Black holes and Holography

The Einstein equation derived from a holographic principle

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abstract

The laws of black hole mechanics are reviewed in order to show that surfaces orthogonal to generators of null hypersurfaces are related to the amount of energy passing through these null hypersurfaces. The energy is related to entropy by $T\mathrm{d}S = \mathrm{d}Q$. The holographic principle is introduced as the recognition of the proportionality of surface areas with entropy. The holographic principle is then assumed to be a fundamental principle of nature. A realisation of the principle is the (generalised) covariant entropy bound. Local light sheets are constructed for the generalised covariant entropy bound. The Einstein equation can be derived if a holographic scaling of entropy is assumed. The Einstein equation is derived in two settings, a thermodynamic setting, which is a generalisation of a derivation given by Jacobson, and a holographic principle setting.

Contents

Chapter 1

Introduction

Quantum field theory knows how to treat all forces except gravity. Quantum field theory is renormalisable on background metrics, but it is not renormalisable when gravity is given field equations of its own. The reason for this lies in the fact that the coupling constant is not dimensionless, and therefore it cannot be absorbed by field redefinition. Apparently gravity is not understood well enough to be treated correctly as a quantised field. In order to acquire a better understanding of gravity it is necessary to look at the regimes where the existing theories break up. In this thesis the theory of general relativity is considered.

The theory of general relativity has as fundamental ingredient the principle of equivalence. The principle of equivalence states that in any point in the space-time a coordinate transformation can be found such that the gravitational field is transformed away. Locally a free particle always sees a flat space-time, and as long as gravity is a negligible force all particle interactions can be described in a locally flat space-time. It is also possible to describe all particle interactions on some background space-time metric if the interaction with gravity is negligible. This is expected to break up at extremely small scales of the Planck length, $l_p = \sqrt{\frac{G\hbar}{c^3}}$. The metric is expected to be quantised and the state of the metric is a superposition of mutually orthogonal states.

At large scales the theory of general relativity breaks up when black holes form. These objects are solutions to the Einstein field equations with a curvature singularity. The physics of black holes is peculiar, because classically what goes in doesn't come out, it ends on the singularity. The classic black hole is an object which can be in only one state. The information that went in appears to be lost, because it can't be encoded in the black hole state. However, it is possible to derive thermodynamic laws for black holes. If the black hole can be described by thermodynamics, then the black hole is a perfect absorber, which means that it must radiate as a black body at some temperature. Quantum mechanically this radiation can be understood as vacuum fluctuations, a negative energy particle drops into the black hole, and a positive energy particle escapes, due to conservation of momentum. Quantum field theory on a black hole background indicates that black holes are unstable, they evaporate if the space-time outside the horizon has lower temperature than the black hole itself.

At the extremely small scales near the Planck length there are quantum field fluctuations of arbitrary high energy. This makes the gravitational field very dynamic and very strong locally. Gravity will then certainly affect the interactions between the fluctuations at these energy scales. Intuitively, high energy fluctuations will collapse to micro-black holes, which are very hot. These micro black hole states are expected to evaporate immediately, or instantaneously. This intuitive picture may not be entirely correct, because as mentioned above the metric is expected to be in a superposition of states when gravity is quantised.

If the existence of black holes at small scales is assumed, then an implication of this is that there too the entropy must scale with black hole surface area. If a fluctuation has enough energy it must collapse into a black hole with an event horizon, and the second law of black hole thermodynamics implies that the entropy of this micro black hole is proportional to its horizon. If a unitary evolution of the quantised fields is assumed, then this means that the number of mutually orthogonal states of the quantised fields is less than quantum field theory indicates, i.e. in quantum field theory the entropy scales with the volume of a system. Not all black hole solutions to the Einstein equation have event horizons, but in this thesis the cosmic censorship conjecture is assumed, i.e. there are no naked singularities, they are all hidden behind an event horizon. The formation of black holes appears to be a natural cut off for the maximum amount of energy present in any region of space-time.

This thesis is divided into two parts, first there is a part about entropy and geometry. Space-time can be interpreted as a manifold, and the laws of black hole mechanics, which can be derived from geometric principles and the Einstein equation, have a thermodynamic interpretation, thus relating entropy and geometry. The second law of black hole thermodynamics can be elevated to a fundamental principle holding for any region of any space-time, this is the holographic principle. It is shown how dynamic local holographic screens and local light sheets can be constructed.

The second part uses the holographic scaling of entropy as fundamental input in derivations of the Einstein equation. In this part the Einstein equation is derived locally in a thermodynamic setting. The derivation presented here is a generalisation of a procedure proposed by Jacobson [10]. The derivation is also carried out in a holographic setting, using local light sheets. These derivations can be performed on arbitrary manifolds. It is found that the holographic principle is consistent with classical general relativity.

Notation

The following notation conventions are used in this thesis

- Greek indices $\kappa, \lambda, ...$ in general run over four space-time coordinates in a general coordinate system
- Greek indices $\alpha, \beta, ...$ in general run over four space-time inertial coordinates, index zero being the time coordinate.
- Repeated indices are summed.
- Cartesian three vectors are indicated by $\vec{v}$.
- All quantities are expressed in Planck units, i.e. $c = \hbar = G = 1$. A conversion table to cgs units can be found in [2].

- Shorthands for the covariant derivative of a tensor $T_{\mu\nu\ldots}$ are $T_{\mu\nu\ldots;\lambda}$ and $D_\lambda T_{\mu\nu\ldots}$.
- Shorthands for the ordinary partial derivative of a tensor $T_{\mu\nu\ldots}$ are $T_{\mu\nu\ldots,\lambda}$ and $\partial_\lambda T_{\mu\nu\ldots}$.

Part I

From black hole mechanics to Holography

Chapter 2

Mechanics of black hole space-times

The purpose of this chapter is to illuminate the four laws of black hole mechanics, which are interpreted as the space-time analogue of the four laws of thermodynamics. The Einstein equation is the fundamental input for these laws of space-time mechanics, therefore the story begins with the original input used in the derivation of the Einstein equation. After that the zeroth law is derived explicitly, in order to show that the surface gravity over the horizon of a stationary black hole is constant, just as the temperature is constant throughout a system in equilibrium. Then the First law is derived in order to show that surface area of a black hole horizon is a measure for the energy contained in the black hole. The second and third law will be introduced only heuristically. Since the laws are derived for stationary axisymmetric space-times which are asymptotically flat, the Kerr-Newman space-time will be introduced to show how it works in a physical environment. More on these subjects can be found in e.g. [2] or [7].

2.1 The Einstein equation

The principle of equivalence states that in an arbitrarily strong gravitational field there exists in every point p a general coordinate transformation such that the metric in this point becomes flat, i.e. the metric $g_{\mu\nu}$ is locally the Minkowski metric, $\eta_{\alpha\beta}$, with vanishing first derivatives, $\eta_{\alpha\beta,\gamma}(p) = 0$. This means that in a point x in the neighbourhood of p the metric $\eta_{\alpha\beta}$ differs at most a factor $(x-p)^2$ so strong gravitational fields are transformed to local weak gravitational fields. The consequences for the local equations of motion are considerable, because in the neighbourhood of p they reduce to ordinary linear differential equations. It is much easier to solve the local ordinary differential equations and the equations of motion in the strong field can be recovered by transforming back to the original coordinates.

In the weak field limit Newton's law of gravitation holds, which can be stated in terms of metric components and energy momentum tensor components by

$$\nabla^2 g_{00} = -8\pi T_{00} \tag{2.1}$$

If the assumption is made that such an equation holds for all components of the energy momentum tensor then one can write down the Lorentz covariant tensor equation

$$G_{\alpha\beta} = -8\pi T_{\alpha\beta} \tag{2.2}$$

with $G_{\alpha\beta}$ a tensor consisting of the metric and its derivatives. Upon transforming this equation back to the strong field coordinate system this equation will read

$$G_{\mu\nu} = -8\pi T_{\mu\nu} \tag{2.3}$$

The tensor $G_{\mu\nu}$ is a tensor by definition, and because the energy momentum tensor $T_{\mu\nu}$ on the right hand side has the properties that it is symmetric and divergence free, the left hand side must have these properties too,

$$G_{\mu\nu} = G_{\nu\mu} \tag{2.4}$$

$$G^{\mu\nu}{}_{;\mu} = 0 \tag{2.5}$$

Furthermore the tensor $G_{\mu\nu}$ is assumed to be scale invariant, which means that $G_{\mu\nu}$ consists of terms containing two derivatives of the metric, i.e. terms quadratic in first derivatives of the metric and linear in second derivatives of the metric. The final requirement imposed on $G_{\mu\nu}$ is that in the weak field limit it reduces to the equation (2.1), so the time time component reads in this limit $G_{00} \approx \nabla^2 g_{00}$.

The only tensor which satisfies these requirements is [1]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R \tag{2.6}$$

where $R_{\mu\nu}$ is the Ricci tensor and R its contraction, see A.3 for their definitions. The Einstein equation for gravitational fields thus reads

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi T_{\mu\nu} \tag{2.7}$$

or equivalently[1],

$$R_{\mu\nu} = -8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}{}_{\lambda}) \tag{2.9}$$

If terms of lower order in the derivatives are allowed, then the only possible addition to $G_{\mu\nu}$ is a term proportional to the metric itself, for it is always possible to transform to a coordinate system where the first derivatives of the metric vanish, the Einstein equation with such an addition is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \lambda g_{\mu\nu} = -8\pi T_{\mu\nu} \tag{2.10}$$

where λ is the cosmological constant. This term must be very small if the tensor $G_{\mu\nu}$ has to satisfy the requirement that the time time component reduces to $\nabla^2 g_{00}$ in the weak field limit

[1] Contraction of the Einstein equation (2.7) yields the identity

$$R = 8\pi T^{\lambda}{}_{\lambda} \tag{2.8}$$

inserting this into the Einstein equation will result in equation (2.9).

2.2 Stationary axisymmetric space-time

In this section a stationary axisymmetric space-time [2] will be considered, because space-times with these properties posses infinitesimal isometries [3] which yield the well known conserved quantities energy and angular momentum.

Consider a metric $g_{\mu\nu}$ associated with some stationary rotating axisymmetric matter configuration and define coordinates

$$(x^0, x^1, x^2, x^3) \equiv (t, \varphi, r, \theta)$$

The metric is stationary,

$$\frac{\partial g_{\mu\nu}}{\partial x^0} = 0 \tag{2.11}$$

and axisymmetric,

$$\frac{\partial g_{\mu\nu}}{\partial x^1} = 0 \tag{2.12}$$

so the metric hasn't got any x^0 and x^1 dependencies and is written

$$g_{\mu\nu} = g_{\mu\nu}(x^2, x^3) \tag{2.13}$$

A stationary axisymmetric metric is symmetric (form invariant) under the mapping that reverses time, $t \to \tilde{t} = -t$, as well as the direction of rotation [4] $\varphi \to \tilde{\varphi} = -\varphi$. The transformation law for a form invariant metric is, see appendix A.4,

$$\tilde{g}_{\mu\nu}(x) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x)$$

and this law yields $g_{02} = g_{03} = g_{12} = g_{13} = 0$, consider e.g.. the component g_{02}, then

$$\begin{aligned} \tilde{g}_{02}(x) &= \frac{\partial x^\rho}{\partial \tilde{x}^0}\frac{\partial x^\sigma}{\partial \tilde{x}^2} g_{\rho\sigma}(x) \\ &= \frac{\partial x^0}{\partial \tilde{x}^0}\frac{\partial x^2}{\partial \tilde{x}^2} g_{02}(x) = (-1)(+1)g_{02}(x) = -g_{02}(x) \end{aligned}$$

but the form invariance of the metric means $\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x)$, and then $g_{02} = \tilde{g}_{02} = -g_{02} = 0$. It is concluded that the coordinate vectors $\frac{\partial}{\partial x^i}$ with $i \in \{2, 3\}$ are orthogonal to the coordinate vectors $\frac{\partial}{\partial x^a}$ with $a \in \{0, 1\}$. The components g_{ij} with $i, j \in \{2, 3\}$ are a metric on a 2-dimensional Riemannian submanifold [5] and can be brought in diagonal form independent of the other components by a coordinate transformation.

[2] It is possible to prove for black holes that the requirement that it is stationary implies axisymmetry, [2].

[3] See appendixA.4 for more on isometries and the Killing vector fields related with them.

[4] Consider e.g.. the case in which only time is reversed, if the object rotated clockwise before this reversal, then it will rotate counter clockwise after the reversal, and therefore time reversal isn't a symmetry. The same holds if only the direction of rotation is reversed.

[5] I.e. a manifold with a metric of signature $(1, 1)$.

The general form of the metric $g_{\mu\nu}$ of a stationary axisymmetric space-time is

$$\mathrm{d}s^2 = -A\mathrm{d}t^2 + B(\mathrm{d}\varphi - \omega\mathrm{d}t)^2 + C\mathrm{d}x^{2^2} + D\mathrm{d}x^{3^2} \tag{2.14}$$

where A, B, C, D, and ω are functions of the coordinates x^2 en x^3 only. These functions are in terms of the metric components

$$\begin{aligned} A &= -g_{tt} + \frac{g_{t\varphi}^2}{g_{\varphi\varphi}} \\ B &= g_{\varphi\varphi} \\ C &= g_{22} \\ D &= g_{33} \\ \omega &= -\frac{g_{t\varphi}}{g_{\varphi\varphi}} \end{aligned}$$

This metric has $g_{\mu\nu,0} = 0$ and $g_{\mu\nu,1} = 0$, therefore the Lie derivative[6] with respect to the coordinate vector fields $\xi = \delta_0{}^\mu \frac{\partial}{\partial x^\mu}$ and $\psi = \delta_1{}^\mu \frac{\partial}{\partial x^\mu}$ of the metric vanishes, so these are the timelike and rotational Killing vector field respectively.

2.2.1 Constants of the motion

The Killing vector fields of a metric give rise to constants of the motion, this can be seen from the equations of motion, consider a free particle with velocity $v^\mu(\tau)$, then the equation of motion is

$$\begin{aligned} 0 &= v_{\mu;\nu}v^\nu && (2.15) \\ &= v_{\mu,\nu}v^\nu - \Gamma^\sigma{}_{\mu\nu}v_\sigma v^\nu && (2.16) \\ &= v_{\mu,\nu}v^\nu - \frac{1}{2}g_{\sigma\nu,\mu}v^\sigma v^\nu && (2.17) \end{aligned}$$

thus a stationary axisymmetric metric gives rise to 2 constants of the motion,

$$\begin{aligned} \frac{\mathrm{d}v_0}{\mathrm{d}\tau} = v_{0,\nu}v^\nu &= \frac{1}{2}g_{\sigma\nu,0}v^\sigma v^\nu = 0 && (2.18) \\ \frac{\mathrm{d}v_1}{\mathrm{d}\tau} = v_{1,\nu}v^\nu &= \frac{1}{2}g_{\sigma\nu,1}v^\sigma v^\nu = 0 && (2.19) \end{aligned}$$

The energy $E = v_0 = g_{0\nu}v^\nu$ associated with the timelike Killing vector field[7] ξ and angular momentum $J = v_1 = g_{1\nu}v^\nu$ associated with the rotational Killing vector field[8] φ.

The third constant of the motion is the proper time τ of the observer, proper time runs constantly for the observer,

$$-C = g_{\mu\nu}v^\mu v^\nu = Ev^t + Jv^\varphi + g_{rr}v^r v^r + g_{\theta\theta}v^\theta v^\theta \tag{2.20}$$

where $C = -1$ for a timelike observer (all physical observers are timelike), $C = 0$ for a null observer (e.g.. a light ray), and $C = -1$ for a spacelike observer (e.g.. a tachyonic particle).

[6]The Lie derivative can be found in section A.1.1 for the Lie derivative.
[7]That is $g_{\mu\nu,0} = g_{\mu\nu,\lambda}\xi^\lambda = 0$.
[8]That is because $g_{\mu\nu,0} = g_{\mu\nu,\lambda}\varphi^\lambda = 0$.

Note that local Killing vector fields give rise to local constants of the motion, i.e. space-time is locally Minkowski, the metric is expressed on a basis of 4 local Killing vector fields, 1 timelike and 3 spacelike, which correspond to local conservation of energy and momentum. Local Minkowski space-time in spherical polar coordinates immediately shows that angular momentum is also conserved locally.

In interactions one expects the energy and momentum to be conserved too, but if for example 2 neutral massless point particles collide this can not be described by a smooth vector field, e.g.. an elastic collision causes a kink in the paths of the particles. The particles can be supposed to travel on geodesics as long as they don't collide, so the energy and momentum are only conserved if the constants of the motion of both particles are taken into account. If this all happens on some non stationary manifold then the energy will not be conserved when the particles move outside each others neighbourhood.

2.2.2 Locally non-rotating observers

A stationary axisymmetric metric has, in general, non zero angular momentum, so it is impossible for an observer to be at rest, the observer is dragged along by the metric and cannot remain static. However there is a close analogy to static observers, locally non rotating observers.

A locally non-rotating observer is an observer that is "static" with respect to the hypersurfaces S which have time[9] $t =$ constant, i.e. the velocity u^μ of a locally non-rotating observer obeys $u_\mu \propto t_{;\mu}$. In other words, the locally non rotating observers move on a world line orthogonal to a hypersurface. The events happening on this hypersurface happen at equal time for the observer at infinity. These "static" observers are instantaneously at rest, but their acceleration is in general non zero, because the timelike vector field is ξ is only orthogonal to an equal time slice at infinity. Now follow some useful properties of locally non-rotating observers.

The angular momentum J of the locally non-rotating observer is zero. That is because the rotational Killing vector field is $\varphi^\mu = \delta_\varphi{}^\mu$

$$J = g_{\mu\nu} u^\mu \varphi^\nu \tag{2.21}$$

$$= u_\varphi \propto t_{;\varphi} = 0 \tag{2.22}$$

The proportionality in the second line is because the locally non rotating observers are defined by $u_\mu \propto t_{;\mu}$.

The angular momentum $J = 0$ is also given by

$$J = g_{\mu\nu} u^\mu \psi^\nu = g_{00} u^0 + g_{01} u^1 = 0 \tag{2.23}$$

and therefore

$$\frac{u^0}{u^1} = -\frac{g_{01}}{g_{11}} \tag{2.24}$$

The components of the velocity are given by

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{2.25}$$

[9] The coordinate is called time coordinate, because static observers at infinity move on the Killing vector field ξ, which is the generator of the time translations in a stationary axisymmetric metric.

where τ is a parameter. Then

$$\frac{u^1}{u^0} = \frac{\mathrm{d}\phi}{\mathrm{d}t} \tag{2.26}$$

This means that the angular velocity $\frac{\mathrm{d}\phi}{\mathrm{d}t}$ of the locally non rotating observers is

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{g_{01}}{g_{11}} \tag{2.27}$$

A final consequence is that a locally non rotating lightlike observer with velocity $u^\mu = \left(u^0, u^1, 0, 0\right)^\mu$ can only exist if it has $E = J = 0$ and $\frac{g_{00}}{g_{01}} = \frac{g_{01}}{g_{11}}$. This is an immediate consequence of the equations of motion (2.20), which can only be solved if both $J = 0$ and $E = 0$, for suppose that only $J = 0$, then

$$0 = g_{\mu\nu}u^\mu u^\nu = Eu^0 + Ju^1 = Eu^0 \tag{2.28}$$

but $u^0 \neq 0$. $E = 0$ means

$$0 = E = g_{00}u^0 + g_{01}u^1 \tag{2.29}$$

so

$$-\frac{g_{00}}{g_{01}} = \frac{u^1}{u^0} = -\frac{g_{01}}{g_{11}} \tag{2.30}$$

2.3 Conserved quantities: Komar integrals

Killing vector fields give rise to constants of the motion, e.g.. the energy of a free particle in a stationary metric is conserved. It is also possible to define conserved quantities for regions of space-time using Killing vector fields, the quantities are charge and current. The conserved charge Q_ξ for a Killing vector field ξ^μ is defined by the *Komar integral*. Consider a space-time volume V on a spacelike hypersurface Σ with boundary ∂V. Every Killing vector field ξ^μ on this hypersurface has a Komar integral associated with it,

$$Q_\xi\left(V\right) = \frac{c_\xi}{16\pi}\oint_{\partial V} \mathrm{d}S^{\mu\nu}\xi_{\nu;\mu} \tag{2.31}$$

where c_ξ is some constant. The Stokes theorem [10] says that this expression is equivalent to

$$Q_\xi\left(V\right) = \frac{c_\xi}{8\pi}\int_V \mathrm{d}S^\mu \xi^\nu{}_{;\mu;\nu} \tag{2.32}$$

Killing vector fields have the property

$$\xi_{\mu;\rho;\sigma} = -R^\lambda{}_{\sigma\rho\mu}\xi_\lambda \tag{2.33}$$

Upon contraction this yields

$$\xi^\mu{}_{;\rho;\mu} = -R^\lambda{}_\rho\xi_\lambda \tag{2.34}$$

[10] The form of the Stokes theorem used here is also called Gauss theorem, see also [2].

and therefore

$$\begin{aligned} Q_\xi(V) &= -\frac{c_\xi}{8\pi}\int_V \mathrm{d}S_\mu R^{\mu\nu}\xi_\nu \\ &\equiv \int_V \mathrm{d}S_\mu J^\mu(\xi) \end{aligned} \tag{2.35}$$

The J^μ defined here is a conserved current, because

$$\begin{aligned} J^\mu{}_{;\mu} &= c_\xi\left(R^{\mu\nu}\xi_\nu\right)_{;\mu} \\ &= c_\xi R^{\mu\nu}{}_{;\mu}\xi_\nu + c_\xi R^{\mu\nu}\xi_{\nu;\mu} \\ &= 0 \end{aligned} \tag{2.36}$$

The first term in this expression is zero because the Bianchi identity for the curvature tensor,

$$\begin{aligned} R^{\mu\nu}{}_{;\mu}\xi_\nu &= \frac{1}{2}g^{\mu\nu}R_{;\mu}\xi_\nu \\ &= \frac{1}{2}R_{,\mu}\xi^\mu = 0 \end{aligned} \tag{2.37}$$

The last equality holds because the metric can be expressed in coordinates such that $\xi^\mu = \delta^\mu_\xi$ is a coordinate vector. The metric is independent of this coordinate $g_{\mu\nu,\lambda}\xi^\lambda = g_{\mu\nu,\xi} = 0$ and then R is independent of the coordinate associated with ξ^μ too. The second term is zero because $R^{\mu\nu} = R^{\nu\mu}$ and $\xi_{\mu;\nu} = -\xi_{\nu;\mu}$, so

$$R^{\mu\nu}\xi_{\nu;\mu} = -R^{\mu\nu}\xi_{\nu;\mu} = 0 \tag{2.38}$$

The fact $J^\mu{}_{;\mu} = 0$ can be stated alternatively as

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{g}J^\mu\right) = 0 \tag{2.39}$$

The current J^μ is conserved so if J^μ vanishes at the boundary ∂V of the domain of integration, then the charge $Q_\xi(V)$ is conserved. By way of the Einstein equation with $\lambda = 0$, equation (2.9) this current can be stated in terms of the energy momentum tensor,

$$J^\mu = -c_\xi(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T^\lambda{}_\lambda)\xi_\nu \tag{2.40}$$

or for non zero cosmological constant, that is using equation (2.10)

$$J^\mu = -c_\xi(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T^\lambda{}_\lambda)\xi_\nu - c_\xi\frac{1}{8\pi}\lambda g^{\mu\nu}\xi_\nu \tag{2.41}$$

In particular asymptotically flat space-times satisfy the requirement that the fields vanish at infinity, and then an asymptotically flat stationary axisymmetric space-time has a Komar integral for total mass associated with ξ^μ and a Komar integral for total angular momentum associated with φ^μ, where the constants are $c_\xi = -2$ and $c_\varphi = 1$ respectively.

The difference between the conserved quantities defined using Komar integrals and the Nöther theorem lies in the fact the Nöther conserved charges and currents are found from a variational principle. A system which posses a continuous local symmetry, for example invariance under infinitesimal time translations, has an action which is stationary under small time variations. The charge is the energy of the system, and the conserved current is the momentum. The Komar integrals relate the Killing vector field on the boundary of a system to a charge contained in that system. Note that for a time translation Killing vector field ξ^μ the energy momentum current is the $p^\mu = T^{\mu 0} - g^{\mu 0} T^\lambda{}_\lambda$, whereas the energy momentum current associated with the Nöther charge is $p^\mu = T^{\mu 0}$.

2.4 Kerr-Newman space-time

The Kerr-Newman metric is the stationary axisymmetric solution to the vacuum Einstein Maxwell equations,

$$\begin{aligned} G_{\mu\nu} &= -8\pi T_{\mu\nu}(F) && (2.42)\\ F^{\mu\nu}{}_{;\mu} &= 0 && (2.43)\\ T_{\mu\nu}(F) &= \frac{1}{4\pi}\left(F_{\mu\rho}F^\rho{}_\nu - \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right) && (2.44)\\ F_{\mu\nu} &= A_{\mu;\nu} - A_{\nu;\mu} && (2.45)\end{aligned}$$

i.e. the Kerr-Newman space-time is empty except for electromagnetic fields.

The Kerr-Newman metric in Boyer-Lindquist coordinates is [7, 2]

$$\begin{aligned} \mathrm{d}s^2 &= -\frac{\Delta - a^2\sin^2\theta}{\Sigma}\mathrm{d}t^2 - 2a\sin^2\theta\frac{r^2+a^2-\Delta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi \\ &+ \frac{\left(r^2+a^2\right)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\mathrm{d}\varphi^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 \qquad (2.46)\end{aligned}$$

where

$$\begin{aligned} \Sigma &\equiv r^2 + a^2\cos^2\theta \\ \Delta &\equiv r^2 - 2Mr + a^2 + e^2 \end{aligned}$$

The Maxwell 1-form of the metric is

$$A_\mu = \frac{Qr\left((\mathrm{d}t)_\mu - a\sin^2\theta(\mathrm{d}\varphi)_\mu\right) - P\cos\theta\left[a(\mathrm{d}t)_\mu - \left(r^2+a^2\right)(\mathrm{d}\varphi)_\mu\right]}{\Sigma}$$

Note that the metric is asymptotically flat, i.e. the limit $r \to \infty$ yields the Minkowski metric. The Kerr-Newman solution to the Einstein equation reduces to the Kerr solution for $e = 0$, it reduces to the Reissner-Nordstrom solution for $a = 0$, and to the Schwarzschild solution for $a = e = 0$.

The Kerr-Newman metric depends on the four parameters M, a and $e^2 = Q^2 + P^2$. The Komar integrals can be used to show that M is the total mass[11] of the space-time,

$$-\frac{1}{8\pi}\oint_{\partial V}\mathrm{d}S^{\mu\nu}\xi_{\nu;\mu} = M \qquad (2.47)$$

[11] In an empty space-time the only matter present is the matter associated with the gravitational field permeating the space-time, which has the black hole as its source. So M is the mass of the black hole.

$a = \frac{J}{M}$ with J the total angular momentum of the space-time,

$$\frac{1}{16\pi}\oint_{\partial V} \mathrm{d}S^{\mu\nu}\varphi_{\nu;\mu} \quad = \quad Ma = J \tag{2.48}$$

and e the total electric and magnetic[12] charge, Q and P respectively, of the space-time,

$$\frac{1}{8\pi}\oint_{\partial V} \mathrm{d}S^{\mu\nu}A_{\nu;\mu} \quad = \quad e \tag{2.49}$$

as long as the boundary ∂V for all these integrals is placed in the asymptotically flat region of the space-time.

The metric has a true singularity at $\Sigma = 0$ and a coordinate singularity at $\Delta = 0$. If $M^2 > a^2 + e^2$, then $\Delta = 0$ has two solutions, which are

$$r_\pm = M \pm \sqrt{M^2 - a^2 - e^2} \tag{2.50}$$

If $M^2 = a^2 + e^2$, then there is one solution, the coordinate singularity is at $r_\pm = M$. In the case $M^2 < a^2 + e^2$ then there are no solutions to $\Delta = 0$. This means that there is no coordinate singularity and therefore no event horizon. The singularity is called naked. This is deemed unphysical, and usually one assumes the cosmic censorship hypotheses to rule out this case as physical solution to the Einstein equation.

The outer event horizon, the one at r_+, will be the only one under consideration, for an observer outside the black hole will never see the inner horizon, unless he goes into the black hole, but he will never be able to report what it is like in there so this horizon is not of practical importance.

The area A of the horizon is

$$A = \int_{r=r_+} \mathrm{d}\theta \mathrm{d}\varphi \sqrt{g_{\varphi\varphi}g_{\theta\theta}} \quad = \quad 4\pi(r_+^2 + a^2) \tag{2.51}$$

2.4.1 Ergosphere

The Kerr-Newman black holes can have an ergosphere, i.e. a region outside the black hole where the Killing vector field ξ^μ becomes spacelike. This vector field is timelike at infinity and is associated with the mass of the space-time. In the region where it becomes spacelike it can no longer be associated with the flow of energy. The norm of the vector field is

$$\xi_\mu \xi^\mu \quad = \quad \frac{a^2 \sin^2\theta - \Delta}{\Sigma} \tag{2.52}$$

and the ergosphere is the region outside the horizon where the vector field is spacelike,

$$r_+ < r < M + \left(M^2 - e^2 - a^2\cos^2\theta\right)^{\frac{1}{2}} \tag{2.53}$$

At r_+ the vector field is spacelike too. In the Schwarzschild case the ergosphere is not present because $a = e = 0$, so the vector field ξ^μ is timelike everywhere

[12] No magnetic charge is known to exist, but it is a theoretical possibility.

except on the horizon, where it is null. An observer always travels on a timelike orbit, the tangent u^μ to a timelike orbit satisfies by definition $u_\mu u^\mu < 0$. The only term on the left hand side which can be negative in the ergosphere is $2g_{01}u^0u^1$, all other terms are positive.

The horizon corotates with the black hole, i.e. the horizon is locally non rotating, and its angular velocity is $\omega_H = -\frac{g_{01}}{g_{11}} = \frac{a}{r_+^2+a^2}$. The horizon is the null hypersurface[13] generated by the null Killing vector field

$$\chi^\mu = \xi^\mu + \omega_H \varphi^\mu \tag{2.54}$$

Often the spatial surface orthogonal to the generators of the null hypersurface is called the horizon too.

2.5 The 4 laws of black hole mechanics

The event horizons in black hole space-times, i.e. stationary axisymmetric space-times, are null hypersurfaces generated by a Killing vector field. In the next sections the zeroth and first laws of black hole mechanics will be derived, and the second and third will be introduced heuristically.

2.5.1 The zeroth law

Consider the black hole horizon, i.e. a null hypersurface, H generated by a Killing vector field χ^μ orthogonal to H. The following statements are true on H

$$\chi_\mu \chi^\mu = 0 \tag{2.55}$$
$$\chi^\mu{}_{;\nu}\chi^\nu = \kappa\chi^\mu \tag{2.56}$$
$$\chi_{\mu;\nu}\chi^\mu = -\kappa\chi_\nu \tag{2.57}$$

where κ is an arbitrary scalar function, is interpreted as the surface gravity [14] The second equality holds because χ^μ is hypersurface orthogonal on[15] H, and the last holds because of the Killing condition. If the Lie derivative [16] Δ of the second equation is taken with respect to χ^μ then it follows immediately that

$$\Delta_\chi \kappa = 0 \tag{2.58}$$

for χ^μ is a Killing vector field. In other words, κ is parallel transported along orbits of χ^μ.

The geodesic congruence k^μ with affine parameter λ generating the null hypersurface can in a local inertial system be defined by [17]

$$k^\mu = e^{-\kappa p}\chi^\mu \tag{2.59}$$

[13]See appendix A.6 for more on null hypersurfaces.

[14]Heuristically this follows from the fact that the second equation is the geodesic equation in non affine parameterisation. The term at the right hand side is the acceleration of a locally non rotating observer on this world line. The norm of χ^μ is zero, it describes a photon which tries to stay on the horizon. A more acceptable line of reasoning may be the fact that the Schwarzschild black hole has $\chi^\mu = \xi^\mu$, and then timelike static observers satisfy near the horizon $\xi^\mu{}_{;\nu}\xi^\nu = \tilde{\kappa}\xi^\mu$. In the limit $r \to r_+$ the $\tilde{\kappa}$ becomes κ.

[15]See appendix A.6.

[16]See A.1.1 for the Lie derivative.

[17]See appendix A.7, where the function f is κ, which can differ from geodesic to geodesic, but it will be shown that it is constant on the entire null hypersurface.

The parameter p of the Killing vector field depends on the affine parameter λ,

$$\frac{\mathrm{d}\lambda}{\mathrm{d}p} \quad \propto \quad e^{\kappa p} \tag{2.60}$$

so

$$\lambda \quad \propto \quad e^{\kappa p} \tag{2.61}$$

An explicit expression for κ on the null hypersurface can be derived using the Frobenius theorem[18], which states that a vector field χ^μ is orthogonal to a hypersurface if and only if it satisfies the condition (on the hypersurface)

$$\chi_{[\mu;\nu}\chi_{\rho]} \quad = \quad 0 \tag{2.62}$$

A Killing vector field has $\chi_{(\mu;\nu)} = 0$ and this equation is equivalent to

$$\chi_{\mu;\nu}\chi_\rho + \chi_{\rho;\mu}\chi_\nu - \chi_{\rho;\nu}\chi_\mu \quad = \quad 0 \tag{2.63}$$

Contracting this with the tensor $\chi^{\mu}{}_{;}{}^{\nu}$ yields

$$\begin{aligned} \chi^{\mu}{}_{;}{}^{\nu}\chi_{\mu;\nu}\chi_\rho &= -2\chi^{\mu}{}_{;}{}^{\nu}\chi_{\rho;\mu}\chi_\nu && (2.64)\\ &= -2\chi^{\mu}{}_{;\nu}\chi^{\nu}\chi_{\rho;\mu} && (2.65)\\ &= -2\kappa\chi_{\rho;\mu}\chi^\mu && (2.66)\\ &= -2\kappa^2\chi_\rho && (2.67) \end{aligned}$$

which only holds on the null hypersurface. The explicit expression for κ on the horizon is

$$\kappa^2 \quad = \quad -\frac{1}{2}\chi^{\mu}{}_{;}{}^{\nu}\chi_{\mu;\nu} \tag{2.68}$$

In a Kerr-Newman space-time the value of κ is

$$\kappa \quad = \quad \frac{\sqrt{M^2 - a^2 - e^2}}{2M^2 + 2M\sqrt{M^2 - a^2 - e^2} - e^2} \tag{2.69}$$

Now consider the flow diagram, see figure 18, of the orbits of χ^μ for $\kappa \neq 0$. The precise expression for χ^μ can be chosen such that[19] $\chi^\mu = \kappa\lambda k^\mu$, with $\lambda = \pm e^{\kappa p}$. If the Killing parameter p runs from $-\infty$ to ∞ either only the area with negative λ or only the area with positive λ is covered. That is, on H, there is a fixed point in the flow diagram at $\lambda = 0$. This fixed point is actually a fixed two sphere. For $\kappa = 0$ this fixed point is not present, so this case will be treated later. For $\kappa \neq 0$ one finds that if there is a fixed point, i.e. where χ^μ vanishes, then κ is constant on every orbit of χ^μ, and all these orbits end on the fixed point. So if κ is constant on the fixed point, then κ is constant on the entire null hypersurface. Parallel transport of κ^2 along an arbitrary tangent vector on the fixed point (the two sphere) v^μ yields

$$\begin{aligned} \kappa^2{}_{;\rho}v^\rho &= -\chi^{\mu}{}_{;}{}^{\nu}\chi_{\mu;\nu;\rho}v^\rho && (2.70)\\ &= \chi^{\mu}{}_{;}{}^{\nu}R^{\sigma}{}_{\rho\nu\mu}\chi_\sigma v^\rho && (2.71)\\ &= 0 && (2.72) \end{aligned}$$

[18] See [2] for a proof of this theorem.
[19] See appendix A.7.

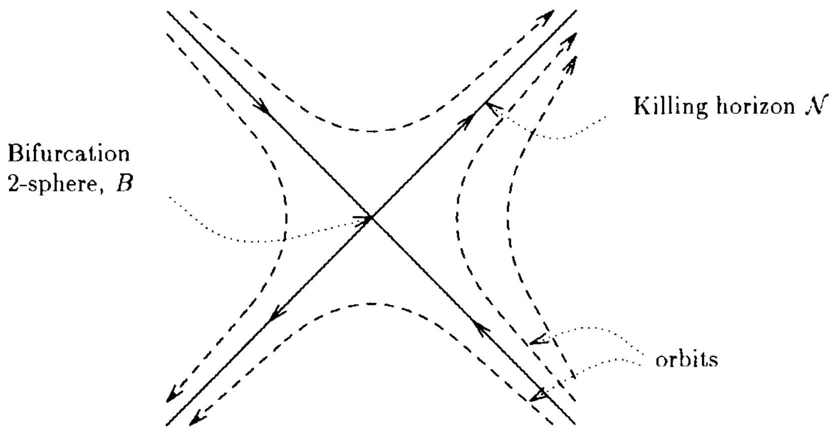

Figure 2.1: The dotted lines are the flow lines of the vector field χ^μ. The fixed point, or bifurcation point is a two-sphere.

for on the fixed point $\chi^\mu = 0$. The tangent vector v^μ to the fixed point is arbitrary and therefore κ is constant on the entire horizon.

If $\kappa = 0$ then χ^μ is a geodesic itself and no fixed point is present. In that case κ is constant on the entire horizon if the energy momentum tensor satisfies the dominant energy condition which states that the vector

$$J^\mu \equiv -T^\mu{}_\nu \chi^\nu \tag{2.73}$$

is timelike or null, $J_\mu J^\mu \leq 0$, and future directed[20]. However a null geodesic congruence satisfying the Killing condition has

$$0 = T_{\mu\nu}\chi^\mu\chi^\nu = -J_\mu\chi^\mu \tag{2.74}$$

because of the Raychaudhuri equation, see equation (3.63) together with the Einstein equation (2.10), and therefore J^μ can be expanded as a vector proportional to χ^μ and an infinitesimal displacement vector in the 2-dimensional space orthogonal to the null geodesic congruence, so $J_\mu J^\mu \geq 0$, and then $J_\mu J^\mu = 0$. This means that $J^\mu \propto \chi^\mu$. Then, because $\chi_\mu\chi_\nu - \chi_\nu\chi_\mu = 0$,

$$\begin{aligned} 0 &= J_\mu\chi_\nu - \chi_\mu J_\nu && (2.75)\\ &= -T_{\mu\rho}\chi^\rho\chi_\nu + \chi_\mu T_{\nu\rho}\chi^\rho && (2.76)\\ &= -R_{\mu\rho}\chi^\rho\chi_\nu + \chi_\mu R_{\nu\rho}\chi^\rho && (2.77)\\ &= -\kappa_{;\mu}\chi_\nu + \kappa_{;\nu}\chi_\mu && (2.78)\\ &= -\chi_{[\mu}D_{\nu]}\kappa && (2.79) \end{aligned}$$

therefore[21] $\kappa_{;\mu} \propto \chi_\mu$ and χ^μ is parallel transported along an arbitrary tangent vector v^μ to H, so $\kappa_{;\mu}v^\mu = 0$, and thus κ is constant on the entire null hypersurface.

[20]This condition means that energy cannot propagate faster than the speed of light.

[21]κ is defined only on the horizon, so all these equations hold only on the horizon. The

This result is known as the zeroth law of black hole mechanics

$$\text{stationary horizon} \Rightarrow \text{surface gravity } \kappa \quad = \quad \text{constant} \tag{2.80}$$

and it is the space-time analogue of the zeroth law of thermodynamics, which states that the temperature throughout a system in thermodynamic equilibrium is constant. Note that a null hypersurface which is generated by vector fields with a fixed point the surface gravity is constant without invoking the Einstein equation.

2.5.2 The first law

With the use of Komar integrals an expression for the mass in a stationary axisymmetric asymptotic flat space-time can be derived. In this derivation the possible electromagnetic charge of the black hole is assumed to be negligible, for a black hole shall selectively attract matter with opposite charge, so that on average the black hole will be electrically neutral. However, a charged particle in the neighbourhood of a local horizon will notice if an electric charge has just crossed the horizon and this small charge associated with the local horizon could determine the fate of the particle, i.e. it could mean the difference between crossing the horizon because of a small attractive electric force, or escaping because of a small repulsive force.

The total mass of the space-time is defined by an observer at infinity by the Komar integral for the Killing vector field ξ^μ, for the observer at infinity is in the asymptotically flat region and will perceive this vector field as the Killing vector field generating time translations.

$$M = -\frac{1}{4\pi}\int_V \mathrm{d}S^\mu \xi^\nu{}_{;\mu;\nu} \tag{2.81}$$

$$= 2\int_V \mathrm{d}V n^\mu J_\mu - \frac{1}{8\pi}\oint_H \mathrm{d}S^{\mu\nu}\xi_{\nu;\mu} \tag{2.82}$$

where J^μ is the matter current associated with ξ^μ, and V is the volume of the space-time with normal n^μ, which has the black hole horizon H as boundary.

Use the horizon generator $\chi^\mu = \xi^\mu + \omega_H \varphi^\mu$ to obtain for the last term of (2.82)

$$\oint_H \mathrm{d}S^{\mu\nu}\xi_{\nu;\mu} = \oint_H \mathrm{d}S^{\mu\nu}\chi_{\nu;\mu} - \omega_H \oint_H \mathrm{d}S^{\mu\nu}\varphi_{\nu;\mu} \tag{2.83}$$

The second term in this equation is proportional to the angular momentum J_H of the horizon[22], for φ is a rotational Killing vector field, and the angular momentum associated with this rotation is defined by the Komar integral, see section 2.3. The second term is then

$$-\omega_H \oint_H \mathrm{d}S^{\mu\nu}\varphi_{\nu;\mu} \equiv -16\pi\omega_H J_H \tag{2.84}$$

equations must be differentiated tangent to the horizon. To resolve this the derivative may be projected on the horizon, but for horizons there exist no unique projection operator. The space-time volume element contracted with the horizon generator, $\sqrt{-\det(g^{\mu\nu})}\chi_\mu$ is tangent to the horizon, $\sqrt{-\det(g^{\mu\nu})}\chi_\mu\chi_\nu = 0$, and thus, see section A.6 , $\sqrt{-\det(g^{\mu\nu})}\chi_\mu D_\nu$ with D_μ the covariant derivative can be applied to all equations holding on the horizon. This expression is equivalent to $\chi_{[\mu}D_{\nu]}$, so if $\chi_{[\mu}D_{\nu]}\kappa = 0$, then κ is constant on the horizon.

[22]This is the angular momentum of the black hole, or rather the angular momentum of the gravitational field of the black hole.

The first term of (2.83) can be evaluated easily with the following choice for the surface element on the horizon [23],

$$\mathrm{d}S^{\mu\nu} = \mathrm{d}A\,(\chi^\mu N^\nu - \chi^\nu N^\mu) \tag{2.85}$$

where $\mathrm{d}A$ is the area of the surface element on the horizon and χ^μ and N^μ are null vector fields orthogonal to the horizon such that $\chi_\mu N^\mu = -1$. The first term of (2.83) is

$$\begin{aligned}
\oint_H \mathrm{d}S^{\mu\nu}\chi_{\nu;\mu} &= \oint_H \mathrm{d}A\,(\chi^\mu N^\nu - \chi^\nu N^\mu)\,\chi_{\nu;\mu} && (2.86)\\
&= 2\kappa \oint_H \mathrm{d}A \chi^\mu n_\mu && (2.87)\\
&= -2\kappa \oint_H \mathrm{d}A && (2.88)\\
&= -2\kappa A && (2.89)
\end{aligned}$$

When (2.84) and (2.89) are inserted in the original expression (2.82) for the total mass M in the space-time, then

$$\begin{aligned}
M &= 2\int_V \mathrm{d}V n^\mu J_\mu - \frac{1}{8\pi}\oint_H \mathrm{d}S^{\mu\nu}\xi_{\nu;\mu} && (2.90)\\
&= 2\int_V \mathrm{d}V n^\mu J_\mu + \frac{1}{4\pi}\kappa A + 2\omega_H J_H && (2.91)
\end{aligned}$$

For a vacuum stationary axisymmetric space-time the energy momentum tensor vanishes, and then equation (2.40) shows that the current $J^\mu = 0$. The mass M_H of the gravitational field of a vacuum space-time is [24] itself is defined by

$$\begin{aligned}
M_H &\equiv -\frac{1}{8\pi}\oint_H \mathrm{d}S^{\mu\nu}\xi_{\nu;\mu} && (2.92)\\
&= \frac{1}{4\pi}\kappa A + 2\omega_H J_H && (2.93)
\end{aligned}$$

By examining the scaling properties of the different quantities it is possible to acquire a local differential equation for the matter near the null hypersurface. As equation (2.92) shows the mass of the horizon is a function of A and J. These parameters describe the electrically neutral rotating object of mass M completely. Note that both parameters have dimension M^2. Therefore for every $\mu > 0$ one has,

$$\mu M(A,J) = M(\mu^2 A, \mu^2 J) \tag{2.94}$$

i.e. M is a generalised homogeneous function [25] Differentiating to parameter μ

[23] See section 35 for the construction of a metric on the horizon.

[24] This is the mass of the black hole.

[25] A function $h(x,y)$ is called a homogeneous function if, for all $\lambda > 0$, it satisfies

$$\lambda h(x,y) = h(\lambda^q x, \lambda^p y) \tag{2.95}$$

for appropriately chosen values for p and q.

yields

$$\begin{aligned} M(A,J) &= \frac{d}{d\mu}M(\mu^2 A,\mu^2 J) && (2.96)\\ &= 2\mu A\frac{\partial}{\partial\mu^2 A}M(\mu^2 A,\mu^2 J)+2\mu J\frac{\partial}{\partial\mu^2 J}M(\mu^2 A,\mu^2 J) && (2.97)\\ &= 2\mu A\frac{\partial}{\partial A}\frac{M(A,J)}{\mu}+2\mu J\frac{\partial}{\partial J}\frac{M(A,J)}{\mu} && (2.98)\\ &= 2A\frac{\partial}{\partial A}M(A,J)+2J\frac{\partial}{\partial J}M(A,J) && (2.99)\\ &= \frac{1}{4\pi}\kappa A+2\omega_H J && (2.100)\end{aligned}$$

Since A and J are independent parameters it is found that

$$\begin{aligned}\frac{\partial M}{\partial A} &= \frac{\kappa}{8\pi} && (2.101)\\ \frac{\partial M}{\partial J} &= \omega_H && (2.102)\end{aligned}$$

The first law of black hole mechanics reads thus

$$\mathrm{d}M = \frac{\kappa}{8\pi}\mathrm{d}A+\omega_H\mathrm{d}J \qquad (2.103)$$

and it represents conservation of energy in a black hole space-time.

2.5.3 The second law

The second law states the surface area of an event horizon can only increase over time.

$$\delta A \geq 0 \qquad (2.104)$$

This statement is certainly true if it holds for each element of the horizon, and the horizon elements increase in size if the expansion θ of the generators of the horizon is greater than or equal to zero, because an area element changes according to[26]

$$\frac{\mathrm{d}a}{\mathrm{d}\lambda} = \theta a \qquad (2.105)$$

It has been discovered that black holes can evaporate over time because of particle creation near the horizon, see 2.6. This would be a violation of this second law. However, this law is interpreted as a statement on entropy, and together with the usual second law of thermodynamics the entropy will increase over time, because the thermal radiation resulting from the evaporation carries more entropy than the original black hole.

Classically the second law of black hole mechanics is true by intuition, no energy can go faster than light, so it is impossible to escape out of a black hole, and then the size cannot decrease. A precise formulation and proof of the Hawking area theorem can be found in [2].

[26] See for a derivation of this equation 35.

2.5.4 The third law

The third law is, just as the zeroth law, a statement about the surface gravity. If the surface gravity goes to zero, then the black hole horizon area goes to a constant finite value, which needn't be zero, e.g. consider an extreme black hole, that is a Kerr-Newman black hole with $M^2 = a^2 + e^2$. Such a black hole has $\kappa = 0$ but a finite surface area.

An alternative version of this law states that it is impossible to reach surface gravity zero by a physical process, but this is easily seen to be violated, for if a black hole evaporates by particle creation near the horizon, then the surface gravity will go to zero, for eventually the black hole will end its existence.

According to Wald [2], calculations show that it is very hard to make an extreme Kerr black hole, i.e. the closer one gets to a Kerr black hole, the harder it is to get even closer. If κ is interpreted as the black hole temperature, then the thermal radiation produced in the evaporation process preserves the generalised third law[27] will still be true, for the emitted radiation has non zero temperature.

2.6 Hawking radiation

The Schwarzschild black hole equilibrium is unstable. Consider quantum fields on the Schwarzschild[28] black hole background. If a Wick rotation is made to imaginary time, $t \rightarrow \tau = it$, then the metric has a Euclidean signature, it becomes

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{2.106}$$

where $\mathrm{d}\Omega^2$ is the metric on the sphere S^2. The metric is singular at $r = 2M$, this is the location of the Schwarzschild black hole horizon. Near the horizon the metric can be approximated by a Euclidean Rindler metric, i.e. the Rindler metric (A.40) with imaginary time, then

$$\mathrm{d}s^2 \approx (\kappa R)^2\,\mathrm{d}\tau^2 + \mathrm{d}R^2 + \frac{1}{4\kappa^2}\mathrm{d}\Omega^2 \tag{2.107}$$

This metric is acquired under the following identifications,

$$r - 2M = \frac{R^2}{8M} \tag{2.108}$$

$$\left(1 - \frac{2M}{r}\right) \approx (\kappa R)^2 \tag{2.109}$$

$$\mathrm{d}r^2 = (\kappa R)^2\mathrm{d}R^2 \tag{2.110}$$

κ is the surface gravity of the Schwarzschild black hole, $\kappa = \frac{1}{4M}$. The metric is Euclidean Rindler times S^2. The Euclidean Rindler part of the metric can be identified with the Euclidean plane in polar coordinates if the imaginary time is identified as an angle,

$$0 \leq \tau < \frac{2\pi}{\kappa} \tag{2.111}$$

[27] Which is the combination of this law with that of thermodynamics.

[28] The Schwarzschild black hole solution is the Kerr-Newman solution with $a = e = 0$.

The singularity at $R=0$ is a coordinate singularity[29].

The Euclidean path integral can now be taken over fields φ with periodic boundary conditions, $\varphi(\tau) \equiv \varphi(\tau + \frac{2\pi}{\kappa})$

$$Z = \int \mathrm{d}[\varphi] e^{-S} \tag{2.112}$$

where S is the action in imaginary time. This is the path integral representation of the partition function for fields with period[30] β

$$Z = \mathrm{Tr} e^{-\beta H} \tag{2.113}$$

In statistical mechanics $\beta = \frac{1}{T}$, but β is the period $\beta = \frac{2\pi}{\kappa}$, so the fields near the black hole have temperature $T = \frac{\kappa}{2\pi}$. This temperature is known as the Hawking temperature.

The black hole is in equilibrium with a heat bath of nearby quantum fields if it is at the Hawking temperature. Is such an equilibrium stable? No, if a quantum field enters the black hole, then the black hole temperature decreases, because the Schwarzschild surface gravity is inversely proportional to the black hole mass. By absorbing heat (the quantum fields) the black hole cools itself, i.e. the black hole has negative specific heat. Therefore the black hole equilibrium is unstable. If the fields near the black hole have lower temperature than the black hole, then the black hole must radiate energy in order to achieve equilibrium with the heat bath which is composed of the quantum fields near the horizon. The black hole will heat up when it radiates and start to radiate more, until it eventually evaporates. The quantum state in equilibrium with the black hole at the Hawking temperature is known as the Hartle-Hawking vacuum.

2.7 Unruh temperature

For self gravitating systems, such as a Schwarzschild black hole a local temperature can be defined. The observers hovering near the horizon move on the Euclidean Rindler geodesics[31] $\xi^\mu = \delta_\tau{}^\mu$. If T_0 is the temperature seen by observers at infinity, then the local observer hovering near the self gravitating system will measure temperature T defined by

$$\sqrt{-g_{\mu\nu}\xi^\mu\xi^\nu} T \equiv T_0 \tag{2.114}$$

For an observer near a Schwarzschild black hole the local temperature is, using the Rindler coordinates near the horizon

$$\kappa R T = \frac{\kappa}{2\pi} \tag{2.115}$$

[29] If an identification is made with a different period, then the 2-dimensional Euclidean Rindler space-time looks like a sheet of paper folded into a cone. The curvature of the metric does not blow up, but at the tip of the cone a point is missing. This point is a 2-sphere in the case presented here. The imaginary time variable is periodic, because the analytic continuation of the time variable in any Green's function of a quantum field theory is symmetric under rotations in 4-dimensional Euclidean space, see [4]. More on quantum fields in curved space can be found in e.g. [5].

[30] See for a derivation e.g. [6].

[31] The geodesics of the metric (2.107)

or

$$T = \frac{R^{-1}}{2\pi} \tag{2.116}$$

For an observer hovering near a black hole $R =$ constant, and $a = R^{-1}$ is the proper acceleration for the observer.

The same result for local temperature is also obtained in quantum mechanics, see [7]. An observer who accelerates uniform with acceleration a observes a heat bath at

$$T = \frac{a}{2\pi} \tag{2.117}$$

This local temperature is known as the Unruh temperature.

An observer at infinity in a space-time in which a Schwarzschild black hole is in equilibrium with quantum fields will measure the Hawking temperature as his local temperature.

Chapter 3

The holographic principle

This chapter is about the holographic principle. It begins with an introduction to the holographic principle, some of the arguments put forward in favour of the principle are discussed and possible arguments against the principle. The covariant entropy bound and the generalised covariant entropy bound are stated as implementations of the principle. These bounds are statements about the amount of entropy passing through geometrical constructs called (local) light sheets. Then the Raychaudhuri equation is derived, that is the equation which describes the expansion of light sheets. Finally local light sheets are constructed.

3.1 Introduction to the holographic principle

The holographic principle is the idea that the information contained in a $(d-1)$-dimensional region in a d-dimensional space-time can be encoded on the $(d-2)$-dimensional surface enclosing the region. This principle was first put forward in 1993 by 't Hooft, [8]. His original statement was for regions surrounding black holes, and the information in this region could be stored on the black hole surface area.

The entropy in a region of space-time is a measure for the information in that region. The entropy of a system is defined as the logarithm of the number of accessible states to that system, [3]. In statistical mechanics the entropy of a system scales with the volume of system, to see this an example will be given.

Consider a volume V and divide the volume in a bunch of cubes of unit volume. Suppose each unit volume can contain either a zero or a one. The number of accessible states available to a unit cube is 2, and the number of accessible states of a volume V composed of these cubes is 2^V. The entropy is $V \ln 2$, it scales with the volume of the system.

This example appears to contradict the holographic principle, but this system is not very physical. In a physical system the zero and one state may be considered as a vacuum state of zero energy and an excited state at some energy $E > 0$. The states with energy E will gravitate, and when enough states in the volume are excited then the system is expected to collapse to a black hole. This black hole will have entropy proportional to its surface area according to the second law of black hole thermodynamics, see section 2.5.3. This can mean 2 things, either information is lost in the black hole, or the entropy of the original system was no more than its surface area dictated.

The laws of quantum mechanics are time reversible. If gravity is to be described by a quantum theory, then the laws of quantum gravity may be expected to be time reversible too. The black hole is expected to evaporate without information loss, unless there is some sort of symmetry breaking.

In his article 't Hooft assumed that quantum gravity is time reversible, and speculated that the information contained in three dimensional space can be stored on a two dimensional surface like a holographic image. The idea evolved in an article of Susskind, [9]. In this article light rays where used to project all information present in a space-time on a distant 2-dimensional screen.

Many indications have been found for a holographic bound on entropy. An example of a cosmological space-time satisfying the bound has been found by Fischler and Susskind, [15], the space-time for which they found the bound to hold was flat and negatively curved space-time. For positively curved close universes the bound was violated. However, the bound implemented was not the covariant entropy bound, which thus far has not been disproved by counterexamples.

The covariant entropy bound was introduced by Bousso [13], the definition of the bound will be given in section 3.2. This bound has been proven for semi-classical systems by Flanagan, Marolf and Wald in [14]. In the same article a generalisation of the covariant entropy bound was proposed. This bound is known as the generalised covariant entropy bound, see section 3.3 for its definition. The advantage of the generalised bound is that it implies black hole thermodynamics.

The holographic principle is to be new input for quantum theories of gravity, constraining the dynamical degrees of freedom of the metric in the hope that a resulting quantum theory of gravity is renormalisable. Another point of view is that the holographic principle should follow from a new theory of gravity. String theory is such a theory, and in the case of $AdS_5 \times S^5$ it is found that 4-dimensional Yang-Mills theory lives on the boundary of AdS_5, see the review of Bousso for more on this, [12].

3.2 The covariant entropy bound

The current version of the holographic principle is the covariant entropy bound, which is stated for space-time manifolds of arbitrary dimension d. It was first stated by Bousso, see his review article [12] for more on this and other bounds.

Let A be the area of an arbitrary $(d-2)$-dimensional spacelike surface B. Consider the null hypersurfaces[1] which are generated by null vector fields (light rays) orthogonal to B. As long as the expansion θ of the generators[2], which start at B, of the null hypersurface is non-positive the null hypersurface is called a light sheet L of B. The light sheet starts at the surface B and terminates when $\theta > 0$. In general there are two null hypersurfaces orthogonal to B, so there are four potential light sheets.

The covariant entropy bound is a statement about the entropy S passing

[1] A null hypersurface is a $(d-1)$-dimensional submanifold, see A.6

[2] The expansion of the generators of a hypersurface is governed by the Raychaudhuri equation, this equation is treated in 3.4

through[3] the light sheet L,

$$S(L) \leq \frac{A(B)}{4l_p^2} \tag{3.1}$$

where $l_p \equiv \sqrt{\frac{G\hbar}{c^3}}$ is the Planck length, given here with the Planck constant $\hbar$, the gravitational constant G and the speed of light c, which are all 1 in the units used in this text.

The spatial surface B can be considered as a holographic screen, the entropy on the light sheet can be projected on B with a density of less than one bit per Planck area.

3.3 The generalised covariant entropy bound

There exists a more general form of the covariant entropy bound. What happens if the light sheets are incomplete, i.e. if they are terminated before $\theta > 0$? For such cases the generalised covariant entropy bound applies, which was first stated by Flanagan Marolf and Wald in their semi-classical proof for the covariant entropy bound [14]. The bound is

$$S(L) \leq \frac{A(B) - \tilde{A}(\tilde{B})}{4l_p^2} \tag{3.2}$$

where $\tilde{A}(\tilde{B})$ is the area of the spacelike surface $\tilde{B}$ on which the light sheet terminates. This generalised bound implies both the covariant entropy bound and the generalised second law of thermodynamics.

The generalised form can be used to construct local holographic screens. The physics on local light sheets can be described by a theory on the boundary of the light sheet. Adding all the local holographic screens may then yield a global holographic screen.

3.4 The Raychaudhuri equation

The Raychaudhuri equation governs the expansion of geodesic congruences. In this section the Raychaudhuri equation is derived for both timelike and null geodesic congruences. First the term congruence is clarified, then the Raychaudhuri equation for timelike geodesic congruences is derived, and after that the Raychaudhuri equation for null geodesic congruences is derived. Timelike geodesic congruences can be used to construct light sheets, the light sheets themselves are generated by null geodesic congruences. Some useful properties of special null geodesic congruences are derived, and for the null geodesic case a practical choice for a projection operator on the orthogonal space will be given. With this projection operator a formula for the expansion of a surface element is derived. Finally some special cases of null geodesic congruences are studied. Some of this material can also be found in [2, 7].

[3]In many references, e.g. [12] authors talk about the entropy on the light sheet. It is unclear what entropy on the light sheet is, if it is the integrated entropy density on the light sheet then it would always be zero. What these authors mean is the entropy passing through the light sheet.

3.4.1 Congruences

It is often useful to slice the space-time manifold into submanifolds, or hypersurfaces. These hypersurfaces can be generated by vector fields, see section A.6. For example equal time slices are useful. If a space-time is symmetric under time translations, then energy is conserved in that space-time. If on top of that the vector field generating the time translations is orthogonal to a family of hypersurfaces, then the space-time is static and then it can be sliced in equal time slices. The paths of static free particles in that space-time are generated by the vector field generating the time translations, and the collection of these paths is an example of a congruence of curves.

A congruence of curves in an open submanifold S of some manifold M is a family of curves such that through every point p in S passes precisely one curve of this family. Two corollaries of this definition are that the tangents to a congruence form a smooth vector field v^μ, and conversely that every smooth vector field v^μ generates a congruence A congruence is a timelike or null congruence if the associated vector field v^μ is timelike, $v_\mu v^\mu < 0$, or null $v_\mu v^\mu = 0$ respectively. A congruence is a geodesic congruence if the associated vector field v^μ satisfies the geodesic equation, ${v^\mu}_{;\nu} v^\nu = 0$.

3.4.2 Geodesic deviations

Consider a geodesic ${v_0}^\mu$ of some geodesic congruence. In general the neighbouring geodesics will not remain parallel, e.g. the meridians on the globe are geodesics, but meridians cross at the poles. The meridians are a geodesic congruence except on the poles. At the equator they are parallel, but the closer one gets to the poles, the closer the geodesics are to each other.

Consider, in order to find out how a geodesic in the family behaves with respect to another geodesic in the family, an infinitesimal displacement vector η^μ measuring the displacement of one geodesic v^μ with respect to the others in the congruence.

Both v^μ and η^μ can be chosen such that they are elements of a coordinate basis, and therefore

$$ {\eta^\mu}_{;\nu} v^\nu - {v^\mu}_{;\nu} \eta^\nu \quad = \quad 0 \tag{3.3} $$

in other words, the Lie derivative of η^μ with respect to v^μ is zero.

This can be stated by

$$ {\eta^\mu}_{;\nu} v^\nu \quad = \quad {B^\mu}_{;\nu} \eta^\nu \tag{3.4} $$

where the tensor field $B_{\mu\nu}$ is defined by

$$ B_{\mu\nu} \quad \equiv \quad v_{\mu;\nu} \tag{3.5} $$

$B_{\mu\nu}$ is a measure of how the infinitesimal displacement vector η^μ changes if it is parallel transported along v^μ, it is a measure of geodesic deviation

A neighbouring geodesic of v^μ can be specified by an infinitesimal displacement vector η^μ, but this specification is not unique, for $\tilde{\eta}^\mu = \eta^\mu + c v^\mu$, with c a constant, is a displacement to the same geodesic. It is possible to give unique specifications for timelike geodesic congruences and for null geodesic congruences in terms of displacement vectors. First I shall treat the unique specification for timelike geodesics, and I will show that this specification doesn't work for null geodesics, then I will give a unique specification for null geodesics.

3.4.3 Expansion of timelike geodesic congruences

Consider a congruence of timelike geodesics ξ^μ with affine parameter τ. The vector field ξ^μ satisfies by definition $\xi^\mu{}_{;\nu}\xi^\nu = 0$ and $\xi_\mu\xi^\mu = -1$. Introduce the notation $B_{\mu\nu} = \xi_{\mu;\nu}$.

The unique specification of a displacement to a neighbouring geodesic in terms of η^μ is acquired by requiring the displacement vector to be orthogonal to the geodesics,

$$\eta_\mu \xi^\mu = 0 \tag{3.6}$$

This yields a clear physical interpretation of the tensor $B_{\mu\nu}$, because this tensor is in both indices orthogonal to the timelike geodesic congruence ξ^μ,

$$B_{\mu\nu}\xi^\nu = \xi_{\mu;\nu}\xi^\nu = 0 \tag{3.7}$$

$$B_{\mu\nu}\xi^\mu = \xi_{\mu;\nu}\xi^\mu = \frac{1}{2}\left(\xi_\mu\xi^\mu\right)_{;\nu} = (-1)_{,\nu} = 0 \tag{3.8}$$

That is, the tensor field is spacelike, so the deviations can be classified in expansion, shear, and twist (or rotation).

The 3-dimensional spacelike subspace orthogonal to the timelike geodesics in which all these deviations happen has as (Euclidean) metric $h_{\mu\nu}$,

$$h_{\mu\nu} = g_{\mu\nu} + \xi_\mu\xi_\nu \tag{3.9}$$

where $g_{\mu\nu}$ is the metric of the 4-dimensional space-time. Thus is $h^\mu{}_\nu = g^{\mu\rho}h_{\rho\nu}$ the projection operator on the subspace orthogonal to ξ^μ. By means of this projection operator the tensor $B_{\mu\nu}$ can be expanded in expansion θ, shear $\sigma_{\mu\nu}$ and twist $\omega_{\mu\nu}$,

$$B_{\mu\nu} = \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} \tag{3.10}$$

$$\theta \equiv h_{\mu\nu}B^{\mu\nu} \tag{3.11}$$

$$\sigma_{\mu\nu} \equiv B_{(\mu\nu)} - \frac{1}{3}\theta h_{\mu\nu} \tag{3.12}$$

$$\omega_{\mu\nu} \equiv B_{[\mu\nu]} \tag{3.13}$$

The expansion measures if the neighbouring geodesics come closer, or move away from the geodesic under consideration. The shear measures the deformation of the congruence, e.g.. if a square is parallel transported along the congruence it will deform to a diamond if the shear does not vanish. The twist measures of neighbouring geodesics twist around the geodesic.

The rate of change of θ, $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ can be derived from the parallel transport of $B_{\mu\nu}$ along ξ^μ,

$$B_{\mu\nu;\rho}\xi^\rho = \xi_{\mu;\nu;\rho}\xi^\rho \tag{3.14}$$

$$= \xi_{\mu;\rho;\nu}\xi^\rho - \xi_\sigma R^\sigma{}_{\mu\nu\rho}\xi^\rho \tag{3.15}$$

$$= \left(\xi_{\mu;\rho}\xi^\rho\right)_{;\nu} - \xi^\rho{}_{;\nu}\xi_{\mu;\rho} - R_{\sigma\mu\nu\rho}\xi^\sigma\xi^\rho \tag{3.16}$$

$$= R_{\mu\sigma\nu\rho}\xi^\sigma\xi^\rho - B^\rho{}_\nu B_{\mu\rho} \tag{3.17}$$

where towards line (3.15) the commutation relation for covariant derivatives was used and in line (3.16) the first term is zero because of the geodesic equation.

The trace of this equation yields the rate of change of the expansion,

$$\theta_{;\rho}\xi^{\rho} = \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} + R_{\mu\nu}\xi^{\mu}\xi^{\nu} \tag{3.18}$$

This is called the Raychaudhuri equation for timelike geodesic congruences.

The traceless symmetric part of (3.17) will yield the rate of change of the shear, whereas the antisymmetric part will yield the rate of change of the twist of the timelike geodesic congruence,

$$\begin{aligned}\sigma_{\mu\nu;\rho}\xi^{\rho} &= -\frac{2}{3}\theta\sigma_{\mu\nu} - \sigma_{\mu\rho}\sigma^{\rho}{}_{\nu} + \omega_{\mu\rho}\omega^{\rho}{}_{\nu} + \frac{1}{3}\left(\sigma_{\rho\sigma}\sigma^{\rho\sigma} - \omega_{\rho\sigma}\omega^{\rho\sigma}\right)\\ &+ C_{\mu\sigma\nu\rho}\xi^{\sigma}\xi^{\rho} + \frac{1}{6}h_{\mu\nu}h_{\rho\sigma}R^{\rho\sigma} - \frac{1}{2}h_{\mu\rho}h_{\nu\sigma}R^{\sigma\rho} \qquad (3.19)\\ \omega_{\mu\nu;\rho}\xi^{\rho} &= -\frac{2}{3}\theta\omega_{\mu\nu} - 2\sigma^{\rho}{}_{[\nu}\omega_{\mu]\rho} \qquad (3.20)\end{aligned}$$

The tensor $C_{\mu\sigma\nu\rho}$ is the Weyl tensor, see A.3. Note that the twist of the congruence either vanishes everywhere or nowhere.

3.4.4 Expansion of null geodesic congruences

Consider a congruence of null geodesics k^{μ} with affine parameter λ. By definition this vector field has the properties $k_{\mu}k^{\mu} = 0$ and $k^{\mu}{}_{;\nu}k^{\nu} = 0$.

The orthogonality condition for the displacement vectors η^{μ} to neighbouring timelike geodesics doesn't yield a unique specification for null geodesic congruences, for the 3-dimensional subspace orthogonal to k^{μ} also contains k^{μ} itself, i.e. if $\tilde{\eta}^{\mu} = \eta^{\mu} + ck^{\mu}$, then

$$\tilde{\eta}_{\mu}k^{\mu} = \eta_{\mu}k^{\mu} + ck_{\mu}k^{\mu} = \eta_{\mu}k^{\mu} \tag{3.21}$$

because $k_{\mu}k^{\mu} = 0$. This problem can be resolved by considering equivalence classes of displacement vectors. Displacement vectors are equivalent if they differ by a multiple c of k^{μ},

$$\eta^{\mu} \sim \eta^{\mu} + ck^{\mu} \tag{3.22}$$

This identification clearly is an equivalence [4].

The orthogonality condition yields a 3-dimensional subspace orthogonal to the null geodesics, the equivalence removes another dimension of this subspace, and therefore together they yield a 2-dimensional spacelike subspace $\hat{V}$ orthogonal to the null geodesics, with metric $\hat{h}_{\mu\nu}$.

To acquire the rate of change of the expansion of the geodesics, again one introduces the tensor $B_{\mu\nu} = k_{\mu;\nu}$, but now it must be projected on $\hat{V}$,

$$\hat{B}_{\mu\nu} = \hat{h}_{\mu}{}^{\rho}B_{\rho\sigma}\hat{h}^{\sigma}{}_{\nu} \tag{3.23}$$

[4] The relation $\sim$ is reflexive for $c = 0$, $\eta^{\mu} \sim \eta^{\mu}$, it is symmetric, i.e. if $\eta^{\mu} + ck^{\mu} \sim \eta^{\mu}$ then $\eta^{\mu} \sim \eta^{\mu} + ck^{\mu}$, and it is transitive, if $\eta^{\mu} + ck^{\mu} \sim \eta^{\mu}$ and $\eta \sim \eta + \tilde{c}k^{\mu}$, then $\eta^{\mu} + ck^{\mu} \sim \eta^{\mu} + \tilde{c}k^{\mu}$. These differences are merely a multiple of k^{μ}.

This tensor is effectively 2-dimensional and can be expressed in terms of the 2-dimensional expansion $\hat{\theta}$, shear $\hat{\sigma}_{\mu\nu}$, and twist $\hat{\omega}_{\mu\nu}$,

$$\hat{B}_{\mu\nu} = \frac{1}{2}\hat{\theta}\hat{h}_{\mu\nu} + \hat{\sigma}_{\mu\nu} + \hat{\omega}_{\mu\nu} \tag{3.24}$$

$$\hat{\theta} \equiv \hat{h}_{\mu\nu}\hat{B}^{\mu\nu} \tag{3.25}$$

$$\hat{\sigma}_{\mu\nu} \equiv \hat{B}_{(\mu\nu)} - \frac{1}{2}\hat{\theta}\hat{h}_{\mu\nu} \tag{3.26}$$

$$\hat{\omega}_{\mu\nu} \equiv \hat{B}_{[\mu\nu]} \tag{3.27}$$

In fact the hat on the expansion can be dropped, but it is kept to indicate that a null geodesic congruence is under consideration.

Along the same lines as in the case of timelike geodesics one obtains the expression for rates of change of the expansion shear and twist, but now the final projection is not on a 3-dimensional space, but a 2-dimensional space, i.e. the equation has hats on the appropriate terms.

$$\hat{B}_{\mu\nu;\rho}k^{\rho} = \widehat{R_{\mu\sigma\nu\rho}k^{\sigma}}k^{\rho} - \hat{B}^{\rho}{}_{\nu}\hat{B}_{\mu\rho} \tag{3.28}$$

The trace of (3.28) again yields the rate of change of the expansion,

$$\hat{\theta}_{;\rho}\xi^{\rho} = \frac{\mathrm{d}\hat{\theta}}{\mathrm{d}\lambda} = -\frac{1}{2}\hat{\theta}^2 - \hat{\sigma}_{\mu\nu}\hat{\sigma}^{\mu\nu} + \hat{\omega}_{\mu\nu}\hat{\omega}^{\mu\nu} + R_{\mu\nu}k^{\mu}k^{\nu} \tag{3.29}$$

and that is the Raychaudhuri equation for null geodesic congruences.

Just as in the timelike case the equation (3.28) yields the rates of change for shear and rotation for the traceless symmetric and antisymmetric parts respectively,

$$\hat{\sigma}_{\mu\nu;\rho}k^{\rho} = \widehat{C_{\mu\sigma\nu\rho}k^{\rho}}k^{\sigma} - \hat{\theta}\hat{\sigma}_{\mu\nu} \tag{3.30}$$

$$\hat{\omega}_{\mu\nu;\rho}k^{\rho} = -\hat{\theta}\hat{\omega}_{\mu\nu} \tag{3.31}$$

Note again that the twist vanishes either everywhere or nowhere in the congruence. Here a few terms extra vanish compared with the timelike case, this has to do with the fact that the vectorspace is 2-dimensional instead of 3-dimensional.

A projection operator

There is no natural way to embed the 2-dimensional subspace. A convenient way to erect the 2-dimensional subspace is by introducing a vector field n^{μ} which is not orthogonal to k^{μ}, but is chosen such that

$$n_{\mu}n^{\mu} = 0 \tag{3.32}$$

$$n_{\mu}k^{\mu} = -1 \tag{3.33}$$

These two expressions have to be satisfied for all λ, so an extra requirement is that n^{μ} is parallel transported along k^{μ},

$$n^{\mu}{}_{;\nu}k^{\nu} = 0 \tag{3.34}$$

The subspace is 2-dimensional if only the displacement vectors η^μ orthogonal to both k^μ and n^μ are considered,

$$n_\mu \eta^\mu = 0 \tag{3.35}$$
$$k_\mu \eta^\mu = 0 \tag{3.36}$$

The associated metric $\hat{h}_{\mu\nu}$ is

$$\hat{h}_{\mu\nu} = g_{\mu\nu} + n_\mu k_\nu + k_\mu n_\nu \tag{3.37}$$

and $\hat{h}^\mu{}_{;\nu} = g^{\mu\rho}\hat{h}_{\rho\nu}$ is a projection operator on the 2-dimensional subspace.

Expansion of a surface

The expansion of a surface element of a null hypersurface generated by a null geodesic congruence can be determined explicitly using the previous choice for the metric. A surface element a, spanned by two linear independent displacement vectors ζ^μ and η^μ is given by

$$a = \varepsilon^{\mu\nu\rho\sigma} k_\mu n_\nu \zeta_\rho \eta_\sigma \tag{3.38}$$

The rate of change of a is

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}\lambda} &= a_{;\kappa} k^\kappa &(3.39)\\ &= \varepsilon^{\mu\nu\rho\sigma} k_\mu n_\nu \left(\zeta_{\rho;\kappa} k^\kappa \eta_\sigma + \zeta_\rho \eta_{\sigma;\kappa} k^\kappa\right) &(3.40)\end{aligned}$$

Note that every displacement vector $\hat{h}^\mu{}_{;\nu}\eta^\nu = \eta^\mu$, which leads, in conjunction with the definition of $\hat{B}_{\mu\nu}$, to

$$\eta^\mu{}_{;\nu} k^\nu = \hat{B}^\mu{}_\nu \eta^\nu \tag{3.41}$$

The rate of change becomes thus

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}\lambda} &= \varepsilon^{\mu\nu\rho\sigma} k_\mu n_\nu \left(\hat{B}_{\rho\kappa}\zeta^\kappa \eta_\sigma + \zeta_\rho \hat{B}_{\sigma\kappa}\eta^\kappa\right) &(3.42)\\ &= \varepsilon^{\mu\nu\rho\sigma} k_\mu n_\nu \left(\hat{B}_{\rho\kappa}\zeta^\kappa \eta_\sigma - \zeta_\sigma \hat{B}_{\rho\kappa}\eta^\kappa\right) &(3.43)\\ &= \hat{\theta} a &(3.44)\end{aligned}$$

The last step can be taken because η^μ and ζ^μ are linear independent vectors spanning a 2-dimensional vectorspace on which $\hat{B}_{\mu\nu}$ is a 2×2 matrix.

Special cases of null geodesic congruences

Consider a congruence of null geodesics k^μ orthogonal to a null hypersurface. In this section it will be proven that such a congruence has vanishing twist, $\hat{\omega}_{\mu\nu} = 0$ on the null hypersurface. Furthermore it will be shown that if $\frac{1}{f}k^\mu = \chi^\mu$, with $(\chi_{\mu;\nu} + \chi_{\nu;\mu})_{\lambda=0} = 0$ and c a constant, then $\hat{B}_{(\mu\nu)}|_{\lambda=0} = 0$ on the null hypersurface.

The Frobenius[5] theorem states that a vector field v^μ is (locally) orthogonal to a hypersurface if and only if

$$v_{[\mu;\nu}v_{\rho]} = 0 \tag{3.45}$$

For the null geodesic congruence k^μ the Frobenius theorem states

$$\begin{aligned} 0 &= k_{[\mu;\nu}k_{\rho]} &(3.46)\\ &= B_{[\mu\nu}k_{\rho]} &(3.47)\end{aligned}$$

Note that

$$B_{[\mu\nu}k_{\rho]} = \hat{B}_{[\mu\nu}k_{\rho]} \tag{3.48}$$

because using the given projection operator $\hat{h}_{\mu\nu}$ one finds,

$$\hat{B}_{\mu\nu} = B_{\mu\nu} + k_\mu\left(n^\rho B_{\rho\nu} + n^\rho B_{\rho\sigma}n^\sigma k_\nu\right) + B_{\mu\rho}n^\rho k_\nu \tag{3.49}$$

where it was also used that $k_{\mu;\nu}k^\nu = 0$ and $k_\mu k^\mu = 0$. So

$$n^\rho B_{\rho[\nu}k_\mu k_{\lambda]} + n^\rho B_{\rho\sigma}n^\sigma k_{[\mu}k_\nu k_{\lambda]} + k_{[\nu}k_\lambda B_{\mu]\rho}n^\rho = 0 \tag{3.50}$$

because the first term is symmetric in two indices so the antisymmetric part of this tensor is equal to zero, the second term is clearly completely symmetric, and the last is again symmetric in two indices. Using this knowledge one obtains

$$\begin{aligned} 0 &= \hat{B}_{[\mu\nu}k_{\rho]} &(3.51)\\ &= \hat{\omega}_{[\mu\nu}k_{\rho]} &(3.52)\end{aligned}$$

The vector n^μ satisfies $n_\mu k^\mu = -1$, as well as $\hat{\omega}_{\mu\nu}n^\mu = 0 = \hat{\omega}_{\nu\mu}n^\mu$, because $\hat{\omega}_{\mu\nu}$ is projected on the subspace orthogonal to n^μ, so contracting (3.52) with n^μ yields,

$$\begin{aligned} 0 &= 3\hat{\omega}_{[\mu\nu}k_{\rho]}n^\mu &(3.53)\\ &= n^\mu\hat{\omega}_{\mu\nu}k_\rho + \hat{\omega}_{\rho\mu}n^\mu k_\nu + \hat{\omega}_{\nu\rho}k_\mu n^\mu &(3.54)\\ &= -\hat{\omega}_{\nu\rho} &(3.55)\end{aligned}$$

in other words, the rotation on the null hypersurface vanishes. Intuitively this result is not unexpected, if a vector field in the plane has rotation around the origin then the vector field has a singularity at the origin.

Now consider the case in which the null geodesics k^μ in the congruence can be written as $k^\mu = f\chi^\mu$. Consider the symmetric part of $\hat{B}_{\mu\nu}$,

$$\begin{aligned} \hat{B}_{(\mu\nu)} &= \hat{h}_\mu{}^\rho\hat{h}^\sigma{}_\nu B_{(\rho\sigma)} &(3.56)\\ &= \hat{h}_\mu{}^\rho\hat{h}^\sigma{}_\nu k_{(\rho;\sigma)} &(3.57)\\ &= \hat{h}_\mu{}^\rho\hat{h}^\sigma{}_\nu f_{;(\sigma}\chi_{\rho)} &(3.58)\\ &= 0 &(3.59)\end{aligned}$$

where in it was used in the step towards line (3.58) that on the null hypersurface $\chi_{(\rho;\sigma)} = 0$, and in the last step it was used that $h_\mu{}^\nu\chi_\nu = 0$, because $h_{\mu\nu}$ projects on the space orthogonal to χ^μ.

[5] See [2] for a proof this theorem

Summarising, null hypersurfaces generated by null congruences which are approximate (local) Killing vector fields have vanishing expansion, shear and rotation on the null hypersurface. Furthermore the expansion disappears in the entire congruence if the vector field satisfies the Killing condition everywhere, for

$$g^{\mu\nu}k_{\mu;\nu} = -g^{\mu\nu}k_{\mu;\nu} = 0 \tag{3.60}$$
$$g^{\mu\nu}k_{\mu;\nu;\rho}k^{\rho} = -g^{\mu\nu}k_{\mu;\nu;\rho}k^{\rho} = 0 \tag{3.61}$$

For congruences orthogonal to null hypersurfaces also the shear and rotation disappear for all λ because of the equations (3.55) and (3.59), this results in

$$0 = \frac{\mathrm{d}\hat{\theta}}{\mathrm{d}\lambda} = R_{\mu\nu}k^{\mu}k^{\nu} \tag{3.62}$$

By means of the Einstein equation (2.10) this is equal to

$$0 = T_{\mu\nu}k^{\mu}k^{\nu} \tag{3.63}$$

Note that $R_{\mu\nu}k^{\mu}k^{\nu} = 0 = T_{\mu\nu}k^{\mu}k^{\nu}$ if along the congruence $\hat{B}_{\mu\nu} = 0$, so if $\hat{B}_{\mu\nu}$ is zero in the points right next to $\lambda = 0$, then no energy flux crosses the horizon near $\lambda = 0$. If only $\hat{B}_{\mu\nu}(0) = 0$, then

$$\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}\lambda}(0) = R_{\mu\nu}k^{\mu}k^{\nu} \neq 0 \tag{3.64}$$

Focusing theorem

For a null hypersurface orthogonal null geodesic congruence $\hat{\omega}_{\mu\nu} = 0$, and all other terms in the Raychaudhuri equation are negative if $T_{\mu\nu}k^{\mu}k^{\nu} > 0$. The condition $T_{\mu\nu}k^{\mu}k^{\nu} > 0$ is known as the weak, strong or null energy condition, and holds for classical matter. The energy momentum density transported along the geodesics is positive. If this condition is met, then the inequality

$$\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} \leq -\frac{1}{2}\theta^2 \tag{3.65}$$

holds. Now the focusing of the generators can be calculated,

$$-\frac{\mathrm{d}\theta^{-1}}{\mathrm{d}\lambda} \leq -\frac{1}{2} \tag{3.66}$$

so for initial value $\theta_i \equiv \theta(0)$,

$$\theta^{-1} \geq \theta_i^{-1} + \frac{1}{2}\lambda \tag{3.67}$$

if $\theta_i < 0$ then θ^{-1} passes through zero for $\lambda \leq \frac{2}{|\theta_i|}$, which means that $\theta \to -\infty$ within finite affine parameter.

3.5 Construction of local light sheets

Local light sheets or local horizons can be constructed by considering the congruence of curves of local uniform accelerated observers. This explains the use of the term horizon, the local light sheet is the past or future horizon for observers which accelerate uniform with respect to local inertial observers. The other advantage of the physical interpretation for the congruence of curves is that it respects the thermodynamic interpretation of the holographic principle.

3.5.1 Light sheets and uniform accelerated observers

Consider a local inertial system in spherical polar coordinates $\zeta^\alpha = (t, r, \theta, \varphi)^\alpha$. The equation of motion for a uniform accelerated observer moving on trajectory $x^\mu(\tau)$ is,

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\kappa\lambda}\frac{\mathrm{d}x^\kappa}{\mathrm{d}\tau}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau} = g^\mu \tag{3.68}$$

The curve is parameterised by observer time τ and g^μ is the acceleration vector. Time runs constantly for the observer, so for all τ

$$g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = -1 \tag{3.69}$$

and this can only be satisfied if

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\left[g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right] = 2g_{\mu\nu}g^\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \tag{3.70}$$

i.e. the acceleration is orthogonal to the velocity [6].

The equation of motion will be looked at in local coordinates, this means that the result is valid to first order in τ. In order to make it valid to higher order a coordinate condition is necessary to fix higher order terms of the metric tensor or the space needs to have symmetries in a larger region.

The observer is accelerated in the radial direction, and the initial velocity is also in this direction. In the local coordinates ζ^α this means that

$$-1 = -\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 \tag{3.71}$$

which implies that the velocity along the curve locally has the appearance

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \pm\cosh(c\tau) \tag{3.72}$$

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm\sinh(c\tau) \tag{3.73}$$

with c some constant, the plus signs are for observers moving to the future and away from the origin. At $\tau = 0$ the orthogonality condition, equation (3.70) is

$$0 = \eta_{\alpha\beta}g^\alpha\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = -g^0\frac{\mathrm{d}t}{\mathrm{d}\tau} \tag{3.74}$$

which means that $g^0 = 0$, the acceleration is purely spacelike. The acceleration is uniform, so at $\tau = 0$

$$\frac{\mathrm{d}g^\alpha}{\mathrm{d}\tau} = (C, 0, 0, 0)^\alpha = C\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} = \frac{\mathrm{d}^3 x^\alpha}{\mathrm{d}\tau^3} \tag{3.75}$$

[6] It is also possible that $g^\mu \propto \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$, and then the equation of motion (3.68) is simply the geodesic equation in non affine parameterisation. In that case a rescaling of the parameter will make $g^\mu = 0$, and then the proper time runs constantly for an observer on a geodesic. A non affine parameterised observer can be interpreted as an observer accelerating along a geodesic.

with C a constant[7]. This is a Lorentz covariant statement, so in local coordinates this holds for all τ in the neighbourhood. The solutions for g^α and x^α are, with $C = c^2$ and A^α, B^α, and C^α constant vectors,

$$g^\alpha = \frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} = c^2 \left(x^\alpha + C^\alpha\right) \tag{3.77}$$

$$x^\alpha = A^\alpha \sinh\left(c\tau\right) + B^\alpha \cosh\left(c\tau\right) - C^\alpha \tag{3.78}$$

The constant vectors A^α and B^α immediately follow from equations (3.72) and (3.73),

$$A^\alpha = (\frac{1}{c}, 0, 0, 0)^\alpha, \qquad B^\alpha = (0, \frac{1}{c}, 0, 0)^\alpha \tag{3.79}$$

The length of the acceleration vector g^α is $\eta_{\alpha\beta} g^\alpha g^\beta = c^2$, and since at $\tau = 0$ the acceleration is purely spacelike this implies that c is the acceleration along the curve of the uniform accelerated observer. The vector C^α can be used to fix the location of the observer at $\tau = 0$, if the observer is at $\tau = 0$ a distance $\frac{d}{c}$ away from the origin, then $C^\alpha = (0, \frac{1-d}{c}, 0, 0)^\alpha$.

Now the coordinate system used by the accelerated observer can be determined. At $\tau = 0$ the time coordinate of the local coordinate system used by the observer at rest coincide, so then the uniform accelerated observer uses the coordinate system $x^\alpha(0, \vec{\zeta})^\alpha$, and at other τ the coordinates are simply the Lorentz boosted coordinates with boost velocity $-c\tau$, so the coordinates are,

$$x^\alpha = ((r - \frac{d}{c}) \sinh(c\tau), (r - \frac{d}{c}) \cosh(c\tau) + d, x^\theta, x^\varphi)^\alpha \tag{3.80}$$

and the line element for the accelerated observer is

$$\eta_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta = -(cr - d)^2 \mathrm{d}\tau^2 + \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\varphi^2 \tag{3.81}$$

This metric is a general form of the Rindler metric in spherical polar coordinates. The Rindler metric in the usual Cartesian form describes observers accelerating away from a plane through the origin with acceleration c,

$$\mathrm{d}s^2 = -c^2 x^2 \mathrm{d}\tau^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{3.82}$$

The singularity of the metric (3.81) at $cr = d$ is a coordinate singularity, because it is always possible to extend this metric back to the local inertial coordinates which are not singular there. The other singularities are the usual coordinate singularities of the spherical polar coordinates.

All the radial accelerated observers together make a congruence of curves, as long as the origin is cut out of the neighbourhood, for that is where the curves intersect. The point of intersection is known as the focal point or caustic. At

[7] For radial motion all affine connection components are zero, i.e. the non zero components of the affine connection contract with the θ and φ components of the vector. The equation of motion in the local inertial spherical polar coordinates for radial motion is therefore

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\tau^2} = g^\alpha \tag{3.76}$$

This is actually the statement that the object is spherically symmetric, i.e. radial motion is nothing more than motion in the 2-dimensional r-t plane.

the points where $cr = d$ the coordinate singularity in the metric occurs, this is the location of the local horizon or holographic screen. The vector field $\xi = \partial_\tau$, which represents (proper) time translations for the accelerated observers is a null vector field at $cr = d$, and it generates the local horizon, or local light sheet. The local holographic screen is the spatial cross section orthogonal to ξ^μ at observer time $\tau = 0$. By choosing an appropriate value for d a local horizon can be constructed with arbitrary positive Gaussian curvature, for the local horizon looks like a surface element of a sphere with radius $\frac{d}{c}$. The physical interpretation of the created metric is a spherically symmetric (artificial) gravitational field.

A flat local horizon can be constructed by considering the congruence of curves of all uniform accelerated observers along an axis in local Cartesian coordinates, the local metric is the Rindler metric (3.82).

3.5.2 Holographic screen with local angular velocity

It is possible to rigidly rotate the metric (3.81) with constant velocity (as seen by the accelerating observer) in order to describe a local rotating horizon. The new metric will describe a stationary axisymmetric rotating (artificial) gravitational field. The new congruence is formed by the curves of observers which accelerate radial to infinity, but with non zero initial angular velocity. The problem will be tackled by rigidly rotating the metric. The local light sheet is constructed at $cr = d$, just as in the non rotating case. The local holographic screen is the spatial surface orthogonal to the light sheet generators at observer time $\tau = 0$.

The infinitesimal rotation is

$$\varphi \quad \rightarrow \quad \tilde{\varphi} = \varphi + \varepsilon\,(\tau) \tag{3.83}$$

inserting this in the metric yields, with $\omega = \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau}$ and dropping the tildes,

$$\begin{aligned} \mathrm{d}s^2 &= -(cr-d)^2\mathrm{d}\tau^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,[\mathrm{d}\varphi - \omega\mathrm{d}\tau]^2 && (3.84)\\ &= -[(cr-d)^2 - \omega^2 r^2\sin^2\theta]\mathrm{d}\tau^2 - 2\omega r^2\sin^2\theta\mathrm{d}\tau\mathrm{d}\varphi \\ &+ \; r^2\sin^2\theta\mathrm{d}\varphi^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 && (3.85)\end{aligned}$$

This is the line element used by the local radial accelerating observers with initial angular velocity ω. Note that it is consistent with the general expression for stationary axisymmetric space-times, (2.14).

The orbit velocity $v = \omega r$ must remain well below the speed of light to if the metric is to be interpreted as a physical (artificial) rotating gravitational field, and such a physical interpretation is necessary in order to do local thermodynamics. However, it is not necessary to have a physical interpretation of the congruence generating the holographic screen in order to do holography. Therefore it is always possible to create a rotating local holographic screen with arbitrary angular velocity.

The metric is singular at $(cr-d)^2 = \omega^2 r^2 \sin^2\theta$, and it is possible to construct a null hypersurface $\tilde{H}$ there, but the tangents to the congruence of the radial accelerating observers are not null there [8], but rather at the horizon or rotating holographic screen H at $cr = d$. In the non rotating case the vector field $\xi = \partial_\tau$

[8]The term $-2\omega r^2 \sin^2\theta < 0$, and this keeps the accelerated observers on timelike curves at the new coordinate singularity.

generates the horizon at $cr = d$, but now this vector is not orthogonal to the horizon H. This is clear if the vector $\varphi = \partial_\varphi$ is considered, which is tangent to the horizon since the surface is φ independent

$$g_{\mu\nu}\xi^\mu\varphi^\nu|_H = -2\omega r_H^2 \sin^2\theta \tag{3.86}$$

As φ is not orthogonal to ξ, except at $\theta = 0$ or $\theta = \pi$, the conclusion must be that ξ is not the horizon generator.

The horizon H in the non rotating case was at $cr = d$, at this location there is a null vector field $\chi^\mu = \xi^\mu + C\varphi^\mu$ in the rotating case, and the constant C follows from

$$0 = g_{\mu\nu}\chi^\mu\chi^\nu|_H \quad = \quad g_{00}\xi^0\xi^0 + 2g_{01}C\xi^0\varphi^1 + g_{11}C^2\varphi^1\varphi^1 \tag{3.87}$$

At $cr = d$ this requirement reduces to

$$0 \quad = \quad \omega^2 - 2\omega C + C^2 \tag{3.88}$$

so there

$$C \quad = \quad \omega \tag{3.89}$$

just as in the Kerr-Newman case. The generator of the holographic screen can be found for arbitrary radius $\frac{d}{c}$, and the vector field exists for all values of ω.

Since the screen is generated by a superposition of the local Killing vector fields ξ and φ with constant coefficients, the vector field χ is also a local Killing vector field. Note however that the constructed smooth vector fields are Killing vector fields at an instant of time, as the local rotating Rindler space-time is constructed in a local inertial system at a point p on the space-time manifold. The true difference between this scenario and the Kerr-Newman scenario is the fact that Kerr-Newman scenario is globally stationary, and this is stationary only in a neighbourhood on an arbitrary manifold.

The generator χ of the holographic screen has zero expansion, shear and twist at $\tau = 0$, because at $\tau = 0$ the vector field coincides with a Killing vector field, and it is hypersurface orthogonal.

Artificial ergosphere

At the new coordinate singularity a null hypersurface $\tilde{H}$ can be constructed, but this is not the natural hypersurface of the congruence of curves under consideration, as the congruence under consideration is not at $\tilde{H}$.

The null vector field generating $\tilde{H}$, must also contain a part in the φ direction. That is because it must be orthogonal to all tangents of $\tilde{H}$, and the location of $\tilde{H}$ is independent of φ. Suppose the vector field orthogonal to $\tilde{H}$ is $\tilde{\chi}^\mu = \xi^\mu + \tilde{C}\varphi^\mu$, then on $\tilde{H}$,

$$0 = g_{\mu\nu}\tilde{\chi}^\mu\tilde{\chi}^\nu|_{\tilde{H}} \quad = \quad g_{00}\xi^0\xi^0 + 2g_{01}\tilde{C}\xi^0\varphi^1 + g_{11}\tilde{C}^2\varphi^1\varphi^1 \tag{3.90}$$

As $g_{00} = 0$ on $\tilde{H}$, the requirement is

$$0 \quad = \quad 2g_{01} + g_{11}\tilde{C} \tag{3.91}$$

so the constant is

$$\tilde{C} = 2\omega \tag{3.92}$$

This vector field is clearly not tangent to the congruence of curves under consideration.

A comparison with the Kerr-Newman metric (2.46) learns that the null hypersurface $\tilde{H}$ looks more like the boundary of the ergosphere than a horizon, i.e. g_{00} vanishes and the θ dependence of $\tilde{H}$ are the same as for the Kerr-Newman metric. The ergosphere of the Kerr-Newman black hole is interpreted as frame dragging due to the concentration of mass in the black hole, but here there is no mass associated with the rotation. The coordinate frames are not dragged by matter in this case, but accelerated by hand.

3.5.3 Other local holographic screens

Two families of local holographic screens have been constructed, rotating and non rotating local holographic screens. Actually this might as well be considered just one family, as the rotating case is the non rotating case for $\omega = 0$. In 4-dimensions, i.e. 1 time and 3 spatial dimensions, the non rotating holographic screens can be divided into two subclasses, the local flat screens and the screens with positive local curvature. Are there screens with a negative local curvature? The answer is no. A null geodesic congruence of curves may perhaps be constructed such that the space orthogonal to the congruence has negative curvature may be found, but then the expansion of the geodesics that extend away from the surface will immediately become positive and therefore no light sheets extend from these surfaces.

The two families of local holographic screens constructed here can be used to construct any finite size holographic screen by patching local screens with the right curvature and right angular velocity together.

Part II

Derivation of the Einstein equation

Chapter 4

A thermodynamic interpretation of space-time

In this chapter the generalised laws of thermodynamics are assumed to hold for all matter in the space-time. The generalised laws will be summarised and the important quantities are defined. Assuming the generalised laws of thermodynamics a derivation is given of a local Einstein equation. If this is the true way to derive the Einstein equation, then it could mean that gravity should not be quantised at all, but that it should be treated as a many particle effect, such as water or sound waves. This issue is left open to debate in this thesis, because any local field description of gravitational interactions at (extremely) small scales can only lead to a better understanding of gravity. Only after the theory is developed which describes local gravitational self-interactions and interactions with matter the answer to this question is known. At this point in time there is absolutely no experimental information to give a solid foundation for any answer to this issue.

The derivation of the Einstein equation presented here is a more general derivation than the one originally proposed by Jacobson in 1995, [10]. Jacobson considered locally static horizons, which he calls local Rindler horizons. He created locally accelerated observers in order to model information entering the space-time through the local past horizon of the local Rindler space. He was led to the conclusion that the Einstein equation does not describe a fundamental force, but rather an effective field description of same more fundamental fields. The effective field description breaks down when the local equilibrium condition fails. In a later article he declared that this interpretation no longer has his favour [11], but this doesn't affect the derivation itself.

In the derivation presented here locally stationary horizons are considered instead of static horizons. These are rotating versions of local Rindler horizons. The equivalence principle, which is usually invoked to transform the gravitational field away, is a starting point for the local thermodynamics. This means that local gravitational waves are described by the model, which propagate on some arbitrary background space-time manifold.

4.1 Observers and information

Consider an observer in some space-time. In general the observer cannot observe the entire space-time, for the information present in the space-time can only travel at finite velocities, velocities which are on average less than the speed of light. Only information within the past light cone of the observer can reach him, and he can only send information to the region within his future light cone. The part of the space-time from which an observer can obtain information is a spacelike hypersurface with boundary, where the boundary is a null hypersurface which is the horizon of the observer. Events happening outside his horizon are a spacelike distance away from the observer. The information actually seen by the observer is the boundary, for only on the light cone travel the photons which reach his eyes.

In particular uniform accelerating observers will be unable to see all of space-time, see figure A.5, and observers in a space-time with a black hole will not be able to acquire information from beyond the black hole horizon unless they enter the black hole themselves. These observers are in space-times in which there are what are sometimes called causality barriers. These are a kind of one way streets for information, in one direction the information will experience no barrier, in the other direction it experiences a wall through which it cannot pass, for they will have to accelerate to velocities greater than the speed of light to get through the barrier, which is possible only for tachyons, which are assumed not to exist.

4.2 The postulates of space-time thermodynamics

In chapter 2 the laws of black hole mechanics have been introduced, and these laws have a thermodynamic interpretation. Together with the usual laws of thermodynamics these form the generalised laws of thermodynamics. In this chapter the generalised laws of thermodynamics are the postulates of what might be called geometric statistical mechanics, and these laws will be used to derive the Einstein equations. The Einstein equations can then be interpreted as equations of state. The original laws of thermodynamics have been explored thoroughly by experimental and theoretic means, but the generalised laws can't yet be tested experimentally and have neither been explored thoroughly by theoretic means. Here the generalised laws are studied near local horizons. Local horizons can be created by considering local uniform accelerated observers, see section 3.5.

The quantities of importance in the laws of thermodynamics are heat Q, temperature T, entropy S, work W, and internal energy U. It is not evident what these quantities are in geometry, but the generalised laws interpret surfaces in space-time to be proportional to entropy and the gravity associated with these surfaces as temperature. The surfaces associated with entropy are supposed to be local event horizons, which form the boundary of the observable space-time.

4.2.1 The laws of thermodynamics

In order to clarify the generalised laws of thermodynamics, i.e. the usual thermodynamic laws and the extra black hole laws, I will summarise both sets of laws here.

The laws of thermodynamics are, the zeroth law,

$$\text{local thermodynamic equilibrium } \Leftrightarrow T = \text{constant} \tag{4.1}$$

so two volume elements in thermal equilibrium have equal temperature.

The first law, which expresses conservation of energy, is

$$\mathrm{d}U = \mathrm{d}Q + \mathrm{d}W \tag{4.2}$$

i.e. the increase of internal energy of a system is equal to the supplied heat to the system plus the work done on the system. The heat transferred to a system is defined by

$$\mathrm{d}Q \equiv T\mathrm{d}S \tag{4.3}$$

Entropy is the logarithm of the number of accessible states. Adding heat to a system means adding degrees of freedom to a system, whereas work does not change the number of accessible states of the system.

The second, with δS the change of entropy in a closed system,

$$\delta S \geq 0 \tag{4.4}$$

or the entropy of a closed system increases over time.

Finally the third law,

$$T \to 0 \Rightarrow \delta S \to 0 \tag{4.5}$$

which means that the entropy will go to a constant value when the temperature goes to zero.

4.2.2 The laws of geometric dynamics

The laws of black hole thermodynamics are assumed to describe the geometry of space-time. The laws are the zeroth,

$$\text{local stationary horizon } \Leftrightarrow \kappa = \text{constant} \tag{4.6}$$

i.e. two pieces of horizon in equilibrium have equal surface gravity. The identification of surface gravity with temperature is made.

The first law, which describes conservation of energy is

$$\mathrm{d}U = \alpha \mathrm{d}A \tag{4.7}$$

where α is a constant of proportionality[1], and A is the area of the horizon. U is the energy hidden by the horizon, sometimes referred to as the energy of the horizon. In chapter 2 the first law was

$$\mathrm{d}M = \frac{\kappa}{8\pi}\mathrm{d}A + \omega_H \mathrm{d}J + \Phi_H \mathrm{d}e \tag{4.8}$$

where M is the mass hidden by or on the horizon, J is the angular momentum of the horizon, or all objects hidden by the horizon, and ω_H the angular velocity

[1] α is constant in equilibrium, i.e. when the surface gravity is constant.

of the horizon. Φ_H is the corotating electric potential of the horizon and e is the total electric charge hidden by the horizon. $\mathrm{d}A$ is an area element of the horizon. The angular momentum term and the electromagnetic charge term are in [2] interpreted as work terms, and the mass is interpreted as the internal energy of a black hole. The area is interpreted as a measure of entropy. The interpretation of work and heat in the scenario presented here is treated in section 4.3, but only in a scenario without electric charges.

The second law is

$$\mathrm{d}a \geq 0 \tag{4.9}$$

which means that the area of an area element $a = \mathrm{d}A$ of a local horizon increases in every process, just as the entropy of a closed system increases in any process. The entropy hidden by horizons is proportional to the surface area of the horizon.

Finally the third law

$$\kappa \to 0 \quad \Rightarrow \quad \mathrm{d}a \to 0 \tag{4.10}$$

The surface of the horizon is constant for zero surface gravity.

The generalised laws of thermodynamics are the same as the laws of thermodynamics with the requirement that the horizon thermodynamics is taken into account, i.e. the temperature of a horizon is proportional to its surface gravity and the entropy of the horizon is proportional to its area. In local equilibrium the heat in a region of space-time with a boundary is the heat hidden by the boundary and the heat carried by matter present in the region,

$$\mathrm{d}Q_{tot} \equiv \mathrm{d}Q_{th} + \mathrm{d}Q_H \tag{4.11}$$
$$= T_{th}\mathrm{d}S_{th} + T_H\mathrm{d}S_H \tag{4.12}$$

where in equilibrium the thermodynamic temperature T_{th} and the horizon temperature T_H are equal, $T_{th} = T_H$. A problem with a thermodynamic interpretation of space-time is how heat and work are defined in a geometric setting. The quantities appearing in the postulates will be explained in section 4.3.

4.3 Interpretation of the thermodynamic quantities

What is the meaning of the postulates of geometric statistical mechanics? A satisfactory interpretation should be given of each quantity appearing in the postulates. The original thermodynamic laws have a clear interpretation in the absence of gravity, so these laws shall not be treated, as the thermodynamics of the interactions of gravity with matter is described by the laws geometric dynamics.

4.3.1 Work

Usually work is associated with external parameters describing the system, and heat is defined as energy flow between the system and a heat bath, or thermal reservoir. The notion of heat and work are not trivial in space-time. The metric can't just be put in a box and it is not clear what the external parameters are if the system can't be isolated. This can be illustrated with the Kerr-Newman space-time.

The Kerr-Newman space-time is asymptotically flat, so an observer at infinity (outside the space-time) will interpret the angular momentum of the space-time as external parameter, which can be changed by somehow adding angular momentum to the space-time. What happens if the outside observer adds angular momentum to the space-time? The initial system is described by the 4 parameters of the Kerr-Newman metric. The outside observer will only change one parameter, the total angular momentum of the space-time J. The external observer[2] adds a little angular momentum δJ. After some time the space-time will return to an equilibrium configuration with angular momentum $\tilde{J} = J + \delta J$. The outside observer is bored[3] with his new space-time, and decides he preferred the original one, therefore he adds angular momentum[4] $-\delta J$. Again after some time the space-time will settle down to an equilibrium, which has angular momentum $\tilde{\tilde{J}} = \tilde{J} - \delta J = J$ and the system has returned to its original configuration without generating entropy.

The outside observer at infinity sees an isolated system and would perhaps naively use the vector field ξ^μ as the vector field generating time translations, but in the ergosphere of the black hole this vector field is no longer a timelike vector field. An observer within the ergosphere will certainly not experience this field as the generator of the time translations. The local observer does not see an isolated black hole and cannot interpret the angular momentum as external variable, for if the angular momentum of the black hole is varied, then also the angular momentum of the gravitational field is changed. The observer in the ergosphere near the horizon would use χ^μ as the timelike vector field, he has to accelerate in that direction to remain at a fixed distance from the black hole, because he is dragged by the rotating black hole. The local observer cannot just increase the angular momentum of the black hole and then decrease it again to its original value without angular momentum dissipating to the gravitational field of the black hole. The local observer would perform a Penrose process to decrease the angular momentum of the black hole, but this is a non local process. The energy of the black hole has decreased if positive energy leaves the space-time at infinity. In other words he cannot return to the original configuration and repeat the process, thus a transfer of angular momentum is an irreversible process.

The Kerr-Newman metric can be interpreted as a closed system by observers at infinity, and can be by external parameters. Observers which are far enough away[5] from the black hole may also give a good description of the black hole with these parameters. An observer which is near the horizon can't use these parameters to describe a closed system, because the gravitational itself carries the mass and angular momentum, which can be assigned to the black hole only by observers at infinity. The conclusion must be that work can only be defined for isolated systems.

[2] This observer is apparently some sort of deity

[3] As usual the creation (the observer) was created as an image of its creator, i.e. in this case me.

[4] Angular momentum can be extracted from a black by a Penrose process, see [2].

[5] Far enough would be the region of space-time where metric is approximately flat

4.3.2 Temperature and surface gravity

It is not manifest what the surface gravity of a horizon is. The horizon of a Rindler space-time for example has a surface gravity that is observer dependent. For a global Rindler space-time the surface gravity must be zero, as Rindler space-time can be extended to Minkowski space-time. Thus no matter is present to generate the gravitational field, although it is possible to define a surface gravity for the observer as his own acceleration. How does a local observer know what the true surface gravity is? It turns out that it doesn't matter what the true surface gravity is, for it will be shown that in the definitions used here both the heat and the temperature contain the surface gravity κ and therefore drop out of the thermodynamic relations.

The temperature of a local patch of space-time in instantaneous equilibrium will be the Unruh temperature, defined by

$$T = \frac{\kappa}{2\pi} \tag{4.13}$$

in which κ is the acceleration of the observer with respect to a vacuum space-time[6]. An accelerated observer with velocity v^μ has as equation of motion $v^\mu{}_{;\nu}v^\nu = \kappa v^\mu$, in other words, he moves on a non affine parameterised geodesic.

4.3.3 Entropy, heat, and internal energy

A local observer hovering near a black hole does not see the horizon, and can't define a black hole as a closed system, unless if he knows the global space-time metric. He can look at matter falling towards the black hole and define a comoving volume element falling with the matter as long as the matter has not entered the black hole. He will see this matter fade to black within a finite time, see [1], so he does notice that matter disappears from the space-time. The comoving volume element appears to flatten as it falls towards the horizon when it is described by the outside observer. An observer falling with the matter does not see this flattening, for he would measure using a local Minkowski metric.

The vector field to be defined as the local (approximate) Killing vector field generating the infinitesimal time translations is the vector field generating the local horizon (null hypersurface), at the horizon this is a null hypersurface orthogonal vector field.

The entropy can only be defined for a closed system, here the closed system is a comoving volume element, a volume element which moves along with the the energy which moves towards the horizon. The entropy leaves the space-time through the local horizon, and the entropy in the volume element scales with the surface area of the local horizon. That is because the generalised second law is assumed and the volume element which falls towards the horizon is flattened as seen by the observer hovering outside the horizon. Just before it fades out of existence the volume element will have vanishing width [7].

[6] One can also call it the gravitational acceleration an observer experiences when he hovers at some point outside the gravitating object.

[7] A volume element falling to the horizon with the shape of a ball will evolve into a volume element which looks like a pancake with the same radius as the ball had for observers which hover near the horizon of a black hole.

There is no work in this scenario, see 4.3.1, and the entropy $\mathrm{d}S$ in the volume element is related to the internal energy U of the volume element by

$$T\mathrm{d}S = \mathrm{d}Q \tag{4.14}$$
$$= \mathrm{d}U \tag{4.15}$$

The internal energy in geometric dynamics is proportional to the surface area of the horizon, and this means that the entropy hidden by the local horizon is

$$T\mathrm{d}S = \alpha \mathrm{d}A \tag{4.16}$$

where $\alpha = \frac{\kappa}{8\pi}$ if the postulated laws of geometric dynamics have exactly the same form as described in chapter 2.

4.3.4 The composition of internal energy

The internal energy of a part of space-time is proportional to the mass and angular momentum content of that part of the space-time. In a more general case also the energy of electromagnetic charges and fields etc. must be added if they are present in the part of space-time under consideration. Locally the internal energy satisfies

$$\mathrm{d}U = \mathrm{d}M - \omega_H \mathrm{d}J \tag{4.17}$$

With equation (4.15) this means that the quantities M and J which have the interpretation of mass and angular momentum respectively are transferred over the horizon as heat.

Infinitesimal geometric quantities mass $\mathrm{d}M$ and angular momentum $\mathrm{d}J$ of a local patch of the horizon may be defined for the [8] by

$$\mathrm{d}M \equiv \xi_{\mu;\nu}\mathrm{d}S^{\mu\nu} \tag{4.18}$$
$$\mathrm{d}J \equiv \varphi_{\mu;\nu}\mathrm{d}S^{\mu\nu} \tag{4.19}$$

with ξ^μ a local stationary Killing vector field and φ^μ a local rotational Killing vector field. The surface element $\mathrm{d}S^{\mu\nu}$ of the horizon can be given by $n^{[\mu}m^{\nu]}\mathrm{d}A$, where n^μ and m^ν are null geodesics orthogonal to the horizon chosen such that $g_{\mu\nu}n^\mu m^\nu = -1$, see section 35 for the construction of a metric on the horizon.

It is also possible to define these geometric quantities for the local metric in a volume near the horizon, but this definition is even less obvious than the previous definition for the horizon.

$$\mathrm{d}M \equiv \xi^\nu{}_{\mu;\nu}\mathrm{d}V^\mu \tag{4.20}$$
$$\mathrm{d}J \equiv \varphi^\nu{}_{\mu;\nu}\mathrm{d}V^\mu \tag{4.21}$$

where $\mathrm{d}V^\mu = n^\mu \mathrm{d}V$ is the volume element of the space-time region with normal n^μ. These definitions are related if the Stokes theorem is applied, but only when the horizon encloses the region V of space-time and the vector fields can be identified with each other.

[8]There are two reasons for these definition, the first is that these Killing vector fields have these constants of the motion associated with them, see 2.2.1, and the second (related) reason is that the Komar integrals will be reproduced if one integrates over a closed horizon which is the boundary of some region of space-time, cf. black holes.

4.4 The Einstein equation

The quantities appearing in the generalised thermodynamic laws have now been identified and in this section the generalised laws will be used to derive the Einstein equation. The derivation is done locally, the resulting Einstein equation describes the local interaction of the metric with matter present locally on the metric.

The principle of equivalence states that in any point p in the space-time it is possible to find a local coordinate system in which the metric is Minkowski at p with vanishing first derivatives. This means that locally the gravitational field, i.e. the affine connection, vanishes in p. In a local Minkowski space-time no horizons are present, but a local horizon can be created by considering Lorentz boosted observers near p, see section 3.5. The horizons created in this way are called local Rindler horizons, and the rotating version may be called a rotating local Rindler horizon. The local Rindler space-time possesses local symmetries, i.e. instead of the usual global Killing vector field generating the Lorentz boost in a global Rindler space-time, the local Rindler space-time has the Killing vector field property only instantaneous [9]. The rotating local Rindler space-time is an instantaneously stationary axisymmetric space-time, with the instantaneous Killing vector fields associated with these symmetries.

In this way not only an event horizon in an arbitrary gravitational field can be described locally, but any local interaction of matter with the metric can be described by the space-time thermodynamics. The only requirement is local thermodynamic equilibrium. As long as the principle of equivalence holds, local thermodynamic equilibrium exists. That is because Minkowski space-time is a flat vacuum space-time. If a transformation can be found such that the space-time is Minkowski at a point, then the space-time is vacuum around that point, and a local vacuum is in local equilibrium.

A locally accelerating observer will see a local Rindler space-time, and a heat bath at the Unruh temperature, see section 2.7. The local Rindler space-time is also in local equilibrium, because it can be transformed to a local Minkowski space-time, but it does have a matter current, and the thermodynamics is non-trivial in the local Rindler space-time.

4.4.1 Einstein equation derived using a local horizon

The Einstein equation will now be derived in the neighbourhood of a local rotating Rindler horizon using the generalised laws of thermodynamics. The original derivation of Jacobson, see [10], was for a non rotating local Rindler horizon, which is a special case of this derivation. The key ingredients in the derivation are the equalities $\mathrm{d}Q = T\mathrm{d}S$ and $\mathrm{d}S = \eta \mathrm{d}A$, where $\eta = \frac{\alpha}{T}$.

Consider an observer in a local rotating Rindler space-time, see section 3.5.2, and let $\chi^\mu = \xi^\mu + \omega_H \varphi^\mu$ be the generator of the local rotating horizon. The vector field ξ^μ generates the motion towards the horizon, and $\omega_H \varphi^\mu$ generates a rigid rotation of the horizon with angular velocity ω_H. The case $\omega_H = 0$ represents the case treated by Jacobson, the non rotating local Rindler horizon. The

[9]The vector field satisfies the Killing property only in a spacelike neighbourhood of p, the vector field exists in the future and past of p, but there it doesn't satisfy the Killing property.

vector field $\chi^\mu$is timelike outside[10] the local horizon. At point p, i.e. the point in the local Minkowski frame where the horizon is constructed, it is normalised such that $\chi_\mu \chi^\mu = -1$ and it is future directed, i.e. it points in the direction of the flow of time. The timelike vector field χ^μ is a geodesic of the local rotating Rindler space-time in non affine parameterisation, $\chi^\mu{}_{;\nu}\chi^\nu = \kappa\chi^\mu$. The vector field χ^μ can be interpreted as a local boost vector field generating the energy (matter) flow over the horizon[11].

On the local rotating horizon χ^μ is null and it is orthogonal to the horizon, for the horizon is the null hypersurface generated by χ^μ. Vector fields which are null hypersurface orthogonal have the property that they satisfy the equation for non affine parameterised geodesics[12]. Therefore on the local rotating horizon the boost vector field satisfies $\chi_\mu\chi^\mu = 0$ as well as $\chi^\mu{}_{;\nu}\chi^\nu = \kappa\chi^\mu$ with the scalar function κ a constant scalar due to the generalised zeroth law which states that the surface gravity is constant on a local stationary horizon. The horizon is in this case generated by the local stationary axisymmetric vector field χ^ν, and is indeed a local stationary horizon.

The χ^μ can be reparameterised such that they are geodesics of the local rotating Rindler space-time. The geodesics k^μ are defined by, see appendix A.7,

$$\chi^\mu = -\kappa\lambda k^\mu \tag{4.22}$$

where λ is the affine parameter, which is zero in the neighbourhood of point p where the local inertial frame is constructed, and is negative to the past of p.

The heat $\mathrm{d}Q$ in the comoving volume element $\mathrm{d}V$ which falls towards the horizon, as seen by the accelerating observers outside the local horizon on the vector field χ^μ, is equal to the energy $\mathrm{d}U$ in that volume element. The energy moving along with the volume element is the energy transported along the vector field χ^μ. The energy current p^μ seen by the observers on χ^μ is

$$\begin{aligned} p^\mu &= T^\mu{}_\nu \chi^\nu &(4.23)\\ &= -\kappa\lambda T_{\mu\nu}k^\nu &(4.24)\end{aligned}$$

The volume $\mathrm{d}V$ is the volume of the spacelike hypersurface generated by the timelike geodesic vector field k^μ, this is the natural normal vector to the volume. The energy in the volume element is

$$\mathrm{d}U = -\kappa\lambda T_{\mu\nu}k^\mu k^\nu \mathrm{d}V \tag{4.25}$$

Near the horizon the volume becomes flatter and flatter, as seen by the observer on χ^μ, the volume element infinitesimally close to the horizon is

$$\mathrm{d}V = \mathrm{d}A\mathrm{d}\lambda \tag{4.26}$$

Infinitesimally close to the horizon the energy or heat moving towards the horizon is given by

$$\mathrm{d}Q = -\kappa\lambda T_{\mu\nu}k^\mu k^\nu \mathrm{d}A\mathrm{d}\lambda \tag{4.27}$$

[10] Outside the local horizon means that the vector field χ^μ is considered to be farther away from the origin of the local metric (3.85) than the local horizon

[11] A rotating black hole drags matter along with it, this scenario is artificially created when local rotating Rindler horizons are constructed.

[12] See appendix A.6.

Although the observer does not see it, the heat is expected to cross the horizon. The amount of entropy that will cross the horizon at observer time λ is the heat divided by the Unruh temperature

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T} \tag{4.28}$$
$$= -2\pi\lambda T_{\mu\nu}k^{\mu}k^{\nu}\mathrm{d}A\mathrm{d}\lambda \tag{4.29}$$

Note that the surface gravity has disappeared from the equation for entropy. The accelerated observer observes a local heat bath which is homogeneous, therefore the scalar quantity $T_{\mu\nu}k^{\mu}k^{\nu}$ has the following expansion in λ,

$$T_{\mu\nu}k^{\mu}k^{\nu}(\lambda) = T_{\mu\nu}k^{\mu}k^{\nu}(0) + \mathrm{O}(\lambda^2) \tag{4.30}$$

The change of entropy is proportional to the change in horizon area, $\mathrm{d}S = \eta\mathrm{d}a$. The change in the horizon area a is[13],

$$\frac{\mathrm{d}a}{\mathrm{d}\lambda} = \theta a \tag{4.31}$$

with θ the expansion of the congruence of null geodesics k^{μ}, which generates the horizon.

The vector field k^{μ} is instantaneously stationary at $\lambda = 0$ and has the property

$$B_{\mu\nu}\big|_{\lambda=0} \equiv h^{\rho}{}_{\mu}h_{\sigma}{}^{\nu}k^{\mu}{}_{;\nu}\big|_{\lambda=0} = 0 \tag{4.32}$$

where $h_{\mu\nu}$ is the projection operator on 2-dimensional space[14] orthogonal to the geodesics k^{μ}.

The horizon area element at affine parameter λ is

$$a(\lambda) = a_0 e^{\int_0^{\lambda}\mathrm{d}\lambda'\theta} \tag{4.33}$$
$$\approx a_0\left(1 + \lambda\theta(0) + \frac{\lambda^2}{2}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}(0)\right) = a_0\left(1 + \frac{\lambda^2}{2}R_{\mu\nu}k^{\mu}k^{\nu}\right) \tag{4.34}$$

where $a_0 = \mathrm{d}A$, the surface element of the horizon at $\lambda = 0$, and to obtain the term $\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}(0)$ the Raychaudhuri equation for null geodesic congruences was used, equation (3.29), in conjunction with the fact $B^{\mu}{}_{\nu}(0) = 0$, see equation (4.32). Thus the local increase of horizon area $\mathrm{d}a$ is

$$\mathrm{d}a = \lambda R_{\mu\nu}k^{\mu}k^{\nu}\mathrm{d}\lambda\mathrm{d}A \tag{4.35}$$

and the increase of the entropy hidden by the horizon at observer time λ is equal to

$$\mathrm{d}S = \eta\mathrm{d}a = \eta\lambda R_{\mu\nu}k^{\mu}k^{\nu}\mathrm{d}\lambda\mathrm{d}A \tag{4.36}$$

and therefore $\frac{\mathrm{d}Q}{T} = \mathrm{d}S$ is equivalent to

$$-2\pi\lambda T_{\mu\nu}k^{\mu}k^{\nu}\mathrm{d}\lambda\mathrm{d}A = \eta\lambda R_{\mu\nu}k^{\mu}k^{\nu}\mathrm{d}\lambda\mathrm{d}A \tag{4.37}$$

[13] A derivation for this equation is in section 35

[14] See section 3.4.4, note that the hats have been dropped here. The equation (4.32) is also proven for the null geodesic congruences under consideration here.

When $\lambda \neq 0$ then the expression can be divided by $\eta\lambda \mathrm{d}A\mathrm{d}\lambda$,

$$-\frac{2\pi}{\eta}T_{\mu\nu}k^\mu k^\nu \mathrm{d}\lambda\mathrm{d}A \quad = \quad R_{\mu\nu}k^\mu k^\nu \mathrm{d}\lambda\mathrm{d}A \tag{4.38}$$

As the k^μ are null vector fields there is freedom to add a term proportional to the metric to this equation. The term $fg_{\mu\nu}k^\mu k^\nu$, with f a scalar function is added to the right hand side,

$$-\frac{2\pi}{\eta}T_{\mu\nu}k^\mu k^\nu \quad = \quad R_{\mu\nu}k^\mu k^\nu + fg_{\mu\nu}k^\mu k^\nu \tag{4.39}$$

or

$$0 \quad = \quad \left(\frac{2\pi}{\eta}T_{\mu\nu} + R_{\mu\nu} + fg_{\mu\nu}\right)k^\mu k^\nu \tag{4.40}$$

This equation holds for all null vector fields on the horizon, for the first law $\mathrm{d}Q = T\mathrm{d}S$ does not depend on any special family of geodesics. Therefore the term between brackets must be zero,

$$0 \quad = \quad \frac{2\pi}{\eta}T_{\mu\nu} + R_{\mu\nu} + fg_{\mu\nu} \tag{4.41}$$

Now the function f can be found by taking the the divergence of this equation [15],

$$0 \quad = \quad R^{\mu\nu}{}_{;\mu} + f_{;\mu}g^{\mu\nu} \tag{4.42}$$

The Bianchi identities, $\left(R^{\mu\nu} - \frac{R}{2}g^{\mu\nu}\right)_{;\mu} = 0$, then yield

$$f \quad = \quad -\frac{R}{2} + C \tag{4.43}$$

with C some constant. The resulting equation relating the metric to the energy momentum density of matter on the metric is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + Cg_{\mu\nu} \quad = \quad -\frac{2\pi}{\eta}T_{\mu\nu} \tag{4.44}$$

Compare this with the Einstein equation (2.10),

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} \quad = \quad -8\pi T_{\mu\nu} \tag{4.45}$$

The Einstein equation is recovered when the constant C is identified with the cosmological constant, $C = -\Lambda$, and η is identified as $\eta = \frac{1}{4}$.

Higher order in λ

Both the change in area and heat flux were given to first order in λ. The solution to the equation of motion for the local observers generating the horizon was also valid to this order. Still it is nice to know if the Einstein equation is recovered if terms at higher order are considered, e.g. if a space-time is considered in which

[15] The energy momentum tensor is divergence free by definition, $T^{\mu\nu}{}_{;\mu} = 0$.

there is a higher order neighbourhood in which the solutions to the equation of motion hold to higher order in λ.

The λ expansion of the area variation is to λ^4 inclusive

$$
\begin{aligned}
a(\lambda) &= \mathrm{d}Ae^{\int_0^\lambda \mathrm{d}\lambda'\theta} && (4.46)\\
&\approx \mathrm{d}A\left(1+\frac{\lambda}{1!}\theta(0)+\frac{\lambda^2}{2!}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}(0)+\frac{\lambda^3}{3!}\frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2}(0)+\frac{\lambda^4}{4!}\frac{\mathrm{d}^3\theta}{\mathrm{d}\lambda^3}(0)\right) && (4.47)
\end{aligned}
$$

These terms are easily evaluated using the expressions for the parallel transport of the $B_{\mu\nu}$, see equation (3.30) together with the fact $B_{\mu\nu}(0)=0$,

$$
\begin{aligned}
\frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2}(0) &= R_{\mu\nu;\rho}k^\rho k^\mu k^\nu && (4.48)\\
\frac{\mathrm{d}^3\theta}{\mathrm{d}\lambda^3}(0) &= R_{\mu\nu;\rho;\sigma}k^\rho k^\sigma k^\mu k^\nu-(R_{\mu\nu}k^\mu k^\nu)^2\\
&- 2C_{\mu\sigma\nu\rho}C^{\mu\kappa\nu\lambda}k^\sigma k^\rho k_\kappa k_\lambda && (4.49)
\end{aligned}
$$

Thus the higher order area variations are

$$
\begin{aligned}
\mathrm{O}(\lambda^2) &= \eta T\frac{\lambda^2}{2!}R_{\mu\nu;\rho}k^\rho k^\mu k^\nu && (4.50)\\
\mathrm{O}(\lambda^3) &= \eta T\frac{\lambda^3}{3!}\left(R_{\mu\nu;\rho;\sigma}-R_{\mu\nu}R_{\rho\sigma}-2C_{\kappa\sigma\lambda\rho}C^{\kappa}{}_{\mu}{}^{\lambda}{}_{\nu}\right)k^\rho k^\sigma k^\mu k^\nu && (4.51)
\end{aligned}
$$

The tensor $C_{\kappa\lambda\mu\nu}$ is the Weyl tensor, i.e. the traceless symmetric part of $R_{\kappa\lambda\mu\nu}$, and it measures the deformation of the manifold.

The heat flux terms to higher order in λ are, under the assumption that $\kappa=\text{constant}$,

$$
\begin{aligned}
\mathrm{O}(\lambda^2) &= -\kappa\frac{\lambda^2}{2!}T_{\mu\nu;\rho}k^\rho k^\mu k^\nu && (4.52)\\
\mathrm{O}(\lambda^3) &= -\kappa\frac{\lambda^3}{3!}T_{\mu\nu;\rho;\sigma}k^\rho k^\sigma k^\mu k^\nu && (4.53)
\end{aligned}
$$

These expressions are simply the terms in the Taylor expansion of the function $F=-\kappa\lambda T_{\mu\nu}k^\mu k^\nu$. If the assumption $\kappa=\text{constant}$ is dropped, then terms with derivatives of κ need to be added, and the relation between χ^μ and k^μ will no longer be a factor λ.

The first order terms gave the Einstein equation which connects $T_{\mu\nu}$ and $R_{\mu\nu}$. In the second order equation there is also the freedom to add new terms. The new equality is, with $C=\frac{2\pi}{\eta}$,

$$
0 = \lambda^2\left[CT_{\mu\nu;\rho}+R_{\mu\nu;\rho}+U_\mu g_{\nu\rho}+V_\rho g_{\mu\nu}+W_\nu g_{\rho\mu}\right]k^\rho k^\mu k^\nu \qquad (4.54)
$$

Contracting the term between brackets with $g^{\mu\nu}$ yields

$$
\begin{aligned}
0 &= CT^\mu{}_{\mu;\rho}+R_{;\rho}+U_\rho+4V_\rho+W_\rho && (4.55)\\
&= (CT^\mu{}_\mu+R)_{;\rho}+X_\rho && (4.56)
\end{aligned}
$$

where $X_\rho\equiv U_\rho+4V_\rho+W_\rho$. The Einstein equation is recovered for

$$
X_\rho = 2R_{;\rho} \qquad (4.57)
$$

A good choice for the three vectors is $U_\mu = W_\mu = 0$, and $V_\mu = \frac{1}{2}R_{;\mu}$.

The third order terms will yield a tensor $Q_{\mu\nu\rho\sigma}$, which is defined as a sum of all permutations of indices of terms $P_{\mu\nu}g_{\rho\sigma}$, because there are now that many ways to contract the metric tensor with the vector field k^μ. The new relation is

$$0 = \lambda^3 \left[CT_{\mu\nu;\rho;\sigma} + R_{\mu\nu;\rho;\sigma} - R_{\mu\nu}R_{\rho\sigma} - 2C_{\kappa\sigma\lambda\rho}C^{\kappa\ \lambda}_{\ \mu\ \nu} + Q_{\mu\nu\rho\sigma} \right] k^\rho k^\sigma k^\mu k^\nu$$

Due to the symmetries of the other terms the tensor $Q_{\mu\nu\rho\sigma}$ has the symmetries

$$Q_{\mu\nu\rho\sigma} = Q_{\mu\nu\sigma\rho} = Q_{\nu\mu\sigma\rho} = Q_{\mu\nu\rho\sigma} \tag{4.58}$$

The Einstein equation is recovered for

$$Q_{\mu\nu\rho\sigma} \equiv R_{;\rho;\sigma}g_{\mu\nu} - R_{\rho\sigma}R_{\mu\nu} - 2C_{\kappa\sigma\lambda\rho}C^{\kappa\ \lambda}_{\ \mu\ \nu} \tag{4.59}$$

The vector X^μ and tensor $Q_{\mu\nu\rho\sigma}$ can be chosen such that the Einstein equation is satisfied, it is expected that this is possible to all orders in λ.

Stability under gravitational disturbances

Consider a gravitational disturbance,

$$g_{\mu\nu} \quad \to \quad \tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu} \tag{4.60}$$

What is the effect of such a disturbance on the Raychaudhuri equation for a null geodesic congruence k^μ? An observer on a geodesic in the congruence will see the other geodesics in the congruence deviate differently than before, the disturbance affects the space orthogonal to the geodesics. The varied Raychaudhuri equation is

$$\frac{\mathrm{d}\delta\hat{\theta}}{\mathrm{d}\lambda} = -\hat{\theta}\delta\hat{\theta} - 2\hat{\sigma}^{\mu\nu}\delta\hat{\sigma}_{\mu\nu} + 2\hat{\omega}^{\mu\nu}\delta\hat{\omega}_{\mu\nu} + \delta R_{\mu\nu}k^\mu k^\nu \tag{4.61}$$

$$= 2\hat{B}^{\nu\mu}\delta\hat{B}_{\mu\nu} + \delta R_{\mu\nu}k^\mu k^\nu \tag{4.62}$$

The variation of the Ricci tensor is well known, this yields the Palatini identity, see e.g. [1], so only the variation of $\hat{B}_{\mu\nu}$ will be treated here,

$$\delta\hat{B}_{\mu\nu} = \delta\hat{h}_{\mu\rho}B^{\rho\sigma}\hat{h}_{\sigma\nu} + \hat{h}_{\mu\rho}B^{\rho\sigma}\delta\hat{h}_{\sigma\nu} \tag{4.63}$$

$$= \delta g_{\mu\rho}\hat{B}^{\rho\sigma}\hat{h}_{\sigma\nu} + \hat{h}_{\mu\rho}\hat{B}^{\rho\sigma}\delta g_{\sigma\nu} \tag{4.64}$$

$$= \delta g_{\mu\lambda}\hat{B}^{\lambda}{}_{\nu} + \hat{B}_{\mu}{}^{\lambda}\delta g_{\lambda\nu} \tag{4.65}$$

This is clearly proportional to $\hat{B}_{\mu\nu}$, and since $\hat{B}_{\mu\nu} = 0$ the only non vanishing term in the Raychaudhuri equation is the disturbed Ricci tensor,

$$\frac{\mathrm{d}\delta\hat{\theta}}{\mathrm{d}\lambda} = \delta R_{\mu\nu}k^\mu k^\nu \tag{4.66}$$

and thus the varied expansion is

$$\delta\hat{\theta} = \lambda\delta R_{\mu\nu}k^\mu k^\nu + C \tag{4.67}$$

with C some constant which is set to zero using appropriate boundary conditions.

The disturbance of the heat transfer is

$$\delta Q \quad \propto \quad \lambda \delta R_{\mu\nu} k^{\mu} k^{\nu} \tag{4.68}$$

so the first order Einstein equations are recovered, i.e. the disturbed metric also satisfies the Einstein equations.

If a second order disturbance is considered nothing changes, now there are terms proportional to $\delta \hat{B}^{\mu\nu} \delta \hat{B}_{\mu\nu}$, $\hat{B}^{\mu\nu} \delta^2 \hat{B}_{\mu\nu}$, and $(\delta^2 R_{\mu\nu}) k^{\mu} k^{\nu}$. The only variation not proportional to $\hat{B}_{\mu\nu}$ is again the varied Ricci tensor, so again nothing new happens.

The conclusion is that the local thermodynamic description of gravity is independent of external gravitational waves passing through the local volume element.

4.4.2 Discussion

The state of a system is only well defined if the system remains infinitesimally close to equilibrium, does this system satisfy that property? Yes, for space-time is locally empty (or a local vacuum) if the local inertial system is considered, so the local infinitesimally accelerated observer observes a space in local equilibrium.

The infinitesimal Lorentz boost can be interpreted as a thermal fluctuation of the vacuum. A fluctuation immediately causes a local acceleration horizon to appear. The surface gravity may be interpreted as the reacting force of the thermal bath which the boosted particle crashes into, thus it experiences a force which pushes it back to its original location. Another interpretation for κ is to interpret it as a measure for the probability for the fluctuation to happen, a larger acceleration is less likely to happen than a small one.

One can also consider the fluctuations as quantum fluctuations, then the origin of the pull back force can be interpreted as the gravitational pull which comes from the antiparticle created at the same time as the particle. The gravitational acceleration felt by the created particle measures the mass of the anti particle. Again the acceleration is a measure of the size of the fluctuation.

Since the local equilibrium is the configuration in which the highest entropy is attained (together with the lowest energy), the entropy seen by the infinitesimally boosted particle must be infinitesimally close to the equilibrium entropy, the process is isentropic.

Forget the second law of black hole thermodynamics for a moment, and consider only the second law of ordinary thermodynamics. This law states that the entropy in a closed system can not decrease. Consider the closed system to be a local Minkowski neighbourhood of space-time. This system is in local equilibrium, and it is a local vacuum, therefore define its temperature and entropy to be zero. A Lorentz boosted particle in this local Minkowski system observes a heat bath, and an entropy transfer over the horizon. This is against the second law, the entropy, which is a scalar and is therefore invariant under coordinate transformations, of the local Rindler space-time observed by the particle must be infinitesimally close to the entropy of the local Minkowski space-time experienced by an inert particle. Apparently the entropy lost over the horizon is equal to the entropy which is created by accelerating, i.e. the entropy of the observed heat bath. The total entropy in the original Minkowski volume was zero, and therefore the entropy hidden by the local horizon of the boosted particle must cancel the entropy of the heat bath observed by the particle. The entropy hidden by the horizon is the entropy which is seen to cross the horizon by the accelerated particle. The entropy which the particle observes to disappear from his local space-time is contained in a volume element, which is flattened to a surface element on the horizon. This gives an upper bound for the total entropy for a system in local equilibrium, it has to be proportional to the area of surfaces rather than volumes, for any fluctuation will generate a local horizon over which entropy is lost. Following this line of reasoning the ordinary second law of statistical mechanics actually implies a scaling of entropy proportional to horizon area for any system with Lorentz symmetries.

The classical statistical mechanics is consistent with this relativistic theory, because for small fluctuations the horizon is far away, and the created horizon lies outside the local system. Therefore entropy is in classical processes, i.e. in low energy processes, proportional to volumes. However, if the fluctuation becomes arbitrarily large, then the horizon comes arbitrarily close to the fluctuation, and then the entropy of the heat bath observed by the fluctuation has to be proportional to an area. Apparently the number of states at high energy reduces at such a high rate that the entropy in a region does no longer scale with the volume, but rather the entropy increases proportional to the area.

Taking this reasoning a step further, a lightlike fluctuation happens on the acceleration horizon, lightlike fluctuations describe the electromagnetic interactions, for these are propagated by photons. All force mediators in the standard model are lightlike (on shell). The space-time a lightlike object experiences is 2-dimensional, i.e. the dimension of the spatial surface orthogonal to the propagation null vector of the photon.

The general theory of relativity can thus be interpreted as a statistical theory, and the principle of equivalence is actually the statement that the neighbourhood of each point in space-time is in local thermodynamic equilibrium. That the general theory of relativity is a statistical theory could have been expected, as Einstein is well known for his work on black body radiation.

Chapter 5

Holographic derivation of the Einstein equation

In this chapter the holographic principle is used to derive the Einstein equation locally, in particular the generalised covariant entropy bound is the holographic bound used in this chapter which is

$$S(L) \leq \frac{A(B) - \tilde{A}(\tilde{B})}{4} \tag{5.1}$$

See section 3.3 for a description of the quantities appearing in the bound.

5.1 infinitesimal light sheet

The generalised covariant entropy bound is supposed to hold for the entropy passing through a local light sheet L connecting the local infinitesimal holographic screens $\mathrm{d}A$ and $\mathrm{d}\tilde{A}$ which are separated by an infinitesimal affine parameter distance λ as measured along geodesics which generate the light sheet. It will be seen that this is a good measure for the length of a light sheet, as the affine parameter on the null geodesic congruence generating the light sheet is associated with the proper time of the natural family of observers which generate the light sheet. The generalised covariant entropy bound for this scenario is

$$\mathrm{d}S \quad \leq \quad \frac{\mathrm{d}A_\lambda - \mathrm{d}A_0}{4} \tag{5.2}$$

The reasoning in this section will be very much like that of section 4.4. First the entropy S passing through the light sheet will be considered. The entropy passing through the light sheet is by definition related to the heat $\mathrm{d}Q$ passing through the light sheet,

$$T\mathrm{d}S \quad = \quad \mathrm{d}Q \tag{5.3}$$

By definition no work is done on the light sheet, i.e. it is a geometrical construction, not a physical object, ergo no work can be done on it. The entropy is related to the energy $\mathrm{d}U$ passing through the light sheet

$$T\mathrm{d}S \quad = \quad \mathrm{d}U \tag{5.4}$$

A local light sheet would be constructed in a local Minkowski space-time. The local temperature and energy of Minkowski space is zero, and this equality is empty. However, the local light sheets constructed in section 3.5.2 come with a natural family of accelerated observers, and the temperature on the light sheet should be the local temperature observed by the accelerated observer, which is the Unruh temperature.

The local holographic screen generator $\chi^\mu = \xi^\mu + \omega\varphi^\mu$ is proportional to the null geodesic k^μ, just as in the thermodynamic scenario, $\chi^\mu = -\kappa\lambda k^\mu$, with λ the affine parameter and κ the acceleration of the observers. The Unruh temperature is

$$T = \frac{\kappa}{2\pi} \tag{5.5}$$

The energy current seen by the observers is

$$\begin{aligned} p^\mu &= T^\mu{}_\nu \chi^\nu && (5.6)\\ &= -\kappa\lambda T^\mu{}_\nu k^\nu && (5.7) \end{aligned}$$

The volume element infinitesimally close to the light sheet containing the energy that will cross it in proper time interval $\mathrm{d}\lambda$ is

$$\mathrm{d}V^\mu = k^\mu \mathrm{d}A\mathrm{d}\lambda \tag{5.8}$$

The heat or energy U passing through the local light sheet is the energy which these observers see leaving the observable part of their space-time at proper time λ is

$$\mathrm{d}U = -\kappa\lambda T_{\mu\nu} k^\mu k^\nu \mathrm{d}A\mathrm{d}\lambda \tag{5.9}$$

The entropy transferred through the light sheet is

$$\mathrm{d}S = \frac{1}{T}\mathrm{d}U = -2\pi\lambda T_{\mu\nu} k^\mu k^\nu \mathrm{d}A\mathrm{d}\lambda \tag{5.10}$$

The accelerated observer sees a local homogeneous heat bath and therefore the term $T_{\mu\nu}k^\mu k^\nu$ can be considered constant for small λ, the first correction being of order λ^2.

Now the area difference of the holographic screens is considered.

$$\begin{aligned} \mathrm{d}A_\lambda - \mathrm{d}A_0 &= \mathrm{d}a && (5.11)\\ &= [R_{\mu\nu}k^\mu k^\nu]_{\lambda=0}\lambda\mathrm{d}\lambda\mathrm{d}A + \mathrm{O}(\lambda^2) && (5.12) \end{aligned}$$

The area variation $\mathrm{d}a = \mathrm{d}A_\lambda - \mathrm{d}A_0$ of an area element is given by equation (4.33). The area element is infinitesimal and therefore the term $R_{\mu\nu}k^\mu k^\nu$ at proper time $\lambda = 0$ can be considered constant on an infinitesimal area element.

The entropy bound is an equality for the holographic screen, the entropy scales proportional to the area because of this bound, and the coefficient must also be the same, otherwise it is possible to either state a tighter bound, or the bound is not true. For example, if the entropy scales with volume, then it is always possible to construct a system with low entropy density large enough such that its total entropy breaks the bound. Also no work is done, all energy passing through the light sheet contributes to the number of accessible states.

The resulting equality is,

$$-2\pi\lambda T_{\mu\nu}k^{\mu}k^{\nu}\mathrm{d}A\mathrm{d}\lambda = \frac{1}{4}R_{\mu\nu}k^{\mu}k^{\nu}\lambda\mathrm{d}\lambda\mathrm{d}A \tag{5.13}$$

For $\lambda \neq 0$ this implies

$$0 = (8\pi T_{\mu\nu} + R_{\mu\nu})k^{\mu}k^{\nu} \tag{5.14}$$

Just as in the thermodynamic derivation there is freedom to add $fg_{\mu\nu}k^{\mu}k^{\nu}$,

$$0 = (8\pi T_{\mu\nu} + R_{\mu\nu} + fg_{\mu\nu})k^{\mu}k^{\nu} \tag{5.15}$$

The generalised covariant entropy bound does not depend on a special family of geodesics, the equation must hold for all null vector fields on the light sheet. This means that the term between brackets must be zero.

$$0 = 8\pi T_{\mu\nu} + R_{\mu\nu} + fg_{\mu\nu} \tag{5.16}$$

The scalar function f can be established by taking the divergence of this equation. The energy momentum tensor is divergence free, $T^{\mu\nu}{}_{;\mu} = 0$, so the divergence of (5.16) is

$$0 = (R^{\mu\nu} + fg^{\mu\nu})_{;\mu} \tag{5.17}$$

The Bianchi identity $\left(R^{\mu\nu} - \frac{R}{2}g^{\mu\nu}\right)_{;\mu} = 0$ now fixes f up to a constant C,

$$f = -\frac{R}{2} + C \tag{5.18}$$

The resulting equation relating the metric to the energy momentum density of matter on the metric is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu} \tag{5.19}$$

and this is exactly the Einstein equation with $\Lambda = -C$ the cosmological constant.

5.1.1 discussion

The derivation, carried out for local holographic screens, is similar to the derivation in section 4.4.1, the generalised covariant entropy bound implies the generalised second law.

What is important is that this holographic derivation is more general than the thermodynamic derivation. The screens constructed here are purely geometric. For example, black hole horizons have a physical interpretation and it may be argued that work can be done on black holes, and therefore also on the local horizons. It remains possible to give a classic physical interpretation of the holographic screen in certain cases, i.e. when the congruence of curves represents physical observers, then the screen might be interpreted as a patch of the horizon of a rotating black hole, even a non stationary one. The local angular velocity is the angular velocity of the curves used to generate the horizon. The mass of the hidden black hole must be determined from the local uniform acceleration along the curves, because the global space-time manifold will not be asymptotically flat.

The great advantage of this derivation of the Einstein equations is that the holographic screens remains a valid construction even if the natural family of curves which generate the light represents unphysical observers which move faster than the speed of light. This fact can be used in the construction of finite dynamical light sheets on a finite patch of the space-time manifold. An arbitrary dynamic finite holographic light sheet may be constructed by smoothly patching together local rotating light sheets with the right angular velocities.

The construction of holographic screens remains valid to arbitrary small length scales, as long as a local inertial system can be found. That is nothing more than requiring space-time to be a smooth manifold and this is not an unusual assumption. Near the Planck length scales the metric is expected to be in a superposition of metrics. In that case there is no transformation to a local inertial frame and the construction of local holographic screens breaks down. A possible way out is to consider the expectation value of the metric, and construct screens on the averaged metric.

5.2 finite holographic screen

The Einstein equation can also be derived from the generalised entropy bound for finite holographic screens A and $\tilde{A}$ which are an infinitesimal distance $\mathrm{d}\lambda$ apart. The bound is then

$$S \leq \frac{A_\lambda - A_0}{4} \tag{5.20}$$

and the bound is again an equality for the holographic screens.

For each patch of the light sheet exists a natural family of accelerated observers, and all these observers see infinitesimal comoving volume elements which will cross the horizon. Now all the energy in these volume elements needs to be added up to calculate the entropy moving through the light sheet. The metric $h_{\mu\nu}$, see section 3.4.3, on the hypersurface orthogonal to the timelike geodesic congruence k^μ is

$$h_{\mu\nu} \quad = \quad g_{\mu\nu} + k_\mu k_\nu \tag{5.21}$$

The local volume element of a spatial volume orthogonal to a timelike geodesic congruence is the determinant of the metric on the hypersurface,

$$\sqrt{h}\mathrm{d}^3x \quad \equiv \quad \sqrt{\det(h_{\mu\nu})}\mathrm{d}^3x \tag{5.22}$$

x are the coordinates on the hypersurface. The vector field k^μ is the normal to the hypersurface.

The total entropy S in a closed comoving finite volume moving towards the horizon is

$$S = -2\pi \int_V \lambda T_{\mu\nu} k^\mu k^\nu \sqrt{h}\mathrm{d}^3x \tag{5.23}$$

The quantity $T_{\mu\nu}k^\mu k^\nu$ can no longer be considered constant, therefore the average will be taken, the average of any scalar ρ in the volume V is defined by

$$< \rho > \quad \equiv \quad \frac{\int_V \rho\sqrt{h}\mathrm{d}^3x}{\int_V \sqrt{h}\mathrm{d}^3x} \tag{5.24}$$

The light sheet under consideration has infinitesimal affine parameter length $\mathrm{d}\lambda$. The entropy passing through the light sheet in observer time λ is the entropy contained in the volume with the horizon as boundary on one side and has thickness $\mathrm{d}\lambda$ orthogonal to the boundary. The direction orthogonal to the boundary is k^μ, and the volume element of this volume near the horizon can be expanded for small λ,

$$\sqrt{h}\mathrm{d}^3x \quad \approx \quad \sqrt{h(x,0)}\mathrm{d}^2x\mathrm{d}\lambda + \mathrm{O}(\lambda^2) \tag{5.25}$$

and $\sqrt{h(x,0)}$ is the metric on the boundary of the volume. The metric $\hat{h}_{\mu\nu}$ on the boundary has been defined in section 35. The λ is coordinate is in the boundary orthogonal direction k^μ.

The average of a scalar in a volume of thickness λ near the holographic screen is

$$< \rho >_0 \quad \equiv \quad \frac{\int_\lambda \int_B \rho\sqrt{\hat{h}}\mathrm{d}^2x\mathrm{d}\lambda}{\int_B \sqrt{\hat{h}}\mathrm{d}^2x \int_\lambda \mathrm{d}\lambda} \tag{5.26}$$

where B is the boundary of V at the holographic screen.

The scalar $T_{\mu\nu}k^\mu k^\nu$ is parallel transported along k^μ,

$$(T_{\mu\nu}k^\mu k^\nu)_{;\rho}k^\rho \quad = \quad 0 + \mathrm{O}(\lambda) \tag{5.27}$$

i.e. $T_{\mu\nu}k^\mu k^\nu$ can be considered constant with respect to λ in a neighbourhood of $\lambda = 0$. The average of $\lambda T_{\mu\nu}k^\mu k^\nu$ near the screen is

$$< \lambda T_{\mu\nu}k^\mu k^\nu >_0 \quad = \quad \frac{\int_\lambda \int_B \lambda T_{\mu\nu}k^\mu k^\nu \sqrt{\hat{h}}\mathrm{d}^2x\mathrm{d}\lambda}{\int_B \sqrt{\hat{h}}\mathrm{d}^2x \int_\lambda \mathrm{d}\lambda} \tag{5.28}$$

$$= \quad \frac{\int_B T_{\mu\nu}k^\mu k^\nu \sqrt{\hat{h}}\mathrm{d}^2x \int_\lambda \lambda\mathrm{d}\lambda}{\int_B \sqrt{\hat{h}}\mathrm{d}^2x \int_\lambda \mathrm{d}\lambda} \tag{5.29}$$

$$= \quad < T_{\mu\nu}k^\mu k^\nu >_0 \frac{\int_\lambda \lambda\mathrm{d}\lambda}{\int_\lambda \mathrm{d}\lambda} \tag{5.30}$$

The entropy passing through the light sheet is found to be

$$S \quad = \quad -2\pi < T_{\mu\nu}k^\mu k^\nu >_0 \int_\lambda \lambda\mathrm{d}\lambda \int_B \sqrt{\hat{h}}\mathrm{d}^2x \tag{5.31}$$

The next step is the area difference, the area A of the spatial surface B orthogonal to the light sheet at proper time λ is

$$A(\lambda) \quad = \quad \int_{B(\lambda)} \sqrt{\hat{h}|_\lambda(x)}\mathrm{d}^2x \tag{5.32}$$

$$= \quad \int_{B(0)} e^{\int \theta\mathrm{d}\lambda}\sqrt{\hat{h}|_{\lambda=0}(x)}\mathrm{d}^2x \tag{5.33}$$

$$\approx \quad \int_B \left(1 + \int_\lambda R_{\mu\nu}k^\mu k^\nu(x)\lambda\mathrm{d}\lambda) + \int_\lambda \mathrm{O}(\lambda^2)\mathrm{d}\lambda\right) \sqrt{\hat{h}}\mathrm{d}^2x$$

In the second line the expression for the rate of change of each area element was used, see section 35, and in the third line[1] it was used that the expansion, shear and twist of the congruence vanish at $\lambda = 0$, and therefore the expansion near $\lambda = 0$ can be approximated by

$$\theta(x,\lambda) \approx \lambda R_{\mu\nu}k^\mu k^\nu(x,0) + \mathrm{O}(\lambda^2) \tag{5.34}$$

The area difference is for screens an infinitesimal distance λ apart,

$$A(\lambda) - A(0) = \int_\lambda \int_B \lambda R_{\mu\nu}k^\mu k^\nu \sqrt{\hat{h}}\mathrm{d}^2x\mathrm{d}\lambda \tag{5.35}$$

The average of $\lambda R_{\mu\nu}k^\mu k^\nu$ for a volume of infinitesimal thickness near the holographic screen is[2]

$$< \lambda R_{\mu\nu}k^\mu k^\nu >_0 = < R_{\mu\nu}k^\mu k^\nu >_0 \frac{\int_\lambda \lambda\mathrm{d}\lambda}{\int_\lambda \mathrm{d}\lambda} \tag{5.36}$$

With that the right hand side of the entropy bound (5.20) has become

$$\frac{A(\lambda) - A(0)}{4} = \frac{1}{4} < R_{\mu\nu}k^\mu k^\nu >_0 \int_\lambda \lambda\mathrm{d}\lambda \int_B \sqrt{\hat{h}}\mathrm{d}^2x \tag{5.37}$$

The equations (5.31) and (5.37) now yield the equality

$$-8\pi < T_{\mu\nu}k^\mu k^\nu >_0 = < R_{\mu\nu}k^\mu k^\nu >_0 \tag{5.38}$$

A zero term $< fg_{\mu\nu}k^\mu k^\nu >_0$ can be added, f is a scalar function, the result is

$$-8\pi < T_{\mu\nu}k^\mu k^\nu >_0 = < R_{\mu\nu}k^\mu k^\nu >_0 + < fg_{\mu\nu}k^\mu k^\nu >_0 \tag{5.39}$$

The sum of averages is the average of a sum, for all the averages here are supposed to be finite,

$$\begin{aligned} 0 &= < 8\pi T_{\mu\nu}k^\mu k^\nu + R_{\mu\nu}k^\mu k^\nu + fg_{\mu\nu}k^\mu k^\nu >_0 && (5.40) \\ &= < (8\pi T_{\mu\nu} + R_{\mu\nu} + fg_{\mu\nu})\, k^\mu k^\nu >_0 && (5.41) \\ &\equiv < H_{\mu\nu}k^\mu k^\nu >_0 && (5.42) \end{aligned}$$

The geodesic generator k^μ of the light sheet is not unique. This relation must hold for all tangents to the light sheet. The average (5.42) is zero for all these tangents,

$$\begin{aligned} 0 &= < H_{\mu\nu}k^\mu k^\nu >_0 && (5.43) \\ &= \frac{\int_B H_{\mu\nu}k^\mu k^\nu \sqrt{\hat{h}}\mathrm{d}^2x}{\int_B \sqrt{\hat{h}}\mathrm{d}^2x} && (5.44) \end{aligned}$$

Since the k^μ are arbitrary, and the surface area of B is non zero and finite, this can only be true if

$$\begin{aligned} 0 &= H_{\mu\nu} && (5.45) \\ &= 8\pi T_{\mu\nu} + R_{\mu\nu} + fg_{\mu\nu} && (5.46) \end{aligned}$$

[1]Note that the third line has only explicit dependence on λ. The surface integral is evaluated at $\lambda = 0$ and the scalar $R_{\mu\nu}k^\mu k^\nu$ too.

[2]The derivation is exactly the same as the one for $\lambda T_{\mu\nu}k^\mu k^\nu$.

The same steps as in the previous derivation [3] are taken again and therefore the most general solution for f is

$$f = -\frac{1}{2}R + \Lambda \tag{5.47}$$

and the Einstein equation is again recovered.

5.2.1 discussion

By considering the average entropy passing through a finite screen an effective description of the behaviour of the metric has been found for finite regions of space-time. Note that the regions have infinitesimal thickness. In order to acquire a volume of finite thickness a whole lot of light sheets of infinitesimal affine parameter length can be constructed. This new light sheet is then bounded by two surfaces a finite distance λ apart, and the entropy passing through the light sheet is the sum of contributions of each light sheet of infinitesimal length. The size of the light sheet is ultimately determined by the convergence condition imposed on the null geodesics generating the light sheets. If two light sheets of infinitesimal length are patched together, then they are patched in such a way that the new light sheet extends orthogonally away from the surface on which the old light sheet terminated orthogonally. The generators of the patched light sheet will be smooth vector fields. The termination surface can always be chosen such that the null geodesic congruence terminates orthogonally, because of equation (3.31), i.e. the vector field will never develop a twist.

Is this derivation of the Einstein field equations more fundamental than the usual derivation, see section 2.1 and [1]? As the boundary area of the light sheet in the generalised covariant entropy bound is described by the Raychaudhuri equation, the fact that the field equations have only second derivatives of the metric and terms proportional to the metric is implicit in the covariant entropy bound. Therefore the derivation is not more general, as the only other assumption in the usual derivation is that it reduces to the Newtonian theory for weak fields, but this requirement is dropped for the Einstein equation with cosmological constant.

If the entropy is supposed to scale with volume, as in ordinary thermodynamics then one might expect that a covariant entropy bound may hold using timelike geodesics, which begin orthogonal on a spatial volume (hypersurface). This will not yield the Einstein equation, unless the trace of the energy momentum tensor is added by hand in the expression for the heat in the volume. That is because in that scenario it is impossible to add a scalar function for free, the inner product of timelike geodesics is not zero. Also the cosmological constant must be added by hand in such a scenario. This may be another indication that the holographic principle is actually correct.

The covariant entropy bound is time reversal invariant, it can equally well describe the decrease of a horizon surface area due to entropy entering a space-time over the past horizon, see figure A.5. It may therefore be used in a unitary microscopic description of quantum gravity.

An advantage of the holographic derivation over the thermodynamic derivation is that the Einstein equations derived in this way is independent of a physical interpretation of the geodesics generating the light sheet.

[3] See the derivation under equation (5.16).

Chapter 6

Conclusion

The Einstein equation and the laws of black hole mechanics imply a scaling of entropy with area. Together with the assumption that black hole formation and evaporation proceed in a time reversible manner, this scaling of entropy with area implies a bound on the total entropy passing through null hypersurfaces, it is less than the surface area orthogonal to the null hypersurface generators. The surface area can contain less than one bit per Planck area if the bound is correct and the number of states of a system can be in is finite.

Reversing the logic, i.e. assuming a holographic entropy scaling either as thermodynamic principle or in the form of the holographic entropy bound, leads to the Einstein equation. This means that a holographic scaling of entropy is consistent with classical general relativity.

There is no a priori reason to assume that the number of degrees of freedom available to the metric at the quantum scale is larger than at large scales. If the number of degrees of freedom available to the metric is reduced, then this reduces the dimension of the divergences in quantum gravity, and it may become renormalisable. Holography is a new symmetry in the theory, but it is not local, a boundary has to be specified on which the information in a region of space-time is projected. the implementation of the symmetry in a quantum theory therefore is nontrivial, it is not possible to define a Nöther current, which exists for local symmetries. It is also debatable if it makes sense to talk about a boundary on such small scales that the metric is in a superposition.

Chapter 7

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What would have become of this thesis without all the people around me during my work on it? Although I do not know the exact answer to this question, I do not think that I would have studied holography if Jan de Boer had not presented it as a subject for a thesis. Since then we had quite a few discussions on this and related subjects. It would be hard to claim that he hasn't influenced it. Then there are my room mates at the ITF, and of course the other master students without whom the lunch and coffee breaks would not have been what they were. You have been sources of inspiration. I should not forget everyone else at the faculty with whom I had discussions on physics in general and holography in particular. Neither should I underestimate the influence of all my friends and family. I do not think I could have been working on my thesis in a good mood if I hadn't had the pleasure of being with them when I wasn't working on the thesis. I shall not name you all, because I believe you know who you are. So to all of you a great big

Thank you!

Appendix A

Tools

In the appendix some facts from general relativity are summarised. Thorough treatments can be found in [1, 2].

A.1 Tensors

Type $\binom{m}{n}$ tensors, i.e. m contravariant and n covariant indices, are defined by their transformation law

$$\tilde{T}^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}\left(\tilde{x}\right) = \prod_{i=1}^{m}\left(\frac{\partial\tilde{x}^{\mu_i}}{\partial x^{\rho_i}}\right)\prod_{j=1}^{n}\left(\frac{\partial x^{\sigma_j}}{\partial\tilde{x}^{\nu_j}}\right)T^{\rho_1\ldots\rho_m}{}_{\sigma_1\ldots\sigma_n}\left(x\right) \quad (A.1)$$

For example, a type $\binom{2}{2}$ tensor transforms according to

$$\tilde{T}^{\kappa}{}_{\lambda}{}^{\mu}{}_{\nu}\left(\tilde{x}\right) = \frac{\partial\tilde{x}^{\kappa}}{\partial x^{\rho}}\frac{\partial x^{\sigma}}{\partial\tilde{x}^{\lambda}}\frac{\partial\tilde{x}^{\mu}}{\partial x^{\tau}}\frac{\partial x^{\upsilon}}{\partial\tilde{x}^{\nu}}T^{\rho}{}_{\sigma}{}^{\tau}{}_{\upsilon}\left(x\right) \quad (A.2)$$

A.1.1 General infinitesimal coordinate transformations

Consider a general infinitesimal coordinate transformation

$$x^{\mu} \to \tilde{x}^{\mu} = x^{\mu} + \mathrm{d}x^{\mu} \quad (A.3)$$

It is often useful to express the transformed tensor, e.g.. $\tilde{T}_{\mu\nu}\left(\tilde{x}\right)$, in the old coordinates, i.e. one would like to know $\tilde{T}_{\mu\nu}\left(x\right)$.

The partial derivatives required to transform the tensor are

$$\begin{aligned}\frac{\partial\tilde{x}^{\mu}}{\partial x^{\nu}} &= \delta^{\mu}{}_{\nu} + \frac{\partial\mathrm{d}x^{\mu}}{\partial x^{\nu}} \\ \frac{\partial x^{\nu}}{\partial\tilde{x}^{\mu}} &= \delta_{\mu}{}^{\nu} - \frac{\partial\mathrm{d}x^{\nu}}{\partial x^{\mu}} + O\left(\mathrm{d}x^2\right)\end{aligned} \quad (A.4)$$

The transformed tensor is

$$\begin{aligned}\tilde{T}_{\mu\nu}\left(\tilde{x}\right) &= \frac{\partial x^{\rho}}{\partial\tilde{x}^{\mu}}\frac{\partial x^{\sigma}}{\partial\tilde{x}^{\nu}}T_{\rho\sigma}\left(x\right) \\ &= T_{\mu\nu}\left(x\right) - T_{\mu\sigma}\left(x\right)\frac{\partial\mathrm{d}x^{\sigma}}{\partial x^{\nu}} - T_{\rho\nu}\left(x\right)\frac{\partial\mathrm{d}x^{\rho}}{\partial x^{\mu}} + O\left(\mathrm{d}x^2\right)\end{aligned} \quad (A.5)$$

The Taylor expansion of the transformed tensor is

$$\begin{aligned}\tilde{T}_{\mu\nu}(\tilde{x}) &= \tilde{T}_{\mu\nu}(x+\mathrm{d}x) \\ &= \tilde{T}_{\mu\nu}(x)+\frac{\partial T_{\mu\nu}(x)}{\partial x^{\lambda}}\mathrm{d}x^{\lambda}+O\left(\mathrm{d}x^{2}\right) \qquad \text{(A.6)}\end{aligned}$$

These facts yield the transformed tensor in old coordinates,

$$\begin{aligned}\tilde{T}_{\mu\nu}(x) &= \tilde{T}_{\mu\nu}(\tilde{x})-\frac{\partial T_{\mu\nu}(x)}{\partial x^{\lambda}}\mathrm{d}x^{\lambda}+O\left(\mathrm{d}x^{2}\right) \\ &= T_{\mu\nu}(x)-T_{\mu\sigma}(x)\frac{\partial \mathrm{d}x^{\sigma}}{\partial x^{\nu}}-T_{\rho\nu}(x)\frac{\partial \mathrm{d}x^{\rho}}{\partial x^{\mu}}-\frac{\partial T_{\mu\nu}(x)}{\partial x^{\lambda}}\mathrm{d}x^{\lambda}+O\left(\mathrm{d}x^{2}\right) \\ &\equiv T_{\mu\nu}(x)-\Delta_{\mathrm{d}x}T_{\mu\nu}(x)+O\left(\mathrm{d}x^{2}\right) \qquad \text{(A.7)}\end{aligned}$$

where $\Delta_{\mathrm{d}x}T_{\mu\nu}(x)$ is the *Lie derivative* with respect to the vector field $\mathrm{d}x^{\mu}$.

The Lie derivative can be expressed in covariant derivatives instead of coordinate derivatives because the affine connections cancel against each other,

$$\begin{aligned}T_{\mu\nu,\lambda}\mathrm{d}x^{\lambda} &= T_{\mu\nu;\lambda}\mathrm{d}x^{\lambda}+\Gamma^{\kappa}{}_{\lambda\mu}T_{\kappa\nu}\mathrm{d}x^{\lambda}+\Gamma^{\kappa}{}_{\lambda\nu}T_{\mu\kappa}\mathrm{d}x^{\lambda} \\ T_{\mu\kappa}\mathrm{d}x^{\kappa}{}_{,\nu} &= T_{\mu\kappa}\mathrm{d}x^{\kappa}{}_{;\nu}-\Gamma^{\kappa}{}_{\lambda\nu}T_{\mu\kappa}\mathrm{d}x^{\lambda} \\ T_{\kappa\nu}\mathrm{d}x^{\kappa}{}_{,\mu} &= T_{\kappa\nu}\mathrm{d}x^{\kappa}{}_{;\mu}-\Gamma^{\kappa}{}_{\lambda\mu}T_{\kappa\nu}\mathrm{d}x^{\lambda} \\ \Delta_{\mathrm{d}x}T_{\mu\nu} &\equiv T_{\mu\nu,\lambda}\mathrm{d}x^{\lambda}+T_{\mu\kappa}\mathrm{d}x^{\kappa}{}_{,\nu}+T_{\kappa\nu}\mathrm{d}x^{\kappa}{}_{,\mu} \qquad \text{(A.8)} \\ &= T_{\mu\nu;\lambda}\mathrm{d}x^{\lambda}+T_{\mu\kappa}\mathrm{d}x^{\kappa}{}_{;\nu}+T_{\kappa\nu}\mathrm{d}x^{\kappa}{}_{;\mu} \qquad \text{(A.9)}\end{aligned}$$

The Lie derivative for tensors of arbitrary type can be derived in the same manner, however note that transforming a contravariant index will yield a sign difference.

A.2 Equation of motion in general relativity

The principle of equivalence states that at every point p in an arbitrary spacetime it is possible to find a local inertial system in a sufficiently small region around p, such that the laws of nature take the same form as in the absence of gravitation. Sufficiently small means that the gravitational field in that region is constant for all practical purposes.

A free particle in a local inertial frame ξ^{α} with local metric $\eta_{\alpha\beta}$ moves in a straight line,

$$\frac{\mathrm{d}^{2}\xi^{\alpha}}{\mathrm{d}\tau^{2}} = 0 \qquad \text{(A.10)}$$

Under a general coordinate transformation to coordinates x^{μ} this equation becomes

$$\frac{\mathrm{d}^{2}x^{\mu}}{\mathrm{d}\tau^{2}}+\Gamma^{\mu}{}_{\kappa\lambda}\frac{\mathrm{d}x^{\kappa}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\lambda}}{\mathrm{d}\tau} = 0 \qquad \text{(A.11)}$$

The affine connection $\Gamma^{\mu}{}_{\kappa\lambda}$ is defined by

$$\Gamma^{\mu}{}_{\kappa\lambda} \equiv \frac{\partial x^{\mu}}{\partial \xi^{\alpha}}\frac{\partial^{2}\xi^{\alpha}}{\partial x^{\kappa}\partial x^{\lambda}} \qquad \text{(A.12)}$$

The metric tensor is defined by

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} \tag{A.13}$$

The metric determines distances in the space-time, but it is not positive definite, distances can be negative, null or positive. The negative distances are timelike distances, the null distances are lightlike distances (the distance traveled by light rays), and the positive distances are spacelike distances.

The proper time τ is defined by

$$\begin{aligned} d\tau^2 &\equiv \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta && \text{(A.14)} \\ &= g_{\mu\nu} dx^\mu dx^\nu && \text{(A.15)} \end{aligned}$$

and it is invariant under general coordinate transformations.

The affine connection can be expressed in the metric tensor

$$\Gamma^\mu{}_{\kappa\lambda} \equiv \frac{1}{2} g^{\mu\nu}\left\{g_{\kappa\nu,\lambda} + g_{\lambda\nu,\kappa} - g_{\kappa\lambda,\nu}\right\} \tag{A.16}$$

The comma in the expression is defined by $g_{\mu\nu,\lambda} = \partial_\lambda g_{\mu\nu} = \frac{\partial}{\partial x^\lambda} g_{\mu\nu}$. The affine connection is the field that determines the gravitational field strength, which can be seen from the equation of motion for a free particle in a general coordinate system. The deviation from movement on a straight line is determined by the affine connection.

The path of a free particle extremises the proper time, a free particle can be interpreted as a particle moving on a geodesic on some manifold. Therefore gravity can be interpreted geometrically, space-time is a manifold on which free particles move along geodesics. The metric on the manifold is $g_{\mu\nu}$, and the metric defines the affine connection, i.e. the gravitational field strength.

The curve x^μ has a tangent vector field v^μ,

$$v^\mu = \frac{dx^\mu}{d\tau} \tag{A.17}$$

The derivative does not transform as a tensor, a general coordinate transformation causes an affine connection to appear. The covariant derivative D_μ of a tensor does transform as a tensor, and is defined for vector fields by

$$\begin{aligned} D_\nu v^\mu &= v^\mu{}_{;\nu} && \text{(A.18)} \\ &= \frac{\partial v^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\nu\kappa} v^\kappa && \text{(A.19)} \\ D_\nu v_\mu &= v_{\mu;\nu} && \text{(A.20)} \\ &= \frac{\partial v_\mu}{\partial x^\nu} - \Gamma^\kappa{}_{\mu\nu} v_\kappa && \text{(A.21)} \end{aligned}$$

The covariant derivative itself is not a tensor. The covariant derivative for vectors v^μ defined only along a curve x^μ is

$$\begin{aligned} \frac{Dv^\mu}{D\tau} &= \frac{dv^\mu}{d\tau} + \Gamma^\mu{}_{\nu\kappa}\frac{dx^\nu}{d\tau} v^\kappa && \text{(A.22)} \\ \frac{Dv_\mu}{D\tau} &= \frac{dv_\mu}{d\tau} - \Gamma^\kappa{}_{\mu\nu}\frac{dx^\nu}{d\tau} v_\kappa && \text{(A.23)} \end{aligned}$$

When v^μ is the tangent to a curve these are simply the equations of motion for a particle on that curve, a free particle has $\frac{Dv^\mu}{D\tau} = 0$.

A.3 Curvature

The Riemann-Christoffel curvature tensor $R^{\lambda}{}_{\mu\nu\kappa}$ is

$$R^{\lambda}{}_{\mu\nu\kappa} \equiv \frac{\partial \Gamma^{\lambda}{}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}{}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}{}_{\mu\nu}\Gamma^{\lambda}{}_{\kappa\eta} - \Gamma^{\eta}{}_{\mu\kappa}\Gamma^{\lambda}{}_{\nu\eta} \tag{A.24}$$

This is the only tensor that can be constructed from the metric tensor and its first and second derivatives. Two other tensors can be obtained by contraction of indices, namely the Ricci tensor and the curvature scalar.

The Ricci tensor $R_{\mu\kappa}$ is

$$R_{\mu\kappa} \equiv R^{\lambda}{}_{\mu\lambda\kappa} \tag{A.25}$$

The curvature scalar (also known as the Ricci scalar) R is the trace of the Ricci tensor

$$R \equiv g^{\mu\kappa} R_{\mu\kappa} \tag{A.26}$$

The trace free symmetric part of the Riemann-Christoffel curvature tensor is called the Weyl tensor $C_{\kappa\lambda\mu\nu}$. This tensor measures the deformation of the space-time, cf. the shear of geodesic congruences (3.12).

For space-times dimension $N \geq 3$ the Weyl tensor is defined by the equation

$$\begin{aligned} R_{\lambda\mu\nu\kappa} &= \frac{1}{N-2}\left(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}\right) \\ &- \frac{R}{(N-1)(N-2)}\left(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}\right) + C_{\lambda\mu\nu\kappa} \end{aligned} \tag{A.27}$$

The Weyl tensor is trace free, that is

$$C^{\lambda}{}_{\mu\lambda\kappa} = 0 \tag{A.28}$$

A.4 Isometries and infinitesimal isometries

In geometry the mappings which leave distances invariant, isometries, are of considerable importance, so what are the coordinate transformations that leave distances in space-time invariant? In this section the isometries and infinitesimal isometries are studied.

Consider a space-time with metric $g_{\mu\nu}$. If space-time is invariant under a coordinate transformation then the metric is said to be form invariant under this mapping. More specific, a metric $g_{\mu\nu}(x)$ is form invariant under a given coordinate transformation $x \to \tilde{x}$ if

$$\tilde{g}_{\mu\nu}(y) = g_{\mu\nu}(y)\ \forall y \tag{A.29}$$

that is $\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x)$ or equivalently $\tilde{g}_{\mu\nu}(\tilde{x}) = g_{\mu\nu}(\tilde{x})$.

In an arbitrary point $p(x) = p(\tilde{x})$ the transformed metric is

$$g_{\mu\nu}(x) = \frac{\partial \tilde{x}^{\rho}}{\partial x^{\mu}} \frac{\partial \tilde{x}^{\sigma}}{\partial x^{\nu}} \tilde{g}_{\rho\sigma}(\tilde{x})$$

Therefore a form invariant metric $\tilde{g}_{\mu\nu}(\tilde{x})$ necessarily satisfies

$$g_{\mu\nu}(x) = \frac{\partial \tilde{x}^{\rho}}{\partial x^{\mu}} \frac{\partial \tilde{x}^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(\tilde{x}) \tag{A.30}$$

Clearly a form invariant metric under some coordinate transformation yields the same distances in both coordinate systems.

A.4.1 Killing vector fields

Consider an infinitesimal isometry, i.e. an isometry generated by an infinitesimal coordinate transformation,

$$\tilde{x}^{\mu} = x^{\mu} + \varepsilon \xi^{\mu} \tag{A.31}$$

where $\varepsilon \in \mathbb{R}$ is an infinitesimal independent parameter. Consider the equation (A.7) for the metric $g_{\mu\nu}$ instead of $T_{\mu\nu}$, then it is clear that the metric is form invariant if and only if the Lie derivative, equation (A.9), vanishes with respect to ξ^{μ}. The Lie derivative is zero if and only if

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \tag{A.32}$$

If a vector field satisfies this condition it is called a Killing vector field.

In general a Killing vector field of the metric $g_{\mu\nu}(x)$ is a vector field $\xi_{\sigma}(x)$ which satisfies the Killing condition

$$0 = \xi_{\sigma;\rho} + \xi_{\rho;\sigma} \tag{A.33}$$

In this thesis smooth vector fields are considered which satisfy the Killing condition only in a neighbourhood on a spatial hypersurface, these vector fields are called local or instantaneous Killing vector fields, because they satisfy the Killing condition at an instant of time.

To acquire all isometries of the space-time it is necessary to discover the Killing vector fields allowed by the metric. A linear combination of Killing vector fields with constant coefficients is again a Killing vector field, so the space spanned by the Killing vector fields determines the isometries of the metric.

Consider the commutator of covariant derivatives of a Killing vector field. The cyclic sum rule for the Riemann-Christoffel curvature tensor leads to the following identity for Killing vector fields,

$$\xi_{\mu;\rho;\sigma} = -R^{\lambda}{}_{\sigma\rho\mu}\xi_{\lambda} \tag{A.34}$$

This identity makes it possible to express all higher order derivatives in terms of $\xi_{\mu}(p)$ and $\xi_{\mu;\nu}(p)$, given $\xi_{\mu}(p)$ and $\xi_{\mu;\nu}(p)$ in some point p. So every Killing vector field can now be expressed as

$$\xi_{\mu}(x) = A_{\mu}{}^{\nu}(x,p)\xi_{\nu}(p) + B_{\mu}{}^{\lambda\nu}(x,p)\xi_{\lambda;\nu}(p) \tag{A.35}$$

A.5 Rindler space-time

Here the transformations are given that lead to the Rindler space-time. Rindler space-time is the way Minkowski space-time is seen by a uniform accelerating observer, the time coordinate is the proper time of this observer. These transformations also apply in the infinitesimal case.

The Minkowski metric in Cartesian coordinates is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{A.36}$$

Consider a coordinate transformation to light cone coordinates,

$$\begin{aligned} t &= \frac{1}{2}(U - V) \\ z &= \frac{1}{2}(U + V) \\ \mathrm{d}t &= \frac{1}{2}(\mathrm{d}U - \mathrm{d}V) \\ \mathrm{d}z &= \frac{1}{2}(\mathrm{d}U + \mathrm{d}V) \end{aligned}$$

The new metric is

$$\mathrm{d}s^2 = \mathrm{d}U\mathrm{d}V + \mathrm{d}x^2 + \mathrm{d}y^2 \tag{A.37}$$

Transform again, restricting the U and V coordinates to $\mathbb{R}_+$

$$\begin{aligned} U &= e^u \\ V &= e^v \\ \mathrm{d}U &= e^u\mathrm{d}u \\ \mathrm{d}V &= e^v\mathrm{d}v \end{aligned}$$

So only the positive wedge of the original Minkowski space-time is now under consideration. The metric is

$$\mathrm{d}s^2 = e^{u+v}\mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2 \tag{A.38}$$

Finally transform to

$$\begin{aligned} u &= -T + \ln Z \\ v &= T + \ln Z \\ \mathrm{d}u &= \left(-\mathrm{d}T + \frac{1}{Z}\mathrm{d}Z\right) \\ \mathrm{d}v &= \left(\mathrm{d}T + \frac{1}{Z}\mathrm{d}Z\right) \end{aligned}$$

The resulting metric is the Rindler metric, which is

$$\mathrm{d}s^2 = -Z^2\mathrm{d}T^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}Z^2 \tag{A.39}$$

In the original Minkowski coordinates this metric covers only the wedge with positive z coordinate, see figure A.5, with Rindler or acceleration horizons given by $z = t$ and $z = -t$. Events happening beyond the line $z = t$ will never be known to the accelerating observer, nor can the observer influence the space-time beyond the line $z = -t$.

The Rindler horizons are generated by the vector field $\xi = \partial_T$, this can easily be verified in the U, V coordinates, then $\xi = V\partial_V - U\partial_U$. The horizons are given by $U = 0$ and $V = 0$ and the vector field ξ satisfies $\xi^2|_{U=0} = 0 = \xi^2|_{V=0}$.

The acceleration along each geodesic of the Rindler space-time can be made explicit by transforming $T \to \tilde{T} = gT$, but this acceleration is not uniquely determined. The resulting general form is

$$\mathrm{d}s^2 = -g^2Z^2\mathrm{d}T^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}Z^2 \tag{A.40}$$

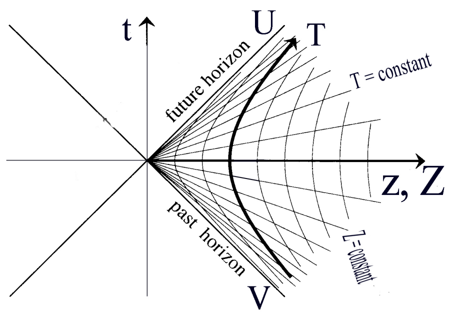

Figure A.1: The Rindler space-time is a wedge of the Minkowski space-time.

These metrics have the horizons intersecting at the origin, but it is trivial to shift this intersection to some other point on the Z axis, the most general Rindler metric is

$$\mathrm{d}s^2 = -(gZ+C)^2\mathrm{d}T^2+\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}Z^2 \qquad \text{(A.41)}$$

with C some constant, this form can be obtained by shifting the $Z \to \tilde{Z} = Z - \frac{C}{g}$. coordinate.

A.5.1 Rotating observer in Rindler space-time

Consider the Rindler metric in spherical polar coordinates,

$$\mathrm{d}s^2 = -(cR-d)^2\mathrm{d}T^2+R^2\mathrm{d}\theta^2+R^2\sin^2\theta\mathrm{d}\varphi^2+\mathrm{d}R^2 \qquad \text{(A.42)}$$

and perform an infinitesimal rotation,

$$\varphi \quad \to \quad \tilde{\varphi} = \varphi + \varepsilon\,(T) \qquad \text{(A.43)}$$

the resulting metric in old coordinates is

$$\mathrm{d}s^2 = -R^2\mathrm{d}T^2+R^2\sin^2\theta\,[\mathrm{d}\varphi-\omega\mathrm{d}T]^2+\mathrm{d}R^2+R^2\mathrm{d}\theta^2$$

where

$$\omega = \frac{\mathrm{d}\varepsilon}{\mathrm{d}T} \qquad \text{(A.44)}$$

A.6 Manifolds and hypersurfaces

A smooth n-dimensional manifold M has an algebraic description. A manifold is given by the function

$$f\left(x^0,x^1,\ldots,x^{n-1}\right)=0 \qquad \text{(A.45)}$$

of which the gradient is required to be non zero,

$$\vec{\nabla}f\left(x^0,x^1,\ldots,x^{n-1}\right)\neq 0 \qquad \text{(A.46)}$$

A family of hypersurfaces[1] S in this manifold M is given by a smooth function

$$S\left(x^0, x^1, \ldots, x^{n-1}\right) = \lambda \tag{A.47}$$

which yields a hypersurface for each value of λ.

The gradient of this function yields a vector field orthogonal to each hypersurface $S = \lambda$. That this does indeed yield a submanifold becomes clear if the function g is considered which gives the algebraic description of the manifold,

$$g = S - \lambda = 0 \tag{A.48}$$

The gradient of g has to be non zero in order to acquire a smooth manifold. However, the inner product of the gradient with itself can be zero for non Euclidean manifolds[2]. If the inner product of the gradient of g with itself is equal to zero then the hypersurface is degenerated and is called a null hypersurface. The general expression for vector fields orthogonal to a family hypersurfaces is

$$n^\mu = h(x) g^{\mu\nu}(x) \partial_\nu S(x) = h g^{\mu\nu} S_{;\nu} \tag{A.49}$$

where the scalar function is non zero, $h(x) \neq 0$.

Vector fields n^μ orthogonal to null hypersurfaces satisfy the geodesic equation in non affine parameterisation. Consider parallel transport of n^μ along itself,

$$\begin{aligned}
{n^\mu}_{;\nu} n^\nu &= \left(h g^{\mu\lambda} S_{;\lambda}\right)_{;\nu} n^\nu \\
&= h_{;\nu} g^{\mu\lambda} S_{;\lambda} n^\nu + h g^{\mu\lambda} S_{;\lambda;\nu} n^\nu \\
&= h_{;\nu} g^{\mu\lambda} S_{;\lambda} n^\nu + h g^{\mu\lambda} S_{;\nu;\lambda} n^\nu \\
&= h^{-1} h_{;\nu} n^\nu n^\mu + h g^{\mu\lambda} \left(h^{-1} n_\nu\right)_{;\lambda} n^\nu \\
&= h^{-1} h_{;\nu} n^\nu n^\mu + g^{\mu\lambda} h {h^{-1}}_{;\lambda} n_\nu n^\nu + g^{\mu\lambda} n_{\nu;\lambda} n^\nu \\
&= h^{-1} h_{;\nu} n^\nu n^\mu + g^{\mu\lambda} h {h^{-1}}_{;\lambda} n_\nu n^\nu + \frac{1}{2} g^{\mu\lambda} \left(n_\nu n^\nu\right)_{;\lambda}
\end{aligned} \tag{A.50}$$

The term proportional to $n_\mu n^\mu$ vanishes by definition, and the last term satisfies for arbitrary tangent t^μ to the null hypersurface the equality $\left(n_\mu n^\mu\right)_{;\nu} t^\nu = 0$, again due to the fact that $n_\mu n^\mu = 0$ everywhere on the null hypersurface[3]. Since t^μ is an arbitrary tangent vector field to the null hypersurface this means that $\left(n_\mu n^\mu\right)_{;\nu}$ is a null hypersurface orthogonal vector field too, and is therefore proportional to n^μ on the null hypersurface,

$$\left(n_\mu n^\mu\right)_{;\nu} = 2 n^\mu n_{\mu;\nu} = -2\eta n_\nu \tag{A.51}$$

where η is some scalar function.

The final expression for parallel transport along itself is on the null hypersurface

$$ {n^\mu}_{;\nu} n^\nu = \left(h^{-1} h_{;\nu} n^\nu - \eta\right) n^\mu \tag{A.52}$$

[1] A hypersurface is a $n-1$ dimensional submanifold $S \subset M$.

[2] E.g. consider the Minkowski metric.

[3] $\left(n_\mu n^\mu\right)_{;\nu} \neq 0$, unless each member in the family hypersurfaces is a null hypersurface.

which is proportional to n^μ, so it is the tangent vector along a geodesic curve x^μ in non affine parameterisation.

The vector field can be brought into affine parameterisation. If the vector field n^μ is in affine parameterisation the scalar function h is such that $n^\mu{}_{;\nu} n^\nu = 0$, and then

$$\eta = h^{-1} h_{;\nu} n^\nu \tag{A.53}$$

A.7 Geodesics in non affine parameterisation

Consider a geodesic in a non affine parameterisation, $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}p}$ and the same geodesic in affine parameterisation, $v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$, the equations of motion are

$$\frac{\mathrm{D}u^\mu}{\mathrm{D}p} = f u^\mu \tag{A.54}$$

$$\frac{\mathrm{D}v^\mu}{\mathrm{D}\tau} = 0 \tag{A.55}$$

with f a scalar function. In a local inertial frame in some point P these equations reduce to the linear differential equations

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}p} = f u^\mu \tag{A.56}$$

$$\frac{\mathrm{d}v^\mu}{\mathrm{d}\tau} = 0 \tag{A.57}$$

Now the relation between u^μ and f is determined, the first equation is for every μ,

$$\frac{\mathrm{d}\ln u^\mu}{\mathrm{d}p} = f \quad \Rightarrow \quad \ln u^\mu = \int \mathrm{d}p f + c^{*\mu} \tag{A.58}$$

with $c^{*\mu}$ a constant (not a vector) belonging to the component μ. The solution is

$$u^\mu = c^\mu e^{\int \mathrm{d}p f} \tag{A.59}$$

where $c^\mu = e^{c^{*\mu}}$ is a constant vector. The solution for v^μ is simple, i.e. it is a constant vector,

$$v^\mu = \tilde{c}^\mu \tag{A.60}$$

The relation between the parameters p and τ can be determined with

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}p} = \frac{\mathrm{d}\tau}{\mathrm{d}p}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{A.61}$$

$$\Rightarrow \quad c^\mu e^{\int \mathrm{d}p f} = \frac{\mathrm{d}\tau}{\mathrm{d}p}\tilde{c}^\mu \tag{A.62}$$

$$\Rightarrow \quad \frac{\mathrm{d}\tau}{\mathrm{d}p} = C e^{\int \mathrm{d}p f} \tag{A.63}$$

with C a constant. For a constant scalar function $f = f_0$ the last equation becomes

$$\frac{\mathrm{d}\tau}{\mathrm{d}p} = Ce^{f_0 p} \tag{A.64}$$

and the solution for τ is

$$\tau = \frac{C}{f_0} e^{f_0 p} + \tilde{C} \tag{A.65}$$

If no boundary conditions are given then a convenient choice for the constants is $C = \pm f_0$, and $\tilde{C} = 0$, for then

$$\frac{\mathrm{d}\tau}{\mathrm{d}p} = \pm f_0 e^{f_0 p} \tag{A.66}$$

$$\tau = \pm e^{f_0 p} \tag{A.67}$$

This choice can be made due to the freedom to translate the parameter p.

Summarising the result for constant function f_0

$$u^\mu = \pm \tau v^\mu \tag{A.68}$$

where the freedom remains to let p run in the same, the $+$ case, or opposite, the $-$ case, direction of τ.

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