

# Knots, Anyons & the Jones Polynomial

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# Summary

This thesis is divided in two parts that treat different subjects, knot theory and anyons. The aim is to give an introduction to the two subjects and show how they are related.

The first part is a short introduction to knot theory. It starts with the definitions of a link and a knot and moves from knot invariants to the Jones polynomial and the braid group.

The second part is about anyons. Anyons are particles in two dimensions with strange properties. If you move two anyons around each other the type of the particle could change. The focus is on the theory of anyonic systems.

In the end a connection between the two subjects is shown. It is shown how the Jones polynomial that is used in knot theory can emerge from a model of anyons.



# Nederlandse samenvatting

Het eerste deel van deze scriptie gaat over knopentheorie. Knopentheorie is een onderdeel van de wiskunde. Pas in de 19de eeuw zijn mensen begonnen systematisch knopen te bestuderen. Het oorspronkelijke doel was een tabel te maken met zoveel mogelijk knopen. Sinds die tijd zijn er meer dan zes miljard knopen gedocumenteerd.

Knopen zijn gesloten lussen in drie dimensies. Twee knopen zijn gelijk als ze in elkaar om te vormen zijn zonder de lus kapot te maken. De simpelste knoop is de unknot (figuur 1). Deze is op geen enkele manier om te vormen in de trefoil knoop, die daarnaast te zien is.

Aan knopen kun je waarden toekennen die niet veranderen als je de knoop vervormt. Zulke waarden heten knoopinvarianten. Het minimale aantal kruisingen waarmee je een knoop kan tekenen is bijvoorbeeld een knoopinvariant. Deze invarianten zijn handig om te bepalen of twee knopen niet in elkaar te vervormen zijn. Als ze een verschillende waarde voor een bepaalde invariant hebben dan zijn het verschillende knopen. Als de waarden gelijk zijn impliceert dat echter niet het omgekeerde.

Het tweede deel van de scriptie gaat over deeltjes in twee dimensies genaamd anyonen. Anyonen hebben eigenschappen die niet voorkomen bij deeltjes in drie dimensies. Ze zijn nog niet in de praktijk waargenomen, maar als ze zouden bestaan kunnen ze fysische eigenschappen van bepaalde systemen verklaren waar nu nog geen goede verklaring voor is. Ook vormen ze de basis van een topolo-

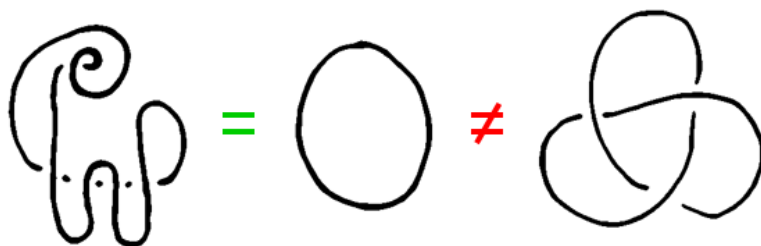


Figure 1: Een knoop die equivalent is met de unknot, maar niet met de trefoil knoop.

gische kwantumcomputer. Dat is een computer die in een seconde berekeningen kan uitvoeren waar de huidige computers jaren voor nodig hebben.

Anyonen en knopen hebben op het eerste gezicht niets met elkaar te maken. Beweging van anyonen door de ruimte en tijd kun je echter zien als lijnen die om elkaar heen draaien. Deze lijnen kun je weer zien als elementen uit de braid groep, een wiskundige groep die gebruikt wordt in de knopentheorie. Ook geven anyon modellen aanleiding to knoopinvarianten, waarover in de scriptie meer te lezen is.

# Contents

<b>Summary</b>	<b>iii</b>
<b>Introduction</b>	<b>ix</b>
<b>1 A Short Introduction to Knot Theory</b>	<b>1</b>
1.1 Knots and links . . . . .	1
1.2 Reidemeister moves . . . . .	2
1.3 Knot invariants . . . . .	2
1.4 The Jones polynomial . . . . .	3
1.5 Braids . . . . .	4
<b>2 Anyons</b>	<b>7</b>
2.1 Non-abelian statistics . . . . .	7
2.2 A model of anyons . . . . .	8
2.3 Fibonacci anyons . . . . .	10
2.4 $SU(2)_k$ . . . . .	10
<b>3 Conclusion</b>	<b>13</b>
<b>A The Jones polynomial of the trefoil knot</b>	<b>17</b>
<b>B Fusion matrices for <math>SU(2)_2</math></b>	<b>19</b>





# Introduction

This bachelor's thesis is divided into two chapters, about subjects that at first sight are not related. The first chapter is an introduction to knot theory. Knot theory is the part of mathematics that studies knots and links (chapter 1.1). Since the 19th century people have been studying knots. The original aim was to tabulate as many knots as possible, and today about 6 billion knots have been tabulated. The main problem in knot theory is to find an easy procedure to determine whether two knots are the same. Since knots are objects in three dimensions this is not an easy problem. To tell if two knots are not the same properties of knots known as invariants (chapter 1.3) can be calculated. These invariants are used to discriminate between knots. If two knots have different values for the same invariant they are different. An important invariant is the Jones polynomial, which is treated in chapter 1.4. The last part of this chapter is about braids.

The second part is about anyons. Anyons are particles in two dimensions that have properties that are not seen in three dimensions. Experimental physicists are working on ways to directly show the existence of anyons, but they have never been observed. They arise in theory just from the assumption a system has two space dimensions. This might seem unreasonable since the world we live in has three dimensions but it is possible to lock out one dimension by a magnetic field, like in the fractional quantum Hall effect, which can be explained by anyonic particles. If it turns out anyons can be created in experiments they can be used to create topological quantum computers (TQC). TQCs are a shortcut to quantum computers. They have the same calculating capabilities [9], ie. TQCs can emulate normal quantum computers and vice versa.

The aim of this thesis is to give small introductions to the two subjects and show how they are related. In most of the literature out there this is done with quantum field theory, but since this is only a bachelor's thesis I will not involve that. As I will show in 2.4 out one particular anyon model gives rise to the Jones polynomial. With this fact it is possible to get more physical intuition about anyons, and get more information about the meaning of the parameter in the Jones polynomial.



# Chapter 1

## A Short Introduction to Knot Theory

The first part of this thesis will be a short introduction into the mathematical theory of knots. Most texts start with some history but since there is already a lot of information out there, and it is not the aim of this thesis to talk about the history of knot theory I will only give some references [1, 2].

### 1.1 Knots and links

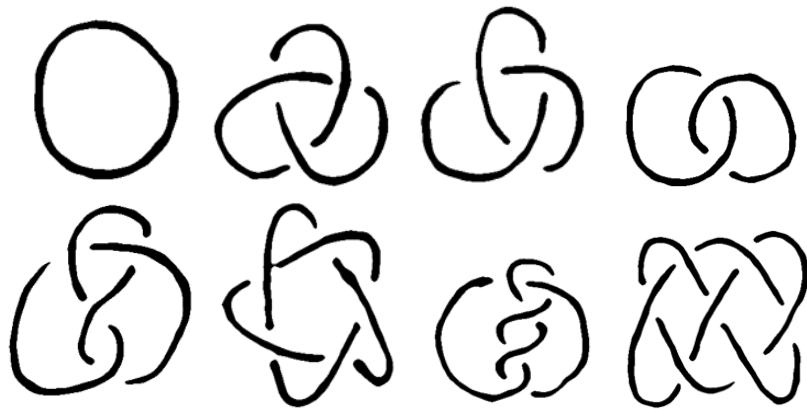


Figure 1.1: The unknot, the trefoil knot and its mirror image, the Hopf link, the figure eight knot, two knots with five crossings and one with seven.

Knot theory is the part of the mathematical theory of topology that studies properties of knots and links. A link is an embedding of one or more circles in the three dimensional Euclidian space  $\mathbb{R}^3$ . Two links are equivalent if they can

be transformed into each other via deformation of  $\mathbb{R}^3$  upon itself. When a link consists of only one component it is called a knot. When you make a projection of a knot (link) to the plane, so that no crossings overlap, and indicating over- and undercrossings by a deleted segment like in figure 1.1 you call it a knot (link) diagram. Such a diagram contains all the information about a knot. If you add a direction to each segment by adding arrows to the edges it is called an oriented diagram.

The most basic knot is the unknot, which is just a circle, shown in figure 1.1. The trefoil knot is the most basic knot with crossings, however there exist two trefoil knots since a trefoil knot is not equivalent to its mirror image. This might seem like a trivial fact, but it is by no means trivial to prove this, we need some more tools to really say something about knots.

## 1.2 Reidemeister moves

If you take two knot diagrams it might be hard to see they if they belong to the same knot. To find out if they are equivalent the most basic way to start is just trying to deform the knot. With some experience and intuition you can see all the moves you make can be reduced to local moves on two or three strands. It is even possible to prove this fact, as Kurt Reidemeister did in 1927. He proved that two different diagrams of the same knot can be related by a sequence of local moves. The three local moves are called the Reidemeister moves. (figure 1.2).

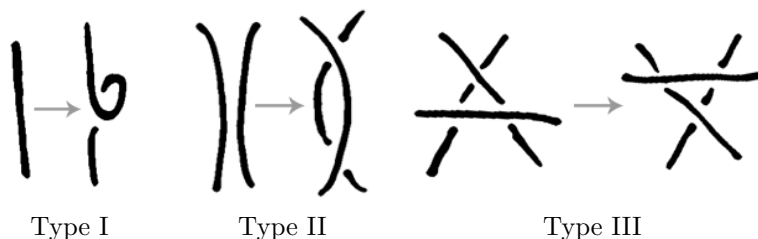


Figure 1.2: The three Reidemeister moves.

Two diagrams that can be related by the moves I, II and III are called ambient isotopic and if only II and III are needed they are called regular isotopic.

## 1.3 Knot invariants

To really tell something about a knot we want to have a value that doesn't change under transformation of the knot. Such a quantity is called a knot invariant. For two equivalent knots the value is the same. Such a quantity could be a number, a polynomial, just a binary value, or anything else. Two knots with a different value of some invariant are different knots, however if they have

the same invariant it doesn't mean they are equivalent. To prove something is a knot invariant it suffices to show it is invariant under the Reidemeister moves.

A simple example of a knot invariant is tricolorability, which is the ability of a knot to be colored with three colors under certain rules. If you can color the edges of a knot diagram one of three colors, with all the colors at a crossing being either all the same color or all different colors and using more than one color the knot is called tricolorable. The unknot is not tricolorable so if a knot is tricolorable you know it is not equivalent to the unknot. It is a simple but nice exercise to check that tricolorability is invariant under the Reidemeister moves. This proves tricolorability is an isotopy invariant of a knot.

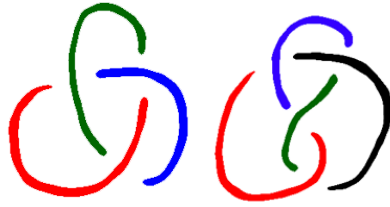


Figure 1.3: The trefoil knot is tricolorable but the figure eight knot is not.

Other important invariants are knot or link polynomials, which assign a polynomial to a link. Famous examples are the Alexander polynomial [4], the HOMFLY polynomial and the Jones polynomial. The next section will be about the last one, and in the second chapter the defining relationship of the Jones polynomial will emerge from a model of anyons. An interesting question is if there exist an invariant that discriminates all knots, so it gives a different value for each knot. No such invariant has been found yet.

## 1.4 The Jones polynomial

The Jones polynomial was discovered by Vaughan Jones in 1983. This polynomial assigns a Laurent polynomial to each oriented knot or link diagram. It can be defined by three properties.

1. If  $K$  and  $K'$  are ambient isotopic then  $V_K(t) = V_{K'}(t)$
2.  $V_{unknot}(t) = 1$
3.  $\frac{1}{t}V_{L_+}(t) - tV_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L_0}(t)$  where  $L_+$ ,  $L_-$  and  $L_0$  indicate diagrams that differ only at one crossing like in figure 1.4. This is called the skein relation.

With this definition it is possible to recursively calculate the Jones polynomial for every knot or link. In appendix A there is an example of how to calculate it for the trefoil knot. It is not obvious these properties are well-defined or that such an invariant even exists, but the Jones polynomial can

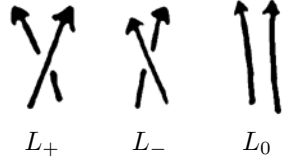
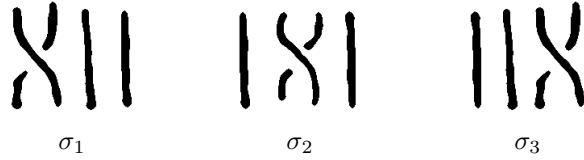


Figure 1.4:

Figure 1.5: Three exchanges and their inverses generate the 4-stranded braid group,  $B_4$ .

also be defined in other ways. One is by the Kauffman bracket, an invariant of regular isotopy, with the proper normalization as shown by Kauffman in [7]. Another way is by braid representation [8]. This is the way Jones originally formulated his polynomial.

The Jones polynomial is not very intuitive. It is hard to tell something about a knot just by looking at its Jones polynomial, other than if the Jones polynomial has a lot of terms it also has a lot of crossings. All knots with seven or less crossings have different values of the Jones polynomial. There are non-equivalent knots that have the same Jones polynomial, however it is not known if there exists non-trivial knots with Jones polynomial 1, like the unknot. It is known there are links with the same Jones polynomial as the unlink (two circles that are not linked). For the simplest example see [8].

## 1.5 Braids

One more thing needed to understand the relation between knots and anyons is the braid group. The braid group on  $n$  strands,  $B_n$ , is a generalization of the symmetric group  $S_n$ . It is generated by the exchange of the  $i$ -th strand,  $\sigma_i$ , with the constraints known as the Artin relations,

$$\begin{aligned}\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i = 1, 2, \dots, n-1 \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \text{ for } |i-j| > 1.\end{aligned}$$

The first one is known as the Yang-Baxter equation. The braid group can be represented quite intuitively, for example the 4-stranded braid group has three generators shown in figure 1.5, and satisfies the two relations (figures 1.6).

A braid is the product of the generators and their inverses. Each braid represents a link if you take the closure. You can take the closure of a braid by

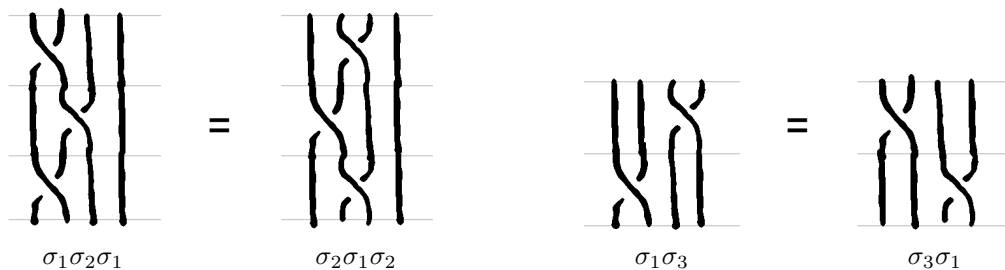


Figure 1.6: The Artin relations, the first one is the Yang-Baxter equation.

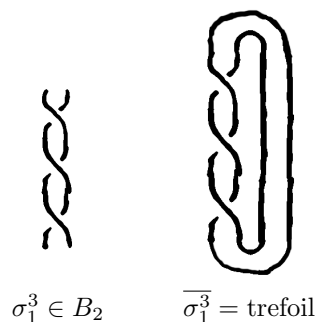


Figure 1.7: The closure of a braid is a knot.

tying the start- and end-points. For example  $\beta = \sigma_1\sigma_1\sigma_1$  is a braid in  $B_2$ . Its closure  $\overline{\beta}$  is the trefoil knot (figure 1.7). In general any link can be represented by the closure of braid. If you try it you will find that it is fairly easy for simple knots, but as the number of crossings increases it gets harder. It is however always possible as Alexander proved in 1923 [5].

There are some intuitive facts about the closure of braids. If  $\beta_1$  and  $\beta_2$  are braids, then  $\overline{\beta_1\beta_2} = \overline{\beta_2\beta_1}$ . Or equivalent,

$$\overline{g\beta g^{-1}} \cong \overline{\beta} \text{ for } g, \beta \in B_n,$$

the closure of the conjugate of a braid is equivalent to the closure of the braid, since  $g$  and  $g^{-1}$  cancel through the closure strands. Also for a  $n$ -stranded braid in  $B_n$ , if you add a disjoint crossing it becomes an  $(n+1)$ -stranded braid. The extra crossing adds an extra loop to the closure knot (Reidemeister move I) so this doesn't change the closure.

$$\overline{\beta} \cong \overline{\sigma_n\beta} \text{ for } \beta \in B_n$$

If two braids have equivalent closures then they are related by a series of these moves. This is known as Markov's theorem, and the two moves of conjugating a braid and adding an extra strain are called the Markov moves.

With this knowledge it is possible to read the second part of this thesis and understand the relation between Anyons and knot theory.



## Chapter 2

# Anyons

Anyons are particles that live in two dimensions. Their properties can differ from particles in three dimensions in a way that they obey neither Bose-Einstein statistics nor Fermi-Dirac statistics. In three dimensions particles can be either bosons or fermions. These two cases were believed to be the only possibilities for almost fifty years, until 1977 when Leinaas and Myrheim showed that in two dimensions more exotic possibilities are possible [6].

### 2.1 Non-abelian statistics

The fact that in three dimensions the only possibilities that exist are bosons and fermions involves some advanced mathematics. To make it more intuitive take a look at two particles in three dimensions that are exchanged two times (figure 2.1 a). The path the particles take can be contracted to one particle not moving at all and one moving around the other (2.1 b). The path can even be contracted like a lasso to no movement at all (2.1 c).

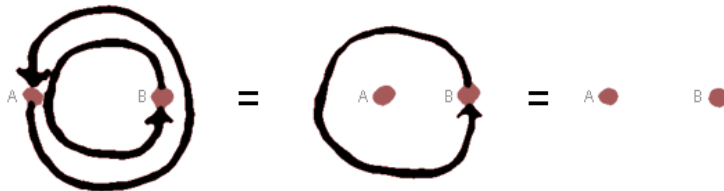


Figure 2.1: Particle exchanges in 3+1 dimensions

This means the wave function of the particles has to remain the same after applying the exchange operator,  $R$ , twice, since these situations are equivalent. So  $R^2\psi = \psi$  which implies  $R\psi = \pm\psi$ , the wave function of two particles can only pick up a minus sign or no sign when the particles are exchanged. This corresponds of course to the particles being fermions and bosons respectively.

In two dimensions (figure 2.2) however the path cannot be contracted without the two particles moving through each other. This means there is no restriction on the wave function like in three dimensions, and there can be more possibilities besides bosons and fermions.

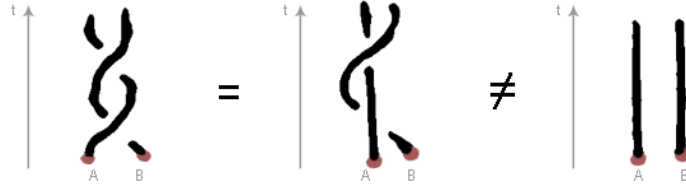


Figure 2.2: Particle exchanges in 2+1 dimensions

It is possible when the particles are exchanged the wave function picks up a phase, ie.  $R\psi = e^{i\theta}\psi$ . This way  $\theta = 0$  corresponds with the particles being bosons and with  $\theta = \pi$  the particles are with fermions. In general the particles are not anyons or bosons but anything in between.

When the wave function just picks up a phase the particles are called abelian anyons. Non-abelian anyons are more complex. Their wave function is a vector, and moving the particles multiplies them with a matrix. These non-abelian anyons could be used for building TQCs. The abelian ones could be used for building topological quantum memory.

## 2.2 A model of anyons

A general theoretical model of anyons can be defined by three properties. Preskill describes this model in detail in his lecture notes [9]. This is a short summary. The three properties that define an anyon model are:

1. A list of particle types.
2. Rules for fusion and splitting.
3. Rules for braiding.

The first property describes which particles can be formed. There are finitely many particles which can be ordered. Particles are denoted by labels, for example a list of particles might be  $\{a, b, c, \dots\}$ . There is always an identity- or vacuum particle, denoted 1, 0 or  $e$  most of the time. Also each particle has an anti-particle. A particle  $a$  can fuse with together with its particle  $\bar{a}$  into the vacuum, or  $a \times \bar{a} = 1$ . The rules for fusion and splitting describe which particles can form when two particles fuse, and in what way one particle can split into two particles. These rules are associative, so if three particles fuse it doesn't matter in what order they fuse;  $(a \times b) \times c = a \times (b \times c)$ . The rules for braiding describe what happens to two particles when they are exchanged.

For each two particles  $a$  and  $b$  there are  $N_{ab}^c$  ways to fuse into  $c$ , so the rules of fusion can be described like

$$a \times b = \sum_c N_{ab}^c c.$$

Each way  $a$  and  $b$  can fuse into  $c$  can be regarded as state in a vector space,  $V_{ab}^c$ . The basis of this so called fusion space can be denoted as

$$\{|ab; c, \mu\rangle, \mu = 1, 2, \dots, N_{ab}^c\}.$$

Exchanging the particles before fusing does not change the total charge of the pair so there is a unitary transformation  $R$ , the braid operator:

$$\begin{aligned} R : \quad V_{ab}^c &\rightarrow V_{ba}^c \\ |ab; c, \mu\rangle &\mapsto |ba; c, \mu'\rangle \end{aligned}$$

Three particles fusing form the space  $V_{abc}^d \cong \bigoplus_e V_{ab}^e \otimes V_{ec}^d \cong \bigoplus_{e'} V_{ae'}^d \otimes V_{bc}^{e'}$ . There are two bases for this space that are related by a unitary transformation  $F$ :

$$\begin{aligned} F : \quad \bigoplus_e V_{ab}^e \otimes V_{ec}^d &\rightarrow \bigoplus_{e'} V_{ae'}^d \otimes V_{bc}^{e'} \\ |(ab)c \rightarrow d; e\mu\nu\rangle &\mapsto \sum_{e'\mu'\nu'} |a(bc) \rightarrow d; e'\mu'\nu'\rangle (F_{abc}^d)_{e\mu\nu}^{e'\mu'\nu'} \end{aligned}$$

Four particles can fuse into one particle in five different ways. The bases can be related by the  $F$ -matrix, as shown in figure 2.3.

There are two unitary transformations to go from the left to the right basis which must be identical. This gives rise to the pentagon equation 2.1c.

$$|\text{left}; a, b\rangle = \sum_{c,d} |\text{right}; c, d\rangle (F_{12c}^5)_a^d (F_{a34}^5)_b^c \quad (2.1a)$$

$$|\text{left}; a, b\rangle = \sum_{c,d,e} |\text{right}; c, d\rangle (F_{234}^d)_e^c (F_{1e4}^5)_b^d (F_{123}^b)_a^e \quad (2.1b)$$

$$(F_{12c}^5)_a^d (F_{a34}^5)_b^c = \sum_e (F_{234}^d)_e^c (F_{1e4}^5)_b^d (F_{123}^b)_a^e \quad (2.1c)$$

Analogous to the pentagon equation you can derive the hexagon equation (2.2).

$$R_{13}^c (F_{213}^4)_a^c R_{12}^a = \sum_b (F_{231}^4)_b^c R_{1b}^4 (F_{123}^4)_a^b \quad (2.2)$$

These two equations must hold for any anyon model.

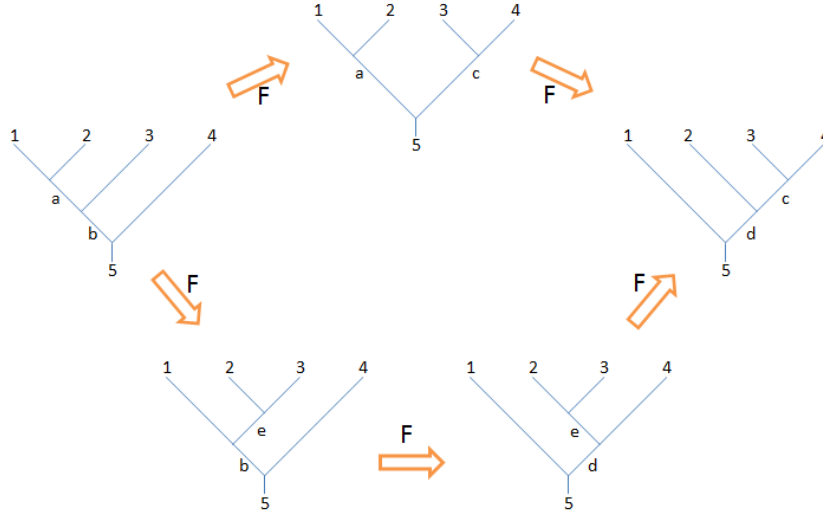


Figure 2.3: The two ways to go from the left base to the right base give the pentagon equation.

### 2.3 Fibonacci anyons

The simplest non-trivial model of anyons is the Fibonacci model [10], also called the Yang-Lee model. The model has only two particle types, the trivial particle 1 and  $\tau$ , the Fibonacci anyon. The braiding rules are given by

$$\tau \times \tau = 1 + \tau.$$

This means two Fibonacci anyon can either fuse into another Fibonacci anyon or they can annihilate into the trivial particle.

Three Fibonacci anyons can fuse into  $\tau$  via  $|\tau\tau; 1\rangle|1\tau; \tau\rangle$ , via  $|\tau\tau; \tau\rangle|\tau\tau; \tau\rangle$ , or they can fuse into 1 in only one way via  $|\tau\tau; \tau\rangle|\tau\tau; 1\rangle$ . This means three Fibonacci's can fuse in three distinguishable ways. Continuing this, the fusion rules for four  $\tau$ 's are  $\tau^4 = 2 \times 1 + 3 \times \tau$ , so that is five ways. Five  $\tau$ 's fuse in 8 ways, and in general for  $n$  Fibonacci anyons the number of ways they can fuse is the  $(n+1)$ -th Fibonacci number. This is of course why this is called the Fibonacci model. Preskill shows this model is sufficient to create a topological quantum computer.

### 2.4 $SU(2)_k$

Chern-Simons theory is a type of quantum field theory introduced by Edward Witten [12]. One particular type of Chern-Simons theory with gauge group  $SU(2)$  at level  $k$  is denoted  $SU(2)_k$ . The anyon model that is an instance of

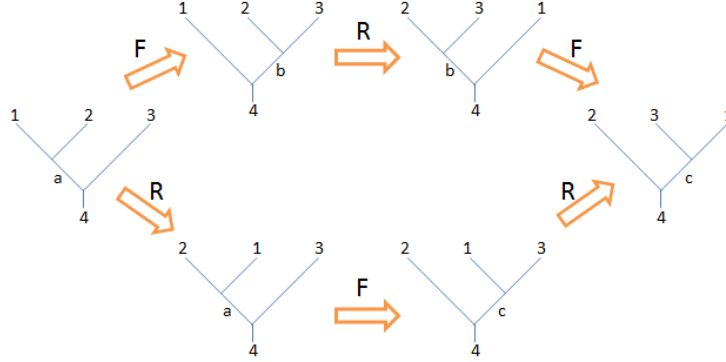


Figure 2.4: Proof of the hexagon equation.

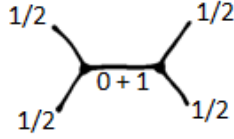


Figure 2.5: The R-matrix acts on a 2 dimensional space.

$SU(2)_k$  has particle types are irreducible representations of the gauge group  $SU(2)$ , truncated by  $k$

$$\text{List of particles} = \{j \mid j \leq \frac{k}{2}\},$$

with fusion rules

$$j_1 \times j_2 = \sum_{\substack{j \in \{|j_2-j_1|, \dots, j_1+j_2\} \\ |j_1+j_2+j \leq k}} j.$$

For an example of  $SU(2)_2$  see appendix B.

One way of looking at the world lines of anyons is as strains in the braid group. If the anyons just move around each other and do not fuse they can be seen as braids. If we calculate properties of these anyons that satisfy the Markov properties it is possible to calculate invariants for links.

$$L(\overline{\alpha\beta}) = L(\overline{\beta\alpha}) \tag{2.3a}$$

$$L(\overline{\sigma_n\beta}) = \tau L(\overline{\beta}) \tag{2.3b}$$

This is possible for any anyon model. The anyons at the bottom should match the anyons at the top of the braid in order to take the closure. In the  $SU(N)_k$  model we can take all the anyons of type  $\frac{1}{2}$ . The fusion space with 4

external lines is 2-dimensional (figure 2.5). Three vectors in a two dimensional space have the property that  $\alpha\psi_1 + \beta\psi_2 + \gamma\psi_3 = 0$  always has a non trivial solution. So with the three vectors

$$\begin{aligned}\psi_+ &= \textcircled{\nearrow}, \\ \psi_0 &= \textcircled{\uparrow\uparrow} = R\psi_+, \\ \psi_- &= \textcircled{\searrow} = R^2\psi_+, \end{aligned}$$

there is a solution to

$$\alpha\psi_+ + \beta\psi_0 + \gamma\psi_- = 0. \quad (2.4)$$

Since  $R$  is a  $2 \times 2$  matrix of full rank it obeys the characteristic equation

$$(\tilde{R} - \lambda_1)(\tilde{R} - \lambda_2) = 0$$

(where  $\tilde{R} = Q^{-1}RQ$  is a diagonal matrix) and thus

$$R^2 - \text{tr}(R)R + \det(R) = 0,$$

$$R^2\psi_+ - \text{tr}(R)R\psi_+ + \det(R)\psi_+ = 0,$$

$$\psi_- - \text{tr}(R)\psi_0 + \det(R)\psi_+ = 0.$$

This implies  $\alpha = \det(R)$ ,  $\beta = -\text{tr}(R)$  and  $\gamma = 1$ . The only things still missing are the eigenvalues of  $R$ . These eigenvalues are given in Witten's article [11] and they are

$$\begin{aligned}\lambda_1 &= \exp\left(\frac{i\pi(1-N)}{N(N+k)}\right), \\ \lambda_2 &= -\exp\left(\frac{i\pi(1+N)}{N(N+k)}\right).\end{aligned}$$

With these values a skein relation can be derived. This relation is

$$q^{N/2}V_{L_+}(q) - q^{-N/2}V_{L_-}(q) = \left(\sqrt{q} - \frac{1}{\sqrt{q}}\right)V_{L_0}(q) \quad (2.5)$$

which is for  $N = 2$  exactly the skein equation of the Jones polynomial.

In general from any anyon model a knot invariant can be derived. Kauffman [13] shows how to create knot invariants from models that satisfy the Yang-Baxter equation.

## Chapter 3

# Conclusion

By now you should have a basic understanding of knot theory and anyons and see the link between the two subjects. I've given only a short introduction to both subjects, but if you want to know more, the references will point you in the good direction. About knot theory there is a lot of material that can be read. There is not one particular book or paper, however [1, 4] are good places to start reading. A lot of the things about anyons in this thesis can be found in more detail in the Preskill lecture notes [9]. In there is even a lot more for the interested reader, so this is a good overview about anyons.

Some things were not mentioned in this thesis. Everything shown here is purely theoretical, but to benefit from the properties of anyons they must be created in experiments. According to Preskill two approaches to realizing anyons in practice have been discussed in recent literature. The theory of anyons is quite young and since they could be a short way to quantum computers the interest is pretty big.

Anyons have not been created in experiments. The theory emerges from just the assumption of particles in two dimensions. My view is, since nothing forbids anyons in practice is must be possible to create them in the real world. It might take some time, but as the interest in them grows a breakthrough might be near.





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## Appendix A

# The Jones polynomial of the trefoil knot

To demonstrate how to recursively calculate the Jones polynomial by the three relations in 1.4 here is an example calculation for the trefoil knot.

If you look at figure A.1 the skein relation of the Jones polynomial tells us

$$\frac{1}{t}V_{trefoil}(t) - tV_{unknot}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{hopf}(t).$$

Since since  $V_{unknot} = 1$  and the skein relation with figure A.2 is  $\frac{1}{t}V_{unknot}(t) - tV_{unknot}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{unlink}(t)$ , it follows

$$V_{unlink}(t) = -(\sqrt{t} + \frac{1}{\sqrt{t}}).$$

Together with  $\frac{1}{t}V_{hopf}(t) - tV_{unlink}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{unknot}(t)$  (figure A.3) this system of equations can be solved, and it turns out the Jones polynomial for the trefoil knot is:

$$V_{trefoil}(t) = t + t^3 - t^4$$



Figure A.1: The trefoil knot, the unknot and the hopf link.



Figure A.2: Two times the unknot and the unlink.



Figure A.3: Hopf link, unlink and unknot.

## Appendix B

# Fusion matrices for $SU(2)_2$

The  $SU(2)_2$  model has three particle types: 0,  $\frac{1}{2}$  and 1. The fusion rules are:

$$\begin{aligned}\frac{1}{2} \times \frac{1}{2} &= 0 + 1 \\ \frac{1}{2} \times 1 &= \frac{1}{2} \\ 1 \times 1 &= 0\end{aligned}$$

The fusion spaces  $V_{\frac{1}{2}\frac{1}{2}x}^y$ ,  $V_{\frac{1}{2}x\frac{1}{2}}^y$  and  $V_{x\frac{1}{2}\frac{1}{2}}^y$  with  $x \in \{0, 1\}$  are one dimensional. For example the space  $V_{\frac{1}{2}\frac{1}{2}1}^0$  has base  $\{|\frac{1}{2}\frac{1}{2}; 0\rangle|01; 1\rangle\}$ , so the transformation to another base is trivial. The only non trivial F-matrix is the matrix with  $V_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}}$  which is  $2 \times 2$ . The pentagon equation becomes

$$1 = ((F_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}})_0^1)^2 + (F_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}})_0^1 (F_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}^{\frac{1}{2}})_1^0.$$

Together with the unitarity conditions F is given up to overall phases and complex conjugation:

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$