comes from the vector field that appears in it.

1.4 $\mathcal{N} = 1$ supersymmetric gauge theory

Having found irreducible representations of the super-Poincaré group for $\mathcal{N} = 1$ supersymmetry, we can now consider gauge transformations. The presence of A_μ in the vector multiplet indicates that if A_μ is to describe a massless vector field, then there should be some underlying gauge symmetry which generalizes the ordinary gauge symmetries alluded to in the beginning of this chapter.

In the case of a $\mathsf{U}(1)$ theory we observe that the superfield $i\Lambda - i\Lambda^\dagger$, where Λ is a chiral superfield, contains a term of the form

$$i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} [i\partial_\mu(\phi(x) + \phi^*(x))] = \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu \alpha(x)$$

where $\alpha(x) = -(\phi(x) + \phi^*(x))$. Therefore, the transformation

$$V \to V + i\Lambda - i\Lambda^\dagger, \quad \bar{D}_{\dot{\alpha}}\Lambda = 0 \tag{1.4.1}$$

induces the transformation (1.1.4) to the gauge field that appears in the vector superfield and, thus, is the correct supersymmetric generalization of a gauge transformation. The corresponding field strength is the gauge-invariant, chiral superfield

$$W_\alpha = -\tfrac{1}{4}\bar{D}^2 D_\alpha V$$

Its form is chosen so that it produces the usual terms which describe the propagation of gauge fields (see equation (1.4.4) below). Now, the gauge freedom enables us to choose a gauge, which in our case will be the so-called Wess–Zumino gauge:

$$C = M = N = 0 \ \text{ and } \ \chi_\alpha = 0$$

in equation (1.3.8). This choice is analogous to the Coulomb gauge of electrodynamics and still allows the usual gauge transformations

$$A_\mu \to A_\mu + \partial_\mu(\phi(x) + \phi^*(x))$$

In the Wess–Zumino gauge the field strength is written as

$$W_\alpha(y,\theta,\bar{\theta}) = -i\lambda_\alpha(y) + \left(\delta_\alpha{}^\beta D(y) - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\beta F_{\mu\nu}(y)\right)\theta_\beta + \theta^2 \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \bar{\lambda}^{\dot{\alpha}}(y) \tag{1.4.2}$$

where y^μ is given by (1.3.4) and $F_{\mu\nu}(y) = \partial_\mu A_\nu(y) - \partial_\nu A_\mu(y)$.

The gauge transformation of a chiral superfield is

$$\Phi \to e^{-iq\Lambda}\Phi$$

where q is the "charge" of the transformation and Λ is a full chiral multiplet, something that guarantees that $e^{-iq\Lambda}\Phi$ is chiral. Now, since $\Lambda \neq \Lambda^\dagger$, we immediately notice that the term $\Phi^\dagger\Phi$ is not gauge-invariant. To fix this we use the vector superfield (1.3.8) with the gauge transformation (1.4.1) to construct the term

$$\Phi^\dagger e^{qV} \Phi$$

This last term is gauge invariant and, as we will see in a while, suitable to serve as the kinetic term for chiral superfields.

Our construction, so far, was limited to Abelian gauge groups but the generalization to the non-Abelian case is straightforward. A transformation on a chiral superfield is given by

$$\Phi \to e^{-i\Lambda}\Phi$$

where Λ is now a matrix-valued chiral superfield. Introducing a matrix-valued superfield V and requiring that

$$\Phi^\dagger e^V \Phi$$

be gauge-invariant, we find that e^V has to transform as

$$e^V \to e^{-i\Lambda^\dagger} e^V e^{i\Lambda}$$

under gauge transformations. From this we can define the gauge-covariant field strength

$$W_\alpha = -\tfrac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V \tag{1.4.3}$$

Summarizing, we have constructed the terms that represent matter fields and their superpartners, included in chiral supermultiplets, and those that represent gauge fields and their superpartners, included in vector supermultiplets. But let us explicitly construct the Lagrangians that describe their propagation. Consider a gauge group G with corresponding (Hermitian) generators $T^a_{\mathcal{R}}$, $a = 1, \ldots, \mathrm{dim}\mathsf{G}$ in a representation $\mathcal{R}$ of G, which satisfy

$$[T^a_{\mathcal{R}}, T^b_{\mathcal{R}}] = i f^{ab}{}_c T^c_{\mathcal{R}}$$

where $f^{ab}{}_c$ are the structure constants of the Lie algebra $\mathfrak{g}$ that corresponds to the Lie group G, which are usually taken to be real by a suitable choice of basis of $\mathfrak{g}$. It is important to stress here that the structure constants depend on the choice of basis of $\mathfrak{g}$. Also, note that in the literature it is common to refer to the structure constants as if they were a characteristic of the Lie group rather than its Lie algebra. This is certainly not so, but the abuse of the terminology is admissible since for simply-connected Lie groups, such as $\mathsf{SU}(n)$ and $\mathsf{Sp}(n)$ but not $\mathsf{SO}(n)$, there exists a natural one-to-one and onto correspondence among the representations of a Lie group and those of its Lie algebra [7, Theorem 3.7]. Now, suppose that G is simple, so that we have only one gauge coupling, g, in our theory. This is introduced by rescaling the vector field V,

$$V \to 2gV$$

and, hence, all its component fields, which results in rescaled definitions for the gauge-covariant derivatives, D_μ,[9]

$$D_\mu \lambda^a = \partial_\mu \lambda^a + g f^{ab}{}_c A_{\mu b} \lambda^c$$

and the component field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{ab}{}_c A_{\mu b} A^c_\nu$$

[9]Those have nothing to do with the superspace-covariant derivatives (1.3.3).

For the vector superfield we are interested in the adjoint representation of G. The adjoint representation of a Lie group G is a real and linear representation of G which acts on the space $\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. If G is a matrix Lie group containing $n \times n$ matrices, then $\mathfrak{g}$ is also realized by $n \times n$ matrices with the commutator as the Lie bracket. Then, the adjoint representation, Ad_G, maps $X \in \mathfrak{g}$ to $GXG^{-1} \in \mathfrak{g}$, where $G \in \mathsf{G}$, and is represented by $\dim \mathfrak{g}$ matrices of dimension $\dim \mathfrak{g} \times \dim \mathfrak{g}$. As an example consider the ubiquitous (matrix) Lie group $\mathsf{SU}(N)$. Its adjoint representation acts on its Lie algebra, $\mathfrak{su}(N)$, which is the space of $N \times N$ Hermitian traceless matrices. Furthermore, the adjoint representation of $\mathsf{SU}(N)$ is realized by $N^2 - 1$ matrices of dimension $(N^2-1) \times (N^2-1)$. Now, if we work in the adjoint representation we take $T^a_{\mathcal{R}} \equiv T^a_{\text{ad}}$ with

$$(T^a_{\text{ad}})^b_c = -if^{ab}{}_c$$

$a, b, c = 1, \ldots, N^2-1$, as the generators of G. Since $f^{ab}{}_c$ can be chosen to be real numbers, we immediately see that the generators T^a_{ad} are matrices with imaginary entries and, hence, $e^{iq_a T^a_{\text{ad}}}$ are matrices with real entries.

If we introduce the complex coupling

$$\tau = \frac{\vartheta}{2\pi} + \frac{4\pi i}{g^2}$$

where ϑ stands for the theta-angle,[10] then the propagation of the gauge field is described by the Lagrangian

$$\begin{aligned}\mathscr{L}_{\text{gauge}} &= \frac{1}{32\pi}\mathrm{Im}\left(\tau \int d^2\theta \, \mathrm{tr}\, W^\alpha W_\alpha\right) \\ &= \mathrm{tr}\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda^\alpha \sigma^\mu_{\alpha\dot\alpha} D_\mu \bar\lambda^{\dot\alpha} + \tfrac{1}{2}D^2\right) + \frac{\vartheta}{32\pi^2} g^2 \mathrm{tr}\, F_{\mu\nu}\tilde F^{\mu\nu}\end{aligned} \tag{1.4.4}$$

where

$$\tilde F^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

is the dual field strength, the trace is used to denote a sum over gauge indices, and

$$\mathrm{tr}\, D^2 \equiv \mathrm{tr}\, D_a T^a_{\text{ad}} D_b T^b_{\text{ad}} = D_a D_b \mathrm{tr}\, T^a_{\text{ad}} T^b_{\text{ad}} = D_a D^a$$

since we use the normalization

$$\mathrm{tr}\, T^a_{\text{ad}} T^b_{\text{ad}} = \delta^{ab} \tag{1.4.5}$$

Note that the single term $\mathrm{tr}\, W^\alpha W_\alpha$ has produced both the properly normalized kinetic term for the gauge field, $-\frac{1}{4}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}$, and the instanton density $\frac{g^2}{32\pi^2}\mathrm{tr}\, F_{\mu\nu}\tilde F^{\mu\nu}$ which multiplies the ϑ-angle.

So far with the gauge fields. We can now add chiral multiplets, Φ^i, transforming in some representation $\mathcal{R}$ of the gauge group G. The generators are represented by matrices, $T^a_{\mathcal{R}}$, and gauge transformations act as

$$\Phi^j \to e^{-i\Lambda}\Phi^j \quad \text{and} \quad \Phi^\dagger_j \to \Phi^\dagger_j e^{i\Lambda^\dagger}$$

[10] We do not use the usual θ here in order to avoid confusion with the superspace anticommuting coordinates.

where $\Lambda = \Lambda_a T^a_{\mathcal{R}}$. Then,

$$\Phi^\dagger_i e^{2gV_a T^a_{\mathcal{R}}} \Phi^i$$

is the gauge-invariant kinetic term and the Lagrangian that describes the propagation of matter fields as well as their possible interactions is

$$\mathscr{L}_{\text{matter}} = \int d^2\theta\, d^2\bar{\theta}\, \Phi^\dagger_i e^{2gV} \Phi^i + \int d^2\theta\, W(\Phi^i) + \int d^2\bar{\theta}\, [W(\Phi^i)]^\dagger \tag{1.4.6}$$

where $W(\Phi^i)$ is the so-called superpotential which is a holomorphic function of Φ^i and does not depend on $\Phi^\dagger_i$.[11] In components the gauge-invariant Lagrangian (1.4.6) becomes

$$\begin{aligned}\mathscr{L}_{\text{matter}} = {} & (D_\mu \phi^i)^\dagger D^\mu \phi^i - i\psi^{i\alpha}\sigma^\mu_{\alpha\dot{\alpha}} D_\mu \bar{\psi}^{\dot{\alpha}}_i + F^\dagger_i F^i \\ & + i\sqrt{2} g \phi^\dagger_i T^a_{\mathcal{R}} \lambda^\alpha_a \psi^i_\alpha - i\sqrt{2} g \bar{\psi}_{i\dot{\alpha}} \bar{\lambda}^{a\dot{\alpha}} T^\dagger_{a\mathcal{R}} \phi^i \\ & + g\phi^\dagger_i D_a T^a_{\mathcal{R}} \phi^i - \left(\frac{\partial W}{\partial \phi^i} F^i + \frac{1}{2} \frac{\partial^2 W}{\partial \phi^i\, \partial \phi^j} \psi^{i\alpha} \psi^j_\alpha + \text{H.c.} \right) \\ & + (\text{total derivative})\end{aligned} \tag{1.4.7}$$

where $D_\mu = \partial_\mu - igA_{a\mu} T^a_{\mathcal{R}}$ and "H.c." means Hermitian conjugate of whatever lies in the containing parentheses. Note that $\mathscr{L}_{\text{matter}}$ contains kinetic terms for the scalar fields, ϕ^i, and the matter fermions, ψ^i_α, as well as specific interactions between ϕ^i, ψ^i_α and the gauginos, λ^a_α. Additionally, there are nonderivative interactions coming from the superpotential. Note that gauge indices carried by matter fields are suppressed in the Lagrangian (1.4.7); for example

$$g\phi^\dagger_i D_a T^a_{\mathcal{R}} \phi^i \equiv g\phi^\dagger_{ib} D_a (T^a_{\mathcal{R}})^b_c \phi^{ic}$$

and

$$\frac{\partial^2 W}{\partial \phi^i\, \partial \phi^j} \psi^{i\alpha} \psi^j_\alpha \equiv \frac{\partial^2 W}{\partial \phi^{ia}\, \partial \phi^{jb}} \psi^{ia\alpha} \psi^{jb}_\alpha$$

Finally, there is another type of term that can appear if the gauge group G contains $\mathsf{U}(1)$ factors. (Of course, if there exists at least a $\mathsf{U}(1)$ factor, then G is certainly not simple (except if $\mathsf{G} = \mathsf{U}(1)$) and we have several gauge couplings.) These are the so-called Fayet-Iliopoulos terms,

$$2g\xi^A V_A$$

where the index A runs in Abelian factors, ξ^A are in general complex numbers and V_A denotes the vector superfield in the Abelian case or the component corresponding to the relevant Abelian factor. The corresponding Lagrangian is

$$\mathscr{L}_{\text{F.I.}} = 2g \int d^2\theta\, d^2\bar{\theta}\, \xi^A V_A = g\xi^A D_A \tag{1.4.8}$$

and, under an Abelian gauge transformation, it is easy to see that it transforms as a full derivative. Moreover, it is easy to show that it transforms as a full derivative under supersymmetry transformations as well.

[11] Of course, if we want the Lagrangian (1.4.6) to be gauge-invariant, we have to choose a gauge-invariant superpotential.

Gathering the terms of equations (1.4.4), (1.4.6) and (1.4.8) we get the $\mathcal{N}=1$ supersymmetric gauge-invariant Lagrangian

$$\begin{aligned}\mathscr{L}_{\mathcal{N}=1} &= \mathscr{L}_{\text{gauge}} + \mathscr{L}_{\text{matter}} + \mathscr{L}_{\text{F.I.}} \\ &= \frac{1}{32\pi}\mathrm{Im}\left(\tau\int d^2\theta\,\mathrm{tr}\,W^\alpha W_\alpha\right) + 2g\int d^2\theta\, d^2\bar\theta\,\xi^A V_A \\ &\quad + \int d^2\theta\, d^2\bar\theta\,\Phi_i^\dagger e^{2gV}\Phi^i + \int d^2\theta\, W(\Phi^i) + \int d^2\bar\theta\,[W(\Phi^i)]^\dagger\end{aligned} \tag{1.4.9}$$

which in components becomes

$$\begin{aligned}\mathscr{L}_{\mathcal{N}=1} = {}&\mathrm{tr}\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda^\alpha\sigma^\mu_{\alpha\dot\alpha}D_\mu\bar\lambda^{\dot\alpha} + \tfrac{1}{2}D^2\right) + \frac{\vartheta}{32\pi^2}g^2\mathrm{tr}\,F_{\mu\nu}\tilde F^{\mu\nu} \\ &+ g\xi^A D_A + (D_\mu\phi^i)^\dagger D^\mu\phi^i - i\psi^{i\alpha}\sigma^\mu_{\alpha\dot\alpha}D_\mu\bar\psi_i^{\dot\alpha} + F_i^\dagger F^i \\ &+ i\sqrt{2}g\phi_i^\dagger T^a_{\mathcal{R}}\lambda_a^\alpha\psi^i_\alpha - i\sqrt{2}g\bar\psi_{i\dot\alpha}\bar\lambda^{a\dot\alpha}T^\dagger_{a\mathcal{R}}\phi^i + g\phi_i^\dagger D_a T^a_{\mathcal{R}}\phi^i \\ &- \left(\frac{\partial W}{\partial\phi^i}F^i + \frac{1}{2}\frac{\partial^2 W}{\partial\phi^i\,\partial\phi^j}\psi^{i\alpha}\psi^j_\alpha + \text{H.c.}\right) \\ &+ (\text{total derivative})\end{aligned} \tag{1.4.10}$$

The equations of motion for the auxiliary fields give the so-called F- and D-terms

$$F_i^\dagger = \frac{\partial W}{\partial\phi^i} \quad\text{and}\quad D^a = -g\phi_i^\dagger T^a_{\mathcal{R}}\phi^i - g\xi^a$$

where $\xi^a = 0$ if the index a does not take values in an Abelian factor of the gauge group G. Substituting those back into the Lagrangian (1.4.10) we get

$$\begin{aligned}\mathscr{L}_{\mathcal{N}=1} = {}&\mathrm{tr}\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda^\alpha\sigma^\mu_{\alpha\dot\alpha}D_\mu\bar\lambda^{\dot\alpha}\right) + \frac{\vartheta}{32\pi^2}g^2\mathrm{tr}\,F_{\mu\nu}\tilde F^{\mu\nu} \\ &+ (D_\mu\phi^i)^\dagger D^\mu\phi^i - i\psi^{i\alpha}\sigma^\mu_{\alpha\dot\alpha}D_\mu\bar\psi_i^{\dot\alpha} \\ &+ i\sqrt{2}g\phi_i^\dagger T^a_{\mathcal{R}}\lambda_a^\alpha\psi^i_\alpha - i\sqrt{2}g\bar\psi_{i\dot\alpha}\bar\lambda^{a\dot\alpha}T^\dagger_{a\mathcal{R}}\phi^i \\ &- \left(\frac{1}{2}\frac{\partial^2 W}{\partial\phi^i\,\partial\phi^j}\psi^{i\alpha}\psi^j_\alpha + \text{H.c.}\right) - V(\phi_i^\dagger,\phi^i) + (\text{total derivative})\end{aligned} \tag{1.4.11}$$

where

$$V(\phi^i,\phi_i^\dagger) = F_i^\dagger F^i + \tfrac{1}{2}D^2 = \sum_i\left|\frac{\partial W}{\partial\phi^i}\right|^2 + \frac{g^2}{2}\sum_a(\phi_i^\dagger T^a_{\mathcal{R}}\phi^i + \xi^a)^2 \tag{1.4.12}$$

is the so-called scalar potential.

The next sensible step is to choose particular superpotentials and analyze the emerging models. However, let us first venture into some general considerations on the physically interesting superpotentials.

1.5 The nonlinear sigma model

1.5.1 Only matter

Having extensively analyzed $\mathcal{N} = 1$ supersymmetric Yang–Mills theory, we should now wonder which are the kinetic terms and superpotentials that describe consistent and sensible physical theories. In other words, we should find the class of supersymmetric theories that result in ordinary theories which are renormalizable. For theories with chiral superfields only, it is very easy to verify that a renormalizable ordinary quantum field theory emerges when the kinetic term is of the form $K^i_j \Phi^\dagger_i \Phi^j$ with some constant Hermitian matrix K,[12] while the superpotential is at most cubic in the chiral superfields, leading to at most quartic scalar potentials. A very well-known example, which we will not analyze though, is the Wess–Zumino model, a theory with a single chiral superfield, Φ, a canonical kinetic term and superpotential

$$W(\Phi) = \tfrac{1}{2} m \Phi^2 + \tfrac{1}{3} \lambda \Phi^3$$

However, it is often the case that the theory we are analyzing is not a microscopic (complete) theory but, instead, a low-energy effective theory, i.e. a theory that describes our system only up to some energy scale. Then, we need not require renormalizability any more and, thus, our freedom in choosing kinetic terms and superpotentials increases. However we should not forget that our effective theories should include terms with at most two spacetime derivatives, since higher-derivative terms are irrelevant at low energies.

The above discussion dictates the form of the most general (nonrenormalizable) $\mathcal{N} = 1$ supersymmetric theory we can consider. If we include chiral superfields only, then

$$\mathscr{L}^{(\text{eff})}_{\text{matter}} = \int d^2\theta \, d^2\bar\theta \, K(\Phi^i, \Phi^\dagger_i) + \int d^2\theta \, W(\Phi^i) + \int d^2\bar\theta \, [W(\Phi^i)]^\dagger \tag{1.5.1}$$

where the function $K(\Phi^i, \Phi^\dagger_i)$ has to satisfy the reality condition

$$K^\dagger(\phi^i, \phi^\dagger_i) = K(\phi^\dagger_i, \phi^i)$$

where ϕ^i is the zero component of Φ^i, since we want to be working with real Lagrangians. It is straightforward, if laborious, to calculate that in components we have

$$\begin{aligned}\int d^2\theta \, d^2\bar\theta \, K(\Phi^i, \Phi^\dagger_i) = {} & K^j_i (F^i F^\dagger_j - \partial_\mu \phi^i \partial^\mu \phi^\dagger_j - i \bar\psi_{j\dot\alpha} \bar\sigma^{\mu\dot\alpha\alpha} \partial_\mu \psi^i_\alpha) \\ & - \tfrac{1}{2} K^i_{jk} F^\dagger_i \psi^{j\alpha} \psi^k_\alpha - \tfrac{1}{2} K^{jk}_i F^i \bar\psi_{j\dot\alpha} \bar\psi^{\dot\alpha}_k \\ & - i K^k_{ij} \bar\psi_{k\dot\alpha} \bar\sigma^{\mu\dot\alpha\alpha} \psi^i_\alpha \partial_\mu \phi^j + \tfrac{1}{4} K^{kl}_{ij} \psi^{i\alpha} \psi^j_\alpha \bar\psi_{k\dot\alpha} \bar\psi^{\dot\alpha}_l \\ & + (\text{total derivative})\end{aligned} \tag{1.5.2}$$

where $K^j_i \equiv \frac{\partial^2 K(\phi,\phi^\dagger)}{\partial\phi^i \, \partial\phi^\dagger_j}$, $K^k_{ij} \equiv \frac{\partial^3 K(\phi,\phi^\dagger)}{\partial\phi^i \, \partial\phi^j \, \partial\phi^\dagger_k}$ and similarly for K^{jk}_i and K^i_{jk} and $K^{kl}_{ij} \equiv \frac{\partial^4 K(\phi,\phi^\dagger)}{\partial\phi^i \, \partial\phi^j \, \partial\phi^\dagger_k \, \partial\phi^\dagger_l}$.

[12] Diagonalization of K and rescaling of the fields leads to the canonical kinetic term $\Phi^\dagger_i \Phi^i$

As we observe there are no holomorphic terms in the Lagrangian (1.5.2), i.e. there are derivatives of $K(\phi, \phi^\dagger)$ with respect to both ϕ and $\phi^\dagger$. This shows that the so-called Kähler transformations,

$$K(\phi, \phi^\dagger) \to K(\phi, \phi^\dagger) + f(\phi) + f^\dagger(\phi^\dagger)$$

where f and $f^\dagger$ are arbitrary analytic functions of ϕ and $\phi^\dagger$ respectively, do not affect the Lagrangian (1.5.2). Moreover, it is easy to see that the Kähler transformations can be generalized to the superfield level, for the transformations

$$K(\Phi, \Phi^\dagger) \to K(\Phi, \Phi^\dagger) + f(\Phi) + f^\dagger(\Phi^\dagger)$$

do not affect the Lagrangian (1.5.1).

Now, note that in the Lagrangian (1.5.2) there is a metric for the kinetic terms for the complex scalars which is obtained by the complex scalar function $K(\phi, \phi^\dagger)$:

$$K_i^j = \frac{\partial^2}{\partial\phi^i \, \partial\phi_j^\dagger} K(\phi, \phi^\dagger)$$

Such a metric is called a Kähler metric and the complex scalar function $K(\phi, \phi^\dagger)$ is called the Kähler potential. Since the Kähler metric is invariant under Kähler transformations of the Kähler potential, we are immediately led to interpret the complex scalars ϕ^i as local coordinates on a Kähler manifold. In that sense the target manifold of the sigma model is Kähler. Also, it is natural to think of fermions as tensors in the tangent space of the Kähler manifold.

Once we identify K_i^j with the metric of the Kähler manifold, we can easily work out the connection and the Riemann tensor of the manifold. As is well known, for a general metric g_{ab} the connection is given by

$$\Gamma^c_{ab} = \tfrac{1}{2} g^{cd}(\partial_a g_{db} + \partial_b g_{ad} - \partial_d g_{ab})$$

where g^{ab} is the inverse of g_{ab}, and the Riemann tensor is defined as

$$(R_{ab})^c_d = \partial_a \Gamma^c_{bd} - \partial_b \Gamma^c_{ad} + \Gamma^c_{ae}\Gamma^e_{bd} - \Gamma^c_{be}\Gamma^e_{ad}$$

In the case of the Kähler metric the only nonvanishing terms are[13]

$$\Gamma^k_{ij} = (K^{-1})^k_l K^l_{ij}, \quad \Gamma^{ij}_k = (K^{-1})^l_k K^{ij}_l$$

where K^{-1} denotes the inverse Kähler metric,

$$(K^{-1})^i_k K^j_i = \delta^j_k$$

[13]This calculation is more easily carried out if we temporarily change our notation by substituting

$$\phi_i^\dagger \to \phi^{\bar{i}} \;\text{ and }\; \frac{\partial K}{\partial\phi^i \, \partial\phi_j^\dagger} \equiv K_i^j \to K_{i\bar{j}} \equiv \frac{\partial K}{\partial\phi^i \, \partial\phi^{\bar{j}}}$$

and similarly for the rest, where bar and dagger both denote complex conjugation. In the end we switch back to our usual notation to get the result mentioned in the text.

and

$$R_{ij}^{kl} = K_{ij}^{kl} - K_{ij}^{m}(K^{-1})_{m}^{n}K_{n}^{kl}$$

and it is worth expressing the component form of the Lagrangian (1.5.1) in this simpler and more geometric language. To achieve this we first need to eliminate the auxiliary fields. Since

$$\int d^2\theta\, W(\Phi^i) + \text{H.c} = (W_i F^i - \tfrac{1}{2}W_{ij}\psi^{i\alpha}\psi_\alpha^j) + \text{H.c} \tag{1.5.3}$$

where $W_i \equiv \partial W/\partial\phi^i$ and $W_{ij} \equiv \partial^2 W/\partial\phi^i\,\partial\phi^j$, by adding (1.5.3) to (1.5.2) we can find the equations of motion for the auxiliary fields F^i:

$$F^i = -(K^{-1})_j^i W^{\dagger j} + \tfrac{1}{2}\Gamma_{jk}^i \psi^{j\alpha}\psi_\alpha^k$$

Those can be used to eliminate the auxiliary fields from the component form of the Lagrangian (1.5.2), which, finally, takes the form

$$\begin{aligned}\mathscr{L}_{\text{matter}}^{(\text{eff})} = &-K_i^j(\partial_\mu\phi^i\partial^\mu\phi_j^\dagger + i\bar{\psi}_{\dot{\alpha}}^j\bar{\sigma}^{\mu\dot{\alpha}\alpha}D_\mu\psi_\alpha^i) + \tfrac{1}{4}R_{ij}^{kl}\psi^{i\alpha}\psi_\alpha^j\bar{\psi}_{k\dot{\alpha}}\bar{\psi}_l^{\dot{\alpha}} \\ &- \tfrac{1}{2}[(W_{ij} - \Gamma_{ij}^k W_k)\psi^{i\alpha}\psi_\alpha^j + (W^{\dagger ij} - \Gamma_k^{ij}W^{\dagger k})\bar{\psi}_{i\dot{\alpha}}\bar{\psi}_j^{\dot{\alpha}}] \\ &- V(\phi^i, \phi_j^\dagger) + (\text{total derivative})\end{aligned} \tag{1.5.4}$$

where

$$V(\phi^i, \phi_j^\dagger) = (K^{-1})_j^i W_i W^{\dagger j} \tag{1.5.5}$$

is the scalar potential and

$$D_\mu\psi_\alpha^i = \partial_\mu\psi_\alpha^i + \Gamma_{jk}^i\partial_\mu\phi^j\psi_\alpha^k$$

is the Kähler-covariant derivative on fermions. From its form we immediately understand that fermions can actually be treated as contravariant vectors on the tangent space of the Kähler manifold.

At this point we have to stress an important characteristic that has to do with the scalar potential (1.5.5) of the effective theories we are analyzing. If we compare that to the scalar potential we found in the case of renormalizable (UV-complete) theories of chiral superfields with $\mathcal{N} = 1$ supersymmetry,

$$V(\phi^i, \phi_i^\dagger) = W_i W^{\dagger i}$$

(that is the superpotential (1.4.12) with zero D-terms) then we immediately see that the Kähler metric changes the way supersymmetry is broken. Indeed, supersymmetry is broken when the scalar potential is larger than zero and, thus, when it comes to supersymmetry breaking, theories with a noncanonical Kähler potential differ substantially from theories with the canonical Kähler potential $K_{\text{can}}(\phi^i, \phi_i^\dagger) = \phi_i^\dagger\phi^i$.

1.5.2 Adding gauge fields

It is now time to add gauge fields in our discussion. The gauge group is G with gauge coupling g and corresponding (Hermitian) generators $T_{\mathcal{R}}^a$, $a = 1, \ldots, \dim\mathsf{G}$ in a representation $\mathcal{R}$ of G.

The first effect of the inclusion of gauge fields is that they change the spacetime derivatives in the term $K(\phi, \phi^\dagger)$ into gauge-covariant derivatives,

$$\partial_\mu \phi^i \to D_\mu \phi^i = \partial_\mu \phi^i - igA_{a\mu} T^a_{\mathcal{R}} \phi^i$$

in the case of scalars, and the Kähler-covariant derivatives into gauge- and Kähler-covariant derivatives,

$$D_\mu \psi^i_\alpha \to \tilde{D}_\mu \psi^i_\alpha = \partial_\mu \psi^i_\alpha - igA_{a\mu} T^a_{\mathcal{R}} \psi^i_\alpha + \Gamma^i_{jk} D_\mu \phi^j \psi^k_\alpha$$

in the case of spinors. In addition, we have to add kinetic terms for the vector superfield containing the gauge fields. Evidently, the discussion we are about to begin will closely follow the steps we took in section 1.4.

As for the matter Lagrangian we need only observe that we can follow the exact steps that brought us to the Lagrangian (1.4.6) to verify that if we just replace

$$\phi^\dagger_i \to (\phi^\dagger e^{2gV})_i$$

in (1.5.1), then the term

$$\int d^2\theta \, d^2\bar{\theta} \, K(\Phi^i, (\Phi^\dagger e^{2gV})_i)$$

is gauge-invariant for any real function K. Therefore, we only need to guarantee that the superpotential is gauge-invariant in order to conclude that the action

$$\mathscr{L}^{(\text{eff})}_{\text{matter}} = \int d^2\theta \, d^2\bar{\theta} \, K(\Phi^i, (\Phi^\dagger e^{2gV})_i) + \int d^2\theta \, W(\Phi^i) + \int d^2\bar{\theta} \, [W(\Phi^i)]^\dagger \tag{1.5.6}$$

is $\mathcal{N} = 1$ supersymmetric and gauge-invariant. Its component form is

$$\begin{aligned}
\mathscr{L}^{(\text{eff})}_{\text{matter}} = &-K^j_i [D_\mu \phi^i D^\mu \phi^\dagger_j + i\bar{\psi}^j_{\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \tilde{D}_\mu \psi^i_\alpha \\
&- (F^i - \tfrac{1}{2}\Gamma^i_{kl} \psi^{k\alpha} \psi^l_\alpha)(F^\dagger_j - \tfrac{1}{2}\Gamma^{mn}_j \bar{\psi}_{m\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}_n)] \\
&- [\sqrt{2} g (T^a_{\mathcal{R}})^i_k \phi^k \bar{\psi}_{j\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}_a + \text{H.c}] \\
&+ [-\tfrac{i}{2} g D^a (T_{a\mathcal{R}})^j_i \phi^i K_j + \text{H.c.}] \\
&+ (W_i F^i - \tfrac{1}{2} W_{ij} \psi^{i\alpha} \psi^j_\alpha + \text{H.c.}) + (\text{total derivative})
\end{aligned} \tag{1.5.7}$$

Proceeding to the generalization of the gauge Lagrangian (1.4.4), we recall that the gauge-covariant field strength is defined by

$$W_\alpha = -\tfrac{1}{4} \bar{D}^2 e^{-2gV} D_\alpha e^{2gV}$$

and it takes the form (1.4.2) times $2g$ in the Wess–Zumino gauge. Since at most two-derivative terms should be included, we end up with the most general possibility

$$\mathscr{L}^{(\text{eff})}_{\text{gauge}} = \int d^2\theta \, f_{ab}(\phi^i) W^{a\alpha} W^b_\alpha + \text{H.c.} \tag{1.5.8}$$

where the holomorphic function $f_{ab} = f_{ba}$ transforms under the gauge group as the symmetrized square of the adjoint representation. Note that with the choice $f_{ab} = (\tau/64\pi i)\mathrm{tr}\, T^a_{\mathrm{ad}} T^b_{\mathrm{ad}}$ we recover the Lagrangian (1.4.4).[14] The component form of (1.5.8) is

$$\begin{aligned}\mathscr{L}^{(\mathrm{eff})}_{\mathrm{gauge}} = &-f_{ab}[i\lambda^{a\alpha}\sigma^{\mu}_{\alpha\dot\alpha}\partial_\mu\bar\lambda^{b\dot\alpha} + i\lambda^{b\alpha}\sigma^{\mu}_{\alpha\dot\alpha}\partial_\mu\bar\lambda^{a\dot\alpha} \\ &+ \tfrac{1}{4}(F^{a\mu\nu} + ig\tilde F^{a\mu\nu})(F^{b}_{\mu\nu} + ig\tilde F^{b}_{\mu\nu}) - D^a D^b] \\ &+ \frac{1}{2\sqrt{2}}\frac{\partial f_{ab}}{\partial\phi^i}\psi^{i\alpha}\left[i\lambda^a_\alpha D^b + i\lambda^b_\alpha D^a + \frac{1}{2}(\sigma^{\mu}_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\beta})(\lambda^a_\beta F^b_{\mu\nu} + \lambda^b_\beta F^a_{\mu\nu})\right] \\ &- \lambda^{a\alpha}\lambda^b_\alpha\left(F^i\frac{\partial f_{ab}}{\partial\phi^i} - \frac{1}{2}\psi^{i\beta}\psi^j_\beta\frac{\partial^2 f_{ab}}{\partial\phi^i\,\partial\phi^j}\right) + \mathrm{H.c.}\end{aligned} \tag{1.5.9}$$

where, here, the Hermitian conjugation applies to all the terms in the right-hand side of the equation.

The sum of the Lagrangians (1.5.7) and (1.5.9) gives the full Lagrangian of the model we are studying, and from that we can work out the equations of motion for the auxiliary fields. Those turn out to be

$$F^i = (K^{-1})^i_j\left[\left(\frac{\partial f_{ab}(\phi)}{\partial\phi^j}\right)^\dagger \bar\lambda^a_{\dot\alpha}\bar\lambda^{b\dot\alpha} + \frac{1}{2}K^j_{kl}\psi^{k\alpha}\psi^l_\alpha - W^{\dagger j}\right]$$

and

$$D^a = \frac{i}{2}(f^{-1})^{ab}\left[g(T_{b\mathcal{R}})^j_i\phi^i K_j - \frac{1}{\sqrt{2}}\frac{\partial f_{bc}}{\partial\phi^i}\psi^{i\alpha}\lambda^c_\alpha\right]$$

and, of course, they can be used to eliminate the auxiliary fields from the Lagrangian of our model.

1.6 $\mathcal{N}=2$ superspace

In order to consider supersymmetric theories with eight supercharges we have to extend the $\mathcal{N}=1$ superspace by adding another pair of anticommuting coordinates. Hence, in addition to the ordinary coordinates, x^μ, $\mathcal{N}=2$ superspace contains the anticommuting coordinates θ^α_i and $\bar\theta^{\dot\alpha}_i$, where $i = 1, 2$ and greek dotted and undotted indices are as in the case of $\mathcal{N}=1$ superspace.

An important characteristic we have to think about at this point is the so-called R-symmetry. This is a global symmetry that transforms different supercharges in a theory with supersymmetry into each other. R-symmetry does not generally commute with supersymmetry and particles in a supermultiplet do not have the same quantum number (R-charge) under it. The reason we did not mention R-symmetry in our treatment of $\mathcal{N}=1$ supersymmetry is that, in that case, the R-symmetry group was just the Abelian group $\mathsf{U}(1)$ which is often broken by quantum anomalies. But in theories with extended supersymmetry the group of R-symmetry becomes non-Abelian and, in the case of $\mathcal{N}=2$ supersymmetry without central

[14] To see this observe that if z is a complex number, then $\mathrm{Re}(z) = \mathrm{Im}(iz)$.

charges, it is the group $\mathsf{SU}(2)$.[15] Now, if we want to respect the $\mathsf{SU}(2)$ R-symmetry, we should think of $(\theta_1^\alpha, \theta_2^\alpha)$ and $(\bar\theta_1^{\dot\alpha}, \bar\theta_2^{\dot\alpha})$ as $\mathsf{SU}(2)_R$ doublets,[16] i.e. so that they transform in the fundamental representation of $\mathsf{SU}(2)_R$.

The $\mathcal{N} = 2$ hypermultiplet $\Psi(x, \theta_i, \bar\theta_i)$ is defined as an $\mathcal{N} = 2$ superfield which is a singlet under the global $\mathsf{SU}(2)_R$ and satisfies the covariant constraints

$$\bar D_{i\dot\alpha}\Psi = 0, \quad i = 1, 2$$

In parallel with $\mathcal{N} = 1$ supersymmetry it is convenient to introduce new bosonic coordinates,

$$\tilde y^\mu = x^\mu + i\sum_{i=1}^{2}\theta_i^\alpha \sigma^\mu_{\alpha\dot\alpha}\bar\theta_i^{\dot\alpha}$$

Now, if we expand a hypermultiplet in powers of θ_2^α, then the components are $\mathcal{N} = 1$ chiral superfields. Indeed,

$$\Psi(x, \theta_i, \bar\theta_i) = \Phi(\tilde y, \theta_1) + i\sqrt{2}\theta_2^\alpha W_\alpha(\tilde y, \theta_1) + \theta_2^2 G(\tilde y, \theta_1)$$

where

$$\Phi(\tilde y, \theta_1) = \Phi(y + \theta_2^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta_2^{\dot\alpha}, \theta_1)$$
$$G(\tilde y, \theta_1) = G(y + \theta_2^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta_2^{\dot\alpha}, \theta_1)$$

and

$$W_\alpha(\tilde y, \theta_1) = W_\alpha(y + \theta_2^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta_2^{\dot\alpha}, \theta_1)$$

are chiral superfields. Consequently, the hypermultiplet contains matter fields.

The physical content of the hypermultiplet is two complex scalars, q and $\tilde q$, and two Weyl fermions, ψ_q and $\psi_{\tilde q}^\dagger$. Since Ψ is an $\mathsf{SU}(2)_R$ singlet, while $(\theta_1^\alpha, \theta_2^\alpha)$ is an $\mathsf{SU}(2)_R$ doublet, it follows that the fermionic component fields are also $\mathsf{SU}(2)_R$ singlets while, on the other hand, the bosonic component fields form an $\mathsf{SU}(2)_R$ doublet. A convenient way which we can use in order to exhibit the $\mathsf{SU}(2)_R$ symmetry is to arrange the two Weyl fermions and the two complex bosons of the hypermultiplet in the diamond

$$\begin{array}{ccc} & \psi_q & \\ q & & \tilde q \\ & \psi_{\tilde q}^\dagger & \end{array}$$

Then, the $\mathsf{SU}(2)_R$ symmetry acts on the rows of the diamond. For further details on the R-symmetry of $\mathcal{N} = 2$ supersymmetry the reader is referred to [8].

The other multiplet of interest in theories with $\mathcal{N} = 2$ supersymmetry is the vector multiplet. This contains an $\mathcal{N} = 1$ vector superfield and an $\mathcal{N} = 1$ chiral superfield, altogether a gauge field, two Weyl fermions, and a complex scalar (all in the adjoint representation of

[15] In fact the R-symmetry group in this case is the group $\mathsf{U}(2)$, but the $\mathsf{U}(1)$ factor of this group is again often broken by quantum anomalies. The whole $\mathsf{U}(2)$ R-symmetry is present in scale-invariant $\mathcal{N} = 2$ theories.

[16] The index R makes it clear that this is the global group that corresponds to the R-symmetry.

the gauge group we consider; for details see section 1.7). Although the hypermultiplet can be either massless or a short massive multiplet (BPS), the vector multiplet can only be massless. The fermions are doublets of $\mathsf{SU}(2)_R$, while the gauge field and the complex scalar are singlets. Again, the diamond structure is useful; we arrange the gauge field, A_μ, the fermionic fields, λ and ψ, and the scalar field, ϕ, of the vector multiplet in the diamond

$$\begin{array}{ccc} & A_\mu & \\ \lambda & & \psi \\ & \phi & \end{array}$$

The $\mathsf{SU}(2)_R$ symmetry acts, just as in the case of the hypermultiplet, on the rows of the diamond. Under $\mathcal{N}=1$ supersymmetry the vector multiplet decomposes into a vector superfield (1.3.8) with the gauge-covariant field strength (1.4.3) and a chiral superfield (1.3.5).

1.7 $\mathcal{N}=2$ supersymmetric gauge theory

Our analysis of the $\mathcal{N}=2$ supersymmetric gauge theory with gauge group G will involve only the vector multiplet and not the hypermultiplet. Using the decomposition of the $\mathcal{N}=2$ vector multiplet into $\mathcal{N}=1$ multiplets, our construction starts by adding the Lagrangians (1.4.4) and (1.4.7).[17] However, we should pay attention to the fact that since our fields belong to the same multiplet they must transform in the same representation of G. The fields in the $\mathcal{N}=1$ vector multiplet transform necessarily in the adjoint representation of G and, thus, the same must hold for the component fields of the chiral superfield. In that case the $\mathcal{N}=2$ matter Lagrangian (1.4.7) becomes

$$\begin{aligned}\mathscr{L}^{(\mathrm{ad})}_{\mathrm{matter}} &= \int d^2\theta\, d^2\bar\theta\; \Phi^\dagger_b (e^{2gV})^b_{\ c}\Phi^c \\ &= \mathrm{tr}\big((D_\mu\phi)^\dagger(D^\mu\phi) - i\psi^\alpha\sigma^\mu_{\alpha\dot\alpha}D_\mu\bar\psi^{\dot\alpha} + F^\dagger F \\ &\quad + i\sqrt{2}g\phi^\dagger\{\lambda^\alpha,\psi_\alpha\} - i\sqrt{2}g\{\bar\psi_{\dot\alpha},\bar\lambda^{\dot\alpha}\}\phi + gD[\phi,\phi^\dagger]\big)\end{aligned} \tag{1.7.1}$$

where

$$\phi = \phi_a T^a_{\mathrm{ad}},\quad \psi_\alpha = \psi_{a\alpha}T^a_{\mathrm{ad}} \ \text{ and } \ F = F_a T^a_{\mathrm{ad}}$$

in addition to

$$\lambda_\alpha = \lambda_{a\alpha}T^a_{\mathrm{ad}},\quad D = D_a T^a_{\mathrm{ad}} \ \text{ and } \ A_\mu = A_{a\mu}T^a_{\mathrm{ad}}$$

Let us see how the anticommutator in the term $\sqrt{2}g\phi^\dagger T^a_{\mathcal{R}}\lambda_a\psi$ arises. Similar steps explain the appearance of the anticommutator in the term $\sqrt{2}g\bar\psi\bar\lambda_a T^a_{\mathcal{R}}\phi$ and the commutator in the term $g\phi^\dagger D_a T^a_{\mathcal{R}}\phi$. In the adjoint representation

$$\phi^\dagger T^a_{\mathcal{R}}\lambda_a\psi \to \phi^\dagger_b (T^a_{\mathrm{ad}})^b_{\ c}\lambda^\alpha_a\psi^c_\alpha$$

[17] Here we do not include the superpotential for reasons we will explain below.

and this becomes

$$\begin{aligned}
\phi_b^\dagger (T^a_{\rm ad})^b_c \lambda^\alpha_a \psi^c_\alpha &= \phi_b^\dagger(-if^{ab}{}_c)\lambda^\alpha_a\psi^c_\alpha \\
&= \phi_b^\dagger(+if^{ba}{}_c)\lambda^\alpha_a\psi^c_\alpha \\
&= \phi_b^\dagger(\operatorname{tr} T^b[T^a,T_c])\lambda^\alpha_a\psi^c_\alpha \\
&= \phi_b^\dagger(\operatorname{tr} T^bT^aT_c)\lambda^\alpha_a\psi^c_\alpha - \phi_b^\dagger(\operatorname{tr} T^bT_cT^a)\lambda^\alpha_a\psi^c_\alpha \\
&= \operatorname{tr}\phi^\dagger\lambda^\alpha\psi_\alpha - \operatorname{tr}\phi^\dagger\psi_\alpha\lambda^\alpha \\
&= \operatorname{tr}\phi^\dagger\lambda^\alpha\psi_\alpha + \operatorname{tr}\phi^\dagger\psi^\alpha\lambda_\alpha \\
&= \operatorname{tr}\phi^\dagger\{\lambda^\alpha,\psi_\alpha\}
\end{aligned}$$

where we used the fact that

$$(T^a_{\rm ad})^b_c = -if^{ab}{}_c$$

the obvious property

$$f^{ba}{}_c = -f^{ab}{}_c$$

the normalization (1.4.5) and the supersymmetric identity

$$\lambda_{a\alpha}\psi^\alpha = -\lambda^\alpha_a\psi_\alpha$$

What remains is to add terms that describe the propagation of our fields. We worked out these terms in section 1.4 (see equation (1.4.4)) and, hence, we just have to add them to the Lagrangian (1.7.1). The resulting Lagrangian is

$$\begin{aligned}
\mathscr{L}_{\mathcal{N}=2} &= \int d^2\theta\, d^2\bar\theta\, \Phi^\dagger e^{2gV}\Phi + \frac{1}{32\pi}\mathrm{Im}\left(\tau\int d^2\theta \operatorname{tr} W^\alpha W_\alpha\right) \\
&= \operatorname{tr}\big(-\tfrac14 F_{\mu\nu}F^{\mu\nu} - i\lambda^\alpha\sigma^\mu_{\alpha\dot\alpha}D_\mu\bar\lambda^{\dot\alpha} - i\psi^\alpha\sigma^\mu_{\alpha\dot\alpha}D_\mu\bar\psi^{\dot\alpha} + \tfrac12 D^2 \\
&\quad + \frac{\vartheta}{32\pi^2}g^2F_{\mu\nu}\tilde F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) + F^\dagger F + gD[\phi,\phi^\dagger] \\
&\quad + i\sqrt2 g\phi^\dagger\{\lambda^\alpha,\psi_\alpha\} - i\sqrt2 g\{\bar\psi_{\dot\alpha},\bar\lambda^{\dot\alpha}\}\phi\big)
\end{aligned} \tag{1.7.2}$$

which is indeed, although far from obviously, $\mathcal{N}=2$ supersymmetric. Now, as we saw in the previous section, theories with $\mathcal{N}=2$ supersymmetry necessarily have a global $\mathsf{SU}(2)_R$ symmetry. From the diamond structure of the vector multiplet we already knew that the R-symmetry rotates the fermionic fields ψ and λ into each other and, hence, it is straightforward to check that indeed the Lagrangian (1.7.2) is invariant under the R-symmetry. Note here that, had we added a superpotential in the beginning of our construction, then we would have a term $\sim W_{ij}\psi^{i\alpha}\psi^j_\alpha$ appearing for the matter fermions, ψ, but not for the gauginos, λ. But that would break the $\mathsf{SU}(2)_R$ invariance unless the superpotential was linear in Φ. Moreover, any superpotential linear in Φ would generate a scalar potential of the form $|W_i|^2 \sim |\text{constant}|^2$ which is strictly positive. However, that would spontaneously break supersymmetry[18] and, thus, the superpotential can only be a constant which we can always

[18] Why this is so will be explained in the next chapter.

set to zero. Also, it has to be stressed that our description of $\mathcal{N}=2$ supersymmetric Yang–Mills theory does not involve a central charge, since the $\mathcal{N}=2$ vector multiplet we have been using carries an off-shell representation of the $\mathcal{N}=2$ supersymmetry algebra in which the central charge is trivial.

Our next step is to eliminate the auxiliary fields from the Lagrangian (1.7.2). Their equations of motion give

$$F^a = 0 \text{ and } D^a = -g[\phi, \phi^\dagger]^a$$

and, thus, the Lagrangian (1.7.2) becomes

$$\begin{aligned}\mathscr{L}_{\mathcal{N}=2} = \operatorname{tr}\big(&-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda^\alpha\sigma^\mu_{\alpha\dot{\alpha}}D_\mu\bar{\lambda}^{\dot{\alpha}} - i\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}D_\mu\bar{\psi}^{\dot{\alpha}} \\ &+ \frac{\vartheta}{32\pi^2}g^2F_{\mu\nu}\tilde{F}^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) \\ &+ i\sqrt{2}g\phi^\dagger\{\lambda^\alpha, \psi_\alpha\} - i\sqrt{2}g\{\bar{\psi}_{\dot{\alpha}}, \bar{\lambda}^{\dot{\alpha}}\}\phi\big) - V(\phi, \phi^\dagger)\end{aligned}$$

where the scalar potential is

$$V(\phi, \phi^\dagger) = \tfrac{1}{2}\operatorname{tr} D^2 = \tfrac{1}{2}g^2\operatorname{tr}[\phi, \phi^\dagger]^2$$

1.8 Effective gauge theories with $\mathcal{N}=2$ supersymmetry

As in the case of gauge theories with $\mathcal{N}=1$ supersymmetry, we can as well work with low-energy effective gauge theories with $\mathcal{N}=2$ supersymmetry. In order to begin we first have to find an appropriate sum of (1.5.6) with $W=0$ and (1.5.8). In anticipation of some connection to the Lagrangian (1.7.2), that is chosen to be

$$\mathscr{L}^{(\text{eff})}_{\mathcal{N}=2} = \int d^2\theta\, d^2\bar{\theta}\, K(\Phi^a, (\Phi^\dagger e^{2gV})_a) + \tfrac{1}{2}\text{Re}\left(\int d^2\theta\, \tau_{ab}(\Phi^i)W^{a\alpha}W^b_\alpha\right)$$

and, after working out its component form and eliminating the auxiliary fields,[19] we see that the kinetic terms for the matter fermions, ψ, and the gauginos, λ, are

$$-iK^b_a\bar{\psi}^b_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\psi^a_\alpha \text{ and } -i\text{Re}(\tau_{ab})\bar{\lambda}^b_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\lambda^a_\alpha$$

respectively. But the $\mathsf{SU}(2)_R$ symmetry demands that the kinetic terms for ψ and λ appear in the same form and, thus, we have to require that

$$K^b_a = \text{Re}\,\tau^b_a$$

where $\tau^b_a = \tau_{ac}\delta^{cb}$. That can be implemented by defining a holomorphic function, $\mathcal{F}(\Phi)$, called the prepotential, such that

$$16\pi K(\phi, \phi^\dagger) = -\frac{i}{2}\phi^{\dagger a}\frac{\partial\mathcal{F}(\phi)}{\partial\phi^a} + \text{H.c.} \equiv -\frac{i}{2}\phi^{\dagger a}\mathcal{F}_a(\phi) + \text{H.c.}$$

[19] Remember that now everything transforms in the adjoint representation of the gauge group

and

$$16\pi\tau_{ab}(\phi) = -i\frac{\partial^2\mathcal{F}(\phi)}{\partial\phi^a\,\partial\phi^b} \equiv -i\mathcal{F}_{ab}(\phi)$$

Then, our effective theory is described by the Lagrangian

$$\mathscr{L}^{(\mathrm{eff})}_{\mathcal{N}=2} = \frac{1}{16\pi}\mathrm{Im}\left[\frac{1}{2}\int d^2\theta\,\mathcal{F}_{ab}(\Phi)W^{a\alpha}W^b_\alpha + \int d^2\theta\,d^2\bar{\theta}\,(\Phi^\dagger e^{2gV})^a\mathcal{F}_a(\Phi)\right]$$

where $\mathrm{Im}\,\mathcal{F}_{ab}$ plays the role of some sort of coupling. As we observe, our theory is completely determined by the prepotential. Note that with the choice $\mathcal{F}(\Phi) = \frac{1}{2}\tau\mathrm{tr}\,\Phi^2$ we obtain the Lagrangian (1.7.2). Actually, for the case of $\mathsf{SU}(2)$ gauge symmetry Seiberg and Witten managed to determine the prepotential, and thus solve the low-energy theory [8].

CHAPTER 2

Supersymmetry breaking

Contents

Even though supersymmetry equips us with powerful methods and gives profound answers to a series of previously insoluble problems, it nevertheless comes in direct conflict with nature as we observe it today. Indeed, the supersymmetric partners of the particles we observe in particle accelerators are absolutely absent in our surroundings. For example, if we consider $\mathcal{N} = 1$ supersymmetry, then the electron belongs to a chiral superfield which also contains a scalar field, the selectron, and an auxiliary field. The selectron is the superpartner of the electron but, although the electron is rather easy to observe, the selectron has not been detected yet.

Therefore, if we want to retain the power of supersymmetry while, at the same time, respect phenomenology, we need supersymmetry to be broken at some intermediate scale, i.e. at some scale below the Planck scale but certainly above the already probed energy of 100 GeV. In this chapter we will present various methods that are useful in the search of supersymmetry breaking. Our treatment contains an explicit example of supersymmetry breaking as well as various criteria that help us find out if supersymmetry is broken. I assume that the reader is familiar with the spontaneous breaking of ordinary global symmetries.

2.1 Spontaneous supersymmetry breaking

Supersymmetry is broken if and only if the supercharge Q_α does not annihilate the vacuum:

$$Q_\alpha|0\rangle \neq 0$$

Given a specific field, bosonic or fermionic, it is

$$\langle 0|\delta(\text{field})|0\rangle = \langle 0|(\epsilon^\beta Q_\beta)(\text{field})|0\rangle$$

and, hence, supersymmetry is broken if the variation of some field, bosonic or fermionic, is such that

$$\langle 0|\delta(\text{field})|0\rangle \neq 0$$

However, since the variation of a bosonic field is fermionic its vacuum expectation value (vev) has to vanish automatically due to Lorentz invariance. Hence, in order to break supersymmetry we need at least one fermionic field, κ_α, whose variation has a nonzero vev:

$$\langle 0|\delta\kappa_\alpha|0\rangle \neq 0$$

But if the fermionic field belongs to a chiral superfield its variation is (see equation (1.3.6)

$$\delta\psi_a = \sqrt{2}\epsilon_\alpha F + \cdots \tag{2.1.1}$$

while, if it belongs to a vector superfield, where, as usual, we call it λ_α to avoid confusion, it is

$$\delta\lambda_\alpha = i\epsilon_\alpha D + \cdots$$

where F and D are the auxiliary fields in the Lagrangian (1.4.10), and the ellipses stand for terms that do not contribute (due to Lorentz invariance) to the vev. Consequently, we observe that nonzero vevs for either F or D signal supersymmetry breaking.

Now, another way to establish supersymmetry breaking is to find a vacuum energy which is greater than zero. To see how this comes about consider the supersymmetry algebra equation

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu$$

Since the Pauli matrices are traceless, taking the trace we obtain

$$\tfrac{1}{4}(Q_1\bar{Q}_{\dot 1} + Q_2\bar{Q}_{\dot 2} + \bar{Q}_{\dot 1}Q_1 + \bar{Q}_{\dot 2}Q_2) = P_0$$

and, hence, the vacuum energy is[1]

$$E_{\text{vac}} = \langle 0|P_0|0\rangle \geq 0$$

From the last equation we immediately see that, since $\langle 0|P_0|0\rangle = 0$ implies $Q_\alpha|0\rangle = 0$, supersymmetry is broken if and only if $E_{\text{vac}} > 0$.

[1] The nonnegativity could be directly seen by looking at the scalar potential (1.4.12).

Let us now consider an example. A class of supersymmetry-breaking models involving only chiral superfields is the class of O'Raifeartaigh models. As a representative consider a theory with three chiral superfields, A, B and X, canonical Kähler potential and superpotential

$$W = \lambda A(X^2 - \mu^2) + mBX$$

The F-terms are[2]

$$F_A = \lambda(X^2 - \mu^2), \quad F_B = mX \text{ and } F_X = 2\lambda AX + mB$$

and, evidently, F_A and F_B cannot be set to zero simultaneously. Therefore, supersymmetry is broken. To find the vacuum energy and the vevs we have to minimize the scalar potential. Since no matter what the vev of X is we can always choose the vevs of A and B such that $F_X = 0$, we just have to minimize the scalar potential (μ and m are assumed to be real)

$$V = |F_A|^2 + |F_B|^2 = |\lambda|^2 |X^2 - \mu^2|^2 + m^2|X|^2$$

The solutions are

$$\langle X \rangle = 0 \text{ or } \langle X \rangle^2 = \mu^2 - \frac{m^2}{2|\lambda|^2}$$

The corresponding vacuum energies are

$$E_{\text{vac},1} = |\lambda|^2 \mu^4 \text{ and } E_{\text{vac},2} = m^2\mu^2 - \frac{m^4}{4|\lambda|^2}$$

Note that the branch $\langle X \rangle^2 = \mu^2 - m^2/2|\lambda|^2$ does not survive if $E_{\text{vac},2} \leq 0$. More specifically, if $E_{\text{vac},2} \leq 0$, then two of the roots of the cubic polynomial $\partial V/\partial X$ are no longer real and, thus, of no relevance to our considerations. Thus, the only physical case is the case $E_{\text{vac},2} > 0$, in perfect agreement with broken supersymmetry.

It is now time to investigate the spectrum of the massless states in the case $\langle X \rangle = 0$. First of all, there is a massless scalar arising from the fact that at this level, which is called the tree or classical level, not all of our fields are determined. Indeed, at $\langle X \rangle = 0$ the F-term for X does not fix the vev of A. We say that we have a vacuum degeneracy at tree level. In the next section we will see how this degeneracy is lifted by loop (quantum) corrections. Furthermore, there is a massless fermion, ψ_X. This could not be absent; it is the sign that supersymmetry is broken and it is called the Goldstino. It is the analog of the Goldstone boson which is observed in the massless spectrum of spontaneously broken ordinary global symmetries and we further analyze its connection to supersymmetry breaking in section 2.4.

Finally, let us consider the massive spectrum of our theory. The importance of doing so is to show that for nonzero energy the bosonic and the fermionic states are not paired. For a general supersymmetric theory with n chiral superfields Q^a and superpotential $W(Q^a)$[3] the mass matrices for scalar and spin-$\frac{1}{2}$ fields are given respectively by the $2n \times 2n$ matrices

$$\mathcal{M}_0^2 = \begin{pmatrix} W^{\dagger ac}W_{cb} & W^{\dagger abc}W_c \\ W_{abc}W^{\dagger c} & W_{ac}W^{\dagger cb} \end{pmatrix} \text{ and } \mathcal{M}_{1/2}^2 = \begin{pmatrix} W^{\dagger ac}W_{cb} & 0 \\ 0 & W_{ac}W^{\dagger cb} \end{pmatrix} \tag{2.1.2}$$

[2]Note that here and in what follows we denote the chiral superfield and its zero component with the same symbol.

[3]Note that any Kähler potential other than the canonical one, $K_{\text{can}}(Q^a, Q_a^\dagger) = Q_a^\dagger Q^a$, alters this result.

with $W_c \equiv \frac{\partial W}{\partial Q^c}$ and similarly for the rest, where the derivatives are to be evaluated at the vevs assumed for the zero components of the chiral superfields. Calculating and diagonalizing the mass matrices in our case shows that the boson mass matrix has two zero eigenvalues, the two eigenvalues

$$m^2 + 2|\lambda\langle A\rangle|^2 + |\lambda|^2\mu^2 \pm |\lambda|\sqrt{4m^2|\langle A\rangle|^2 + 4|\lambda|^2|\langle A\rangle|^4 + |\lambda|^2\mu^4 + 4\mu^2|\lambda\langle A\rangle|^2}$$

and the two eigenvalues

$$m^2 + 2|\lambda\langle A\rangle|^2 - |\lambda|^2\mu^2 \pm |\lambda|\sqrt{4m^2|\langle A\rangle|^2 + 4|\lambda|^2|\langle A\rangle|^4 + |\lambda|^2\mu^4 - 4\mu^2|\lambda\langle A\rangle|^2}$$

while the fermion mass matrix also has two zero eigenvalues but the two double eigenvalues

$$m^2 + 2|\lambda\langle A\rangle|^2 \pm 2|\lambda|\sqrt{m^2|\langle A\rangle|^2 + |\lambda|^2|\langle A\rangle|^4}$$

which are different from the ones found for the boson mass matrix. Therefore, our expectation that we would find a mass splitting between bosons and fermions is fulfilled. As a mild consistency check note that at $\lambda = 0$ there is no mass splitting, something that was to be expected since, then, supersymmetry remains unbroken. Also, note that

$$\mathrm{STr}\,\mathcal{M}^2 \equiv \mathrm{Tr}\,\mathcal{M}_0^2 - \mathrm{Tr}\,\mathcal{M}_{1/2}^2 = 0$$

i.e. the mass splitting in each supermultiplet vanishes. Of course this should be expected because of the form of the mass matrices (2.1.2).

2.2 Loop corrections

In the previous section we analyzed an O'Raifeartaigh model at the classical level and we found that in the case $\langle X\rangle = 0$ there is a large vacuum degeneracy: No matter what the vev of the zero component of the chiral superfield A was, the vacuum energy was exactly the same. In general, vevs of zero components of superfields that are undetermined in the vacuum of a theory are called moduli, and the space they parametrize is called moduli space.

In our particular case $\langle A\rangle$ is an approximate modulus since a potential for A is generated at one-loop, thus lifting the vacuum degeneracy. To see this we have to integrate out all the massive fields in order to get the effective action for A. The one-loop correction to the vacuum energy is given by the Coleman–Weinberg potential [9], which in the case of supersymmetric theories where quadratic divergences cancel among bosons and fermions,

$$\mathrm{STr}\,\mathcal{M}^2 \equiv \mathrm{Tr}\,\mathcal{M}_0^2 - \mathrm{Tr}\,\mathcal{M}_{1/2}^2 = 0$$

takes the form

$$\Delta V(\langle A\rangle) = \frac{1}{64\pi^2}\mathrm{STr}\,\mathcal{M}^4 \ln\frac{\mathcal{M}^2}{\Lambda^2} \equiv \frac{1}{64\pi^2}\left(\mathrm{Tr}\,\mathcal{M}_0^4 \ln\frac{\mathcal{M}_0^2}{\Lambda^2} - \mathrm{Tr}\,\mathcal{M}_{1/2}^4 \ln\frac{\mathcal{M}_{1/2}^2}{\Lambda^2}\right) \tag{2.2.1}$$

where Λ is the scale below which our theory is valid. The minus sign in the fermionic contribution arises because fermionic path integrals of Gaussians give a result proportional to the determinant of the matrix coefficient in the exponent, in contrast to bosonic path integrals which give a result proportional to the inverse of this determinant. Note that if our theory is supersymmetric this correction vanishes automatically due to the pairing of bosons and fermions.

Now, to compute the correction exactly we need the spectrum as a function of $\langle A \rangle$. But this was done in the previous section and, thus, we can find the result which, in fact, is too long to be presented here. However, plotting the correction to the scalar potential for some values of the parameters as a function of $\langle A \rangle$ can tell us what is the behavior of the correction as we vary $\langle A \rangle$. The plot is shown in Fig. 2.1 and it can be seen that its crucial characteristics do not depend on the particular values we assumed for the parameters.[4] Thus, we say that the potential is stable under deformations.

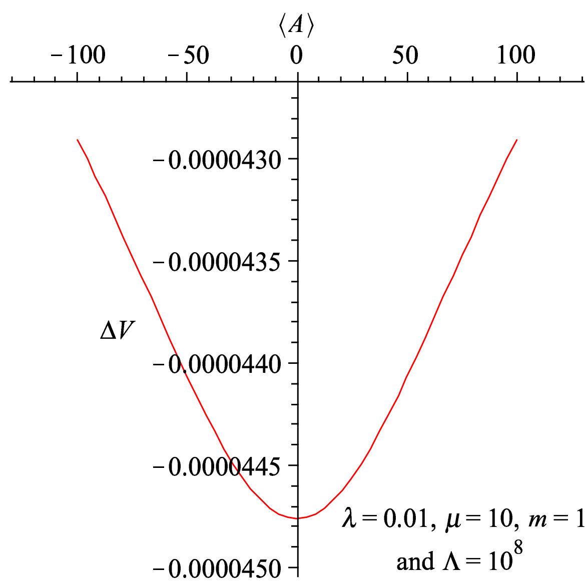


Fig. 2.1: The one-loop correction to the vacuum energy

We observe an absolute minimum at $\langle A \rangle = 0$ where the correction to the vacuum energy is slightly smaller than zero. As expected, quantum corrections lifted the vacuum degeneracy we found in the previous section.

2.3 Dynamical supersymmetry breaking

In the example we just analyzed we saw the spontaneous breaking of supersymmetry taking place at tree-level. However, we often happen to consider a theory where at tree-level supersymmetry remains an exact symmetry of the vacuum, i.e. where supersymmetry is not spontaneously broken. Then, there exists a wide class of theorems, known as nonrenormal-

[4] This can be done, for instance, by drawing the 3d plots of ΔV with respect to $\langle A \rangle$ and one of λ, μ, m and Λ at a time. Note that λ is dimensionless while μ, m Λ and $\langle A \rangle$ have mass dimension one.

ization theorems, which guarantee that supersymmetry cannot be broken at any loop order. That is a remarkable result and although it was originally proved using Feynman diagrams in superspace (supergraphs) [10], it was later found to arise solely by considerations regarding the holomorphy of the superpotential [11]. Also, it can be shown that if supersymmetry is broken at tree-level, then nothing can be done to restore it in perturbation theory. Let us see an example, namely the Wess–Zumino model. As we saw in section 1.5 this is a theory with a single chiral superfield, Φ, a canonical kinetic term and tree-level superpotential

$$W_{\text{tree}}(\Phi) = \tfrac{1}{2}m\Phi^2 + \tfrac{1}{3}\lambda\Phi^3$$

For $m = \lambda = 0$ the theory has a $\mathsf{U}(1) \times \mathsf{U}(1)_R$ symmetry under which Φ transforms as $(1, 1)$. Note that under R-symmetry $d^2\theta$ has R-charge -2 and, thus, in order for the term $\int d^2\theta\, W(\Phi)$ to be invariant under $\mathsf{U}(1)_R$, the superpotential has to have R-charge $+2$. Now, if W_{tree} is invariant under the $\mathsf{U}(1)\times\mathsf{U}(1)_R$ symmetry even if the couplings are nonzero, then we have to impose the condition that the charges of m and λ under $\mathsf{U}(1)\times\mathsf{U}(1)_R$ are $(-2, 0)$ and $(-3, -1)$ respectively. Now, the most general renormalized superpotential invariant under $\mathsf{U}(1) \times \mathsf{U}(1)_R$ is

$$W_{\text{eff}} = m\Phi^2 f\left(\frac{\lambda\Phi}{m}\right)$$

where f is an arbitrary holomorphic function. Considering a small λ in order to give meaning to perturbation theory, we expand f around $\lambda = 0$ and we find

$$W_{\text{eff}} = \sum_{n=0}^{\infty} a_n \frac{\lambda^n \Phi^{n+2}}{m^{n-1}}$$

In the limit $m \to 0$ we have to require that W_{eff} be free of singularities. That is so because we use the Wilsonian effective action[5] and gives the condition $n < 2$. In addition, as $\lambda \to 0$ we have to recover the tree-level superpotential. This fixes the undetermined constants a_0 and a_1 and shows that W_{eff} does not receive any corrections. Thus, we proved the standard perturbative nonrenormalization theorem and, moreover, we extended it beyond perturbation theory. However, the Wess–Zumino model probably does not exist as an interacting field theory[6] and, hence, this nonperturbative result is of little interest.

Actually, as we will see, nonrenormalization theorems can be violated by nonperturbative mechanisms, thus making theories with supersymmetric vacua at tree-level subject to possible spontaneous supersymmetry breaking. This kind of supersymmetry breaking, that is spontaneous supersymmetry breaking by a nonperturbative mechanism, is called dynamical supersymmetry breaking [13]. Also, note that in case our theory has a noncanonical Kähler potential, then that is corrected by quantum effects since it is not protected by holomorphy.

The most important theory in which dynamical supersymmetry breaking occurs is the $\mathcal{N} = 1$ supersymmetric extension of QCD, a theory which we will extensively analyze in the following chapter.

[5] For further details the reader is referred to [12].

[6] Unless we work in two dimensions where the Wess–Zumino model is an asymptotically free theory and, so, probably exists nonperturbatively.

2.4 The Goldstino

A direct consequence of the breaking of any bosonic global symmetry is the appearance of massless scalars, $\pi(p)$, called Goldstone bosons, which couple linearly to the symmetry current j^μ. Due to Lorentz-invariance it is

$$\langle 0|j^\mu(x)|\pi(p)\rangle = f p^\mu e^{-ipx}$$

and, correspondingly, the current takes the form

$$j_\mu(x) = f\partial_\mu \pi(x) + \cdots$$

where the ellipsis stands for terms quadratic in the fields and for potential derivative terms.[7]

Similarly, the breaking of supersymmetry in any theory is communicated by the appearance of a Goldstone fermion, more commonly referred to as the Goldstino, which couples linearly to the supersymmetry current [14]. The Goldstino coupling to the supersymmetry current can be expressed as

$$J^\mu_\alpha = f\sigma^\mu_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} + \cdots \tag{2.4.1}$$

where, again, the ellipsis stands for terms quadratic in the fields and for potential derivative terms. As we will now show, when supersymmetry is broken f is nonzero.

Proceeding to the proof of the above statement we can consider for simplicity a theory with n chiral superfields in which we have the usual F-term supersymmetry breaking, i.e. the F-term, F, of say one chiral superfield cannot be set to zero consistently with all other F-terms being zero. Our proof will be based on the Källén–Lehmann spectral representation, that is the fact that the Fourier-transformed two-point function exhibits a pole at the mass of the one-particle state, which for fermions takes the form

$$\int d^4x\, e^{ipx}\langle 0|T\bar{\psi}(x)\psi(0)|0\rangle = \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} + \cdots$$

where the ellipsis stands for terms contributed by multi-particle states and T denotes time-ordering. Evidently, what we need to prove is that in case supersymmetry is broken, then the Fourier-transformed two-point function for fermions has a pole at zero mass. Let us, therefore, study the Green's function $\langle 0|T\epsilon^\beta J^\mu_\beta(x)\psi^\alpha(0)|0\rangle$, where ϵ^β is an infinitesimal constant Weyl spinor. Our starting point is the equation

$$\int d^4x\, \partial_\mu e^{ipx}\langle 0|T\epsilon^\beta J^\mu_\beta(x)\psi^\alpha(0)|0\rangle = 0 \tag{2.4.2}$$

which is true since the integral of a total divergence is identically zero (when there are no surface terms). However, if we act with the derivative before performing the integration, then we take two contributions: One from the derivative acting on the exponential,

$$\int d^4x\, (\partial_\mu e^{ipx})\langle 0|T\epsilon^\beta J^\mu_\beta(x)\psi^\alpha(0)|0\rangle = ip_\mu \int d^4x\, e^{ipx}\langle 0|T\epsilon^\beta J^\mu_\beta(x)\psi^\alpha(0)|0\rangle$$

[7] Those are not taken into account in the low-energy effective theory.

and one from the derivative acting on the Green's function. This can be calculated by expressing the time-ordering in terms of theta functions and remembering that time derivatives of theta functions are delta functions. The result is

$$\int d^4x\, e^{ipx} \partial_\mu \langle 0|T\epsilon^\beta J^\mu_\beta(x)\psi^\alpha(0)|0\rangle = \int d^4x\, e^{ipx} \delta^{(4)}(x)\langle 0|\delta\psi^\alpha(0)|0\rangle$$

where the variation of ψ^α is[8]

$$\delta\psi^\alpha(0) \equiv \left[\int d^3x\, \epsilon^\beta J^0_\beta(x), \psi^\alpha(0)\right]$$

which, remembering equation (2.1.1) and dropping derivative contributions since those have vanishing vevs due to Lorentz invariance, becomes

$$\delta\psi^\alpha(0) = \sqrt{2}\epsilon^\alpha F$$

Hence, by equation (2.4.2) we get

$$\sqrt{2}\epsilon^\alpha\langle F\rangle = -ip_\mu \int d^4x\, e^{ipx}\langle 0|T\epsilon^\beta J^\mu_\beta(x)\psi^\alpha(0)|0\rangle$$

The last equation can obviously be brought to the form

$$\int d^4x\, e^{ipx}\langle 0|T\epsilon^\beta J^\mu_\beta(x)\psi^\alpha(0)|0\rangle = \epsilon^\alpha \frac{fp^\mu}{p^2}$$

where $f = i\sqrt{2}\langle F\rangle$, which, by the Källén–Lehmann spectral representation, indicates that there exists a massless fermionic one-particle state in the spectrum. Therefore, our initial assumption that $\langle F\rangle \neq 0$, i.e. that supersymmetry is broken, resulted in the appearance of the Goldstino in the spectrum.

The Goldstino coupling to the supersymmetry current is

$$J^\mu_\alpha = f\sigma^\mu_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} + \cdots$$

and, indeed, when supersymmetry is broken f is nonzero. For completeness, note that it can be shown using the Noether technique that the supersymmetry current of a theory with n chiral superfields is given by

$$J^\mu_\alpha = \sqrt{2}\sigma^\nu_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\psi^i_\beta\partial_\nu\phi^\dagger_i + i\sqrt{2}F^i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}_i \tag{2.4.3}$$

where $i = 1, \ldots, n$.

[8] Note that the space integral of the zero component of the supersymmetry current is the supercharge:

$$\epsilon^\alpha Q_\alpha = \int d^3x\, \epsilon^\alpha J^0_\alpha(x)$$

and

$$\bar{\epsilon}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}} = \int d^3x\, \bar{\epsilon}_{\dot{\alpha}}\bar{J}^{0\dot{\alpha}}(x)$$

2.5 The Witten index

A very powerful tool which helps us see whether a given theory breaks supersymmetry is the Witten index [15]. The Witten index is essentially based on the fact that supersymmetry breaking depends on the existence of zero-energy states. To be more specific, consider the supersymmetry generators and their action on bosonic and fermionic states (in finite volume):

$$Q|\text{boson}\rangle = \sqrt{E}|\text{fermion}\rangle \quad \text{and} \quad Q|\text{fermion}\rangle = \sqrt{E}|\text{boson}\rangle$$

where $E > 0$ is the energy of either of the states. Evidently, states of nonzero energy are constrained by supersymmetry to appear in boson-fermion pairs. However, zero-energy states are not subject to this constraint since, then,

$$Q|\text{boson}\rangle = 0 \quad \text{and} \quad Q|\text{fermion}\rangle = 0$$

But if there exists some zero-energy state, then supersymmetry is unbroken since, then, the scalar potential has zero as its minimum. On the other hand, if there are no zero-energy states, then supersymmetry is broken.

However, in supersymmetric theories it is very hard to count the exact number of bosonic and fermionic states at each energy level. A somewhat easier, but still hard, job is to count their difference. Thus, we can calculate the quantity

$$\sum_E \left(n_B^E - n_F^E\right) \tag{2.5.1}$$

where n_B^E is the number of bosonic states with energy E and n_F^E the number of fermionic states with energy E. But for $E > 0$ bosons and fermions come in pairs and, hence, the nonzero energy contributions in the quantity (2.5.1) cancel. What remains is the so-called Witten index

$$\mathrm{Tr}(-1)^F = n_B^{E=0} - n_F^{E=0}$$

If the Witten index is nonzero, then supersymmetry is obviously unbroken. On the other hand though, if the Witten index is zero we cannot reach a conclusion, since either $n_B^{E=0} = n_F^{E=0} \neq 0$ and supersymmetry is unbroken, or $n_B^{E=0} = n_F^{E=0} = 0$ and supersymmetry is broken.

The usefulness of the Witten index lies on the fact that it is a topological invariant of the theory. It may be calculated for some convenient choice of the parameters of the theory but the result is true in general. A very important application of this fact is that we can obtain results for the strongly-coupled regime of a theory by calculating the Witten index at weak coupling. More specifically, under mild variations of the parameters of the theory the states may move to or from zero energy. But the pairing of positive-energy states guarantees that they do so in pairs, thus not changing the value of the Witten index. Now, by mild variation of the parameters we mean that as long as a parameter of a theory is nonzero and varied to another nonzero value, then we do not expect the Witten index to change. The change can only happen if new states of zero energy appear, which is a possibility if there is a change in the asymptotic behavior of the potential in field space. This can occur if some parameter of the theory is set to zero or is turned on, in which case states may come in from or move out to infinity.

Witten calculated the index of several theories. For example, he found that the index of pure supersymmetric Yang–Mills theory is nonzero and, thus, he proved that those theories do not break supersymmetry spontaneously. In addition, supersymmetric Yang–Mills theories with massive matter do not break supersymmetry either, for, at least at weak coupling, one can take all masses to be large in which case there are no massless states beyond those of pure supersymmetric Yang–Mills theory and, so, the value of the Witten index is the same as in the case of pure superglue.

However, we can easily be carried away here so we better be careful. The theory with zero mass has flat directions along which the scalar potential is classically zero. On the contrary, the theory with nonzero mass for the matter fields does not have classical flat directions. Hence, as the mass is taken to zero the asymptotic behavior of the scalar potential changes and so might the Witten index. In fact the Witten index is ill-defined in the presence of flat directions since, then, there exist zero-modes associated with the flat directions giving rise to a continuous spectrum of states, while to calculate the Witten index we actually need to consider the theory in a finite volume so that we have a discrete spectrum. Therefore, we cannot draw any conclusion regarding supersymmetry breaking in massless, nonchiral theories based on the Witten index of the corresponding pure supersymmetric Yang–Mills theory.

2.6 Global symmetries and supersymmetry breaking

Another means of checking whether supersymmetry is broken is provided by the connection between global symmetries and supersymmetry breaking. Consider a theory with an exact, nonanomalous global symmetry and no flat directions, i.e. without moduli. If the global symmetry is spontaneously broken, then there is a massless scalar field, the Goldstone boson, without a potential. Now, if supersymmetry is an exact symmetry of the vacuum, then the Goldstone boson belongs to a chiral superfield which also contains another massless scalar, namely the parity-reflected state of the Goldstone boson, without a potential. But then, this second scalar is a modulus and, therefore, we have a contradiction with our initial assumption that there are no moduli. The only way to avoid the contradiction is to drop the assumption of unbroken supersymmetry.[9] The aforementioned criterion for supersymmetry breaking first appeared in a paper by Affleck, Dine and Seiberg [16] and it is often referred to as the ADS criterion for supersymmetry breaking.

In general, the spontaneous breaking of a global symmetry requires a detailed knowledge of the potential of the theory under consideration and, thus, it is as difficult to decide as determining whether the vacuum energy vanishes. Moreover, in the case where the theory is strongly coupled at the scale of supersymmetry breaking, then neither of those questions can be directly answered. However, in some cases, we can see if a global symmetry is broken based on the so-called 't Hooft anomaly-matching conditions: If a continuous global symmetry is

[9] This result can be invalidated if the parity-reflected state of the Goldstone boson is a Goldstone boson itself.

unbroken in the vacuum, then the massless fermions of the low-energy theory should reproduce the global triangle anomalies of the microscopic theory [17], [18]. Therefore, if a global symmetry is unbroken, then there should exist a set of fields, with appropriate charges under the global symmetry, that give a solution to the anomaly-matching conditions. Now, if we find that in order to satisfy the 't Hooft anomaly-matching conditions we need a large set of fields, then it is plausible to conclude that the global symmetry is spontaneously broken.

In supersymmetric theories we have a ubiquitous continuous global symmetry, namely the R-symmetry. The ADS criterion, then, says that if we find that the R-symmetry is spontaneously broken and there are no noncompact flat directions, then supersymmetry is broken. If the scale of supersymmetry breaking is much lower that the strong-coupling scale, then we can study supersymmetry breaking in the low-energy effective theory. This contains only chiral superfields and, typically, some terms in the superpotential acquire vevs. But then, the fact that the superpotential has R-charge two indicates that the R-symmetry is spontaneously broken. The above argument makes it clear that the spontaneous breaking of the R-symmetry is much more easily established compared to that of any other global symmetry.

In fact, it is known [19] that the existence of an R-symmetry is a necessary condition for supersymmetry breaking. Furthermore, if the effective Lagrangian is a generic Lagrangian consistent with the symmetries of the theory (no fine tuning), and if the low energy theory can be described by a supersymmetric Wess–Zumino effective Lagrangian without gauge fields, then a spontaneously broken R-symmetry is a sufficient condition supersymmetry breaking.

2.7 Gaugino condensation

Finalizing this chapter let us introduce a criterion for supersymmetry breaking which is based on gaugino condensation. Suppose that a certain chiral superfield or a linear combination of chiral superfields does not appear in the superpotential and, yet, all the moduli are stabilized. In such a case the Konishi anomaly [20] implies that

$$\bar{D}^2(\Phi^\dagger e^V \Phi) \sim \mathrm{tr}\, W^\alpha W_\alpha \tag{2.7.1}$$

where $\bar{D}^{\dot{\alpha}}$ is the superspace-covariant derivative, Φ is the chiral superfield that does not appear in the superpotential and V is a vector superfield. From the component form of equation (2.7.1) we find

$$\{Q_\alpha, \psi^\alpha \phi\} \sim \mathrm{tr}\, \lambda^\alpha \lambda_\alpha \tag{2.7.2}$$

where ϕ and ψ^α are the zero and fermionic components of Φ respectively, λ^α is the gaugino and tr denotes a sum over gauge indices. From this equation we observe that supersymmetry is broken by a nonzero vev of the lowest component of $\mathrm{tr}\, W^\alpha W_\alpha$, that is by nonzero $\langle \mathrm{tr}\, \lambda^\alpha \lambda_\alpha \rangle$. Therefore, the existence of gaugino condensation in this case implies that supersymmetry is broken.

If the superfield Φ appears in the superpotential, then the right-hand sides of equations (2.7.1) and (2.7.2) are modified in such a way so that the gaugino condensate forms without violating supersymmetry. As an example, consider the case where the superpotential contains a mass term for Φ. In components this gives the term $m\phi^\dagger\phi$ for the scalar component and, then, equation (2.7.2) becomes

$$\{Q_\alpha, \psi^\alpha\phi\} \sim -m\phi^\dagger\phi + \frac{g^2}{32\pi^2}\mathrm{tr}\,\lambda^\alpha\lambda_\alpha$$

The last equation is not necessarily incompatible with supersymmetry and, moreover, determines the vevs of the scalar fields in terms of the gaugino condensate.

CHAPTER 3

Supersymmetric QCD

Contents

In this chapter we will analyze the supersymmetric extension of quantum chromodynamics. We will study its classical and quantum dynamics and we will make use of an important duality in order to find nonsupersymmetric vacua in the strong-coupling regime of the massive theory.

3.1 $\mathcal{N} = 1$ supersymmetric QCD

$\mathcal{N} = 1$ SQCD with N_c colors and N_f flavors is an $\mathcal{N} = 1$ $\mathsf{SU}(N_c)$ gauge theory with N_f quark flavors Q^i (left-handed quarks) which are chiral superfields transforming in the $\mathbf{N}_c$ of $\mathsf{SU}(N_c)$ and N_f quark flavors $\tilde{Q}_{\tilde{i}}$ (left-handed antiquarks) which are chiral superfields transforming in the $\overline{\mathbf{N}}_c$ of $\mathsf{SU}(N_c)$, where $i, \tilde{i} = 1, \ldots, N_f$ are flavor indices. Since the gauge group does not contain $\mathsf{U}(1)$ factors there are no Fayet–Iliopoulos terms.

In order to start with, consider the theory without superpotential for the quarks. Its Lagrangian can be written down immediately with the aid of the Lagrangian (1.4.9) and it turns out to be

$$\begin{aligned} \mathscr{L} = &\int d^2\theta\, d^2\bar{\theta}\, Q_i^\dagger e^{2gV} Q^i + \int d^2\theta\, d^2\bar{\theta}\, \tilde{Q}_{\tilde{i}} e^{2gV} \tilde{Q}^{\tilde{i}\dagger} \\ &+ \frac{1}{32\pi} \mathrm{Im}\left(\tau \int d^2\theta\, \mathrm{tr}\, W^\alpha W_\alpha \right) \end{aligned} \tag{3.1.1}$$

or, in components,

$$\begin{aligned}\mathscr{L} = {} & (D_\mu Q^i)^\dagger D^\mu Q^i - i\psi^{i\alpha}\sigma^\mu_{\alpha\dot\alpha}D_\mu\bar\psi_i^{\dot\alpha} + i\sqrt{2}gQ_i^\dagger T^A\lambda_A^\alpha\psi_\alpha^i \\ & - i\sqrt{2}g\bar\psi_{i\dot\alpha}\bar\lambda_A^{\dot\alpha}T^AQ^i + (D_\mu\tilde Q_{\tilde i})^\dagger D^\mu\tilde Q_{\tilde i} - i\tilde\psi_{\tilde i}^\alpha\sigma^\mu_{\alpha\dot\alpha}D_\mu\bar{\tilde\psi}^{\tilde i\dot\alpha} \\ & - i\sqrt{2}g\tilde Q^{\tilde i\dagger}T^A\lambda_A^\alpha\tilde\psi_{\tilde i\alpha} + i\sqrt{2}g\bar{\tilde\psi}^{\tilde i}_{\dot\alpha}\bar\lambda_A^{\dot\alpha}T^A\tilde Q_{\tilde i} - V(Q^i,\tilde Q_{\tilde i},Q_i^\dagger,\tilde Q^{\tilde i\dagger}) \\ & + \operatorname{tr}\left(-\tfrac14F_{\mu\nu}F^{\mu\nu} - i\lambda^\alpha\sigma^\mu_{\alpha\dot\alpha}D_\mu\bar\lambda^{\dot\alpha}\right) + \frac{\vartheta}{32\pi^2}g^2\operatorname{tr}F_{\mu\nu}\tilde F^{\mu\nu}\end{aligned} \tag{3.1.2}$$

where the scalar potential is

$$V(Q^i,\tilde Q_{\tilde i},Q_i^\dagger,\tilde Q^{\tilde i\dagger}) = \tfrac12 g^2\sum_{A=1}^{N_c^2-1}(Q_i^\dagger T^AQ^i - \tilde Q_{\tilde i}T^A\tilde Q^{\tilde i\dagger})^2 \tag{3.1.3}$$

and the Qs and $\tilde Q$s that appear in the component expansion are called squarks and are the zero components of the chiral multiplets that represent the quark multiplets of SQCD. The use of the same letter for both a multiplet and its zero component, with the meaning hopefully clear from the context, is common practice in supersymmetric gauge theory considerations. Note that T^A, $A = 1,\dots,N_c^2-1$, are the generators of $\mathsf{SU}(N_c)$ in the fundamental representation, Q^i and $\tilde Q_{\tilde i}$ are $N_c\times 1$ and $1\times N_c$ matrices respectively and, also, note that we do not write the total derivative that appears in the component expansion.

Our theory (3.1.1) has a large global symmetry since

- We can transform the Qs and $\tilde Q$s by separate $\mathsf{SU}(N_f)$ transformations
- We can multiply the Qs and $\tilde Q$s by different phases
- There is the usual R-symmetry $\mathsf{U}(1)_{R'}$

The relevant representations and charge assignments are shown in the following table.

	$\mathsf{SU}(N_f)$	$\mathsf{SU}(N_f)$	$\mathsf{U}(1)_B$	$\mathsf{U}(1)_A$	$\mathsf{U}(1)_{R'}$
Q	$\mathbf{N}_f$	$\mathbf{1}$	1	1	1
$\tilde Q$	$\mathbf{1}$	$\overline{\mathbf{N}}_f$	-1	1	1

However, there is an anomaly of the $\mathsf{U}(1)_A\times\mathsf{U}(1)_{R'}$ symmetry. Despite that, a single $\mathsf{U}(1)$ symmetry which we will denote $\mathsf{U}(1)_R$ survives and is a full quantum symmetry. Therefore, the global symmetry of the quantum theory is

$$\mathsf{SU}(N_f)\times\mathsf{SU}(N_f)\times\mathsf{U}(1)_B\times\mathsf{U}(1)_R$$

and by the usual results about anomalies we find that the appropriate charge assignment is

	$\mathsf{SU}(N_f)$	$\mathsf{SU}(N_f)$	$\mathsf{U}(1)_B$	$\mathsf{U}(1)_R$
Q	$\mathbf{N}_f$	$\mathbf{1}$	1	$1-N_c/N_f$
$\tilde Q$	$\mathbf{1}$	$\overline{\mathbf{N}}_f$	-1	$1-N_c/N_f$

To make our notation more convenient we define the matrices

$$Q = \begin{array}{c} \\ a \\ \downarrow \end{array}\!\begin{array}{c} i \rightarrow \\ \left(\begin{pmatrix} \\ Q^1 \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ Q^{N_f} \\ \\ \end{pmatrix} \right) \end{array} \qquad Q^\dagger = \begin{array}{c} i \\ \downarrow \end{array}\!\begin{array}{c} a \rightarrow \\ \begin{pmatrix} (\quad Q^\dagger_1 \quad) \\ \vdots \\ (\quad Q^\dagger_{N_f} \quad) \end{pmatrix} \end{array}$$

and

$$\tilde{Q} = \begin{array}{c} i \\ \downarrow \end{array}\!\begin{array}{c} a \rightarrow \\ \begin{pmatrix} (\quad \tilde{Q}_1 \quad) \\ \vdots \\ (\quad \tilde{Q}_{N_f} \quad) \end{pmatrix} \end{array} \qquad \tilde{Q}^\dagger = \begin{array}{c} \\ a \\ \downarrow \end{array}\!\begin{array}{c} i \rightarrow \\ \left(\begin{pmatrix} \\ \tilde{Q}^{1\dagger} \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ \tilde{Q}^{N_f\dagger} \\ \\ \end{pmatrix} \right) \end{array}$$

where $a = 1, \ldots, N_c$ is a color index. Evidently, Q is an $N_c \times N_f$ matrix, $Q^\dagger$ is an $N_f \times N_c$ matrix, while the opposite holds for $\tilde{Q}$ and $\tilde{Q}^\dagger$. In this notation the Lagrangian (3.1.2) becomes

$$\begin{aligned} \mathscr{L} = \mathrm{Tr}\,[&(D_\mu Q)^\dagger D^\mu Q - i\psi^\alpha \sigma^\mu_{\alpha\dot\alpha} D_\mu \bar\psi^{\dot\alpha} + i\sqrt{2} g Q^\dagger T^A \lambda^\alpha_A \psi_\alpha \\ &- i\sqrt{2} g \bar\psi_{\dot\alpha} \bar\lambda^{\dot\alpha}_A T^A Q + (D_\mu \tilde{Q})^\dagger D^\mu \tilde{Q} - i\tilde\psi^\alpha \sigma^\mu_{\alpha\dot\alpha} D_\mu \bar{\tilde\psi}^{\dot\alpha} \\ &- i\sqrt{2} g \tilde{Q}^\dagger T^A \lambda^\alpha_A \tilde\psi_\alpha + i\sqrt{2} g \bar{\tilde\psi}_{\dot\alpha} \bar\lambda^{\dot\alpha}_A T^A \tilde{Q}] - V(Q, \tilde{Q}, Q^\dagger, \tilde{Q}^\dagger) \\ &+ \mathrm{tr}\left(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\lambda^\alpha \sigma^\mu_{\alpha\dot\alpha} D_\mu \bar\lambda^{\dot\alpha}\right) + \frac{\vartheta}{32\pi^2} g^2 \mathrm{tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu} \end{aligned} \tag{3.1.4}$$

where

$$V(Q, \tilde{Q}, Q^\dagger, \tilde{Q}^\dagger) = \tfrac{1}{2} g^2 \sum_{A=1}^{N_c^2 - 1} (\mathrm{Tr}\, Q^\dagger T^A Q - \mathrm{Tr}\, \tilde{Q} T^A \tilde{Q}^\dagger)^2 \tag{3.1.5}$$

Note that Tr denotes a sum over both gauge and flavor indices,

$$\mathrm{Tr}\, Q^\dagger T^A Q \equiv Q^\dagger_{ib} (T^A)^b_c Q^{ic} \;\text{ and }\; \mathrm{Tr}\, \tilde{Q} T^A \tilde{Q}^\dagger \equiv \tilde{Q}_{\tilde{i}b} (T^A)^b_c \tilde{Q}^{\tilde{i}c\dagger}$$

and we immediately see the equivalence of the scalar potentials (3.1.5) and (3.1.3).

Now, it turns out that in order to give masses to the quark flavors consistently with the gauge symmetry of our theory, we have to use the unique gauge-invariant chiral superfield we can construct from Q^i and $\tilde{Q}_{\tilde{i}}$, namely the mesonic superfield

$$M^i_{\tilde{i}} = \tilde{Q}_{\tilde{i}} Q^i$$

$M^i_{\tilde{i}}$ is gauge invariant since Q is in the fundamental while $\tilde{Q}$ in the antifundamental representation of $\mathsf{SU}(N_c)$ but, evidently, it is not invariant under the global symmetry of our theory. The superpotential that results in mass terms for the zero components of the quark flavors is

$$W_{\text{tree}}(Q, \tilde{Q}) = \mathrm{tr}'\, mM \tag{3.1.6}$$

where m is a nondegenerate $N_f \times N_f$ mass matrix and tr' is not the same as tr, since it denotes a sum over flavor indices, while tr denotes a sum over gauge indices. With the inclusion of W_{tree} the Lagrangian (3.1.4) becomes[1]

$$\begin{aligned}\mathscr{L}' = \mathrm{Tr}\,[&(D_\mu Q)^\dagger D^\mu Q - i\psi^\alpha \sigma^\mu_{\alpha\dot\alpha} D_\mu \bar\psi^{\dot\alpha} + i\sqrt{2} g Q^\dagger T^A \lambda^\alpha_A \psi_\alpha \\ &- i\sqrt{2} g \bar\psi_{\dot\alpha} \bar\lambda^{\dot\alpha}_A T^A Q + (D_\mu \tilde Q)^\dagger D^\mu \tilde Q - i\tilde\psi^\alpha \sigma^\mu_{\alpha\dot\alpha} D_\mu \bar{\tilde\psi}^{\dot\alpha} \\ &- i\sqrt{2} g \tilde Q^\dagger T^A \lambda^\alpha_A \tilde\psi_\alpha + i\sqrt{2} g \bar{\tilde\psi}_{\dot\alpha} \bar\lambda^{\dot\alpha}_A T^A \tilde Q] - (\tfrac{1}{2}\mathrm{Tr}\, m \psi^\alpha \tilde\psi_\alpha + \text{H.c.}) \\ &+ \mathrm{tr}\left(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\lambda^\alpha \sigma^\mu_{\alpha\dot\alpha} D_\mu \bar\lambda^{\dot\alpha}\right) + \frac{\vartheta}{32\pi^2} g^2 \mathrm{tr}\, F_{\mu\nu} \tilde F^{\mu\nu} \\ &- V'(Q, \tilde Q, Q^\dagger, \tilde Q^\dagger)\end{aligned}$$

where the scalar potential is

$$\begin{aligned}V'(Q, \tilde Q, Q^\dagger, \tilde Q^\dagger) &= \mathrm{Tr}\, m^2 Q^\dagger Q + \mathrm{Tr}\, m^2 \tilde Q \tilde Q^\dagger \\ &\quad + \tfrac{1}{2} g^2 \sum_{A=1}^{N_c^2-1} (\mathrm{Tr}\, Q^\dagger T^A Q - \mathrm{Tr}\, \tilde Q T^A \tilde Q^\dagger)^2\end{aligned}$$

with

$$\mathrm{Tr}\, m^2 Q^\dagger Q \equiv m_i^{\tilde i} (m^\dagger)^j_{\tilde i} Q^\dagger_{jc} Q^{ic}$$

and

$$\mathrm{Tr}\, m^2 \tilde Q \tilde Q^\dagger \equiv (m^\dagger)^i_{\tilde j} m_i^{\tilde i} \tilde Q_{\tilde i c} \tilde Q^{\dagger \tilde j c}$$

Indeed, the choice (3.1.6) for the superpotential resulted in the anticipated mass terms for the squarks.

3.2 The classical moduli space

As we saw in the previous section, in the absence of mass terms the scalar potential is given by equation (3.1.5). We now want to study the vacuum structure of SQCD, i.e. we want to find vevs for the squarks that make the scalar potential (3.1.5) vanish.[2] A necessary condition for that to happen is

$$D^A \propto \mathrm{Tr}\, Q^\dagger T^A Q - \mathrm{Tr}\, \tilde Q T^A \tilde Q^\dagger = 0 \tag{3.2.1}$$

for every $A = 1, \ldots, N_c^2 - 1$. Now, define

$$D^L_M = Q^\dagger_{ia} (A^L_M)^a_b Q^{ib} - \tilde Q_{\tilde i a} (A^L_M)^a_b \tilde Q^{\tilde i b \dagger} \tag{3.2.2}$$

where A^L_M are the real generators of $\mathsf{GL}(N_c)$ and $L, M = 1, \ldots, N_c$ count them. Since Q^i is in the fundamental and $\tilde Q_{\tilde i}$ in the antifundamental representation of $\mathsf{SU}(N_c)$ we have to choose

$$(A^L_M)^a_b = \delta^a_M \delta^L_b$$

[1] This is easily seen by looking at the terms a (nonconstant) superpotential contributes to the Lagrangian (1.4.11).

[2] Note that in what follows we denote the squarks and their vevs with the same symbol.

Then, equation (3.2.2) becomes

$$\begin{aligned} D^L_M &= Q^\dagger_{ia}\delta^a_M\delta^L_b Q^{ib} - \tilde{Q}_{\tilde{i}a}\delta^a_M\delta^L_b\tilde{Q}^{\tilde{i}b\dagger} \\ &= Q^\dagger_{iM}Q^{iL} - \tilde{Q}_{\tilde{i}M}\tilde{Q}^{\dagger\tilde{i}L} \end{aligned}$$

which helps us see that

$$\begin{aligned} (T^A)^M_L D^L_M &= (T^A)^M_L Q^\dagger_{iM}Q^{iL} - (T^A)^M_L\tilde{Q}_{\tilde{i}M}\tilde{Q}^{\dagger\tilde{i}L} \\ &= \frac{2}{g^2}D^A \end{aligned} \tag{3.2.3}$$

where the roles of Q and $Q^\dagger$ and those of $\tilde{Q}$ and $\tilde{Q}^\dagger$ have been interchanged, i.e. Q and $\tilde{Q}^\dagger$ are now $N_f \times N_c$ matrices and $Q^\dagger$ and $\tilde{Q}$ are $N_c \times N_f$ matrices. Note that this change will not be made explicit in what follows since our notation will remain exactly as it was.

Now (using color indices instead of the indices L, M) we observe that D^c_b is a Hermitian matrix and, thus, it can be uniquely expanded as

$$D^c_b = \lambda\delta^c_b + \lambda_E(T^E)^c_b \tag{3.2.4}$$

where T^E are the generators of $\mathsf{SU}(N_c)$, λ and λ_E are constants and δ^c_b represents the unit matrix. Therefore, since the matrices T^E are traceless, we conclude that if we need $D^E = 0$ for every $E = 1, \ldots, N_c^2 - 1$, then we better have

$$D^c_b = \lambda\delta^c_b$$

for the second term in the right-hand side of equation (3.2.4) would contribute the nonzero term

$$\operatorname{tr} T^A T^E = \tfrac{1}{2}\delta^{AE}$$

upon contraction with $(T^A)^b_c$. Thus, the equation we have to solve in order to reveal the vacuum structure of SQCD is

$$Q^\dagger_{ib}Q^{ia} - \tilde{Q}_{\tilde{i}b}\tilde{Q}^{\dagger\tilde{i}a} = \lambda\delta^a_b \tag{3.2.5}$$

Now, the matrix Q has rank[3] $\min(N_c, N_f)$ and so does $Q^\dagger$. Therefore, since

$$\operatorname{rank}(Q^\dagger Q) \le \min(\operatorname{rank}(Q^\dagger), \operatorname{rank}(Q))$$

we conclude that the rank of $Q^\dagger Q$ is at most $\min(N_c, N_f)$. Of course, the above discussion can be repeated in exactly the same way for the matrix $\tilde{Q}\tilde{Q}^\dagger$, proving that the rank of $\tilde{Q}\tilde{Q}^\dagger$ is at most $\min(N_c, N_f)$ as well. Note that $Q^\dagger Q$ and $\tilde{Q}\tilde{Q}^\dagger$ are $N_c \times N_c$ Hermitian, positive semi-definite matrices, i.e. they can be diagonalized by an $\mathsf{SU}(N_c)$ transformation (not necessarily by the same one) and their eigenvalues are nonnegative.

[3] Remember that the rank of a matrix A is given by the maximal number of linearly independent columns of A, which is equal to the maximal number of linearly independent rows of A.

3.2.1 Fewer flavors than colors

In this case the rank of $Q^\dagger Q$ is at most N_f and so is the rank of $\tilde{Q}\tilde{Q}^\dagger$. If we diagonalize $Q^\dagger Q$ by an $\mathsf{SU}(N_c)$ transformation, then we would get at least $N_c - N_f$ zero eigenvalues. But then, equation (3.2.5) says that $\tilde{Q}\tilde{Q}^\dagger$ is also diagonal in this basis and, hence, the existence of at least one zero eigenvalue with the rest positive guarantees that $\lambda = 0$. Therefore, up to an $\mathsf{SU}(N_c)$ transformation, equation (3.2.5) is true if

$$Q^\dagger Q = \tilde{Q}\tilde{Q}^\dagger = \begin{pmatrix} |a_1|^2 & & & \\ & \ddots & & \\ & & |a_{N_f}|^2 & \\ & & & \\ \end{pmatrix} \tag{3.2.6}$$

where a_i, $i = 1, \ldots, N_f$, are complex numbers, i.e. if[4]

$$Q = \tilde{Q}^\dagger = \begin{pmatrix} a_1^* & & & \\ & \ddots & & \\ & & a_{N_f}^* & \end{pmatrix}$$

up to global transformations. Now if we remember the interchange that took place in equation (3.2.3), then we find the solution to equation (3.2.1) in terms of the original variables:

$$Q = \tilde{Q}^\dagger = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{N_f} \\ & & \\ \end{pmatrix} \tag{3.2.7}$$

up to global and gauge rotations.

The discussion we presented so far is obviously not gauge-invariant. However, it turns out that we can combine the squarks into the gauge-invariant combinations

$$M^i_{\tilde{i}} = \tilde{Q}_{\tilde{i}} Q^i, \quad i, \tilde{i} = 1, \ldots, N_f$$

which we call mesons. As we observe the mesonic matrix contains N_f^2 complex entries. Actually, one can show that the complex dimension of the classical moduli space is exactly N_f^2 and, therefore, we can parametrize the whole moduli space with arbitrary vevs of the mesons up to global symmetry transformations. For further details the reader is referred to [18] and [21]

[4] Undisplayed entries in matrices are zero and the asterisk denotes complex conjugation.

3.2.2 More flavors than colors

In this case the rank of $Q^\dagger Q$ is at most N_c and so is the rank of $\tilde{Q}\tilde{Q}^\dagger$. Again due to equation (3.2.5) we can see that $Q^\dagger Q$ and $\tilde{Q}\tilde{Q}^\dagger$ are diagonal in the same basis but, now, they do not have any zero eigenvalues. This means that λ need not vanish and, hence, up to an $\mathsf{SU}(N_c)$ transformation, equation (3.2.5) becomes

$$Q^\dagger Q - \tilde{Q}\tilde{Q}^\dagger = \begin{pmatrix} |a_1|^2 & & \\ & \ddots & \\ & & |a_{N_c}|^2 \end{pmatrix} - \begin{pmatrix} |\tilde{a}_1|^2 & & \\ & \ddots & \\ & & |\tilde{a}_{N_c}|^2 \end{pmatrix} = \lambda \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

where $a_i, \tilde{a}_{\tilde{i}}$, $i, \tilde{i} = 1, \ldots, N_c$, are complex numbers. The solution to the last equation is

$$Q = \begin{pmatrix} a_1^* & & \\ & \ddots & \\ & & a_{N_c}^* \\ & & \\ & & \end{pmatrix} \quad \text{and} \quad \tilde{Q}^\dagger = \begin{pmatrix} \tilde{a}_1^* & & \\ & \ddots & \\ & & \tilde{a}_{N_c}^* \\ & & \\ & & \end{pmatrix}$$

with

$$|a_i|^2 - |\delta_i^{\tilde{i}} \tilde{a}_{\tilde{i}}|^2 = \lambda, \quad \text{for every } i, \tilde{i} = 1, \ldots, N_c$$

up to global transformations and, consequently, in terms of the original variables, equation (3.2.1) is satisfied if

$$Q = \begin{pmatrix} a_1 & & & & \\ & \ddots & & & \\ & & a_{N_c} & & \end{pmatrix} \quad \text{and} \quad \tilde{Q}^\dagger = \begin{pmatrix} \tilde{a}_1 & & & & \\ & \ddots & & & \\ & & \tilde{a}_{N_c} & & \end{pmatrix}$$

with

$$|a_i|^2 - |\delta_i^{\tilde{i}} \tilde{a}_{\tilde{i}}|^2 = \lambda, \quad \text{for every } i, \tilde{i} = 1, \ldots, N_c$$

up to global and gauge rotations.

Obviously, the chain of reasoning we followed in this case can be easily followed to solve the case $N_f = N_c$ as well. But, again, we have a description of the moduli space which is not gauge-invariant. The situation is once again amended by combining the moduli in gauge-invariant combinations. If $\lambda = 0$, we can again use the mesons

$$M_{\tilde{i}}^i = \tilde{Q}_{\tilde{i}} Q^i, \quad i, \tilde{i} = 1, \ldots, N_c$$

On the other hand, when $\lambda \neq 0$ the mesons are not enough to describe the moduli space since new flat directions arise. This can be seen by the observation that although for $\lambda = 0$ one can see that the complex dimension of the moduli space is N_c^2, if $\lambda \neq 0$ this changes to $2N_cN_f - N_c^2$ which is larger than N_c^2 if $N_f > N_c$. In that case, the missing $2N_c(N_f - N_c)$ complex parameters are provided by the baryons

$$B_{i_{N_c+1} \ldots i_{N_f}} = \frac{1}{N_c!} \epsilon^{a_1 \ldots a_{N_c}} \epsilon_{i_1 \ldots i_{N_f}} Q_{a_1}^{i_1} \cdots Q_{a_{N_c}}^{i_{N_c}}$$

and

$$\tilde{B}^{\tilde{i}_{N_c+1}\ldots\tilde{i}_{N_f}} = \frac{1}{N_c!}\epsilon_{a_1\ldots a_{N_c}}\epsilon^{\tilde{i}_1\ldots\tilde{i}_{N_f}}\tilde{Q}^{a_1}_{\tilde{i}_1}\cdots\tilde{Q}^{a_{N_c}}_{\tilde{i}_{N_c}}$$

where the is and $\tilde{i}$s are flavor indices while the as are color indices. Therefore, up to global symmetry transformations, the classical moduli space is labeled by vevs of the mesons and the baryons. For further details the reader is again referred to [18] and [21].

3.3 Dynamics of SQCD

In the previous section we found a large vacuum degeneracy of massless SQCD at the classical level. Now, as we mentioned before, there are nonrenormalization theorems which prove that if there is no superpotential in the classical theory, then loop corrections cannot generate one. Therefore, the vacuum degeneracy we found can only be lifted by a dynamically generated superpotential, i.e. by a superpotential generated by nonperturbative mechanisms. In this section we will investigate this possibility. For a thorough review of these ideas and methods the reader is referred to [12].

But before we begin our discussion let us briefly discuss the idea of the running of a coupling constant. This phenomenon arises when we require that the physical quantities predicted by our theory do not depend on the scale at which we impose the renormalization conditions, that is on the scale at which we observe the theory. Indeed, if we use the renormalization scale μ and then we change it to μ', then the physical quantities should not change. As we will see, this requirement is fulfilled if the Callan–Symanzik equation holds. Then, solving the Callan–Symanzik equation we find the renormalization group equation, an equation which proves that couplings run, i.e. they depend on the momentum or, equivalently, length scale at which we study our theory.

Consider for simplicity a theory with n scalar fields and bare coupling λ_0. The bare n-point Green's function,

$$\langle 0|T\phi_0(x_1)\cdots\phi_0(x_n)|0\rangle$$

has no dependence on the renormalization scale μ, but this is not so for the renormalized n-point Green's function,

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = [Z(\mu)]^{-n/2}\langle 0|T\phi_0(x_1)\cdots\phi_0(x_n)|0\rangle$$

where $Z(\mu)$ is the field strength renormalization at scale μ.

Let us now see what is the effect of a shift of μ. Let

$$G^{(n)}(x_1,\ldots,x_n) = \langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle_{\text{connected}}$$

be the connected renormalized n-point Green's function. Now, if we shift μ by $\delta\mu$, then there is a corresponding shift in the renormalized coupling constant and the rescaled field such that the bare Green's functions do not change:

$$\mu \to \mu + \delta\mu$$

$$\lambda \to \lambda + \delta\lambda$$

and

$$\phi \to (1 + \delta\eta)\phi$$

Then, the shift in $G^{(n)}$ is the one induced by the field strength shift:

$$G^{(n)} \to (1 + n\delta\eta)G^{(n)}$$

Thinking of $G^{(n)}$ as function of μ and λ, we can write the shift in $G^{(n)}$ as

$$\delta G^{(n)} = \frac{\partial G^{(n)}}{\partial \mu}\delta\mu + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda$$

and since this is equal to $n\delta\eta G^{(n)}$ we obtain the famous Callan–Symanzik equation:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right] G^{(n)}(\{x_i\}; \mu, \lambda) = 0$$

where

$$\beta(\lambda) \equiv \frac{\mu}{\delta\mu}\delta\lambda$$

is the beta function and

$$\gamma(\lambda) \equiv -\frac{\mu}{\delta\mu}\delta\eta$$

is known as the anomalous dimension, owing its name to the fact that the mass dimension for ϕ is $1 + \gamma(\lambda)$. The Callan–Symanzik equation asserts that there are two functions, $\beta(\lambda)$ and $\gamma(\lambda)$, related to the shifts in the coupling constant and field strength respectively, which depend only on the coupling and which are responsible to compensate for the shift induced to the bare Green's functions by the shift in the renormalization scale μ. In fact, since

$$\delta\eta = \frac{[Z(\mu + \delta\mu)]^{-1/2}}{[Z(\mu)]^{-1/2}} - 1$$

we get

$$\gamma(\lambda) = \frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z$$

In addition, if λ is thought of as a function of μ it is

$$\delta\lambda = \frac{\partial\lambda}{\partial\mu}\delta\mu$$

and, thus,

$$\beta(\lambda) = \mu\frac{\partial}{\partial\mu}\lambda$$

From the last equation we immediately observe that the beta function gives information about the behavior of the renormalized coupling constant as we vary the renormalization scale μ.

Let us now solve the Callan–Symanzik equation for the two-point Green's function in a theory with a single massless scalar field. Since $G^{(2)}(p)$ has mass dimension -2 we can express its dependence on p^2 and μ^2 as

$$G^{(2)}(p) = \frac{i}{p^2} g(-p^2/\mu^2)$$

Replacing the derivative with respect to μ with a derivative with respect to $p \equiv \sqrt{-p^2}$ we can write the Callan–Symanzik equation as

$$\left[p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial \lambda} + 2 - 2\gamma(\lambda)\right] G^{(2)}(p,\lambda) = 0$$

which is solved by

$$G^{(2)}(p,\lambda) = \frac{i}{p^2}\mathcal{G}(\bar{\lambda}(p,\lambda)) \exp\left[2\int_{p'=\mu}^{p'=p} d\left(\ln\frac{p'}{\mu}\right)\gamma(\bar{\lambda}(p',\lambda))\right] \tag{3.3.1}$$

where $\bar{\lambda}(p,\lambda)$, the running coupling, solves the equation

$$\frac{d}{d\ln(p/\mu)}\bar{\lambda}(p,\lambda) = \beta(\bar{\lambda}(p,\lambda)), \quad \text{with } \bar{\lambda}(\mu,\lambda) = \lambda \tag{3.3.2}$$

The function $\mathcal{G}(\bar{\lambda})$ can be determined by computing $G^{(2)}(p,\lambda)$ as a perturbation series in λ and match terms in the expansion of equation (3.3.1) as a series in λ. Equation (3.3.2) is known as the renormalization group equation. Obviously, if the beta function for a coupling is positive, then the theory becomes more weakly-coupled at low momenta (large distances). In that case the theory is called IR (infrared) free. In contrast, if the beta function is negative the theory becomes more weakly-coupled at high momenta and it is said to be asymptotically free.

If we consider, for example, an $\mathsf{SU}(N_c)$ gauge theory with N_f flavors and coupling g, then it can be shown that at one-loop the beta function is

$$\beta_{1-\text{loop}}(g) = -b_0\frac{g^3}{16\pi^2}$$

where

$$b_0 = \tfrac{11}{3}N_c - \tfrac{2}{3}N_f$$

Therefore, for an ordinary $\mathsf{SU}(3)$ gauge theory the one-loop beta function is negative if there are no more than sixteen flavors. In particular, in QCD there are six flavors in total and, hence, QCD is asymptotically free. In contrast, QED can be shown to be IR free.

The solution to the renormalization group equation at one-loop exhibits a disturbing behavior, namely the existence of a finite momentum scale at which the coupling diverges.

This is the so-called Landau pole. In the case of an $\mathsf{SU}(N_c)$ gauge theory with N_f flavors and coupling g the solution to the renormalization group equation at one-loop is

$$\frac{1}{\bar{g}^2(p)} = \frac{1}{g^2} + \frac{b_0}{8\pi^2} \ln \frac{p}{\mu}$$

and the Landau pole lies at

$$\Lambda_{\text{Landau}} = \mu e^{-8\pi^2/b_0 g^2}$$

If $b_0 > 0$, i.e. if the theory is asymptotically free, the Landau pole lies at very low energies. On the contrary, if $b_0 < 0$, i.e. if the theory is IR free, we encounter the Landau pole at high energy.[5] The position of the Landau pole in a gauge theory is referred to as the dynamically generated scale of the theory. (It is dynamically generated because of the e^{-1/g^2} dependence, a dependence which cannot be seen in perturbation theory. That is so because the functions $T(g)$ and $T(g) + e^{-1/g^2}$ have exactly the same perturbation expansion.)

Of course, couplings are expected to be running in supersymmetric theories as well. It can be shown that for SQCD, as we have described it so far, the exact beta function is [22], [23], [24]

$$\beta(g) = -\frac{g^3}{16\pi^2} \frac{3N_c - N_f + N_f \gamma(g^2)}{1 - N_c \frac{g^2}{8\pi^2}} \tag{3.3.3}$$

where

$$\gamma(g^2) = -\frac{g^2}{8\pi^2} \frac{N_c^2 - 1}{N_c} + \mathcal{O}(g^4)$$

3.3.1 The case $N_f < N_c$

In this case the global symmetries of our theory turn out to be a very useful tool in the search for a dynamically generated superpotential. Any superpotential that might be generated has to be invariant under the full nonanomalous global symmetry. In fact, there is a unique superpotential that is compatible with the symmetries of our theory [25]:

$$W_{\text{eff}} = C_{N_c,N_f} \left(\frac{\Lambda^{3N_c - N_f}}{\det M} \right)^{1/(N_c - N_f)} \tag{3.3.4}$$

where Λ is the dynamically generated scale of the theory and C_{N_c,N_f} are constants which depend on the subtraction scheme. Indeed, the generated superpotential is nonperturbative and, hence, there is no conflict with the nonrenormalization theorems.

The superpotential (3.3.4) is further constrained in the limit of large $M^{N_f}_{\tilde{N}_f}$ or, equivalently, in the limit of large a_{N_f} in equation (3.2.6). In this limit, $\mathsf{SU}(N_c)$ with N_f flavors is broken to $\mathsf{SU}(N_c - 1)$ with $N_f - 1$ flavors by the Higgs mechanism at energy a_{N_f}. The scale of the low energy theory is

$$\Lambda_L^{3(N_c-1)-(N_f-1)} = \frac{\Lambda^{3N_c - N_f}}{a_{N_f}^2}$$

[5] Note here that in the above calculation we took into account only the one-loop result for the beta function and, therefore, we cannot be sure that the Landau pole persists once we use the exact beta function.

in the so-called $\overline{DR}$ subtraction scheme. Requiring the superpotential (3.3.4) to produce the correct result for the low-energy theory in this limit gives the condition

$$C_{N_c,N_f} = C_{N_c-N_f} \tag{3.3.5}$$

Furthermore, if we give a very large mass to the N_f-th flavor by adding the superpotential $W_{\text{tree}} = mM^{N_f}_{N_f}$ then the resulting low-energy theory is $\mathsf{SU}(N_c)$ SQCD with $N_f - 1$ flavors and scale

$$\Lambda_L^{3N_c-(N_f-1)} = m\Lambda^{3N_c-N_f}$$

(in the $\overline{DR}$ scheme) a condition that appears by matching the running gauge coupling at the transition scale m. By requiring invariance under the symmetries we can prove that the exact superpotential is of the form

$$W_{\text{exact}} = \left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{1/(N_c-N_f)} f(t)$$

where

$$t = mM^{N_f}_{N_f}\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{-1/(N_c-N_f)}$$

Now, in the limit of small mass and weak coupling we know that $f(t) = C_{N_c,N_f} + t$. But all values of t can be obtained in this limit and, thus, the function $f(t)$ in this limit is exactly $f(t) = C_{N_c,N_f} + t$ for all t. This conclusion shows that the exact superpotential with the added mass term is

$$W_{\text{exact}} = C_{N_c,N_f}\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{1/(N_c-N_f)} + mM^{N_f}_{N_f} \tag{3.3.6}$$

But the requirement that the superpotential (3.3.6) should give the correct superpotential upon integrating out $M^{N_f}_{N_f}$ relates C_{N_c,N_f} to C_{N_c,N_f-1} which, when combined with the condition (3.3.5), gives

$$C_{N_c,N_f} = (N_c - N_f)C^{1/(N_c-N_f)}$$

where C is a numerical constant. Actually, in the case $N_f = N_c - 1$ we can carry out a detailed one-instanton calculation which shows that, in the $\overline{DR}$ scheme, $C = 1$ [26]. Hence, the dynamically generated superpotential in the case $N_f < N_c$ is

$$W_{\text{eff}} = (N_c - N_f)\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{1/(N_c-N_f)} \tag{3.3.7}$$

We should now investigate the vacuum structure of the full quantum theory. It is

$$\left.\frac{\partial W_{\text{eff}}}{\partial M^i_{\tilde{i}}}\right|_{\langle M^i_{\tilde{i}}\rangle} = -\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{1/(N_c-N_f)} (M^{-1})^{\tilde{i}}_i\Big|_{\langle M^i_{\tilde{i}}\rangle}$$

and, thus, the superpotential (3.3.7) results in a potential for the squarks which slopes to zero as $\det M \to \infty$. Therefore, the quantum theory does not have a ground state. It is rather surprising that we started with the infinite set of vacua (3.2.7) in the classical theory and we ended up in a quantum theory without a vacuum.

Nevertheless, the addition of masses for the N_f flavors results in the appearance of vacua. To see this, suppose that we add the superpotential (3.1.6) to W_{eff}. Then, the N_f flavors all get a mass and the exact superpotential is

$$W_{\text{exact}} = (N_c - N_f)\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{1/(N_c-N_f)} + \mathrm{tr}'\, mM$$

This gives N_c vacua at

$$\langle M^i_{\tilde{i}}\rangle = \left(\Lambda^{3N_c-N_f}\det m\right)^{1/N_c}(m^{-1})^i_{\tilde{i}}$$

corresponding to the N_c branches of the N_c-th root. If the masses of the flavors are very large, then the massive fields decouple leaving a low-energy $\mathsf{SU}(N_c)$ supersymmetric Yang–Mills theory. The low-energy theory has confinement with a mass gap and N_c vacua.

3.3.2 The case $N_f = N_c$

In the case $N_f = N_c$ the superpotential (3.3.7) obviously cannot be generated. Consequently, we expect classical flat directions to persist in the quantum theory, although the geometry of the classical moduli space might change. We will refer to the moduli space of the quantum theory as the quantum moduli space.

The classical moduli space of $N_f = N_c$ SQCD is parametrized by vevs of mesons,

$$M^i_{\tilde{i}} = \tilde{Q}_{\tilde{i}} Q^i$$

and baryons,

$$B = \frac{1}{N_c!}\epsilon^{a_1\ldots a_{N_c}}\epsilon_{i_1\ldots i_{N_c}} Q^{i_1}_{a_1}\cdots Q^{i_{N_c}}_{a_{N_c}}$$

and

$$\tilde{B} = \frac{1}{N_c!}\epsilon_{a_1\ldots a_{N_c}}\epsilon^{\tilde{i}_1\ldots\tilde{i}_{N_c}} \tilde{Q}^{a_1}_{\tilde{i}_1}\cdots \tilde{Q}^{a_{N_c}}_{\tilde{i}_{N_c}}$$

subject to the algebraic constraint

$$\det M - \tilde{B}B = 0 \tag{3.3.8}$$

The generic point of this space does not have any unbroken gauge symmetry. However, at $B = \tilde{B} = 0$ and if $\mathrm{rank}(M) \le N_c - 2$, i.e. if at least one of the N_c eigenvalues of M is zero, then the classical moduli space has a singular submanifold. This is immediately seen by the fact that the function

$$F(M^i_{\tilde{i}}, B, \tilde{B}) = \det M - \tilde{B}B$$

has Jacobian matrix

$$J = \begin{pmatrix} \dfrac{\partial F}{\partial M^i_{\tilde{i}}} & \dfrac{\partial F}{\partial B} & \dfrac{\partial F}{\partial \tilde{B}} \end{pmatrix} = \begin{pmatrix} (M^{-1})^{\tilde{i}}_i \det M & -\tilde{B} & -B \end{pmatrix} \tag{3.3.9}$$

which has rank zero when $B = \tilde{B} = 0$ and $\mathrm{rank}(M) \leq N_c - 2$. Physically, in the singular points of the moduli space we have the appearance of extra massless fields.

The quantum moduli space is parametrized by the same fields but the constraint (3.3.8) is modified to [27]

$$\det M - \tilde{B}B = \Lambda^{2N_c} \tag{3.3.10}$$

Obviously the quantum moduli space does not have singular points; all the singularities of the classical moduli space have been smoothed out by quantum effects.

The constraint (3.3.10) can be implemented with the superpotential

$$W = \lambda(\det M - \tilde{B}B - \Lambda^{2N_c}) \tag{3.3.11}$$

where λ is a Lagrange multiplier. The form of the superpotential (3.3.11) is motivated by requiring that if we give mass to the N_c-th flavor and then integrate it out, then we should obtain a low-energy theory with $N_f = N_c - 1$ flavors and superpotential

$$W_{\text{eff}} = \frac{\Lambda^{2N_c+1}}{\det M}$$

which is exactly the superpotential (3.3.7) for $N_f = N_c - 1$.

3.3.3 The case $N_f = N_c + 1$

In the case $N_f = N_c + 1$ there are two kinds of gauge invariant objects: The mesons,

$$M^i_{\tilde{i}} = \tilde{Q}_{\tilde{i}} Q^i$$

and the baryons

$$B_i = \frac{1}{N_c!} \epsilon^{a_1 \dots a_{N_c}} \epsilon_{i j_1 \dots j_{N_c}} Q^{j_1}_{a_1} \cdots Q^{j_{N_c}}_{a_{N_c}}$$

and

$$\tilde{B}^{\tilde{i}} = \frac{1}{N_c!} \epsilon_{a_1 \dots a_{N_c}} \epsilon^{\tilde{i} \tilde{j}_1 \dots \tilde{j}_{N_c}} \tilde{Q}^{a_1}_{\tilde{j}_1} \cdots \tilde{Q}^{a_{N_c}}_{\tilde{j}_{N_c}}$$

From these we can build the superpotential

$$W = \frac{1}{\Lambda^{2N_c - 1}} (\operatorname{Tr} BM\tilde{B} - \det M) \tag{3.3.12}$$

which is invariant under the global symmetry, which, in this case, has representations and charge assignments

	$\mathsf{SU}(N_f)_L$	$\mathsf{SU}(N_f)_R$	$\mathsf{U}(1)_B$	$\mathsf{U}(1)_R$
Q	$\mathbf{N}_f$	$\mathbf{1}$	1	$1/(N_c+1)$
$\tilde{Q}$	$\mathbf{1}$	$\overline{\mathbf{N}}_f$	-1	$1/(N_c+1)$

It is easy to see that if we give a mass to the (N_c+1)-th flavor and, then, integrate it out, we obtain the quantum moduli space with constraint (3.3.10) in the low energy theory with $N_f = N_c$.

3.3.4 The case $N_f > N_c + 1$

From the form of the beta function of SQCD, (3.3.3), we can immediately recognize that for $N_f > N_c + 1$ there are various ranges of N_f with different dynamics. In our treatment we will consider the case $N_c + 1 < N_f < 3N_c$ only, since in the range $N_f > 3N_c$ the dynamics is trivial, in the sense that the IR theory is weakly-coupled. As we will see in the next section, in the range $N_c + 1 < N_f \leq \frac{3}{2}N_c$ the physics of our theory has a dual description which will turn out to be very helpful in the search for supersymmetry breaking.

For now, we turn our attention to the range $\frac{3}{2}N_c < N_f < 3N_c$, where SQCD is asymptotically free. However, the coupling does not grow to infinity at long distances but, instead, it reaches a finite value. Our theory, therefore, reaches a fixed point of the renormalization group. Indeed, because for the beta function of SQCD there are values of N_f and N_c such that the one-loop beta function is negative, while the two-loop contribution is positive, there might exist a nontrivial fixed point of the renormalization group. In [28] it was argued that such a fixed point exists for any number flavors such that $\frac{3}{2}N_c < N_f < 3N_c$. Therefore, for this range of N_f, called the conformal window, the infrared theory is a nontrivial four-dimensional superconformal field theory. The elementary quarks and gluons are not confined but appear as interacting massless particles. Furthermore, as the number of flavors decreases, the fixed-point coupling increases. In fact, for N_f at or below $\frac{3}{2}N_c$ the theory is very strongly coupled at the IR fixed-point.

3.4 Seiberg duality

The idea of analyzing the same physics using two different descriptions is particularly appealing. The reason, of course, is that problems that are not easy (or even impossible) to analyze in the one description, might turn out to be trivial in the dual description. To be more specific consider a gauge theory with coupling constant $g > 1$. At this case the only tool we have in our disposal in order to analyze the theory, namely perturbation theory, is no longer meaningful. But imagine that the same physics could be described by another gauge theory with gauge coupling, g', related to g by

$$g' \sim \frac{1}{g}$$

Then, perturbation theory in this dual description can be used to analyze the physics and the results can be translated back to the original strongly-coupled theory. The above duality is often called S-duality or strong-weak duality. For example, the coupling constant of QED is the fine-structure constant which is proportional to the fundamental electric charge, e, squared. Now, the Dirac quantization condition states that if there are monopoles in the universe, then

$$eg = 2\pi n$$

where g is the fundamental magnetic charge and n is an integer. Therefore, there might exist a theory with coupling proportional to e^{-2} which describes the same physics as QED. However, this theory would be strongly-coupled and, so, not really useful, but there is always the possibility that we can find a weakly-coupled dual to a strongly-coupled theory.

The basic result discovered by Seiberg is that low-energy[6] "electric" SQCD with $N_c+1 < N_f < 3N_c$ has a dual description in terms of "magnetic variables" [28]. More specifically, the physics of the interacting fixed point in the range $N_c+1 < N_f < 3N_c$ has a dual description based on the gauge group $\mathsf{SU}(N_f - N_c)$ with the same number of flavors, an elementary (singlet) magnetic mesonic field, $(M_m)^i_{\tilde{i}} = M^i_{\tilde{i}}/\hat{\Lambda}$,[7] and superpotential

$$W = \frac{1}{\hat{\Lambda}} M^i_{\tilde{i}} q_i \tilde{q}^{\tilde{i}} \tag{3.4.1}$$

where the scale $\hat{\Lambda}$ is inserted in order to guarantee that the superpotential has the correct mass dimensions. $\mathsf{SU}(N_f - N_c)$ is called the magnetic gauge group and the quarks q_i and $\tilde{q}^{\tilde{i}}$ are referred to as magnetic quarks in order to avoid confusion with the electric quarks Q^i and $\tilde{Q}_{\tilde{i}}$. Note that if $\frac{3}{2}N_c < N_f < 3N_c$, then $\frac{3}{2}(N_f - N_c) < N_f < 3(N_f - N_c)$ and, thus, the magnetic theory indeed flows to an interacting conformal fixed point. The claim is that this fixed point is exactly the one to which the electric theory flows. However, if $N_c + 1 < N_f < \frac{3}{2}N_c$ the magnetic theory has a positive beta function, i.e. it is IR-free. It is exactly in the range $N_c+1 < N_f < \frac{3}{2}N_c$, called the free-magnetic range, that Seiberg duality is an example of S-duality, albeit only in the IR.

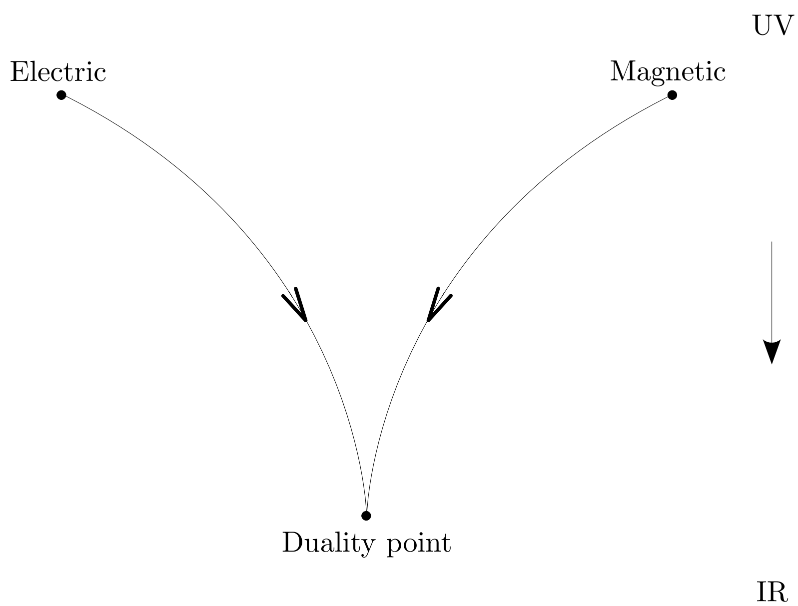


Fig. 3.1: The flow of the electric and magnetic theories to the same infrared physics

An important fact that needs to be stressed at this point is that adding tree-level masses for the quark flavors does not spoil the duality we established. Indeed, the superpotential (3.1.6) which gives masses to the quark flavors in the electric theory is interpreted as a term

[6]Note that Seiberg duality applies to the low-energy and not the UV-complete theory.
[7]In our treatment we will use M and $\hat{\Lambda}$ instead of M_m.

linear in the fundamental magnetic mesonic field in the superpotential of the magnetic theory. Again, those two theories describe the same physics in the IR.

At the more technical level the magnetic theory has scale $\tilde{\Lambda}$ which is related to the scale Λ of the electric theory by

$$\Lambda^{3N_c-N_f}\tilde{\Lambda}^{3(N_f-N_c)-N_f}=(-1)^{N_f-N_c}\hat{\Lambda}^{N_f} \tag{3.4.2}$$

Equation (3.4.2) shows that as the electric theory becomes more strongly-coupled the magnetic theory becomes more weakly-coupled and vice-versa. Due to the presence of the phase $(-1)^{N_f-N_c}$ equation (3.4.2) does not look dual. But if we perform another duality transformation it becomes

$$\Lambda^{3N_c-N_f}\tilde{\Lambda}^{3(N_f-N_c)-N_f}=(-1)^{N_c}\hat{\Lambda}'^{N_f} \tag{3.4.3}$$

and, therefore, the requirement that equations (3.4.2) and (3.4.3) have the same form enables us to relate $\hat{\Lambda}'$ and $\hat{\Lambda}$:

$$\hat{\Lambda}'=-\hat{\Lambda}$$

In fact, the minus sign is important when we dualize the magnetic theory. Then, we obtain an $\mathsf{SU}(N_c)$ theory with scale Λ, quarks p^i and $\tilde{p}_{\tilde{i}}$ and additional singlets $M^i_{\tilde{i}}$ and $N^{\tilde{i}}_i=q_i\tilde{q}^{\tilde{i}}$ with superpotential

$$W=\frac{1}{\hat{\Lambda}'}N^{\tilde{i}}_i\tilde{p}_{\tilde{i}}p^i+\frac{1}{\hat{\Lambda}}N^{\tilde{i}}_iM^i_{\tilde{i}}=\frac{1}{\hat{\Lambda}}N^{\tilde{i}}_i(-\tilde{p}_{\tilde{i}}p^i+M^i_{\tilde{i}})$$

Since M and N are massive they can be integrated out by use of their equations of motion:

$$N^{\tilde{i}}_i=0\ \text{ and }\ M^i_{\tilde{i}}=\tilde{p}_{\tilde{i}}p^i$$

The equation of motion for M shows that we can identify p^i and $\tilde{p}_{\tilde{i}}$ with the original quarks, Q^i and $\tilde{Q}_{\tilde{i}}$ and since, additionally, the superpotential vanishes, we conclude that the dual of the magnetic theory is the electric theory.

Seiberg duality says that the electric and magnetic theories, two theories with different gauge symmetries, both describe the same IR fixed point. This is possible since gauge symmetries are not true symmetries but have actually to do with a redundant description of the physics. In that sense, having two different redundant descriptions of the same physics is not a problem. On the other, hand global symmetries are real symmetries and they have to be the same in both the electric and the magnetic description. Indeed, the magnetic theory has the same anomaly-free global symmetry as the electric theory with $M^i_{\tilde{i}}$ transforming as $\tilde{Q}_{\tilde{i}}Q^i$ and with representations and charge assignments

	$\mathsf{SU}(N_f)_L$	$\mathsf{SU}(N_f)_R$	$\mathsf{U}(1)_B$	$\mathsf{U}(1)_R$
q	$\overline{\mathbf{N}}_f$	$\mathbf{1}$	$\frac{N_c}{N_f-N_c}$	$\frac{N_c}{N_f}$
$\tilde{q}$	$\mathbf{1}$	$\mathbf{N}_f$	$-\frac{N_c}{N_f-N_c}$	$\frac{N_c}{N_f}$

for the magnetic quarks.

In order for the magnetic dual to describe the same physics as the original electric theory there must be a mapping of all gauge-invariant operators of the electric theory to those of the magnetic theory. Indeed, the electric mesons $M^i_{\tilde{i}}=\tilde{Q}_{\tilde{i}}Q^i$ become identical to the magnetic singlets $M^i_{\tilde{i}}$ in the IR. Likewise, magnetic baryons can be written down and shown to be related to the electric baryons.

3.5 Metastable vacua in SQCD

As we saw in the previous section massive SQCD in the free-magnetic range, $N_c + 1 < N_f < \frac{3}{2}N_c$, is an asymptotically free theory which flows to a nontrivial fixed point. For this range of N_f there is also another theory which flows to exactly the same fixed point, namely the dual magnetic theory, which is IR free and, thus, its vacuum structure is rather easy to analyze. In fact, using this duality, Intriligator, Seiberg and Shih found that supersymmetry in SQCD with small masses for the flavors is broken in a metastable vacuum [29].

3.5.1 A toy model

To begin with consider a theory of chiral superfields Φ_{ij}, φ_c^i and $\tilde{\varphi}^{ic}$, where $i = 1, \ldots, N_f$ and $c = 1, \ldots, N$ with $N < N_f$, canonical Kähler potential and superpotential

$$W = h\mathrm{Tr}\,\varphi\Phi\tilde{\varphi} - h\mu^2\mathrm{tr}'\,\Phi \tag{3.5.1}$$

The F-terms for φ and $\tilde{\varphi}$ can be set to zero by choosing $\Phi_{ij} = 0$, but the F-terms for Φ cannot all be set to zero. To see this note that the F-terms

$$F_{\Phi_{ij}} = \frac{\partial W}{\partial \Phi_{ij}} = h\varphi_c^i\tilde{\varphi}^{jc} - h\mu^2\delta^{ij}$$

can be written schematically in the matrix form

$$F_\Phi = h\begin{pmatrix} a_1 & & & \\ & \ddots & & \\ & & a_N & \\ & & & \\ \end{pmatrix} - h\mu^2\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

where we diagonalized the matrix $\varphi\tilde{\varphi}$ which has rank N and, thus, at least $N_f - N$ zero eigenvalues. Now, we can choose $a_1 = \cdots = a_N = \mu^2$ but we cannot do much with the zero eigenvalues. Therefore, not all $F_{\Phi_{ij}}$s can be set to zero and, thus, supersymmetry is spontaneously broken. This is the rank-condition mechanism of supersymmetry breaking.

Without the term $h\mu^2\mathrm{tr}'\,\Phi$ in the superpotential the theory we described above has the global symmetry

$$\mathsf{SU}(N) \times \mathsf{SU}(N_f) \times \mathsf{SU}(N_f) \times \mathsf{U}(1)_B \times \mathsf{U}(1)' \times \mathsf{U}(1)_R$$

and matter content

	$\mathsf{SU}(N)$	$\mathsf{SU}(N_f)$	$\mathsf{SU}(N_f)$	$\mathsf{U}(1)_B$	$\mathsf{U}(1)'$	$\mathsf{U}(1)_R$
Φ	$\mathbf{1}$	$\mathbf{N}_f$	$\overline{\mathbf{N}}_f$	0	-2	2
φ	$\mathbf{N}$	$\overline{\mathbf{N}}_f$	$\mathbf{1}$	1	1	0
$\tilde{\varphi}$	$\overline{\mathbf{N}}$	$\mathbf{1}$	$\mathbf{N}_f$	-1	1	0

The term $h\mu^2 \mathrm{tr}' \, \Phi$ breaks the global symmetry to

$$\mathsf{SU}(N) \times \mathsf{SU}(N_f) \times \mathsf{U}(1)_B \times \mathsf{U}(1)_R$$

where the unbroken $\mathsf{SU}(N_f)$ is the diagonal subgroup of the original $\mathsf{SU}(N_f)^2$. The minimum of the scalar potential is

$$V_{\min} = (N_f - N)|h^2\mu^4|$$

and it occurs along a classical moduli space of vacua which, up to global symmetries, is given by

$$\Phi = \begin{pmatrix} 0 & 0 \\ 0 & \Phi_0 \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_0 \\ 0 \end{pmatrix} \quad \text{and} \quad \tilde{\varphi}^T = \begin{pmatrix} \tilde{\varphi}_0^T \\ 0 \end{pmatrix}$$

with $\varphi\tilde{\varphi}^T = \mu^2 \mathbb{1}_N$, where Φ_0 is an arbitrary $(N_f - N) \times (N_f - N)$ matrix and φ_0 and $\tilde{\varphi}_0^T$ are $N \times N$ matrices. Up to unbroken flavor rotations the vacua of maximal unbroken global symmetry are

$$\Phi_0 = 0 \quad \text{and} \quad \varphi_0 = \tilde{\varphi}_0^T = \mu^2 \mathbb{1}_N \tag{3.5.2}$$

and by calculating the one-loop effective potential around the vacua (3.5.2) it can be shown that they are stable, i.e. they do not develop any tachyonic directions.

So far we have considered a theory that breaks supersymmetry spontaneously. Our next step is to gauge the $\mathsf{SU}(N)$ symmetry. We will be interested in the case $N_f > 3N$, where the $\mathsf{SU}(N)$ theory is IR-free and, thus, has a scale Λ_m above which it is strongly coupled. Then, the running of the gauge coupling, g, is

$$e^{-8\pi^2/g^2 + i\vartheta} = \left(\frac{E}{\Lambda_m}\right)^{N_f - 3N}$$

and, indeed, g runs to zero in the IR. Note that here we take E and Λ_m to be complex numbers and that is why we can have the term $e^{i\vartheta}$ multiplying $e^{-8\pi^2/g^2}$. Our theory has a Landau pole at $E = \Lambda_m$ and, hence, our description will be valid up to energies $E \sim \Lambda_m$.

As we have already encountered before, the gauging of a global symmetry results in the appearance of D-terms in the scalar potential, terms that arise because of the gauge fields that are introduced with the gauging. The contribution of the D-terms to the scalar potential is (cf. equation (1.4.12) and use equation (3.1.5))

$$V_D = \tfrac{1}{2} g^2 \sum_{A=1}^{N^2-1} (\mathrm{Tr}\, \tilde{\varphi}^\dagger T^A \tilde{\varphi} - \mathrm{Tr}\, \varphi T^A \varphi^\dagger)^2 \tag{3.5.3}$$

The D-term potential (3.5.3) vanishes in the vacua (3.5.2) and, so, (3.5.2) remains a minimum of the tree-level potential

$$V_{\text{tree}} = V_F + V_D$$

where

$$V_F = \sum_{i,j} \left| \frac{\partial W}{\partial \Phi_{ij}} \right|^2$$

The $\mathsf{SU}(N)$ gauge theory is completely Higgsed in the vacua (3.5.2). Again, the one-loop correction to the scalar potential after the gauging brings no surprises since, even in this case, the vacua (3.5.2) do not develop any tachyonic directions. The reason for this is that the tree-level spectrum of the massive $\mathsf{SU}(N)$ vector multiplet is supersymmetric and, thus, its contributions to the supertrace of the Coleman–Weinberg potential (2.2.1) cancel.

Although the gauging of $\mathsf{SU}(N)$ does not affect the nonsupersymmetric vacua it has an important effect elsewhere in field space, namely it leads to the appearance of supersymmetric vacua. If we give Φ a nonzero vev, then the first term in the superpotential (3.5.1) gives mass $\langle h\Phi\rangle$ to the fundamental flavors φ and $\tilde{\varphi}$ and, below the energy scale $\langle h\Phi\rangle$, we can integrate out these massive flavors and take a low-energy pure $\mathsf{SU}(N)$ Yang–Mills theory with scale Λ_L. Matching of the running couplings at scale $\langle h\Phi\rangle$ gives

$$e^{-8\pi^2/g^2+i\vartheta} = \left(\frac{\Lambda_L}{E}\right)^{3N} = \frac{h^{N_f}\det\Phi}{\Lambda_m^{N_f-3N}E^{3N}} \tag{3.5.4}$$

Now, gaugino condensation results in the superpotential

$$W_{\text{gaugino}} = N\Lambda_L^3$$

in the low-energy theory and, therefore, after eliminating Λ_L in favor of Λ_m with the aid of equation (3.5.4), we obtain the superpotential of the low-energy pure $\mathsf{SU}(N)$ Yang–Mills theory:

$$W_{\text{low}} = N\left(h^{N_f}\Lambda_m^{-(N_f-3N)}\det\Phi\right)^{1/N} - h\mu^2\operatorname{tr}'\Phi \tag{3.5.5}$$

Extremizing the superpotential (3.5.5) leads to $N_f - N$ supersymmetric vacua at

$$\langle h\Phi\rangle = \Lambda_m\epsilon^{2N/(N_f-N)}\mathbb{1}_{N_f} = \mu\frac{1}{\epsilon^{(N_f-3N)/(N_f-N)}}\mathbb{1}_{N_f}$$

where $\epsilon = \mu/\Lambda_m$. Note that our analysis is reliable when $|\epsilon| \ll 1$.

Therefore, we find that gauging the $\mathsf{SU}(N)$ symmetry results in the emergence of supersymmetric vacua without spoiling the supersymmetry breaking vacua we had found in the case of only global symmetries.[8] Therefore, supersymmetry is dynamically broken in a metastable vacuum. Note that our result is in accordance with the conclusions of section 2.6. More specifically, the global theory has an R-symmetry which is broken spontaneously when the zero components of she superfields acquire vevs. (In addition, the global theory has no moduli. Hence, the global theory breaks supersymmetry.) In contrast, once we gauge the $\mathsf{SU}(N)$ global symmetry the R-symmetry is anomalous under the gauged $\mathsf{SU}(N)$. Correspondingly, there are supersymmetric vacua. However, for $\langle\Phi\rangle$ near the origin, the $\mathsf{SU}(N)$ gauge theory is IR-free. Consequently, the $\mathsf{U}(1)_R$ symmetry returns as an accidental symmetry of the infrared theory and, thus, the nonsupersymmetric vacuum near the origin is related to the accidental R-symmetry there.

[8]For $\Lambda_m \to \infty$ with μ fixed the theory breaks supersymmetry. For Λ_m large but finite, corresponding to small but nonzero ϵ, a supersymmetric vacuum comes in from infinity.

Finally, we have to mention that the perturbative and nonperturbative calculations we carried out so far are completely under control and lead to the dominant contributions to the low-energy dynamics. Corrections can be safely neglected and, thus, our result of metastable supersymmetry breaking is robust.

3.5.2 Supersymmetric QCD

We now move on to the interesting case of massive $\mathsf{SU}(N_c)$ SQCD with scale Λ. This theory has N_c supersymmetric ground states given by

$$\langle M^i_{\tilde{i}} \rangle = \left(\Lambda^{3N_c - N_f} \det m\right)^{1/N_c} (m^{-1})^i_{\tilde{i}}$$

All these supersymmetric states preserve baryon number and, correspondingly, the vevs of all the baryonic operators vanish. The eigenvalues of the mass matrix, m, are positive numbers, m_i, $i = 1, \ldots, N_f$. We will be interested in the free magnetic range, $N_c + 1 < N_f < \frac{3}{2} N_c$, with

$$m_i \ll |\Lambda| \text{ and } \frac{m_i}{m_j} \sim 1$$

Then, the expectation values $\langle M^i_{\tilde{i}} \rangle$ approach the origin.

Seiberg duality dictates that, in the free magnetic range, the region around the origin can be easily analyzed by using the dual magnetic theory, a theory which is IR-free. In the free magnetic range the metric of the moduli space is smooth around the origin. Therefore, the Kähler potential is regular there and can be expanded as

$$K = \frac{1}{\beta} \mathrm{Tr}(q q^\dagger + \tilde{q}^\dagger \tilde{q}) + \frac{1}{\alpha |\Lambda|^2} \mathrm{tr}'\, M^\dagger M$$

where α and β are dimensionless positive numbers of order one. The superpotential of the magnetic theory is

$$W_{\text{dual}} = \frac{1}{\hat{\Lambda}} \mathrm{Tr}\, \tilde{q} M q + \mathrm{tr}'\, m M \tag{3.5.6}$$

where the matching of the scales is given by equation (3.4.2). Now, if all m_is are equal, $m_i = m_0$, $i = 1, \ldots, N_f$, then the theory we are analyzing is exactly the toy model of the previous subsection under the dictionary

$$\varphi = q, \quad \tilde{\varphi} = \tilde{q}, \quad \Phi = M/\sqrt{\alpha}\Lambda$$
$$h = \sqrt{\alpha}\Lambda/\hat{\Lambda}, \quad \mu^2 = -m_0 \hat{\Lambda}, \quad \Lambda_m = \tilde{\Lambda} \text{ and } N = N_f - N_c$$

Here we have chosen $\beta = 1$ by rescaling the magnetic quarks and expressed our answers in terms of $\tilde{\Lambda}$ and $\hat{\Lambda}$.

In the case $N_f = N_c + 1$ we do not set $\beta = 1$ but, instead, we scale the magnetic quarks to be the same as the electric baryons of subsection 3.3.3. More specifically $q_i \to B_i$ and $\tilde{q}^{\tilde{i}} \to \tilde{B}^{\tilde{i}}$ and, then, the Kähler potential is

$$K = \frac{1}{\beta |\Lambda|^{2N_c - 2}} \mathrm{Tr}(B B^\dagger + \tilde{B}^\dagger \tilde{B}) + \frac{1}{\alpha |\Lambda|^2} \mathrm{tr}'\, M^\dagger M$$

where again β is a dimensionless positive parameter and α is the same as before. The superpotential in this case is given by equation (3.3.12) with the addition of the mass terms:

$$W = \frac{1}{\Lambda^{2N_c-1}}(\mathrm{Tr}\, BM\tilde{B} - \det M) + \mathrm{tr}'\, mM$$

For $N_c > 2$ the additional determinant term is irrelevant in the IR and, thus, negligible near the origin. Hence, this theory is the same as the $N = 1$ version of the theory in the previous subsection.

Therefore, borrowing all the results from the previous subsection, we conclude that for N_f in the range $N_c + 1 \leq N_f < \frac{3}{2}N_c$ and for suitable tree-level masses, SQCD has a moduli space of nonsupersymmetric metastable vacua near the origin. It can actually be proved that the dynamical supersymmetry breaking in metastable vacua persists if the quark masses are different, although, still, well below $|\Lambda|$. Finally, it can be shown that the metastable vacua we found can be made arbitrarily long-lived.

Qualitatively, the potential can be thought of as resembling the one shown in Fig. 3.2. Although quantum tunneling can lead the system to the supersymmetric vacuum, the fact

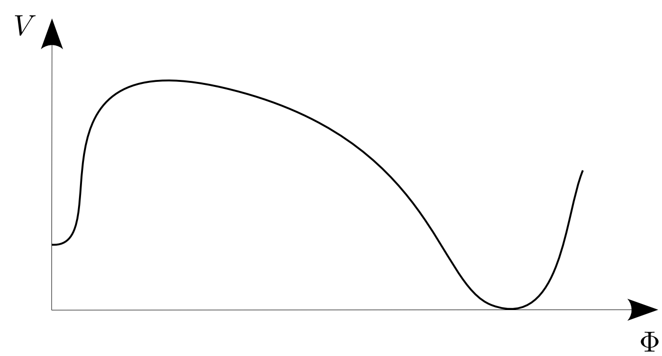


Fig. 3.2: Qualitative form of the potential of massive SQCD in the free magnetic range

that $\epsilon \ll 1$ shows that the two sets of vacua are widely separated in field space and, thus, the lifetime of the vacuum can be made arbitrarily long by making ϵ arbitrarily small.

3.5.3 Metastable vacua in $N_f = N_c$ SQCD

It was conjectured in [29] that metastable vacua exist in $N_f = N_c$ SQCD. In this subsection we will try to check this conjecture. As we found before, $N_f = N_c + 1$ SQCD experiences dynamical supersymmetry breaking in metastable vacua near the origin. Beginning with the superpotential

$$W = \frac{1}{\Lambda^{2N_c-1}}(\mathrm{Tr}\, BM\tilde{B} - \det M) + \mathrm{tr}'\, mM$$

bringing M, m to the form

$$M = \begin{pmatrix} M & \\ & M_{\widetilde{N_c+1}}^{N_c+1} \end{pmatrix}, \quad m = \begin{pmatrix} m & \\ & m_{N_c+1}^{\widetilde{N_c+1}} \end{pmatrix}$$

$B, \tilde{B}$ to the form

$$B = \begin{pmatrix} B & \\ & B_{N_c+1} \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{B} & \\ & \widetilde{B^{N_c+1}} \end{pmatrix}$$

and separating the terms referring to the (N_c+1)-th flavor, we can write

$$W = \frac{1}{\Lambda^{2N_c-1}}(\operatorname{Tr} BM\tilde{B} - \det M) + \operatorname{tr}' mM + \frac{1}{\Lambda^{2N_c-1}} B_{N_c+1} A \tilde{B}^{\widetilde{N_c+1}} + \eta A$$

where traces and determinants count only up to N_c and $\eta = m_{N_c+1}^{\widetilde{N_c+1}}$ and $A = M_{\widetilde{N_c+1}}^{N_c+1}$. The F-terms for $M_{\tilde{i}}^{i}$ and A are

$$F_{M_{\tilde{i}}^{i}} = \frac{1}{\Lambda^{2N_c-1}} \left(B_i \tilde{B}^{\tilde{i}} - A(M^{-1})_i^{\tilde{i}} \det M \right) + m_i^{\tilde{i}}$$

and

$$F_A = \frac{1}{\Lambda^{2N_c-1}} (B_{N_c+1} \tilde{B}^{\widetilde{N_c+1}} - \det M) + \eta$$

and the conditions $B_i = 0$ and $\tilde{B}^{\tilde{i}} = 0$ which set the other two F-terms to zero result in

$$F_{M_{\tilde{i}}^{i}} = m_i^{\tilde{i}} - \frac{1}{\Lambda^{2N_c-1}} A(M^{-1})_i^{\tilde{i}} \det M$$

and

$$F_A = \eta - \frac{1}{\Lambda^{2N_c-1}} A$$

$F_{M_{\tilde{i}}^{i}}$ can be set to zero by

$$\langle M \rangle = \eta^2 \frac{1}{\epsilon^{(2N_c-1)/N_c}} \mathbb{1}_{N_c}$$

where, now, m is the mass of each of the N_c flavors and $\epsilon = \eta/\Lambda$. Then, F_A can be set to zero with the choice

$$\langle A \rangle = \eta^2 \delta \frac{1}{\epsilon^{(2N_c-1)/N_c}}$$

where $\delta = m/\eta$. Note that, as expected, if $\eta = m$, then $\langle A \rangle$ becomes equal to the diagonal entries of $\langle M \rangle$.

Therefore, we found supersymmetric vacua for the above choices of $\langle M \rangle$ and $\langle A \rangle$. Now, the nonsupersymmetric vacua lie in the origin of field space, $\langle M \rangle = 0$ and $\langle A \rangle = 0$, and they have energy

$$V_+ = N_c m^2 + \eta^2 = \eta^2 (N_c \delta^2 + 1)$$

In order to find the lifetime of the nonsupersymmetric vacua we can model the needed calculation of the bounce action by a triangle potential barrier. Then, using the results of [30], we find that the bounce action is

$$S \sim \frac{\left[\sqrt{N_c(\Delta M)^2 + (\Delta A)^2}\right]^4}{V_+} = \frac{\eta^6}{\epsilon^{4(2N_c-1)/N_c}} \frac{(N_c + \delta^2)^2}{N_c \delta^2 + 1}$$

or

$$S \sim \frac{1}{\epsilon^{2(N_c-2)/N_c}}$$

As we observe, as long as $\epsilon \ll 1$, i.e. $\eta \ll \Lambda$, our nonsupersymmetric vacua can become parametrically long-lived. However, this behavior is spoiled when $\eta \to \Lambda$, i.e. as we flow down to $N_f = N_c$ SQCD.

To be more specific, it is very hard to reach a verdict on the existence of metastable supersymmetry breaking in $N_f = N_c$ massive SQCD. The rank condition breaks down at $N_f = N_c$, while the corrections to the Kähler potential are not easy to control. That is because the Kähler potential does not belong to the holomorphic information of the theory and, thus, it is not protected by supersymmetry. Our analysis, however, seems to suggest that metastable supersymmetry breaking occurs in this case as well.

Part II

String theory and M-theory

CHAPTER 4

Brane configurations and $\mathcal{N} = 2$ $\mathsf{U}(N_c)$ gauge theory

Contents

In this chapter we will use branes, extended objects in string theory, in order to describe four-dimensional $\mathcal{N} = 2$ $\mathsf{U}(N_c)$ supersymmetric gauge theories. Our treatment will make use of brane configurations in type-IIA superstring theory. It will be seen that these classical brane configurations give rise to the corresponding classical gauge-theory dynamics. Quantum effects in gauge theory are introduced through the lift of the classical brane configurations to M-theory. For a general and detailed exposition of the subject the reader is referred to [31] and references therein. I assume some familiarity of the reader with superstring theory.

4.1 Dp-branes and NS5-branes

Superstring theories not only contain one-dimensional fundamental strings, but, also, extended p-dimensional objects called p-branes. The world-volume of these objects is $(p+1)$-dimensional and they fall in two categories according to their behavior at weak string coupling, g_s:

- Solitonic or Neveu–Schwarz (NS) branes, whose tension is proportional to g_s^{-2}
- Dirichlet or D-branes, whose tension is proportional to g_s^{-1}

Obviously, at weak string coupling, $g_s \to 0$, the NS-branes are much heavier than the D-branes.

In a superstring theory at weak coupling branes are stable BPS-saturated objects which couple naturally to a corresponding field in spacetime. By natural coupling we mean the equivalent of the usual coupling, $q\int A_\mu\, dx^\mu$, of a particle (zero-dimensional object) to a vector field. The immediate generalization is that a p-dimensional brane couples naturally to a $(p+1)$-form potential. Moreover, each $(p+1)$-form potential, A_{p+1}, gives rise to a $(p+2)$-form field strength,

$$F_{p+2} = \boldsymbol{d} A_{p+1}$$

where $\boldsymbol{d}$ denotes the exterior derivative. But then, by the usual Hodge-star operator, we can find the magnetic dual of F_{p+2} which, in d spacetime dimensions, is a $(d-p-2)$-form which solves the Bianchi identities, and whose corresponding potential is a $(d-p-3)$-form. Therefore, the magnetic dual of a $(p+1)$-form potential, A_{p+1}, is a $(d-p-3)$-form potential, $\tilde{A}_{d-p-3}$. Consequently, we establish the existence of a $(d-p-4)$-brane, the magnetic dual of the p-brane we started off with, which couples naturally to $\tilde{A}_{d-p-3}$. Having said all that and knowing the massless bosonic field content of the different critical, i.e. ten-dimensional, superstring theories, one can now see which branes appear to each superstring theory at weak coupling (Table 4.1).

	Massless bosonic fields		Branes
	NS-NS sector	RR sector	
Type-I	$G_{\mu\nu}$, Φ	$C_{\mu\nu}$	D1, D5, D9
Type-IIA	$G_{\mu\nu}$, Φ, $B_{\mu\nu}$	$H^{(0)}$, C_μ, $C_{\mu\nu\rho}$	NS5, D0, D2, D4, D6, D8
Type-IIB	$G_{\mu\nu}$, Φ, $B_{\mu\nu}$	C, $C_{\mu\nu}$, $C_{\mu\nu\rho\sigma}$	NS5, D(-1), D1, D3, D5, D7, (D9)
Heterotic	$G_{\mu\nu}$, Φ, $B_{\mu\nu}$		NS5

Table 4.1: Massless bosonic fields and p-branes appearing in the five critical superstring theories at weak coupling

Type-IIA superstring theory has a constant nonpropagating zero-form field, $H^{(0)}$, a vector potential, C_μ, and a three-form potential, $C_{\mu\nu\rho}$, all in the RR sector. In addition there is the dilaton, Φ, the graviton, $G_{\mu\nu}$, and the antisymmetric field, $B_{\mu\nu}$, in the NS-NS sector. On the other hand, type-IIB superstring theory has the same field content in the NS-NS sector, but a zero-form, C, a two-form potential, $C_{\mu\nu}$, and a four-form potential, $C_{\mu\nu\rho\sigma}$, in the RR sector.

The D(-1)-brane sits at a particular point in spacetime (its world-volume is zero-dimensional) and has to be interpreted as a (Euclidean) instanton. The space-filling D9-brane indicates that open strings can propagate freely in spacetime, while the D8-brane is a domain-wall coupling to the magnetic dual of the nonpropagating field $H^{(0)}$. Note that we can introduce a D9-brane in type-IIB superstring theory as well. The rest of the D-branes in Table 4.1 all couple, electrically or magnetically, to the appropriate field in the RR sector. For example, the D2-brane of type-IIA superstring theory couples electrically to $C_{\mu\nu\rho}$, while the D4-brane couples magnetically to it, since the magnetic dual of $C_{\mu\nu\rho}$ is a 5-form field. In general, the tension of a Dp-brane is given by

$$T_p = \frac{1}{g_s l_s^{p+1}}$$

where l_s is the fundamental string scale. (The tension of the fundamental string is $T = l_s^{-2}$.) The tension of the Dp-brane is equal to its RR charge. An anti-Dp-brane carries the opposite RR-charge.

The NS5-brane in Table 4.1 couples magnetically to the antisymmetric field $B_{\mu\nu}$ and, thus, can be thought of as the magnetic dual of the fundamental string of type-II and heterotic theories. Its tension is

$$T_{NS} = \frac{1}{g_s^2 l_s^6}$$

The introduction of D-branes in a superstring theory has a major impact on the symmetries of the theory. First of all, the $\mathsf{SO}(1,9)$ Lorentz group is broken by the introduction of a Dp-brane to $\mathsf{SO}(1,p) \times \mathsf{SO}(9-p)$, i.e. to the Lorentz group in the world-volume of the Dp-brane times the (global) rotation group in the $9-p$ transverse directions. Secondly, half of the supersymmetry of spacetime is broken on the brane. To see this, remember that the thirty-two supercharges in the bulk of type-II superstring theory are arranged in Majorana–Weyl spinors.[1] In type-IIA we have a left-handed Majorana–Weyl spinor, Q_L, and a right-handed one, Q_R, while, in type-IIB superstring theory, the two Majorana–Weyl spinors have the same chirality. Now, open strings whose both endpoints lie on the same brane induce the reflection of right-movers to left-movers and vice versa at the boundary of the world-sheet. Correspondingly, one can see that a Dp-brane stretched in the hyperplane $(x^1, \ldots, x^p)$ preserves supercharges of the form $\epsilon_L Q_L + \epsilon_R Q_R$ with

$$\epsilon_L = \Gamma^0 \Gamma^1 \cdots \Gamma^p \epsilon_R$$

where Γs are the 32×32 Dirac matrices in ten dimensions. Therefore, the world-volume theory of a D-brane has half the number of supercharges of the superstring theory at whose spacetime it is introduced. Furthermore, two parallel D-branes of the same dimensionality preserve the same supercharges. If we introduce two D-branes of different dimensionality, then, in general, they break all the supersymmetry of spacetime. However, one can see that they can be arranged in such a way so as to preserve one-quarter of the spacetime supersymmetry. We will see an example of a system containing D4- and D6-branes in the next chapter. The generalization to larger numbers of D-branes is straightforward and has no surprises.

The low-energy world-volume theory of an infinite Dp-brane is a $(p+1)$-dimensional supersymmetric field theory with sixteen supercharges, which describes the dynamics of open

[1] In ten dimensions and, in general, in dimensions d such that $d \mod 8 = 2$, we can have spinors which are both Majorana and Weyl. (Note that this holds if we have only one time dimension.) Hence, the real dimension of the irreducible spinor representation of the Lorentz group is $2^{[d/2]-1}$, instead of $2^{[d/2]}$ if $d \mod 8 \neq 2$, where $[d/2]$ denotes the integer part of the division $d/2$. Therefore, a ten-dimensional superstring theory can have up to $\mathcal{N} = 2$ spacetime supersymmetry. Type-II superstring theories have $\mathcal{N} = 2$ supersymmetry, hence the II in their name, while the type-I and heterotic superstring theories have half of it.

strings whose both endpoints lie on the brane. At the massless level it contains a $(p+1)$-dimensional $\mathsf{U}(1)$ gauge field, $A^\mu(x)$, $9-p$ scalars, $X^I(x)$, which parametrize the transverse fluctuations of the brane, and fermions required by supersymmetry. ($\mu = 0, \ldots, p$, $I = p+1, \ldots, 9$ and x is a point on the brane.) The bosonic part of the low-energy world-volume action is[2]

$$S = \frac{1}{g^2} \int d^{p+1}x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{l_s^4} \partial_\mu X^I \partial^\mu X_I \right) \tag{4.1.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, obtained by dimensional reduction of $\mathcal{N} = 1$ $\mathsf{U}(1)$ gauge theory from ten to $p+1$ dimensions. The $\mathsf{U}(1)$ gauge coupling, g, can be shown to be related to the string coupling, g_s, and the fundamental string length, l_s, as

$$g = g_s l_s^{p-3}$$

At higher energies the interaction with the infinite tower of open-string massive states and that of closed strings of the bulk of spacetime has to be taken into account. However, we will mainly be interested in the supersymmetric gauge theory on the brane, for the study of which we have to decouple the gauge theory degrees of freedom from gravity and massive string modes. This is achieved in the limit $l_s \to 0$ with g kept fixed. In the case $p < 3$, keeping g fixed while $l_s \to 0$ means $g_s \to 0$. Then, we have a consistent theory on the Dp-brane, whose UV behavior is that of a supersymmetric $(p+1)$-dimensional gauge theory. On the contrary, if $p > 3$ then $g_s \to \infty$ if $l_s \to 0$, something that means that the description of the theory on the Dp-brane as a supersymmetric $(p+1)$-dimensional gauge theory is valid only in the IR. Finally, if $p = 3$, g is independent of l_s and, thus, in the $l_s \to 0$ limit we obtain a consistent $\mathcal{N} = 4$ $\mathsf{U}(1)$ supersymmetric gauge theory in four dimensions.

The transition to the non-Abelian case is achieved by placing N parallel Dp-branes close to each other. The fact that D-branes are BPS-saturated objects implies that parallel Dp-branes do not exert forces on each other. More precisely, in the case of parallel Dp-branes the attractive gravitational ($G_{\mu\nu}$) and dilaton (Φ) forces cancel against the repulsive electromagnetic force ($B_{\mu\nu}$). Therefore, a stack of N nearby parallel Dp-branes is a stable configuration which, as discussed before, preserves sixteen supercharges. Now, open strings whose endpoints lie on the Dp-branes give rise to a $(p+1)$-dimensional supersymmetric $\mathsf{U}(N)$ gauge theory at low energies. The bosonic part of the low-energy world-volume action is

$$S = \frac{1}{g^2} \int d^{p+1}x \, \mathrm{tr} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{l_s^4} D_\mu X^I D^\mu X_I - \frac{1}{l_s^8} [X^I, X^J]^2 \right)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ and $D_\mu X^I = \partial_\mu X^I - i[A_\mu, X^I]$. The $9-p$ transverse scalars, X^I, are now $N \times N$ matrices transforming in the adjoint of $\mathsf{U}(N)$, and so are the N^2 gauge fields. The off-diagonal elements of those matrices, (a, b) and (b, a), $a, b = 1, \ldots, N$ with $a \neq b$, arise from the two orientations of a fundamental string connecting the a-th and b-th brane. The N photons in the Cartan subalgebra of the algebra of $\mathsf{U}(N)$ and the diagonal components of the matrices X^I correspond to strings whose endpoints lie on the same brane. Again, supersymmetry unambiguously determines the fermions that will appear in our theory.

[2]Here we write down only the bosonic part of the action, since supersymmetry unambiguously defines the fermionic part.

Let us driefly discuss now the symmetry- and gauge-theory-related matters for the case of the NS5-brane. The inclusion of an NS5-brane in the spacetime of a superstring theory results in the breaking of the $\mathsf{SO}(1,9)$ Lorentz group to an $\mathsf{SO}(1,5)\times\mathsf{SO}(4)$ symmetry. As for supersymmetry, an infinite NS5-brane stretched in the hyperplane $(x^1,\ldots,x^5)$ preserves supercharges of the form $\epsilon_L Q_L+\epsilon_R Q_R$ with

$$\epsilon_L=\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\epsilon_L \text{ and } \epsilon_R=\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\epsilon_R$$

for the type-IIA NS5-brane and

$$\epsilon_L=\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\epsilon_L \text{ and } \epsilon_R=-\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\epsilon_R$$

for the type-IIB NS5-brane. As we observe the nonchiral type-IIA superstring theory gives rise to a six-dimensional chiral theory with $(2,0)$ supersymmetry in the world-volume of the NS5-brane, while from the chiral type-IIB superstring theory we obtain a six-dimensional nonchiral theory with $(1,1)$ supersymmetry in the world-volume of the NS5-brane. In any occasion, as in the case of D-branes, the world-volume theory of an NS5-brane has half the number of supercharges of the superstring theory at whose spacetime it is introduced.

The light fields that appear in the world-volume of a single type-IIA NS5-brane, namely a self-dual field, $B_{\mu\nu}$, five scalars and the corresponding fermions required by supersymmetry, belong to a tensor multiplet of $(2,0)$ supersymmetry. Four of the five scalars describe fluctuations of the type-IIA NS5-brane in the transverse directions, while the fifth lives on a circle of radius l_s, thus giving a hint for a hidden extra dimension. On the other hand, the world-volume of a single type-IIB NS5-brane has a six-dimensional gauge field, four scalars, describing fluctuations of the type-IIB NS5-brane in the transverse directions, and the corresponding fermions. The gauge coupling, g, in this case is given by

$$g^2=l_s^2$$

Finally let us describe what happens when we place N NS5-branes on top of each other. (This configuration is stable since NS5-branes are BPS-saturated objects.) If we have type-IIB NS5-branes, the low-energy theory is a six-dimensional $(1,1)$ supersymmetric $\mathsf{U}(N)$ gauge theory, arising from the ground states of D1-branes (D-strings) stretching between the type-IIB NS5-branes. In the case of type-IIA NS5-branes the low-energy theory is a nontrivial six-dimensional field theory with $(2,0)$ supersymmetry, arising by D2-branes (membranes) stretching between the type-IIA NS5-branes. (For details see [31].)

4.2 M-theory interpretation

The five different ten-dimensional superstring theories can be thought of as asymptotic expansions around different vacua of a single quantum theory [32]. This theory is known as M-theory and it is eleven-dimensional. Its low-energy limit is eleven-dimensional supergravity [33] and its only parameter is the Planck length, l_P. The physics is weakly coupled and well approximated by semiclassical supergravity for length scales much larger then l_P, and strongly coupled at scales smaller than l_P.

The spectrum of M-theory, besides the graviton G_{MN}, $M, N = 0, \ldots, 10$, includes a three-form potential, A_3. The theory possesses a membrane and a five-brane,[3] commonly referred to as M2- and M5-brane respectively. An Mp-brane, $p = 2$ or 5, has tension

$$T_p = \frac{1}{l_P^{p+1}}$$

and, when it is stretched in $(1 \ldots p)$, preserves supercharges ϵQ with[4]

$$\Gamma^0 \Gamma^1 \cdots \Gamma^p \epsilon = \epsilon$$

where Γs are the 32×32 Dirac matrices in eleven dimensions. Therefore, the number of supercharges preserved by the Mp-brane is sixteen.

The ten-dimensional type-IIA vacuum can be thought of as a compactification of M-theory on $\mathbb{R}^{1,9} \times S^1$, where S^1 is the circular direction (10) with radius R_{10}. The parameters of type-IIA superstring theory (l_s, g_s) are then related to those of M-theory (R_{10}, l_P) by

$$\frac{R_{10}}{l_P^3} = \frac{1}{l_s^2} \quad \text{and} \quad R_{10} = g_s l_s$$

Thus, the strong coupling limit of type-IIA theory, $g_s = R_{10}/l_s \to \infty$, is described by the eleven-dimensional Minkowski vacuum of M-theory.

Type-IIA branes have a natural interpretation in M-theory:

- A fundamental type-IIA string stretched, say, along (1), can be thought of as an M2-brane wrapped around (10) and (1). It is charged under the gauge field $B_{\mu 1} = A_{10 \mu 1}$.
- A D-particle corresponds to a Kaluza–Klein mode of the graviton carrying momentum R_{10}^{-1} along the compact direction and electric charge under $G_{\mu,10}$.
- A D2-brane corresponds to an M2-brane unwrapped around (10). It is charged under $A_{\lambda\mu\nu}$.
- A D4-brane corresponds to an M5-brane wrapped around (10). It is charged under $\tilde{A}_{10,\mu_1,\ldots,\mu_5}$.
- An NS5-brane corresponds to an M5-brane and is thus charged under $\tilde{A}_{\mu_1,\ldots,\mu_6}$.
- A D6-brane is a Kaluza–Klein monopole magnetically charged under the gauge field $A_\mu = G_{\mu 10}$.
- A D8-brane can is conjectured to correspond to an eight-dimensional M-theory brane.

[3] Since we are in eleven dimensions the magnetic dual of a 2-brane is a 5-brane.

[4] In eleven dimensions there are only Dirac and Majorana spinors. Majorana spinors carry an irreducible spinor representation of the Lorentz group of real dimension thirty-two. As a result, we can arrange the thirty-two supercharges of spacetime in one Majorana spinor and, hence, M-theory has $\mathcal{N} = 1$ spacetime supersymmetry.

Ten-dimensional type-IIB superstring theory has a complex coupling

$$\tau = a + \frac{i}{g_s}$$

where a is the vev of the massless RR scalar, C. The ten-dimensional type-IIB vacuum corresponds to M-theory compactified on a two-torus of complex structure τ and vanishing area. The theory appears to be nine-dimensional but, in fact, as the area of the torus goes to zero the wrapping modes of the M2-brane become light and give rise to a noncompact direction, x_B. If $a = 0$ the M-theory torus is rectangular with sides R_9 and R_{10}. Then, the mapping of the M-theory parameters (R_9, R_{10}, l_P) to those of type-IIB theory (R_B, g_s, l_s) is

$$\frac{R_{10}}{l_P^3} = \frac{1}{l_s^2}, \quad \frac{R_9}{l_P^3} = \frac{1}{g_s l_s^2} \quad \text{and} \quad \frac{R_9 R_{10}}{l_P^3} = \frac{1}{R_B}$$

Type-IIB branes have a natural interpretation in M-theory:

- A fundamental type-IIB string corresponds to an M2-brane wrapped around (10).
- A D-string wrapped around x^B corresponds to a Kaluza–Klein mode of the eleven-dimensional supergraviton carrying momentum in (10). An unwrapped (around x^B) D-string arises by an M2-brane wrapped around (9).
- A D3-brane wrapped around x^B corresponds to an M2-brane. An uwrapped D3-brane arises by an M5-brane wrapped around (9,10).
- A D5-brane wrapped around x^B arises by an M5-brane wrapped around (10). An unwrapped D5-brane arises by a Kaluza–Klein monopole charged under the gauge field $G_{\mu,10}$ and wrapped around (9).
- An NS5-brane wrapped around x^B arises by an M5-brane wrapped around (9). An unwrapped NS5-brane corresponds to a Kaluza–Klein monopole charged under the gauge field $G_{\mu,9}$ and wrapped around (10).
- A D7-brane wrapped around x^B arises by a Kaluza–Klein monopole charged under the gauge field $G_{\mu,10}$. An unwrapped D7-brane is related to the conjectured eight-dimensional M-theory brane.

4.3 $\mathcal{N} = 2$ $\mathsf{U}(N_c)$ supersymmetric gauge theory

As we saw in section 4.1 D-branes are defined with the property that open strings can end on them. However, it can be shown using dualities that this is not the only possibility; in fact branes can end on branes. This is a very interesting result and it opens new ways for realistic model-building. As an example consider the setup of Fig. 4.1: A type-IIA superstring theory configuration of two parallel NS5-branes a distance L_6 apart in the direction (6) stretched in the directions (12345) and at the same point in the directions (789), and N_c nearby parallel

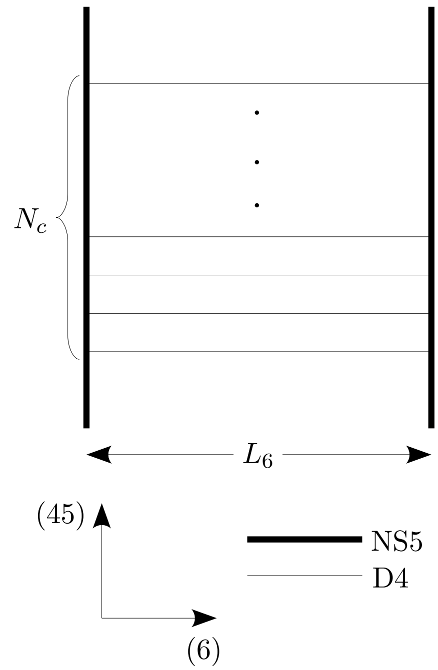


Fig. 4.1: Four-dimensional $\mathcal{N} = 2$ $\mathsf{U}(N_c)$ supersymmetric gauge theory without flavors

D4-branes stretched in the directions (1236), with both sides ending on the NS5-branes in the direction (6).

This configuration has global symmetry $\mathsf{SO}(1,3) \times \mathsf{SO}(3) \times \mathsf{SO}(2)$, where the $\mathsf{SO}(1,3)$ acts on (0123), the $\mathsf{SO}(3)$ on (789) and the the $\mathsf{SO}(2)$ on (45). Furthermore, the preserved supercharges are of the form $\epsilon_L Q_L + \epsilon_R Q_R$ subject to the conditions

$$\epsilon_L = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\epsilon_L \quad \text{and} \quad \epsilon_R = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\epsilon_R \tag{4.3.1}$$

from the NS5-branes and

$$\epsilon_L = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^6\epsilon_R \tag{4.3.2}$$

from the D4-branes. Conditions (4.3.1) cut in half the number of preserved supercharges and condition (4.3.2), which is independent of conditions (4.3.1), further reduces the number of conserved supercharges in half. Therefore, the configuration of Fig. 4.1 preserves one-quarter of the spacetime supersymmetry, i.e. eight supercharges. In four dimensions this corresponds to $\mathcal{N} = 2$ supersymmetry. However, we are not there yet, for we did not establish the existence of an R-symmetry, the ubiquitous automorphism of supersymmetry algebras (remember the discussion in section 1.6). In fact, the breaking of the $\mathsf{SO}(1,9)$ Lorentz group provides us with the global symmetry $\mathsf{SO}(3) \times \mathsf{SO}(2)$ which can be immediately identified with the $\mathsf{SU}(2)_R \times \mathsf{U}(1)_R = \mathsf{U}(2)_R$ symmetry of $\mathcal{N} = 2$ supersymmetry.

In general, for an observer on a Dp-brane, modes that live in the bulk of spacetime and modes that live on the higher-dimensional branes are nondynamical background degrees of freedom in the low-energy theory (at least in infinite volume). They are frozen at their classical values by infinite volume factors, as it can be seen by the example of a $\mathsf{U}(N)$ gauge field living on a D$(p+4)$-brane on which a Dp-brane is connected; its coupling on the $(p+1)$-dimensional world-volume of the Dp-brane, g_{p+1}, is related to its coupling in the $p+5$-dimensional world-volume of the D$(p+4)$-brane, g_{p+5}, by

$$\frac{1}{g_{p+1}^2} = \frac{V_{p+1,\ldots,p+4}}{g_{p+5}^2}$$

where $V_{p+1,\ldots,p+4}$ is the world-volume of the D$(p+4)$-brane transverse to that of the Dp-brane. Hence, from the point of view of the Dp-brane, the $\mathsf{U}(N)$ gauge symmetry of the D$(p+4)$-brane is a global symmetry. Of course, modes of strings stretched between the Dp- and D$(p+4)$-brane have to be included in the discussion of the low-energy theory on the Dp-brane.

Now, fundamental strings of type-IIA theory do not end on NS5-branes and, thus, the modes we encounter in the low-energy theory of the configuration of Fig. 4.1 are those of stretched strings ending on the N_c nearby D4-branes. Since the D4-branes are finite in (6), those modes are dynamical in four dimensions only.[5] Therefore, the configuration of Fig. 4.1 gives, at low-energy, a four-dimensional $\mathcal{N} = 2$ $\mathsf{U}(N_c)$ supersymmetric gauge theory in the four dimensions (0123). The gauge coupling of the five-dimensional gauge theory on the D4-branes is

$$g_{D4}^2 = g_s l_s$$

and, if we use Kaluza–Klein reduction in order to single out direction (6), we find that the gauge coupling of the four-dimensional $\mathcal{N} = 2$ $\mathsf{U}(N_c)$ supersymmetric gauge theory is

$$\frac{1}{g^2} = \frac{L_6}{g_s l_s}$$

So far in our discussion we uncovered only vector supermultiplets of $\mathcal{N} = 2$ supersymmetry in four dimensions. However, it is often the case that we want to obtain a low-energy theory which contains hypermultiplets as well. If we want them to transform in the fundamental representation of the gauge group, then the relevant configuration is shown in Fig. 4.2.

The addition is N_f semi-infinite D4-branes to the left of the leftmost NS5-brane. Supersymmetry is not affected by the introduction of this new set of branes. Open strings with one end on the N_c D4- and the other on the N_f D4-branes describe hypermultiplets transforming in the $\mathbf{N}_c$ of $\mathsf{U}(N_c)$. The position of each of the N_f D4-branes in (45), labeled by the complex number m_i, $i = 1, \ldots, N_f$, can be thought of as mass for the corresponding hypermultiplet.

Note here that the configuration of Fig. 4.2 is not enough to describe the complete dynamics of the low-energy $\mathcal{N} = 2$ gauge theory. More precisely, the configuration of Fig. 4.2 cannot describe the Higgs branch of the moduli space of the low-energy $\mathcal{N} = 2$ gauge theory. In order to obtain a full description of the moduli space one needs a set of N_f semi-infinite D4-branes, that is a set of D4-branes in (1236), each ending on the right to the leftmost NS5-brane and on the left to a different D6-brane stretched in (123789). We will see such a configuration in the next chapter.

[5]Note that there is also an infinite tower of modes which, however, we do not take into account in our analysis of the low-energy theory.

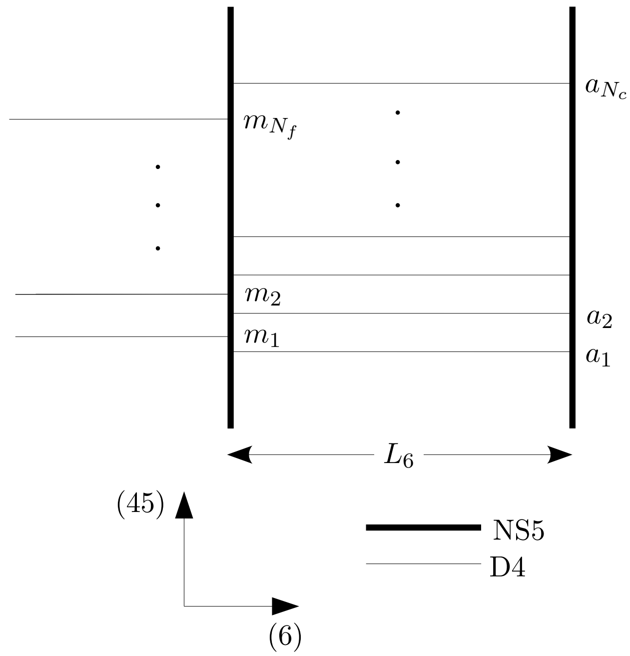


Fig. 4.2: Four-dimensional $\mathcal{N}=2$ $\mathsf{U}(N_c)$ supersymmetric gauge theory with N_f flavors

4.4 Quantum effects (pure $\mathcal{N}=2$)

The brane configuration depicted in Fig. 4.1 gives rise to the classical $\mathcal{N}=2$ gauge-theory dynamics. In general, loop (quantum) effects are incorporated by lifting the classical brane configurations to M-theory, in which case we interpret the superstring-theory branes as M-theory branes. For finite g_s the type-IIA superstring theory becomes eleven-dimensional at small distances. The eleventh dimension is circular and its radius is $R_{10}=g_s l_s$. In the case of Fig. 4.1, D4-branes stretched between NS5-branes are interpreted as a single M5-brane with a curved world-volume [34]. Since all type-IIA branes are stretched in (123) and sit at a single point in (789), the world-volume of the M5-brane is $\mathbb{R}^{1,3}\times\Sigma$, with Σ a two-dimensional surface embedded in the four-dimensional space $Q=\mathbb{R}^3\times S^1$, where (456) span $\mathbb{R}^3$ and (10) S^1. Evidently, our goal is to find the exact shape of Σ.

If we parametrize Q with the complex coordinates

$$s=x^6+ix^{10} \quad\text{and}\quad v=x^4+ix^5$$

then, in the classical type-IIA limit, the D4-branes correspond to constant v, while the NS5-branes to constant s. Therefore, placing two NS5-branes at s_1, s_2 and N_c D4-branes at $v_1,\ldots,v_{N_c}$ results in a classical complex curve, Σ_{cl}, described by

$$(s-s_1)(s-s_2)\prod_{a=1}^{N_c}(v-v_a)=0, \quad \mathrm{Re}s_1\leq\mathrm{Re}s\leq\mathrm{Re}s_2$$

Σ_{cl} is a singular surface with different components which meet at the singular points $s = s_i$, $i = 1, 2$, and $v = v_a$, $a = 1, \ldots, N_c$.

To determine the shape of the smooth complex curve Σ we can consider its large-v asymptotics. At this point it is important to recognize that, in the quantum theory, the end of a brane ending on another brane looks like a charged object in the world-volume theory of the latter. Consider for example a fundamental string ending on a Dp-brane. In the world-volume of the Dp-brane the endpoint of the string provides a point-like source for a Dp-brane world-volume gauge field with Coulomb potential

$$A_0 = \frac{Q}{r^{p-2}}$$

where Q is the charge provided by the endpoint of the string in the world-volume of the Dp-brane and r the distance from the endpoint on the Dp-brane,

$$r = \sqrt{(x^1)^2 + \cdots + (x^p)^2}$$

In that case, minimizing the action of the Dp-brane world-volume theory[6] would require a scalar, say X^{p+1}, to satisfy the p-dimensional equation

$$\left(\frac{\partial^2}{\partial (x^1)^2} + \cdots + \frac{\partial^2}{\partial (x^p)^2} \right) X^{p+1} = Q\delta(r)$$

in the large-v limit. For $p \neq 2$ the last equation is satisfied if

$$X^{p+1} = \frac{Q l_s^2}{r^{p-2}} \tag{4.4.1}$$

and, therefore, the string bends the Dp-brane: Its position at large v becomes r-dependent, $X^{p+1} = X^{p+1}(r)$, approaching the classical value $X^{p+1} = 0$ at large r (for $p > 2$). Likewise, in the quantum theory a brane is bent according to equation (4.4.1) when another brane ends on it, where p is now the codimension of the intersection of the first brane and r the p-dimensional distance from the end of the second brane in the world-volume of the first.

In our case $p = 2$ (five dimensions from the NS5-brane minus three dimensions from the intersection with the D4-brane in (123)) and the Laplace equation we obtain in the large-v limit is the two-dimensional one. Its solution is not of the form of equation (4.4.1), but of the form

$$X^6 = l_s g_s \sum_{i=1}^{q_L} \ln |v - a_i| - l_s g_s \sum_{i=1}^{q_R} \ln |v - b_i| \tag{4.4.2}$$

which describes q_L D4-branes ending on the NS5-brane from the left at the points $v = a_1, \ldots, a_{q_L}$ and q_R D4-branes ending on the NS5-brane from the right at $v = b_1, \ldots, b_{q_R}$. However, from the point of view of the four-dimensional $\mathcal{N} = 2$ supersymmetry, X^6 is the real part of a complex scalar field which belongs to a vector multiplet. The imaginary part

[6] This is equivalent to preserving the original supersymmetry in the world-volume theory with the added gauge field.

of that complex scalar field is a real scalar field propagating on the NS5-brane. But this can only be X^{10} and, thus, equation (4.4.2) must be generalized to the holomorphic equation

$$s = R_{10} \sum_{i=1}^{q_L} \ln(v - a_i) - R_{10} \sum_{i=1}^{q_R} \ln(v - b_i) \tag{4.4.3}$$

with the real part being equation (4.4.2) and the imaginary part indicating that x^{10} jumps by $\pm 2\pi R_{10}$ when we circle a_i or b_i in the complex v-plane. Defining

$$t = e^{-s/R_{10}}$$

we avoid the multi-valuedness induced to s by the compact dimension x^{10} and we can write equation (4.4.3) as

$$t = \frac{\prod_{i=1}^{q_R} \ln(v - b_i)}{\prod_{i=1}^{q_L} \ln(v - a_i)}$$

Using the variables t and v we are now in position to determine the shape of Σ. Supersymmetry requires Σ to be a holomorphic curve in the two-complex-dimensional space Q labeled by t and v. Therefore, we need one condition on t and v in order to define it,

$$F(t, v) = 0$$

for some function F. Now, viewing F as a function of t for large v we have to see two branches,

$$t_1 = v^{N_c} \text{ and } t_2 = v^{-N_c}$$

corresponding to the two NS5-branes. Therefore, Σ should be described by setting to zero a polynomial of second degree in t,

$$A(v)t^2 + B(v)t + C(v) = 0 \tag{4.4.4}$$

where A, B and C are polynomials of degree N_c in v. A root of $A(v)$ means that $t \to \infty$, something that happens only when $x^6 \to -\infty$. This indicates that roots of $A(v)$ correspond to locations of semi-infinite D4-branes to the left of the leftmost NS5-brane in Fig. 4.1. Such a brane does not exist in the classical configuration of Fig. 4.1 and, thus, $A(v)$ should have no roots in the first place, i.e. it should be a constant. Furthermore, a root of $C(v)$ means that $t \to 0$ or, equivalently, $x^6 \to \infty$, and this indicates that the root corresponds to the location of a semi-infinite D4-brane to the right of the rightmost NS5-brane. But such a brane again does not exist and, hence, $C(v)$ should be a constant as well. Since v and t^{1/N_c} scale like energy, we can, consistently with dimensional analysis, choose $A(v) = 1$ and $C(v) = \Lambda_{\mathcal{N}=2}^{2N_c}$, where $\Lambda_{\mathcal{N}=2}$ is the dynamically generated scale of the four-dimensional $\mathcal{N} = 2$ $\mathsf{SU}(N_c)$ supersymmetric gauge theory. Then, equation (4.4.4) becomes

$$t^2 + B(v)t + \Lambda_{\mathcal{N}=2}^{2N_c} = 0$$

Now, once the polynomial

$$B(v) = \lambda(v - v_0) \cdots (v - v_{N_c - 1}), \quad \lambda \neq 0$$

is expanded, it contains the term $-\lambda v^{N_c-1}(v_0 + \cdots + v_{N_c-1})$ which can be canceled by an appropriate shift of v:

$$v \to v + \frac{v_0 + \cdots + v_{N_c-1}}{N_c}$$

If in addition we absorb λ in a rescaling of v we obtain

$$B(v) = v^{N_c} + u_2 v^{N_c-2} + u_3 v^{N_c-3} + \cdots + u_{N_c}$$

where $u_2, \ldots, u_{N_c}$ are constants parameterizing the polynomial B.

At this point we have found the exact shape of Σ. It depends on $N_c - 1$ complex numbers, the constants $u_2, \ldots, u_{N_c}$. Interestingly, we have $N_c - 1$ moduli, something that is due to the fact that the $\mathsf{U}(1)$ factor of $\mathsf{U}(N_c)$ has a vanishing coupling and, thus, its gauge field appears frozen at its classical value. Consequently, once the brane configuration of Fig. 4.1 is lifted to M-theory, the resulting low-energy quantum theory is a four-dimensional $\mathcal{N} = 2$ $\mathsf{SU}(N_c)$ supersymmetric gauge theory.

Let us summarize what we have done so far. First of all we established in the previous section the existence of a one-to-one correspondence between configurations of D4-branes stretched between NS5-branes and vacua of classical four-dimensional $\mathcal{N} = 2$ $\mathsf{U}(N_c)$ supersymmetric gauge theory. Then, in this section, we established a one-to-one correspondence between vacua of the quantum four-dimensional $\mathcal{N} = 2$ $\mathsf{SU}(N_c)$ supersymmetric gauge theory and supersymmetric configurations of an M5-brane with world-volume $\mathbb{R}^{1,3} \times \Sigma$, with Σ described above. The classical limit of the quantum theory corresponds to $x^{10} \to 0$, i.e. to the limit where the extra dimension of M-theory vanishes.

CHAPTER 5

SQCD from brane configurations

Contents

In the previous chapter we presented brane configurations that preserve eight supercharges and, thus, describe $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions. In this chapter we will further break supersymmetry. More specifically, we will consider brane configurations in type-IIA superstring theory that preserve four supercharges. These configurations describe classical $\mathcal{N} = 1$ supersymmetric gauge theory in four dimensions. Next, the lift to M-theory will be performed in order to obtain the full quantum $\mathcal{N} = 1$ supersymmetric gauge theory. Again the reader is referred to [31] for a general and detailed exposition of the subject.

5.1 Classical $\mathcal{N} = 1$ SQCD

In order to describe $\mathcal{N} = 1$ $\mathsf{U}(N_c)$ supersymmetric gauge theory in four dimensions we consider the type-IIA configuration of Fig. 5.1: An NS5-brane stretched in (12345), a rotated NS5-brane, denoted NS5$'$, stretched in (12389), N_c D4-branes stretched in (123) and of length L_6 in (6) (they lie between NS5 and NS5$'$) and N_f D4-branes stretched in (123) to the left of the NS5-brane and each ending in (6) on a different D6-brane stretched in (123789).

The N_c D4-branes preserve supercharges of the form $\epsilon_L Q_L + \epsilon_R Q_R$ with

$$\epsilon_L = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^6\epsilon_R$$

and so do the N_f D4-branes. The NS5- and NS5′-brane preserve supercharges of the form $\epsilon_L Q_L + \epsilon_R Q_R$ with

$$\epsilon_L = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\epsilon_L \quad \text{and} \quad \epsilon_R = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\epsilon_R$$

and

$$\epsilon_L = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^8\Gamma^9\epsilon_L \quad \text{and} \quad \epsilon_R = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^8\Gamma^9\epsilon_R$$

respectively. Finally, the D6-branes preserve supercharges of the same form but with

$$\epsilon_L = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^7\Gamma^8\Gamma^9\epsilon_R$$

Hence, we have four independent conditions, each of which cuts in half the number of pre-

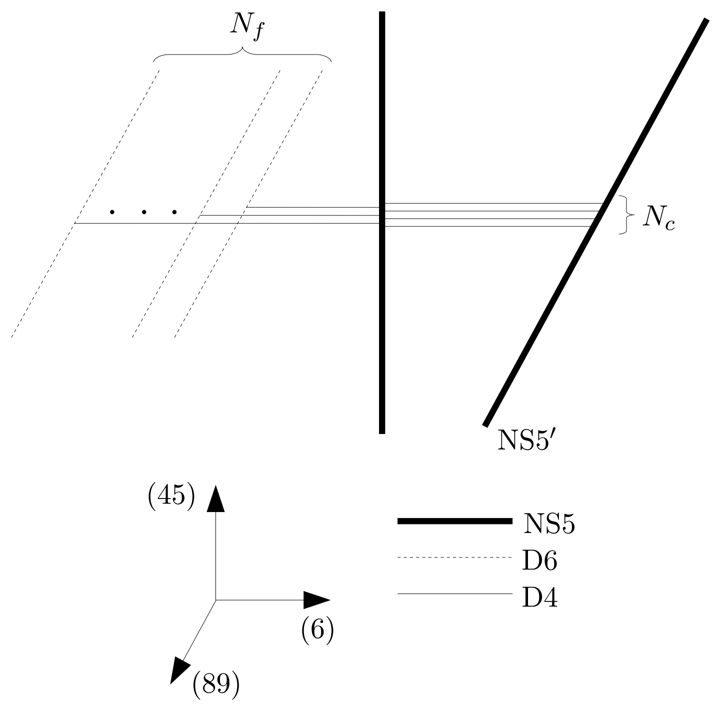


Fig. 5.1: Four-dimensional $\mathcal{N} = 1$ $\mathsf{U}(N_c)$ supersymmetric gauge theory with N_f massless flavors

served supercharges. Therefore, the configuration of Fig. 5.1 preserves four supercharges (one-eighth of the thirty-two supercharges of type-IIA theory) corresponding to $\mathcal{N} = 1$ supersymmetry in four dimensions.[1] Note that if $0 < N_f < N_c$, then we can fully describe the dynamics of the low-energy gauge theory by attaching N_f semi-infinite D4-branes to the left of the leftmost NS5-brane. However, the extra directions of the moduli space we found in subsection 3.2.2 in the case $N_f > N_c$ can be described only if we add the D6-branes.

[1] We should mention here the s-rule of brane dynamics [35]: A configuration in which an NS5-brane is connected to a D6-brane by more than one D4-branes is not supersymmetric.

There is another configuration of type-IIA branes which is equivalent to the configuration of Fig. 5.1 and is related to it by a series of Hanany–Witten transitions [35]. If we move a D6-brane to the right in Fig. 5.1, there will be a value of x^6 where it will coincide with the NS5-brane, for their orientation is such that they cannot avoid each other. As they approach, the D4-brane that connects them becomes very short in (6) and, in fact, disappears when they cross.

Therefore, when the D6-brane is in the right of the NS5-brane there is no D4-brane connecting them. Conversely, if the D6- and NS5-brane that approach each other are not connected by a D4-brane, then one is created once they cross and switch positions. The phenomenon that takes place once a D6- and an NS5-brane switch positions in the manner described above is called a Hanany–Witten transition. The resulting configuration is shown in Fig. 5.2.

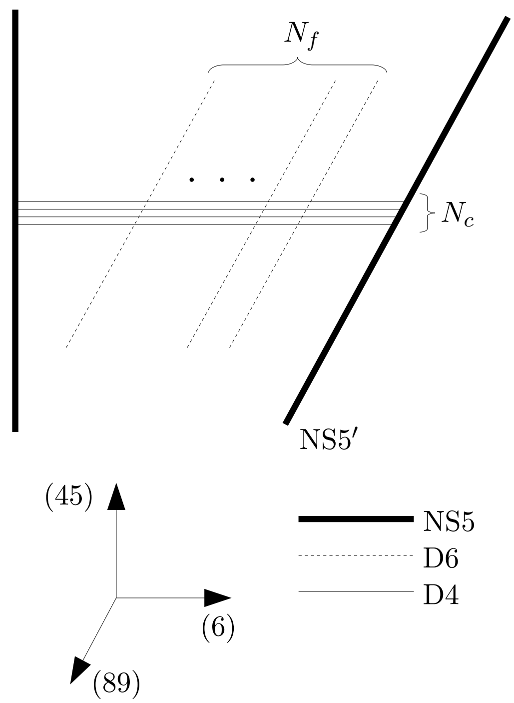


Fig. 5.2: Four-dimensional $\mathcal{N} = 1$ $\mathsf{U}(N_c)$ supersymmetric gauge theory with N_f massless flavors, obtained by the configuration of Fig. 5.1 after N_f Hanany–Witten transitions

These configurations describe the dynamics of classical $\mathcal{N} = 1$ $\mathsf{U}(N_c)$ supersymmetric gauge theory in four dimensions. In the quantum theory the Abelian factor of $\mathsf{U}(N_c)$ has vanishing coupling and decouples, leaving us exactly with the $\mathcal{N} = 1$ SQCD theory described in section 3.1 (without mass terms for the squarks).

The color D4- and the D6-branes of Fig. 5.1 sit at the same point in (45) and so do the D6-branes and the NS5′-brane in Fig. 5.2. Masses for the squarks are then introduced by relative displacements of the D6- and D4-branes (Fig. 5.1) or, equivalently, D6-branes and

NS5′-brane (Fig. 5.2) in (45). The resulting $N_f \times N_f$ mass matrix m satisfies the constraint

$$[m, m^\dagger] = 0$$

and, hence, m and $m^\dagger$ can be diagolized simultaneously. The locations of the D6-branes in the v-plane (remember $v = x^4 + ix^5$) are the eigenvalues of m.

5.2 Quantum $\mathcal{N} = 1$ SQCD

We will start the treatment of quantum effects by considering pure $\mathcal{N} = 1$ $\mathsf{U}(N_c)$ supersymmetric gauge theory, i.e. by lifting to M-theory the configuration of Fig. 5.2 (without the D6-branes) at nonzero g_s [36], [37]. As in the case of $\mathcal{N} = 2$ supersymmetry the interpretation is that of a single M5-brane whose world-volume is $\mathbb{R}^{1,3} \times \Sigma$. However, Σ is now a complex curve embedded in the three-complex-dimensional space $\mathbb{R}^5 \times S^1$ parametrized by

$$v = x^4 + ix^5, \quad w = x^8 + ix^9 \quad \text{and} \quad s = x^6 + ix^{10}$$

The extra complex dimension compared to the $\mathcal{N} = 2$ case arises because one of the NS5-branes is now rotated.

The shape of Σ can be determined by studying the asymptotics of v and w. Defining

$$t = e^{-s/R_{10}}$$

as in the $\mathcal{N} = 2$ case, we know that as we approach the region of the NS5-brane, i.e. as $v \to \infty$ and $w \to 0$ on Σ, it is $t = v^{N_c}$. On the other hand, as we approach the region of the NS5′-brane, i.e. as $v \to 0$ and $w \to \infty$ on Σ, it is $t = w^{-N_c}$. More generally, t should be a function of t without poles or zeros except at $v = 0$, which is $w = \infty$, and $v = \infty$. The unique solution to all the constraints is

$$v^{N_c} = t, \quad w^{N_c} = \zeta^{N_c} t^{-1} \quad \text{and} \quad vw = \zeta \tag{5.2.1}$$

where ζ is an undetermined constant. Of course, this is a redundant description of Σ. Now, if we start with the configuration of Fig. 4.1 and rotate the rightmost NS5-brane appropriately, then we end up with the configuration of Fig. 5.2 without the D6-branes. Therefore, the shape of Σ in the $\mathcal{N} = 1$ case can be obtained by a flow from the $\mathcal{N} = 2$ case. With an explicit calculation we find again equations (5.2.1) with $\zeta = N_c \Lambda^3$, where Λ is the scale of pure four-dimensional $\mathcal{N} = 1$ $\mathsf{U}(N_c)$ supersymmetric gauge theory. Again, the $\mathsf{U}(1)$ factor of $\mathsf{U}(N_c)$ decouples and, hence, we obtain quantum pure $\mathcal{N} = 1$ SQCD.

In the case where flavors are included the results are rather different [36]. More specifically, the inclusion of $0 < N_f < N_c$ massless flavors, described adequately by the addition of N_f semi-infinite D4-branes to the left of the NS5-brane, leads to a singular complex curve Σ, infinitely elongated in (6). Therefore, the corresponding brane configuration does not describe a four-dimensional field theory. This is consistent with the field theory analysis of subsection 3.3.1, where we found that the dynamically generated superpotential (3.3.7) results in a quantum theory without a vacuum. However, masses for the quarks lift the singularity of

Σ and, in accordance with the purely field-theoretic analysis of subsection 3.3.1, lead to the description of a well-behaving field theory. More specifically, if the i-th quark flavor has mass m_i, $i = 1, \ldots, N_f$, then Σ is described by

$$v^{N_c} = t \prod_{i=1}^{N_f} \left(1 - \frac{v}{m_i}\right) \quad \text{and} \quad vw = \zeta$$

where

$$\zeta^{N_c} = \Lambda^{3N_c - N_f} \prod_{i=1}^{N_f} m_i$$

The inclusion of $N_f \geq N_c$ massless flavors can be studied by lifting to M-theory the configuration of Fig. 5.2. In this case the corresponding field theory has a quantum moduli space of vacua parametrized by vevs of mesons and baryons (see section 3.3). In the brane picture the new ingredient [38] is that the presence of the D6-branes results in a complex curve Σ which is embedded not in $\mathbb{R}^5 \times S^1$ but in $\mathrm{TN} \times \mathbb{R}^2$, where TN is the Taub-NUT space with asymptotic radius R_{10} and charge N_f, parametrized by $(\mathbf{r}, x^{10}) = (x^4, x^5, x^6, x^{10})$ with metric

$$ds^2 = V d\mathbf{r}^2 + V^{-1}(dx^{10} + \boldsymbol{\omega} \cdot d\mathbf{r})^2$$

where

$$V = 1 + \frac{N_f R_{10}}{r} \quad \text{and} \quad \boldsymbol{\nabla} \times \boldsymbol{\omega} = \boldsymbol{\nabla} V$$

and $\mathbb{R}^2$ is parametrized by x^8 and x^9. The complex curve describing the baryonic branch of the quantum moduli space in this case splits into two components,

$$\Sigma_L : \quad t = v^{N_c - N_f} \quad \text{and} \quad w = 0$$

and

$$\Sigma_R : \quad t = \Lambda^{3N_c - N_f} w^{-N_c} \quad \text{and} \quad v = 0$$

The addition of masses in the last case is postponed until section 5.4. There, we will discuss the possibility of using brane configurations to describe massive SQCD in the free magnetic range, the theory at which we found metastable nonsupersymmetric vacua in section 3.5. The piece that is missing, however, is the realization of Seiberg duality in the brane picture, something to which we now turn.

5.3 Seiberg duality in the brane picture

The magnetic brane setup describing four-dimensional $\mathcal{N} = 1$ $\mathsf{U}(N)$ supersymmetric gauge theory with N_f massless flavors is shown in Fig. 5.3.

As in the case of the electric theory, the gauge bosons come from strings stretched between the color D4-branes. Likewise, the N_f flavors of magnetic quarks, q_i and $\tilde{q}^{\tilde{i}}$, $i, \tilde{i} = 1, \ldots, N_f$, arise from strings connecting color D4-branes and flavor D4-branes. However, in this case there are additional modes, namely the magnetic mesons, denoted $(M_m)^i_{\tilde{i}}$, coming from strings

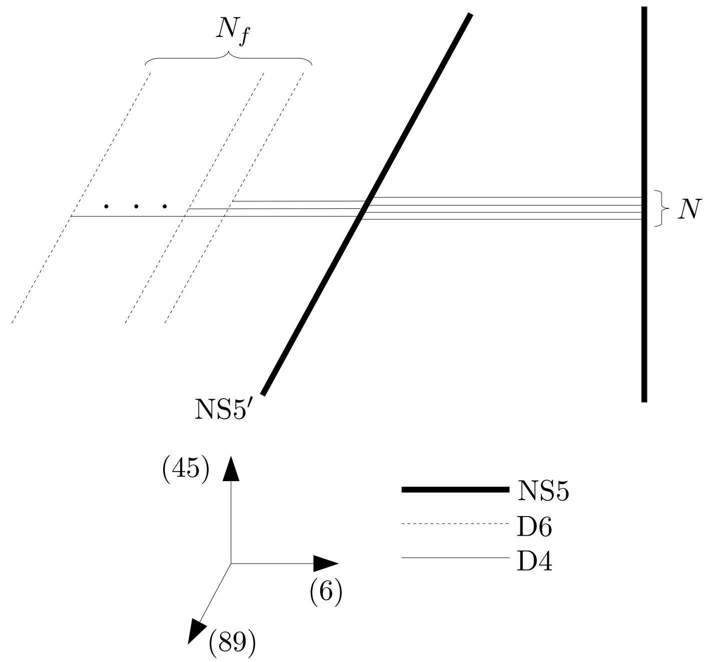


Fig. 5.3: The magnetic description of four-dimensional $\mathcal{N} = 1$ $\mathsf{U}(N)$ supersymmetric gauge theory with N_f massless flavors

stretched between flavor branes. These are singlets under the gauge group and we can see that the standard coupling of three open strings gives rise to the superpotential (3.4.1),

$$W = (M_m)^i_{\tilde{i}} q_i \tilde{q}^{\tilde{i}} = \frac{1}{\hat{\Lambda}} M^i_{\tilde{i}} q_i \tilde{q}^{\tilde{i}}$$

which couples magnetic mesons and magnetic quarks. Therefore, this is exactly the magnetic theory discussed in section 3.4.

Since we found the relevant description of the magnetic theory in terms of branes, it only remains to connect it to the electric description. In other words, in order to uncover Seiberg duality we have to connect the electric description of four-dimensional $\mathcal{N} = 1$ $\mathsf{U}(N_c)$ supersymmetric gauge theory with N_f massless flavors to the magnetic description of four-dimensional $\mathcal{N} = 1$ $\mathsf{U}(N_f - N_c)$ supersymmetric gauge theory with N_f massless flavors. Note, here, that although Seiberg duality in field theory involves the special unitary gauge groups $\mathsf{SU}(N_c)$ and $\mathsf{SU}(N_f - N_c)$, the generalization to the unitary gauge groups $\mathsf{U}(N_c)$ and $\mathsf{U}(N_f - N_c)$ is straightforward and achieved by gauging the baryon number symmetry, $\mathsf{U}(1)_B$. The required procedure is described in Fig. 5.4 and, obviously, it is valid only when $N_f \geq N_c$.

Starting from the configuration of Fig. 5.1 we connect the N_c color D4-branes to N_c of the N_f flavor branes. The result is shown in Fig. 5.4(a) and, in order to reach the magnetic description, we have to pass the NS5-brane to the right of the NS5′-brane avoiding their meeting in space. This can be done by turning on a Fayet–Iliopoulos term in the gauge theory, something we can do since the gauge group $\mathsf{U}(N_c)$ has an Abelian factor. In the

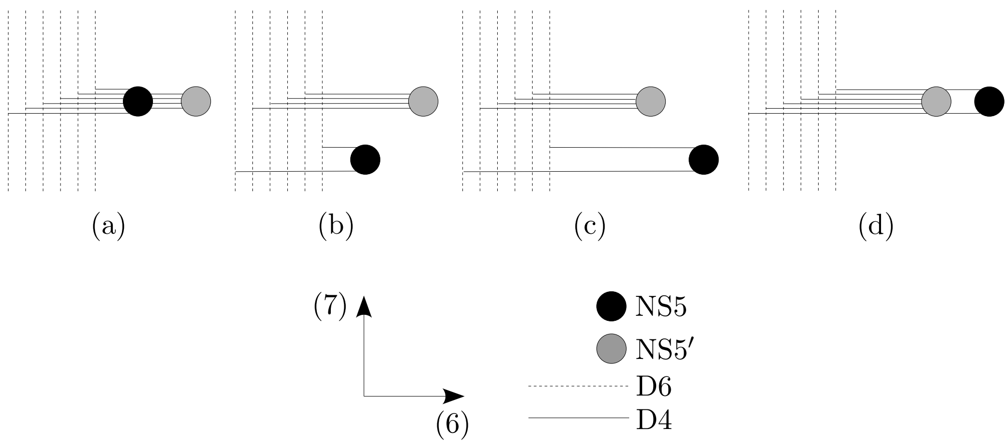


Fig. 5.4: Continuous connection of electric and magnetic brane configurations

brane picture a Fayet–Iliopoulos term corresponds to moving the NS5-brane and its attached D4-branes in (7). Then, we reach the configuration of Fig. 5.4(b), in which we can be immediately exchange the NS5- and NS5′ branes in (6), thus arriving in the configuration of Fig. 5.4(c). Finally, turning off the Fayet–iliopoulos term we obtain the configuration of Fig. 5.4(d) which contains $N_f - N_c$ branes connecting the NS5- to the NS5′ brane. This is exactly the configuration of Fig. 5.3 with $N = N_f - N_c$.

The above procedure shows that the classical moduli space of vacua of the electric theory with gauge group $\mathsf{U}(N_c)$ and N_f massless quark flavors and the classical moduli space of vacua of the magnetic theory with gauge group $\mathsf{U}(N_f - N_c)$ and the same number of quark flavors can be thought of as providing different descriptions of a single classical moduli space of supersymmetric brane configurations.

However, we have to underline the fact that a Fayet–Iliopoulos term causes the complete breaking of the gauge groups. Therefore, the duality actually appears at the classical level, since, at that level, we do not have strong infrared gauge dynamics. As the gauge symmetry is restored, i.e. as we turn off the Fayet–Iliopoulos term, we find a discrepancy. More specifically, in the electric theory nothing special happens since, although the restoration of the gauge symmetry results in additional massless degrees of freedom, the moduli space does not develop new branches we can access. On the contrary, in the magnetic theory, turning off the Fayet–Iliopoulos term causes an enlargement of the moduli space. Consequently, we cannot really rely on the classical "Seiberg duality" we have found. In order to gain a deeper understanding we have to study the quantum dynamics. In fact, quantum mechanically the jump in the dimension of the magnetic moduli space disappears and, hence, the quantum moduli space of electric SQCD with gauge group $\mathsf{SU}(N_c)$ and N_f quark flavors and that of magnetic SQCD with gauge group $\mathsf{SU}(N_f - N_c)$ and the same number of quark flavors can be thought of as providing different descriptions of a single quantum moduli space.

5.4 Metastable vacua in the brane picture

After the discovery of metastable supersymmetry-breaking vacua in massive SQCD, a type-

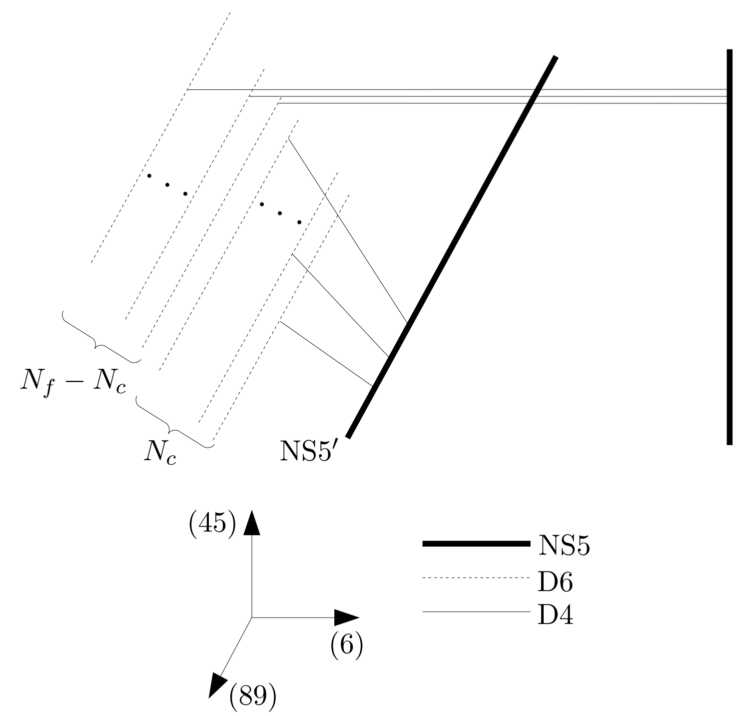


Fig. 5.5: The nonsupersymmetric magnetic brane configuration

IIA brane configuration that reproduces the observed phenomena was found [39], [40], [41]. The supersymmetry breaking vacuum in the magnetic theory with superpotential (3.5.6) is obtained by the brane configuration of Fig. 5.5.

The fact that the D4-branes are not parallel makes the nonsupersymmetric nature of this configuration explicit. Furthermore, as we mentioned in the end of section 5.1, masses for the quark flavors correspond to different positions of the D6-branes in (45). Now, the N_c D4-branes are stretched between the NS5′-brane and the D6-branes that are closest to the origin of (45), i.e. to those that correspond to the N_c smallest flavor masses. The rest of the D4-branes are stretched between the NS5-brane and the furthest D6-branes. This choice is justified if we observe that the position of the latter set of D4-branes in (45) is related to the vevs of the magnetic quarks. Then, if the vev of a magnetic quark is given by one of the N_c smaller masses, we encounter the situation of one D6-brane on which two D4-brane pieces coincide. But the fact that the two D4-branes are not supersymmetric with respect to each other results in the appearance of an open string tachyon at the intersection.

As for the electric theory, moving the flavor D4-branes of Fig. 5.1 in (45) does not break supersymmetry if we keep them parallel (Fig. 5.6(a)). The nonsupersymmetric vacuum in the electric description is shown in Fig. 5.6(b), which is obtained as the configuration that under (classical) Seiberg duality gives that of Fig. 5.5.

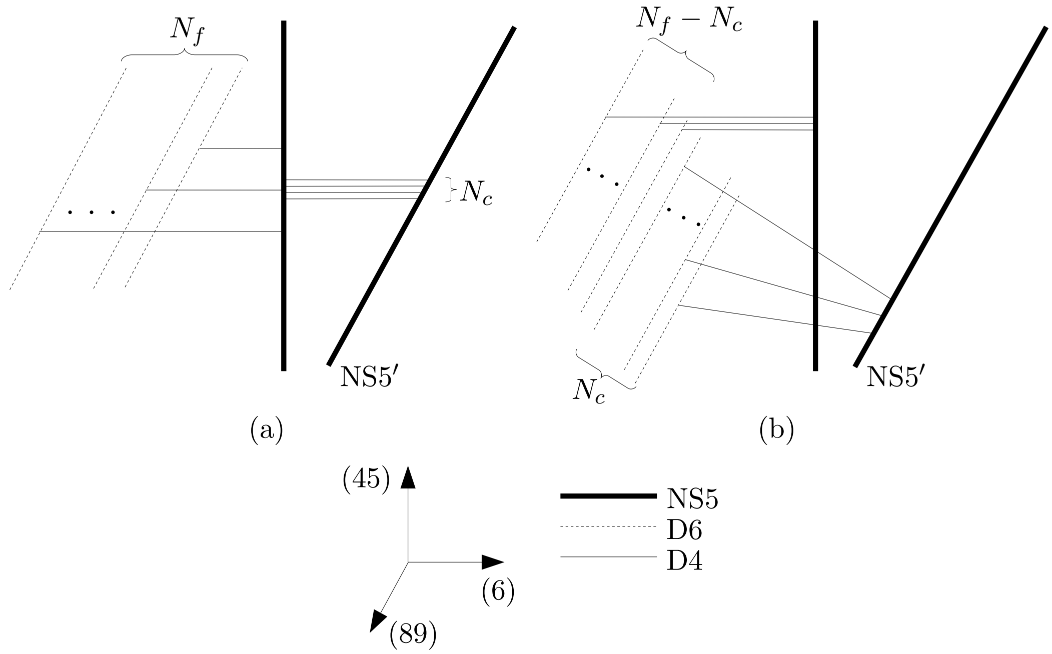


Fig. 5.6: The supersymmetric (a) and nonsupersymmetric (b) electric brane configurations

At this point we have to lift the configurations we found to M-theory in order to be able to explore the full quantum dynamics of the low-energy theory. What we have done so far is just the classical approximation, $g_s \to 0$, and, of course, this is far from a satisfactory treatment. More specifically, the effects of nonzero g_s have to be studied, since those will reproduce the effects of quantum mechanics in the low-energy theory.

But in an attempt for an M-theory lift we find an obstruction due to the phenomenon of brane bending (see section 4.4) [41]. The crucial observation is that the proper way to define a theory on the branes at $g_s \neq 0$ is not in terms of the detailed positions of the branes but, instead, in terms of the asymptotic behavior of the branes that stretch to infinity. These are the boundary conditions on the system, while the branes in the interior are dynamical and free to adjust themselves. Now, the supersymmetric brane configuration of Fig. 5.6(a) gives a specific set of boundary conditions and, of course, any state of this theory, stable or metastable, must have exactly this set of boundary conditions at infinity. Therefore, we expect the M-theory lift of the configuration of Fig. 5.6(b) to have exactly the same boundary conditions as the M-theory lift of the configuration of Fig. 5.6(a). In the opposite case the metastable nonsupersymmetric vacuum it describes is not a state of the system where the supersymmetric vacuum belongs.

In fact, the relevant calculations were carried out in [41] and it was found that the boundary conditions do not match. Thus, the supersymmetric and nonsupersymmetric vacua do not belong to the same theory. Actually, a very small g_s causes infinite deviation from the SQCD limit at infinity and this means that the Hilbert space of states is drastically altered.

Therefore, we observe that the qualitative nonsupersymmetric features of SQCD, like the metastable vacuum of section 3.5, cannot be reproduced by the brane description of this theory.

Conclusions and outlook

The discovery of dynamical supersymmetry breaking in metastable vacua in [29] was followed by a great deal of excitement in the high-energy-physics scientific community. The phenomenon immediately seemed to be generic and, indeed, soon after the appearance of [29], many models where metastable supersymmetry breaking occured were found. Furthermore, it was recently proved that perturbed Seiberg–Witten theories, that is $\mathcal{N} = 1$ theories obtained by perturbing $\mathcal{N} = 2$ theories with some superpotential, contain nonsupersymmetric metastable vacua [42]. All these results show that there is a large class of field theories which experience metastable supersymmetry breaking for some ranges of their parameters.

Based on the abovementioned results, metastability attracted great attention in model-building. In fact, metastability simplifies model-building since, in many examples, supersymmetric vacua are unavoidable as can be seen by arguments based on the Witten index (see section 2.5) and the so-called Nelson–Seiberg theorem [19]. Therefore, the only way to break supersymmetry spontaneously is in a long-lived metastable vacuum. Then, we acquire much more flexibility in constructing meaningful physical theories and, thus, we can hope to find a model which incorporates metastability and, at the same time, correctly describes our nonsupersymmetric universe. (For some recent attempts in that direction see [43] and references therein.)

We should stress here that, for cosmological arguments, it is very important that we found a large moduli space of nonsupersymmetric vacua in section 3.5. Indeed, as the energy of the universe decreases, the existence of a very large configuration of nonsupersymmetric vacua makes it much more likely that the universe will end up there, rather than in one of the isolated N_c supersymmetric vacua further away in field space. Hence, it seems that cosmology is not incompatible with our results so far.

Another important question which arose after the appearance of [29] pertains to the existence of metastable nonsupersymmetric vacua in string/M-theory. In chapter 5 we saw a particular approach failing to do so, but there is intensive research and several other directions in search for metastability in string theory. A very alluring approach is offered by engineering quiver gauge theories with several interesting features by studying D-branes at a simple Calabi–Yau singularity [44], [45].

Metastable supersymmetry-breaking vacua would have a large impact on the landscape of string theory. Actually, the landscape would be enhanced by theories with metastable vacua.

Indeed, theories which have supersymmetric vacua would have to be taken into account, since nothing precludes them from having nonsupersymmetric vacua as well.

Concluding, it is worth underlining again the fact that nonsupersymmetric metastable vacua appear to be ubiquitous in supersymmetric field theories and, furthermore, once they are utilized we gain greater flexibility in realistic model-building. The search for metastability in string theory is widely believed to give positive results, and may even lead to new and unexpected advances in string theory itself. Therefore, it is likely that important and beautiful results still remain uncovered.

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