Relativistic Strings in a Classical Gravitational Potential

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Abstract

In this report the motion of a closed string in a classical gravitational potential is analyzed. We chose for a description of the forcefield in 3+1 dimensions because this matches with our everyday intuition. We were interested in the difference between the motion of a free string and that of a string in a gravitational field. First a brief review on relativistic mechanics and Lagrangian formalism was given. Then we introduced a method to describe the motion of a string using the Nambu-Goto action and the variational principle, which we both tried to describe comprehensively. Then the variational principle was used to derive an equation that describes the motion of a string in a classical gravitational field. The imposed gravitational field is a constant vector field working in one direction. This field describes the gravitational field as we experience it on the earths surface. We tried to avoid using general relativity in our the description of the problem. Since in the problem a classical gravitational potential was analyzed we thought this approached would work out. However it can be concluded that our approach was wrong and that it is probably easier and more accurate to involve general relativity into our description. As an extension of our research one could take a look at a radial vector field which drops with $\frac{1}{r^2}$ and finally at the behavior of a string close to the event horizon of an black hole.

Contents

1 Introduction

In modern physics there are two theories that describe physics at two different scales. General relativity describes gravity at a macroscopic scale. But at small scales, gravity is such a weak force in comparison to, for example, the electromagnetic force (magnitude differences in the order of 10^{40}) that it is negligible. The standard model describes forces between subatomic particles; the electromagnetic-, strong- and weak force and is valid only at a microscopic scale. String theory is an attempt to unify general relativity and the standard model.

In string theory matter is described by tiny (at least smaller then 10^{-16} m) vibrating strings instead of infinite small point particles with an infinite mass density. Each vibrating mode of the strings would describe another particle as we know them today. Strings have the property to either be open or closed. Open strings do have two endpoints which are always connected to a surface. The endpoints move over the surface, while the rest of the string vibrates between these moving point. Closed strings on the contrary do not have end points at all and can be seen as continues loops. Both types of strings are described by the same physics. That their motions differ is because they obey different boundary conditions, for example the endpoints of an open string should stay attached to the surface while a closed string does not have endpoints at all. The solution of a closed string should therefore be periodically.

In this paper we are going to derive the equations of motion for a one dimensional closed relativistic string moving in a classical potential. In general gravitational force field are radial and drop with $\frac{1}{r^2}$. We are going to use a simplified model of the gravitational field by assuming that it is constant over the distance the string travels. In fact this approximation describes the gravitational force on earths surface pretty well and it holds for kilometers above it. Only if we look at larger scales then kilometers this approximation loses its validity. We have depicted the force field of this potential in figure (1). We are interested in this problem because we

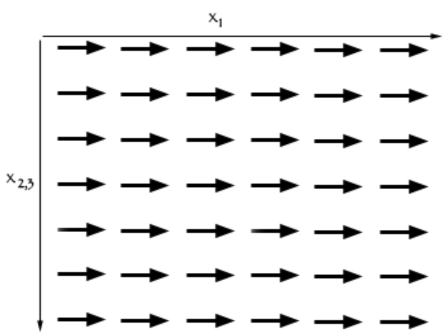

Figure 1: 2D Visualization of the gravitational field will use in our calculations.

want to know if a string behaves differently in a gravitational field than it would do in free space. If we will ever be able to perform measurements on string or to make them visible it will always be in presence of the earths gravitational field. Therefore it would be very good to know if the theory predicts some extraordinary motion for a string in such a gravitational field.

We start with a quick review of Special Relativity and Lagrangian Formalism. Then we derive the equations of motion for a relativistic string in a potential as depicted above. Finally we try to solve these equations of motion.

2 Basic review of Special Relativity and Lagrangian formalism

In this paragraph we give a short review of special relativity theory and Lagrangian mechanics. The main goal is to familiarize the reader with some of the notation we are going to use often. We also try to give some motivation why we make use them.

We begin with Special Relativity[1]. In Special Relativity, events are marked by four-vectors: (ct, x, y, z), where the first coordinate time is multiplied with the speed of light. This makes that all coordinates have units of length. Now we introduce some new notation, we write:

$$x^\mu = (x^0, x^1, x^2, x^3) \equiv (ct, x, y, z)$$

Of course, the superscript μ runs over the values $0, 1, 2, 3$. This is a very general notation which allows you to add as many dimensions as your theory needs to describe physics. Therefore special relativity is very useful to describe strings in string theory. Since we want string theory to be a grand unifying theory, it should not only describe matter in a non relativistic limit. Thats the other reason we use relativity theory to describe relativistic strings.

We define Δs^2 as the invariant interval, the value of this interval is equal to all observers. In our notation:

$$-\Delta s^2 = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 \tag{1}$$

Note the minus sign in front of x^0. This sign represents the fundamental difference between '*timelike*' coordinates and '*spacelike*' coordinates. Another useful notation can be obtained by simplifying the expression for the invariant interval. We define

$$\Delta x_0 \equiv -\Delta x^0, \quad \Delta x_1 \equiv \Delta x^1, \quad \Delta x_2 \equiv \Delta x^2, \quad \Delta x_3 \equiv \Delta x^3.$$

With this definition equation (1) becomes

$$-\Delta s^2 = \sum_{\mu=0}^{3} \Delta x_\mu \Delta x^\mu. \tag{2}$$

This is the last time we use the sigma-notation, throughout the rest of the paper we use Einsteins summation convention. In this notation we get $-\Delta s^2 = \Delta x_\mu \Delta x^\mu$, where summation over all possible values of (in this case) μ is implied. In a more mathematical way we can also write the invariant interval in terms of the Minkowski metric. This is a very important metric used to describe spacetime. We get

$$-\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu, \qquad \eta_{\mu\nu} = \eta_{\nu\mu}. \tag{3}$$

From (2) and (3) we can see that the Minkowski metric is defined by the matrix

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Another important concept in Special relativity is that of *proper time*. This is a measurement that is Lorentz invariant of time. A quantity is Lorentz invariant if observers in different inertial-systems agree on the value of that quantity. Consider a particle moving in some direction and imagine we mark two events along its trajectory. If the particle is carrying a clock, the proper time is the elapsed time between the two events on that clock. Logically this is invariant because all observers that can 'see' the clock *must* agree on the elapsed time. We can quantify proper time by considering equation (1) in a the Lorentz frame attached to the particle. In this frame the particle does not move, so the spatial coordinates are equal to zero. This implies that

$$-ds^2 = -c^2 dt^2, \tag{4}$$

or

$$dt = \frac{ds}{c}. \tag{5}$$

We will use this later on to derive the action for a relativistic particle.

The history of a particle is represented in space-time as a *curve*, this curve is called the $world - line$ of the particle. A (1-D) string however, traces out a *sheet* in spacetime, called the $world - sheet$. Ultimately, we want to derive the equations of motion in terms of the parameterization of the world-sheet. But first we review the Lagrangian formalism.

[1]Based on Zwiebach page 15-18.

The Lagrangian formalism is based on the assumption that the action is stationary under infinitesimal variation of the path the particle takes. The action is given in terms of the 'Lagrangian' $L = T - V$:

$$S = \int_{t_0}^{t_1} L dt, \tag{6}$$

with T the kinetic energy, which for a non relativistic particle is given by $T = \frac{1}{2}m\dot{x}(t)^2$ and with $V = V(x(t))$, the potential. Equation (6) gives the action for a point-particle. From this, we can derive the equations of motion using the variational principle. To illustrate how this is done in principle we work out a short derivation of the motion of a point particle. We vary the action by varying $x(t)$ an infinitesimal amount δ, we get

$$\delta S = \int_{t_0}^{t_1} \left[\frac{1}{2} m(\dot{x}(t) + \delta\dot{x}(t))^2 - V(x(t) + \delta x(t)) \right] dt. \tag{7}$$

Expanding V(x) as a Taylor series, we can write

$$S(x + \delta x) = S(x) + \int_{t_0}^{t_1} \left[m\dot{x}(t) \frac{d}{dt} \delta x(t) - V'(x(t))\delta x(t) \right] dt, \tag{8}$$

where we have neglected the terms $(\delta x)^2$ and higher. We can now write 8 as $S(x) + \delta S(x)$, so we see that

$$\delta S(x) = \int_{t_0}^{t_1} \left[m\dot{x}(t) \frac{d}{dt} \delta x(t) - V'(x(t))\delta x(t) \right] dt. \tag{9}$$

To get the equations of motion we must make sure that no derivatives are acting on $\delta x(t)$. We can achieve this by using integration by parts, we derive (making use of the fact that $\delta x(t_i) = \delta x(t_f) = 0$)

$$\delta S = \int_{t_0}^{t_1} \delta x(t) \left[-m\ddot{x} - V'(x(t)) \right] dt. \tag{10}$$

We can now see that the action is stationary if δS vanishes for every variation $\delta x(t)$ which we van only achieve if $-m\ddot{x} - V'(x(t))$ is equal to zero. We now have the equations of motion for a point-particle moving in a potential!

$$-m\ddot{x} = V'(x(t)). \tag{11}$$

This equation of motion however, describes the motion of a nonrelativistic particle. We can see this by noting that a free particle, according to the above equation, has no constraints on its velocity: a particle in free space is allowed to move with an arbitrary speed. To give a generalization we must define the action differently. In particular, we want an action that is *Lorentz invariant*, which means that all observers of the particle must agree on its action. If ℓ represents the world line of the particle, the the quantity related to ℓ is the elapsed proper time. The infinitesimal proper time is given by $\frac{ds}{c}$. The integral over this quantity gives the elapsed proper time on ℓ. To get the action we should get the units right. We see that if we integrate $\frac{ds}{c}$ over ℓ, we get a quantity with units of time. To get the units of action we must multiply with a factor of units mass times velocity squared. This quantity must also be lorentz invariant. We make a guess by claiming that this term is mc^2. As it seems, the action is

$$S = mc \int_{\ell} ds. \tag{12}$$

combining equation 1 with 12 implies that

$$S = mc^2 \int_{\ell} dt \sqrt{1 - \frac{v^2}{c^2}}. \tag{13}$$

So the Lagrangian of the relativistic particle is given by

$$L = mc^2 \sqrt{1 - \frac{v^2}{c^2}}. \tag{14}$$

But there is still one problem, in the limit $c >> v$ we see that

$$L \approx mc^2 \left(1 - \frac{1}{2} \frac{v^2}{c^2} \right) = mc^2 - \frac{1}{2} mv^2, \tag{15}$$

this is incorrect because a Lagrangian is always given in terms of $L = T - V$. Equation (15) however, is of the form $L = V - T$. A minus sign should be added to the Lagrangian to correct for this. The action for a relativistic particle is thus given by

$$S = -mc \int_{\ell} ds. \tag{16}$$

A (1-D) string however, does not trace out a path but a sheet in spacetime. We therefore have to define the action differently. In the next paragraph we derive the equation of motion for a string.

3 Lagrangian formalism for a 1D-String

In this paragraph we are going to use the concepts introduced in de preceding section to derive the action for a 1D-string. Recall that a particle traces out a path in spacetime. For a string, one could imagine that every single point on that string traces out path as well. All the worldlines together span a surface in spacetime, which normally is called the world-sheet of the string. Therefor, we have to define the action and the Lagrangian differently. We start with taking $X^\mu(\tau, \sigma)$ as a parameterized function that describes the world-sheet of the string. Here, (τ, σ) represent points in parameter space. So X^μ is a function which maps (τ, σ) from parameterspace to spacetime-coordinates as it is shown in figure (2). We will call X^μ the

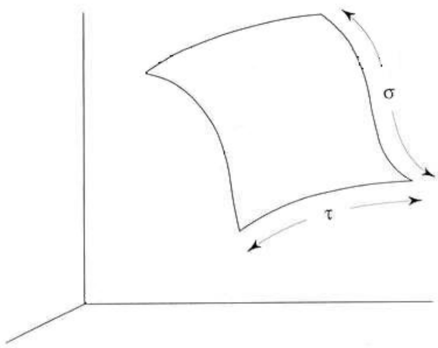

Figure 2: Parameterization of the world sheet by τ and σ

string coordinates. To reduce the size of our equations we will define $\dot{X} \equiv \frac{dX}{d\tau}$ and $X' \equiv \frac{dX}{d\sigma}$.

Now we have to derive the *proper area* of the world-sheet. For that we need to find an expression for the area element in terms of the vectors that span the element as indicated in figure 3. The area is a parallelogram which can be split in to two triangles with equal size. The area of a triangle is given by $\frac{1}{2}$ times base times height. Let $\|v_1\|$ be the base of the triangle. We then have to express the height of the triangle in terms of the two vectors v_1 and v_2. The height of the triangle is equal to $\sin\theta \|v_2\|$, where θ is the angle between the two vectors. $\sin\theta$ can be substituted by $\sqrt{1 - \cos\theta^2}$. We can express the cosine in terms of the two vectors v_1 and v_2 by using the definition of the dotproduct $\cos\theta = \frac{v_1 \cdot v_2}{\|v_1\|\|v_2\|}$. The area dA of the parallelogram can thus be written as

$$dA = 2 \cdot \frac{1}{2} \cdot b \cdot h = \|v_1\|\|v_2\| \sqrt{1 - \frac{(v_1 \cdot v_2)^2}{\|v_1\|^2 \|v_2\|^2}}. \tag{17}$$

Finally we just rearrange the expression and write out the definition of the norm of a vector and find that

$$dA = \sqrt{(v_1 \cdot v_1) \cdot (v_2 \cdot v_2) - (v_1 \cdot v_2)^2}. \tag{18}$$

In case of the world-sheet the vectors that span an area element are $\frac{\partial X^\mu}{\partial \tau} d\tau$ and $\frac{\partial X^\mu}{\partial \sigma} d\sigma$. If we substitute v_1 an v_2 by these vectors and integrate the over τ and σ we find an expression for the area of the total worksheet of a string

$$A = \int d\tau \int d\sigma \sqrt{\left(\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma}\right)^2 - \left(\frac{\partial X}{\partial \tau}\right)^2 \left(\frac{\partial X}{\partial \tau}\right)^2}. \tag{19}$$

Normally, one would think that the sign of the expression under the square root is negative (because of the Cauchy-Schwartz inequality), but we also have to encounter the fact that one of the vectors is timelike and the minus sign in the Minkowski-metric changes the sign of the expression. This is the reason why the two term are switched.

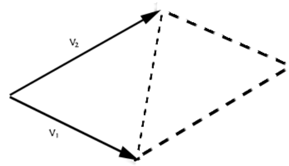

Figure 3: Area spanned by two vectors

Now, the action is proportional to the proper area. The action must have units of $\frac{kgm^2}{s}$. Therefore we must multiply (19) with a constant with units of mass divided by units of time. A string in free space has energy in the form of tension, so we make an educated gues and multiply the proper area with the tension in the string T_0. Note that tension (with units of force) divided by velocity (with units of speed) has the desired units. We therefore define the action to be

$$S_{NG} = -\frac{T_0}{c} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2} d\sigma. \tag{20}$$

which is called the Nambu-Goto action. The minus sign comes from the fact that the Lagrangian is defined as $L = T - V$, but in equation (19) we had to change the sign of the term in the square root. We have to multiply with a minus sign again to compensate. To simplify our expressions we define

$$\mathcal{L} = \frac{T_0}{c} \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}, \tag{21}$$

the so called Lagrangian density. Using the variational principle and following the same steps as for the one dimensional action we get

$$\delta S_{NG} = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left[\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \frac{\partial \delta X^\mu}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial X'^\mu} \frac{\partial \delta X^\mu}{\partial \sigma}\right]. \tag{22}$$

If we use integration by parts equation 22 can be written as

$$\delta S_{NG} = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \left[\frac{\partial}{\partial \tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \delta X^\mu\right) + \frac{\partial}{\partial \sigma}\left(\frac{\partial \mathcal{L}}{\partial X'^\mu} \delta X^\mu\right) - \delta X^\mu \left(\frac{\partial}{\partial \tau} \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} + \frac{\partial}{\partial \sigma} \frac{\partial \mathcal{L}}{\partial X'^\mu}\right)\right].$$

We can work out the first two terms of this equation a bit further. Equation 22 then becomes

$$\begin{aligned} \delta S_{NG} = & \int_0^{\sigma_1} d\sigma \left[\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \delta X^\mu\right]_{\tau_i}^{\tau_f} + \int_{\tau_i}^{\tau_f} d\tau \left[\frac{\partial \mathcal{L}}{\partial X'^\mu} \delta X^\mu\right]_0^{\sigma_1} \\ & - \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \delta X^\mu \left[\frac{\partial}{\partial \tau} \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} + \frac{\partial}{\partial \sigma} \frac{\partial \mathcal{L}}{\partial X'^\mu}\right]. \end{aligned} \tag{23}$$

Most of the times it is trivial to parameterize τ as t. If we do so, $X^\mu(t_i)$ and $X^\mu(t_f)$ are the initial and the final state of the string. We only consider variations δX^μ between these two points. With other words $\delta X^\mu(\tau_i, \sigma) = 0$ and $\delta X^\mu(\tau_f, \sigma) = 0$. We can therefore forget about the first integral. The integrand of this integral will always vanish. The outcome of the second integral depends on the boundary conditions. Whatever the boundary conditions are, the variation in action should be 0. Therefore the last integral must be equal to 0. Since δX^μ is non zero, $\frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\dot{X}} + \frac{\partial}{\partial\sigma}\frac{\partial\mathcal{L}}{\partial X'}$ must be equal to zero.

We have now derived that the equations of motion for a (1-D) string are

$$\frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\dot{X}} + \frac{\partial}{\partial\sigma}\frac{\partial\mathcal{L}}{\partial X'} = 0. \tag{24}$$

In the next paragraph we will derive the equation of motion for a two dimensional closed string moving in a classical gravitational potential.

4 Derivation of the equations of motion

In this paragraph we derive the equations of motion for a closed string in a classical -earth-surface like- gravitation field. With this we mean that the gravitational force is always pointing in one direction and has a constant magnitude. To derive the equations of motion we will make use of the Nambu-Goto action and the variational principle.

There are several ways to add the imposed gravitational potential into our action integral. One possibility is, to look at this potential in a classical way. In such way that the presence of a gravitational field affects the potential energy of the string in the Lagrangian. We chose the gravitational force so, that it acts in the X_1 direction. Then an infinitesimal small length dl of string increases the total potential energy of the string by $\mu_0 g X_1 dl$, with μ_0 the mass density of the string and g a constant, which gives the magnitude of the gravitational field. The addition to the action of the string would be the additional potential energy integrated over the time t. So the term we would like to add to the Lagrangian density will look like $\mu_0 g X_1 dldt$. $dldt$ would be de area spanned by the string in a non relativistic way. For a relativistic description of the string we replace the $dldt$ with the area dA we used to describe the surface of the world sheet of a free string. Remember from the previous paragraph that dA is given by $\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}$. The additional term for the Lagrangian density of the whole world sheet is therefore given by $\mu_0 g X_1\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}$. Ultimately this is not a correct way to add the potential, because by multiplying with this square root we also multiply with the kinetic energy of the string. We will discus the influence of this on our calculations in the discussion section. As said before, there are other ways to think about adding a potential. The accuracy and the precision of the different methods will be discussed later on, but from now on we will chose to work the problem out as discussed above.

The Nambu-Goto Action is the integral of the Lagrangian density $\mathcal{L}$ over the parameterization variables τ and σ. With the Lagrangian density given in terms of the string coordinates $X^\mu(\tau,\sigma)$ and their derivations $\frac{dX}{d\tau}$, $\frac{dX}{d\sigma}$. To get the get the action of the string, the term discussed above is now added to the Lagrangian density. The Nambu-Goto Action for a string is thus given by

$$S = -\int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} \left[\frac{T_0}{c} + \mu_0 g X_1\right]\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}d\sigma. \tag{25}$$

Just to be clear, the first term of the integrand is the contribution of the because of the tension in the string. The second term in this action is an addition of the gravity potential. Where the gravitation is chosen to act in the X_1 direction.

To get the equations of motion for the relativistic string from the action we make use of the variational principle. A small variation in X should not result in a variation in action. We vary X and we impose that the variation in action is equal to 0. The most general way to write down the variation in the action is

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} \left[\frac{d\mathcal{L}}{d\dot{X}^\mu}\frac{d(\delta X^\mu)}{d\tau} + \frac{d\mathcal{L}}{dX^{\mu\prime}}\frac{d(\delta X^\mu)}{d\sigma} + \frac{d\mathcal{L}}{dX^\mu}\delta X^\mu\right] d\sigma, \tag{26}$$

where $\mathcal{L}$ for this problem is given by

$$\mathcal{L} = -\left[\frac{T_0}{c} + \mu_0 g X_1\right]\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}. \tag{27}$$

By substituting $\mathcal{L}$ into equation (26) we get an equation in terms of $X, \dot{X}, X'$. For convenience we define

$$\mathcal{P}^{\tau}_{\mu} \equiv \frac{d\mathcal{L}}{d\dot{X}^{\mu}} = -\left[\frac{T_0}{c} + \mu_0 g X_1\right] \frac{(\dot{X}\cdot X')X'_{\mu} - (X')^2\dot{X}_{\mu}}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}}, \tag{28}$$

$$\mathcal{P}^{\sigma}_{\mu} \equiv \frac{d\mathcal{L}}{dX^{\mu\prime}} = -\left[\frac{T_0}{c} + \mu_0 g X_1\right] \frac{(\dot{X}\cdot X')\dot{X}_{\mu} - (\dot{X})^2 X'_{\mu}}{\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}} \tag{29}$$

and

$$\mathcal{P}^{X}_{\mu} \equiv \frac{d\mathcal{L}}{dX^{\mu}} = \begin{pmatrix} 0 \\ -\mu_0 g\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2} \\ 0 \\ 0 \end{pmatrix}.$$

We can now rewrite the variation in the Nambu-Goto action δS in terms of $\mathcal{P}^{\tau}_{\mu}$, $\mathcal{P}^{\sigma}_{\mu}$ and $\mathcal{P}^{X}_{\mu}$

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} \left[\mathcal{P}^{\tau}_{\mu}\frac{d(\delta X^{\mu})}{d\tau} + \mathcal{P}^{\sigma}_{\mu}\frac{d(\delta X^{\mu})}{d\sigma} + \mathcal{P}^{X}_{\mu}\delta X^{\mu}\right] d\sigma = 0. \tag{30}$$

We then make use of the chain rule to separate the equation into a part that is easy to integrate and a part that depends only on δX and not on its derivative, we get

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} \left[\frac{d(\mathcal{P}^{\tau}_{\mu}\delta X^{\mu})}{d\tau} + \frac{d(\mathcal{P}^{\mu}_{\sigma}\delta X^{\mu})}{d\sigma} + \delta X(\mathcal{P}^{X}_{\mu} - \frac{d\mathcal{P}^{\tau}_{\mu}}{d\tau} - \frac{d\mathcal{P}^{\sigma}_{\mu}}{d\sigma})\right] d\sigma = 0. \tag{31}$$

The first term of the integrand can easily be integrated over τ. The result is a term which must be evaluated in τ_i and τ_f. X is fixed in τ_i and τ_f, since we will only look at variations in X *between* these two points. The variations $\delta X(\tau_i, \sigma) and \delta X(\tau_f, \sigma)$ are therefore set to be 0. So this term vanishes.

The second term can be integrated over σ first and be evaluated in zero and σ_1. It depends on the boundary conditions what the outcome of this will be. We try to find solutions for the motion of a closed string, this implies periodic boundary conditions. We can identify the point $X^{\mu}(\tau, \sigma)$ with the point $X^{\mu}(\tau, \sigma + \sigma_1)$. The variation in the points 0 and sigma are equal: $\delta X^{\mu}(\tau, 0) = \delta X^{\mu}(\tau, \sigma_1)$. Therefore, this whole term will vanish as well for any solution to our problem. There is only one term left in the variation of the action. Since we state that the variation in action under a small variation of X must be 0 this last term must be equal to zero too. Since δX represents the small variation, it is definitely not equal to zero. Therefore $\frac{d\mathcal{P}^{\tau}_{\mu}}{d\tau} + \frac{d\mathcal{P}^{\sigma}_{\mu}}{d\sigma} - \mathcal{P}^{X}_{\mu}$ must be 0. So the equations of motion can be written as:

$$\frac{d\mathcal{P}^{\tau}_{\mu}}{d\tau} + \frac{d\mathcal{P}^{\sigma}_{\mu}}{d\sigma} = \mathcal{P}^{X}_{\mu}. \tag{32}$$

5 Solving the equation of motion

Choose a parametrization

In the previous paragraph the following partial differential equations were found. In this paragraph we try solve these equations.

$$\frac{d\mathcal{P}^{\tau}_{\mu}}{d\tau} + \frac{d\mathcal{P}^{\sigma}_{\mu}}{d\sigma} = \mathcal{P}^{X}_{\mu}. \tag{33}$$

The solution of this differential equation describes the world-sheet of the string. This solution is parameterized by the two parameters τ and σ. Until now we did not specify what τ and σ are, which give us the freedom to chose how to parameterize. First we fix τ to be X^0. Since X^0 represents the timecoordinate t, $\frac{dX}{d\tau}$ becomes $\frac{dX}{dt}$ which is recognizable as the (transversal) velocity. It can be shown that, with this parameterization of τ, the following relation between the area element dA and the transversal velocity is true[2].

$$\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2} = c\frac{ds}{d\sigma}\sqrt{1 - \frac{v^2}{c^2}} \tag{34}$$

[2]See Zwiebach page 120-122.

The parameterization of σ is still free to choose. Since τ represents only a displacement in time it seems logical to choose σ as a spatial-parameter. From that parametrization it follows that $\frac{dX}{dt} \cdot \frac{dX}{d\sigma} = 0$. This result simplifies (33) and (34) because one whole term is 0. We now write down the equations of motion with τ and σ parameterized as explained above:

$$\frac{1}{c^2}\frac{d}{dt}\left[(T_0 + \mu_0 gcX_1)\frac{\frac{ds}{d\sigma}\dot{X}^\mu}{\sqrt{1-\frac{v^2}{c^2}}}\right] - \frac{d}{d\sigma}\left[(T_0 + \mu_0 gcX_1)\sqrt{1-\frac{v^2}{c^2}}\frac{X'^\mu}{\frac{ds}{d\sigma}}\right] = \begin{pmatrix} 0 \\ -\mu_0 gc\sqrt{1-\frac{v^2}{c^2}} \\ 0 \\ 0 \end{pmatrix} \tag{35}$$

There is still some freedom in the parameterization because we did not totally fix σ. Because we chose τ to be perpendicular to σ only a function of σ only can be parameterized in a desired way. A function that also depend on t could not be parametrized to any desired function because we already fixed τ. To get a time independent function we will look at the differential equation with $\mu = 0$. We already know what the solution of this equation should be, since we chose our parameterization $X_0 = t$. Derivatives with respect σ are 0 and $\frac{dX_0}{dt} = 1$ which leave us with the equation

$$\frac{1}{c^2}\frac{d}{dt}\underbrace{\left[(T_0 + \mu_0 gcX_1)\frac{\frac{ds}{d\sigma}}{\sqrt{1-\frac{v^2}{c^2}}}\right]}_{f} = 0. \tag{36}$$

This equation tells us that the derivative of some function, lets call it f, with respect to time is equal to 0. Of course this implies that the function f is independent of time. We now fix σ in such way that $\underbrace{\left[\frac{T_0}{c^2} + \frac{\mu_0 g}{c}X_1\right]\frac{\frac{ds}{d\sigma}}{\sqrt{1-\frac{v^2}{c^2}}}}_{f} = 1$. After applying this extra parameterization condition we can simplify the equations of motions to

$$\frac{1}{c^2}\frac{d^2X^\mu}{dt^2} - \frac{d}{d\sigma}\left[(T_0 + \mu_0 gcX_1)^2\frac{dX^\mu}{d\sigma}\right] = \begin{pmatrix} 0 \\ -\mu_0 gc\sqrt{1-\frac{v^2}{c^2}} \\ 0 \\ 0 \end{pmatrix}. \tag{37}$$

$$\frac{1}{c^2}\frac{d^2X^\mu}{dt^2} - (T_0 + \mu_0 gcX_1)^2\frac{d^2X^\mu}{d\sigma^2} - 2(T_0 + \mu_0 gcX_1)\mu_0 gc\frac{dX_1}{d\sigma}\left(\frac{dX^\mu}{d\sigma}\right) = \begin{pmatrix} 0 \\ -\mu_0 gc\sqrt{1-\frac{v^2}{c^2}} \\ 0 \\ 0 \end{pmatrix}. \tag{38}$$

To be clear, X^μ is a vector, so we have to solve a set of differential equations. One of the equations is different from the others. For $\mu = 1$, it looks as if this is a differential equation that depends on only one unknown function X_1. However, we must be aware that the v in this equation is the transversal velocity, which is not a constant, but a function of the vector X^μ and derivatives of this vector. It is important to know how this whole term depends on X^μ and its derivatives. This maybe simplifies the equation, but it can also make it even more complicated if it turns out that it also depends on other components of the vector X^μ then just X_1. Recall that

$$\begin{gathered}\left[\frac{T_0}{c^2} + \frac{\mu_0 g}{c}X_1\right]\frac{\frac{ds}{d\sigma}}{\sqrt{1-\frac{v^2}{c^2}}} = 1 \\ \text{and } \sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2X'^2} = c\frac{ds}{d\sigma}\sqrt{1-\frac{v^2}{c^2}}.\end{gathered} \tag{39}$$

First multiply these two equations with each other and then divide both sides with $\frac{\frac{ds}{d\sigma}}{\sqrt{1-\frac{v^2}{c^2}}}$.

This results in

$$\left[\frac{T_0}{c^2} + \frac{\mu_0 g}{c}X_1\right]\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2X'^2} = c\left(1-\frac{v^2}{c^2}\right). \tag{40}$$

Remember that we choose the parameterization in such way that $\frac{dX}{dt} \cdot \frac{dX}{d\sigma} = 0$, which allows to simplify the equation as

$$\left[\frac{T_0}{c} + \mu_0 g X_1\right] \sqrt{\dot{X}^2 X'^2} = c^2 \left(1 - \frac{v^2}{c^2}\right). \tag{41}$$

We are now very close to the desired result. As a last step we take the square root on both sides of the equation and multiply by $-\mu_0 g$ and find that

$$-\mu_0 gc \sqrt{1 - \frac{v^2}{c^2}} = -\sqrt{\left(\frac{T_0}{c} + \mu_0 g X_1\right) \|\dot{X}\| \|X'\|}. \tag{42}$$

One of the differential equations can be written as

$$\frac{1}{c^2} \frac{d^2 X^1}{dt^2} - (T_0 + \mu_0 gc X_1)^2 \frac{d^2 X^1}{d\sigma^2} - 2 (T_0 + \mu_0 gc X_1) \mu_0 gc \left(\frac{dX^1}{d\sigma}\right)^2 = -\sqrt{\left(\frac{T_0}{c} + \mu_0 g X_1\right) \|\dot{X}\| \|X'\|}. \tag{43}$$

We can now conclude that even this differential equation depends on more then one unknown function and so did the others. Now we are left with three coupled differential equations. The equations are all very different that we cannot even combine the equations to get a solvable one. Several attempts to solve these equations failed and we conclude that there is no analytical solution to the problem. There maybe is a numerical solution to this set of equations. However, we did not tried to find this numerical solutions since these difficult equations indicates that we might be on the wrong way with our approach. We will discuss this in the next section.

6 Discussion

We were able to derive equations of motion for a one dimensional string in a constant gravitational force field. The derived equations however are very hard to solve. The differential equations we found are all coupled differential equations. There is no equation that depends only on one of the functions for which we try to solve the equation. The equations are coupled in such a complicated way, that we failed to find an analytical solution to this problem. We could have tried to find an numerical solution to the equations of motion. However, we stopped at this point to question ourselves, wether we were on the right track with this approach.

General relativity is based on the equivalence principle, which states that a observer in a gravitational field has equivalent observations as an observer that moves with constant acceleration. This is a powerful statement, which directly leads to the conclusion that a string in a constant gravitational field should move with constant acceleration. Since this analytical solution is definitely not a solution to the equations of motion we know that our approach is wrong.

We could ask ourself where we made a wrong assumption. We have to remember that the way we added the potential to our equations was not totally justified. We already mentioned before that the multiplication of the additional potential with the surface area also multiplies with the kinetic energy of the string. So apparently we cannot approximate the action in this way. We should define the action more carefully for reasons mentioned above. If we did not also multiply with the kinetic energy we might find the expected outcome.

To make sure the motion of the string is derived from correct equations we should involve general relativity in our description of the gravitational field. In general relativity the addition of a gravitational field (the presence of mass) is not described by adding a potential field but by a curved spacetime, which in mathematical sense means that the Minkowski-metric should be replaced by another metric $g_{\mu\nu}()x$, which depends on the position of the particle. In other words: a curved spacetime. For a weak gravitational field one could argue that the metric should be very similar to the Minkowski-metric. Therefore it should be possible to give a good linear approximation of the field by writing $g_{\mu\nu}$ as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. With $h_{\mu\nu}$ a small variation on the metric. For the gravitational field of the problem we tried to solve, the variation $h_{\mu\nu}$ is described by the metric

$$h_{\mu\nu} = \begin{pmatrix} dX_1 & 0 & 0 & 0 \\ 0 & dX_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{44}$$

Describing the gravitational field this way lead to a more accurate solution for the motion of the string, since it is more general then the classical approach. The classical description is only valid for limited situations.

As an extension of our research one could think of a more interesting gravitational field. A radial gravitational force with a magnitude that drops with $1/r^2$ since this is a more realistic model for the gravitational force. We think the best approach for this problem would be to again, describe the curved spacetime with a metric. First a correct metric should be found and then new equations of motion can be derived. When this is accomplished our next step would be to look at the behavior of a string close to the event horizon of a black hole. The metric that describes a black hole is a well known one. It is described by the Schwarzshild metric. In the article from de Vega and Egusquiza [3] the motion of a string in such a curved space is computed numerical and the result is being visualized. In the same article multiple variations on this curved spacetime are described as well. From this article we learn how powerful the metric description from general relativity is also in the describing strings. Finally, It would also be interesting to generalize this problem to higher dimensions since the string theory needs more then three spatial dimensions to describe physics.

7 Conclusion

In this paper we derived the equations of motion for a one dimensional relativistic string moving in a classical gravitational potential. This was done by varying the Nambu-Goto action for a relativistic string. Then we tried to solve the obtained differential equations. We tried to solve these equations but we failed to find an analytical solution. We can conclude that approach that we used to define the action of the string was wrong. The wrong assumption led to equations that were too hard to solve. The best continue this research would be to involve general relativity in to our description of the problem.

References

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