The Information Loss Paradox

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Master's Thesis

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Abstract

The subject of this Master's thesis is the information loss paradox. In the first chapters we treat the theoretical background needed to understand the original calculation carried out by Hawking that shows that black holes radiate. A detailed discussion of this calculation, its implications for information loss and its validity is provided in the chapters 3 to 5. Then we focus on the suggested solutions to the paradox. In particular, we investigate one of these proposals that was found by Mathur using string theory. We subject this so-called 'fuzzball' picture to a test that should give some insight regarding the question whether or not Mathur's proposal is likely to solve the paradox in the long run. We were unable to draw a definite conclusion. We do, however, obtain a couple of intermediate results that can be useful in successive research. Some suggestions for a follow up are given explicitly.

Acknowledgments

Writing a thesis was undoubtedly the most trying endeavor I have faced during my college years. I am glad that I did not need to face this challenge alone. I sincerely appreciate the academic support, friendship and love I have received during these trying times.

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Contents

Introduction

In the mid seventies Stephen Hawking discovered a remarkable feature of black holes. Classically, black holes are defined as objects so dense that even light gets trapped in their gravitational field. However, Hawking showed that when quantum effects are taken into account black holes have a finite temperature. They radiate and eventually evaporate when placed in a cooler environment. This radiation is called Hawking radiation and it is easiest to picture it as follows. In a quantum field theory there are vacuum fluctuations, that is virtual pairs of particles can be produced by the vacuum, one with a negative and the other with a positive energy. In a flat spacetime setting these virtual particle pairs may exist for an instance to annihilate again immediately. But now imagine such a pair production to occur just outside a black hole, then if the virtual particle with a negative energy E falls through the event horizon, whereas the particle with positive energy does not, the two can no longer rejoin to annihilate. In this scenario the negative energy particle causes the black hole mass to reduce with an amount E that is carried away by the particle with positive energy E. This is roughly how quantum effects allow black holes to evaporate. We must emphasize, however, that this picture serves merely to sketch the idea and should not be taken too literally.

In his calculation Hawking shows that the radiation coming from the black hole is exactly thermal, which means that it does not carry any information. As long as the black hole exists the information about everything that fell into it is inaccessible to an observer that stays outside the black hole, but could in principle be retrieved by going after it. However, when the black hole evaporates all that remains is the information-less radiation. So Hawking concluded that information is lost in the process of black hole formation and evaporation, but this is in contradiction with one of the fundaments of quantum mechanics (unitary time evolution). This constitutes the information loss paradox.

Over the past three decades many a physicist has attempted to solve the paradox and numerous articles appeared on the subject (Hawking's 1975 article is cited over 2200 times!). Although a satisfactory solution to the paradox has not yet been provided, everyone seems to agree on two things. Firstly, that information is not really lost and secondly, that Hawking was led to draw this false conclusion because, in the lack of a full theory of quantum gravity, he had to do a semi-classical approximation.

To find this full theory of quantum gravity is probably the most pressing problem in theoretical physics. Due to non-renormalizabitlity gravity cannot be incorporated in a quantum field theory along with the other three forces of nature; the electromagnetic force, the weak force and the strong force. Apart from being non-renormalizable, another striking feature of gravity is that it is extremely weak compared to all other forces. In fact, there are only two cases in which gravitational effect may be strong enough to affect quantum phenomena. These are the origin of the universe and the very instances afterwards, as well as the close vicinity of a black hole. Consequently, both black holes and the first instances of the universe are widely studied in the quest for a theory that unifies all four forces of nature. And not surprisingly, the information loss paradox plays a main role in the black hole studies, since a better understanding of the paradox may provide a better understanding of the full theory and vice versa. A nice example of how studies of the paradox and searches for a theory of quantum gravity are intertwined is the following. In 1993 Gerard 't Hooft deduced from detailed studies of the paradox that the world is in principle holographic. Five years later Juan Maldacena found that string theory indeed exhibits this holographic feature, called the AdS/CFT correspondence. This correspondence tells us that if string theory is the correct theory of quantum gravity, information is not lost.

The goal of this thesis is first of all to give a clear explanation of the information loss paradox that should be accessible to students who have basic knowledge of general relativity and quantum field theory. Secondly, we focus on the validity of Hawking's calculation to try and answer the question "Where is the loophole in Hawking's calculation?". Thirdly, we study some of the most important attempts at resolving the paradox to understand how the common belief in the physics community has developed towards the conclusion that information is not lost. At this point we are fully equipped to motivate why we selected the so-called 'fuzzball' proposal for further studies. The fuzzball proposal is an idea based on string theory arguments that was put forward by Mathur as a possible solution to the paradox. We shall see that the motivation for our choice is twofold. On the one hand, our studies led us to believe that solving the paradox requires a full theory of quantum gravity and string theory is such a theory. On the other hand, Mathur's work is attractive for the fact that his approach enables one to do actual calculations, whereas many attempts fail to go beyond speculation. The final question we shall try to answer is "Do the geometries, that Mathur claims to describe black holes more accurately than the classical black hole geometry, exhibit the property crucial for restoring information, namely non-locality?".

We shall start by reviewing the theory of black holes. Special attention will be paid to the analogy between black hole mechanics and thermodynamics. In the chapters 3 to 5 we consecutively give a thorough review of Hawking's original calculation, discuss how Hawking's results lead to information loss and finally elaborate on the validity of Hawking's calculation. Chapter 6 is dedicated to the attempts that have been made to solve the paradox. Here we shall encounter concepts such as the holographic principle and the AdS/CFT correspondence mentioned before. In the final chapter we focus on a possible resolution put forward by Mathur. We investigate his fuzzball proposal and also present some original work that was carried out to test if this fuzzball can in fact leak out information over its horizon. Although, the answer was inconclusive the results we obtain may be useful for further studies of Mathur's work. We conclude by providing some ideas for successive research that may give a conclusive answer to the question whether or not the fuzzball proposal can restore information.

Chapter 1

General Relativity

At the start of the 20th century Albert Einstein forced us to radically change our view on space and time. The theory of special relativity, which he wrote down in 1905, was the first step. And in 1915 he made it complete with the theory of general relativity, which really took our understanding of space, time and gravity to another level.

In special relativity spacetime has the structure of an $\mathbb{R}^4$ manifold with Lorentzian signature. In this spacetime our idea of simultaneity is drastically changed since a natural notion (i.e. observer independent) of the event p occurring "at the same time as" the event q no longer exists. This is of course very contradictory to what we seem to experience in everyday life. Our ability to ignore our intuition is even more challenged by general relativity in which spacetime is no longer flat. Instead, Einstein's equation of general relativity tells us that spacetime is curved due to the presence of mass. Gravitational attraction is not a consequence of a force anymore, but of the curvature of spacetime. When the energy-momentum tensor, sometimes called stresstensor, $T_{\mu\nu}$ describes the distribution of energy and momentum in the universe, the Einstein equation determines the metric that is generated by this energy-momentum tensor in the following way[1]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu},$$

where $R_{\mu\nu}$ and R are the Ricci tensor and scalar respectively, G is the gravitational constant and $g_{\mu\nu}$ is the metric. In this thesis we shall use the $(-+++)$ sign convention for spacetime metrics.

We assume that the reader is familiar with the basics of general relativity[2], nevertheless, the following sections can help you refresh your memory as we will quickly introduce the geodesic equation and Killing vectors to be able to discuss in detail a particular solution to the Einstein equation, namely the Kerr metric. The Kerr metric describes the spacetime around a spinning black hole. We chose to discuss this case because in the next chapter we will treat the mechanics of a spinning black hole to show the remarkable resemblance between black hole mechanics and thermodynamics.

[1] In the remainder of this thesis I will work in units where the gravitational constant G, the velocity of light c and Planck's constant $\hbar$ are set to unity.

[2] If not we can recommend [1] for a quick introduction or [2], [3] for a thorough discussion of the entire theory.

1.1 Geodesic Equation

Geodesics in curved spacetime are the analogs of straight lines in flat spacetime. They are given by

$$\frac{D^2x^\mu}{D\lambda^2} \equiv \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0, \tag{1.1}$$

where $\Gamma^\mu_{\nu\rho}$ is the Christoffel connection

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}). \tag{1.2}$$

The meaning of the parameter λ in (1.1) depends on the type of geodesic, which is determined by the number

$$e \equiv -g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{1.3}$$

which is constant along the geodesic. When $e > 0$ the geodesic is called timelike and it describes the path a massive particle follows when no external force acts on it, note that the same is true for straight timelike lines in flat spacetime. For timelike geodesics the parameter λ is proportional to the proper time, which is extremized along the path. This can be seen from the action principle. The propertime τ is given by

$$d\tau = \sqrt{-dx^\mu dx^\nu g_{\mu\nu}} = \sqrt{-\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}d\lambda, \tag{1.4}$$

where the dot denotes derivation with respect to λ. The action of a particle of mass m is then

$$S = -m\int d\tau = -m\int d\lambda\sqrt{-\dot{x}^\mu\dot{x}^\nu g_{\mu\nu}}. \tag{1.5}$$

And the variation principle ($\delta S = 0$) yields the geodesic equation (1.1).

When $e < 0$ the geodesic is called spacelike and λ is proportional to $\imath$ times the propertime. Finally, for $e = 0$ we call it a null geodesic and λ is called the affine parameter which is not related to propertime. Massless particles move along null geodesics.

1.2 Conserved Charges

In this section we will use the symmetry of the action (1.5) of a massive particle to derive its conserved charges. Consider, therefore, the following coordinate transformation

$$x^\mu \to x^\mu - \alpha k^\mu(x), \tag{1.6}$$

this leaves the action invariant to first order in α if k^μ obeys

$$k^\lambda\partial_\lambda g_{\mu\nu} + (\partial_\mu k^\lambda)g_{\lambda\nu} + (\partial_\nu k^\lambda)g_{\lambda\mu} = 0, \tag{1.7}$$

we then call $k^\mu(x)$ a Killing vector field and (1.7) is known as Killing's equation. When we define $Q \equiv k^\mu p_\mu$ with

$$p_\mu = \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}, \tag{1.8}$$

where $\mathcal{L}$ is the Lagrangian: $\mathcal{L} = -m\sqrt{-dx^\mu dx^\nu g_{\mu\nu}}$. It is easy to show that Q is a conserved charge

$$\begin{aligned} 0 &= \delta\mathcal{L} \\ &= \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}\delta \dot{x}^\mu \\ &= \frac{d}{d\lambda}\Big[\frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}\Big]\delta x^\mu + \frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}\delta \dot{x}^\mu \\ &= \frac{d}{d\lambda}\Big[\frac{\partial\mathcal{L}}{\partial \dot{x}^\mu}\delta x^\mu\Big] \\ &= -\alpha\frac{d}{d\lambda}[p_\mu k^\mu] \end{aligned} \tag{1.9}$$

where we used the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial x^\mu} - \frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial \dot{x}^\mu} = 0$$

in the third line.

If we define a Killing vector as $k \equiv k^\mu\partial_\mu$, then one can always find local coordinates such that $k = \partial_\xi$, where ξ is one of the coordinates. In these coordinates the Killing equation gives: $\partial_\xi g_{\mu\nu} = 0$. So that it can be seen immediately from the components of the metric what its Killing vectors, and consequently its conserved charges, are.

1.3 Kerr Metric

The metric of a spacetime with a certain energy and momentum distribution is determined by Einstein's equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1.10}$$

where $R_{\mu\nu}$ and R are the Ricci tensor and scalar respectively, $g_{\mu\nu}$ is the metric and $T_{\mu\nu}$ is the energy-momentum tensor. The Ricci tensor and scalar are obtained from the Riemann tensor by the following contractions

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \quad \text{and} \quad R = g^{\mu\nu}R_{\mu\nu}, \tag{1.11}$$

the Riemann tensor, also called curvature, is defined by

$$[D_\mu, D_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma, \tag{1.12}$$

where V^ρ is a vector and D_μ is the covariant derivative

$$D_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\sigma}A^\sigma. \tag{1.13}$$

The explicit form of the Riemann tensor is

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{1.14}$$

In general it is very difficult to solve the Einstein equation given a certain energy-momentum tensor. Just a few solutions have been found that solve the Einstein equation analytically. And they do so for very special values of the energy-momentum tensor. For example the Friedman-Robertson-Walker metrics solve the Einstein equation for a spatially homogeneous and isotropic fluid[3] and the Schwarzschild, Kerr and Kerr-Newmann metric are solutions of the Einstein equation in vacuum, i.e. for $T_{\mu\nu} = 0$. The Schwarzschild solution is probably the most well known analytic solution. It is the unique spherically symmetric vacuum solution of Einstein's equation, which describes the empty space outside a spherical star or black hole. The Schwarzschild metric is given by

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (1.15)$$

where M is the mass of the star or black hole that sits at $r = 0$.

As we learned in the previous section we can see immediately from the Schwarzschild metric that we have two conserved charges, since the metric is independent of t and ϕ. The conserved charges for the Killing vectors $k = \partial_t$ and $n = \partial_\phi$ are respectively the energy and angular momentum

$$\begin{aligned} E &= -k^\mu p_\mu \\ L &= n^\mu p_\mu, \end{aligned} \qquad (1.16)$$

where the minus sign in the definition of the energy serves to make it positive everywhere outside the horizon (there both the Killing vector field k^μ and the momentum p^μ are timelike so their inner product is negative).

Furthermore, we see from (1.15) that the components of the metric become singular at $r = 0$ and $r = 2M$. The second is not a real singularity, which can be seen by performing the coordinate change $r \to r^* = r + 2M \ln|\frac{r}{2M} - 1|$. The coordinate r^* is called the tortoise coordinate and the metric has its only singularity at $r^* = 0$. An interesting thing does happen at $r = 2M$, though; when you cross $r = 2M$ in the direction of decreasing r, r becomes timelike, whereas t becomes spacelike. In other words the light cones tilt over and you are forced to continue moving towards $r = 0$. The surface at $r = 2M$ is called an event horizon, it causally disconnects two regions of spacetime in the sense that anything that falls through the horizon can no longer influence anything in the outer region. The singularity at $r = 0$ is real, but it is hidden from us by the event horizon.

The metric that describes the spacetime outside a spinning black hole[4] was much harder to find, because the condition of spherical symmetry had to be given up. It was found only in 1963 by Kerr and is appropriately called the Kerr metric

$$\begin{aligned} ds^2 = -dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + \\ \frac{2Mr}{\rho^2}(a\sin^2\theta d\phi - dt)^2, \end{aligned} \qquad (1.17)$$

[3]In cosmology one often looks at such large scales that it seems legitimate to describe the universe as a homogeneous and isotropic fluid. See [2] for more on Friedman-Robertson-Walker universes.

[4]Of course it also describes the metric outside a spinning star but we will focus on the black hole from now on.

where

$$\Delta(r) = r^2 - 2Mr + a^2 \quad \text{and} \quad \rho^2(r,\theta) = r^2 + a^2\cos^2\theta,$$

and a is related to the angular momentum J of the black hole by $J = Ma$. This metric can be extended to the metric of the spacetime outside a spinning charged black hole, with angular momentum J and charge Q by replacing $2Mr$ with $2Mr - Q^2$. It is then called the Kerr-Newman metric.

When a is set to zero in (1.17) we recover the Schwarzschild metric, on the other hand we would expect to recover flat Minkowski spacetime when M is set to zero. Although this is the case, it may not be obvious immediately, because (t, r, ϕ, θ) are not ordinary polar coordinates but ellipsoidal coordinates (see Fig. 1.1), in which Minkowski spacetime looks like this

$$ds^2 = -dt^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2.$$

Note that the Kerr metric also has two Killing vectors $k = \partial_t$ and $n = \partial_\phi$. Since the metric is independent of t it is stationary, but it is not static; it does not change with time, but it is spinning.

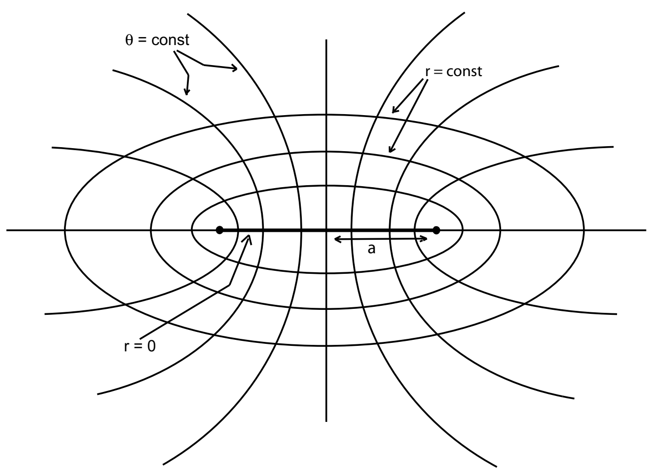

Figure 1.1: Flat space in ellipsoidal coordinates.

Now let us look at where the coefficients of the metric become singular. This happens for $\rho = 0$ and $\Delta = 0$. We shall see that the first corresponds to a real singularity in spacetime whereas the second is an artifact of the coordinates. We will first discuss $\Delta = 0$, which happens at two radii

$$r_\pm = M \pm \sqrt{M^2 - a^2},$$

in the case that $M^2 > a^2$. This is the case we are interested in, since for $M^2 < a^2$ the spacetime contains a naked singularity. A naked singularity is a singularity that it is not hidden from our view by an event horizon, which means that things can come out of the singularity and influence our universe. Since the laws of physics do not apply at a singularity, we have no idea of what can come out of it. Consequently, we loose the possibility of saying anything reasonable about the future in the presence of a naked singularity. This is of course very disturbing, but fortunately numerous thorough studies of collapsing bodies have made it reasonable to believe that naked singularities do not form in such processes (see [2]). The fact that singularities seem to be shielded from our view is often referred to as "cosmic censorship" and is thought to apply to all naked singularities except the Big Bang singularity at the beginning of our universe. Although we shall also adopt this view here and therefor omit the case for which $M^2 < a^2$, we should emphasize that cosmic censorship has not been proven.

Finally, the case for which the mass of the black hole exactly equals a (or $-a$) is called the extremal case. It is highly unstable, since adding just the slightest bit of mass will change it into the first case, where $M^2 > a^2$.

At r_+ we find that we have a usual event horizon[5] as we have in the Schwarzschild metric at $r = 2M$. At r_- the opposite happens; since the light cones tilt again r switches back to being spacelike and t to being timelike. So you can choose whether to continue moving in the direction of decreasing r and eventually hit the singularity (or pass through it as we will see in a bit) or return in the direction you came from and this time when you cross $r = r_-$ you will be forced to move towards r_+, since the time direction has been inverted. Eventually, you will be spit out past the outer horizon into a different universe than the one you originally came from[6]. So in this new region of spacetime there is a naked singularity, but for the region you started from the singularity is shielded by the event horizon at $r = r_+$.

In the Kerr spacetime there is yet another surface at which interesting things happen. In the Schwarzschild metric the Killing vector $k = \partial_t$ is timelike outside the event horizon and spacelike inside the event horizon, at the horizon it is null[7]. Now let us see where this happens in the Kerr metric

$$\begin{aligned} k_\mu k^\mu &= g_{tt} \\ &= \frac{2Mr}{\rho^2} - 1 \\ &= \frac{1}{\rho^2}(a^2 \sin^2\theta - \Delta). \end{aligned}$$

This vanishes at

$$r_{K\pm} = M \pm \sqrt{M^2 - a^2\cos^2\theta}.$$

So $r_{K+} \geq r_+$ (they coincide for $\theta = 0, \pi$) and at r_+ the Killing vector is already spacelike. The surface at which the Killing vector is null is called the Killing horizon and the region between the Killing horizon and the outer horizon

[5] I.e. you can choose coordinates that are not singular at r_+.

[6] See next section.

[7] So at the event horizon $g_{tt} = 0$, physically this means that it is impossible for massive particles to stand still at the event horizon. This can be seen from the equation for the constant e (1.3), which is positive for the path of a massive particle:

$$\begin{aligned} 0 &< -g_{\mu\nu} dx^\mu dx^\nu \\ &< g_{tt} dt^2 - g_{ij} dx^i dx^j. \end{aligned}$$

At the event horizon g_{tt} vanishes so the dx^i cannot be all zero there.

is called the ergosphere (see Fig. 1.2). Since the Killing vector is spacelike inside the ergosphere, but the momentum of a massive particle is still timelike, particles can have negative energy there. In the next chapter we will discuss the remarkable consequence hereof, namely that energy can be extracted from a spinning black hole via a so-called Penrose process.

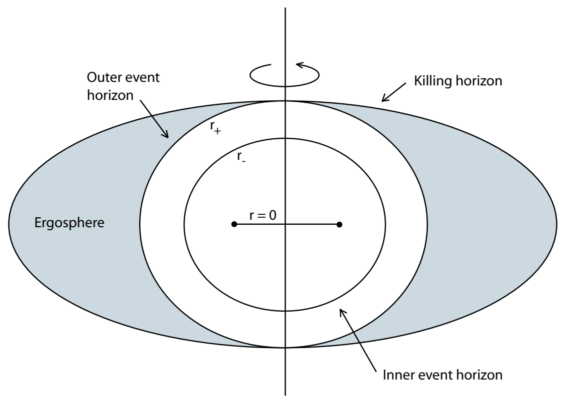

Figure 1.2: Spinning black hole with ergosphere.

At $\rho = 0$ the spacetime truly becomes singular, but as opposed to what you see in the Schwarzschild case this is not a point in spacetime. Remember that in ellipsoidal coordinates $r = 0$ is not a point but a disc (see Fig. 1.1) so that the singularity which occurs at

$$r = 0 \quad \text{and} \quad \theta = \frac{\pi}{2},$$

is actually a ring. What happens when you enter this ring is more science fiction than science, but it can be shown that you will find yourself in another asymptotically flat spacetime, described by the Kerr metric but now with $r < 0$, so Δ never vanishes and there are no event horizons.

1.4 Penrose Diagrams

The last thing we want to discuss in this chapter is a very useful way of depicting curved spacetime. It was discovered by Roger Penrose and therefore we call them Penrose diagrams. They are obtained by rewriting your metric such that it has the form

$$ds^2 = A(\alpha, \beta)(-d\alpha^2 + d\beta^2) + B(\alpha, \beta)d\Omega^2$$

with Ω a function of the angular coordinates. So that null geodesics make a 90^o angle and all the coordinates have finite ranges.

As an example we will draw the Penrose diagram for the Schwarzschild metric (1.15). First we have to do some smart coordinate changes, starting with going to Kruskal coordinates

$$\begin{aligned} \tilde{u} &= -4Me^{-u/4M} \\ \tilde{v} &= 4Me^{v/4M}, \end{aligned} \tag{1.18}$$

with $u = t - r^*$ and $v = t + r^*$ and the tortoise coordinate $r^* = r + 2M \ln|\frac{r}{2M} - 1|$. This enables us to write the metric as

$$ds^2 = \frac{2M}{r} e^{-r/2M} d\tilde{u} d\tilde{v} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1.19}$$

We can define the angular part as $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and we will set this to zero, which essentially does not change things, since the Schwarzschild solution is spherically symmetric. When we set $d\Omega$ to zero and change to Kruskal-Szekeres coordinates

$$\begin{aligned} u' &= (\tilde{u} - \tilde{v})/4 \\ v' &= (\tilde{u} + \tilde{v})/4, \end{aligned} \tag{1.20}$$

and use a conformal transformation ($g_{\mu\nu} \to \alpha(x) g_{\mu\nu}$) we can write the metric as

$$ds^2 = dv'^2 - du'^2$$

Finally, this can be compactified by making the range of the coordinates finite with a last coordinate change

$$\begin{aligned} u'' &= 2\tan^{-1} u' \\ v'' &= 2\tan^{-1} v', \end{aligned} \tag{1.21}$$

with $-\pi \leq u'', v'' \leq \pi$.

The Penrose diagram for the maximally extended[8] Schwarzschild metric can now be drawn and is shown in Fig. 1.3. Note that the boundaries of the Penrose diagram represent infinity and that ingoing and outgoing null geodesics always make a 90^o angle. $\mathcal{I}^+$ and $\mathcal{I}^-$ are defined as future and past null infinity respectively, furthermore, we write future and past timelike infinity as $\imath^+$ and $\imath^-$ respectively and spacelike infinity as $\imath^0$. Fig. 1.4 shows the Penrose diagram of a spherically collapsing body. The shaded area represents the collapsing body and its surface is a timelike geodesic. Outside the surface of this body the spacetime is described by the Schwarzschild metric, whereas the metric inside the body is completely different. The regions III and IV do not exist and neither do the past event horizon and singularity. Finally, the vertical line represents the origin of the collapsing body and is also a timelike geodesic.

[8]Maximally extended means for $-\infty < r, t < \infty$.

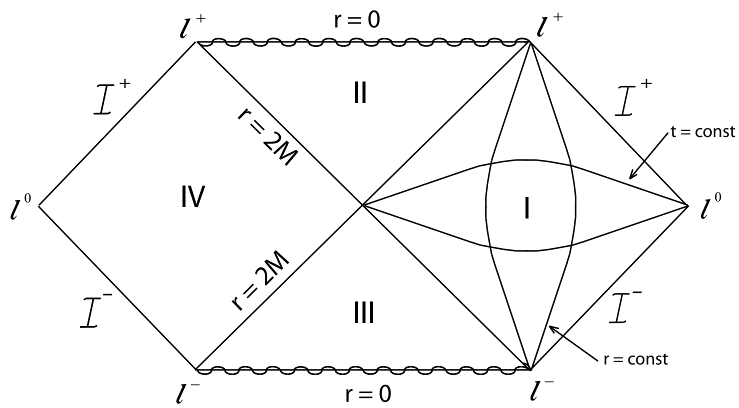

Figure 1.3: Penrose diagram of the extended Schwarzschild metric.

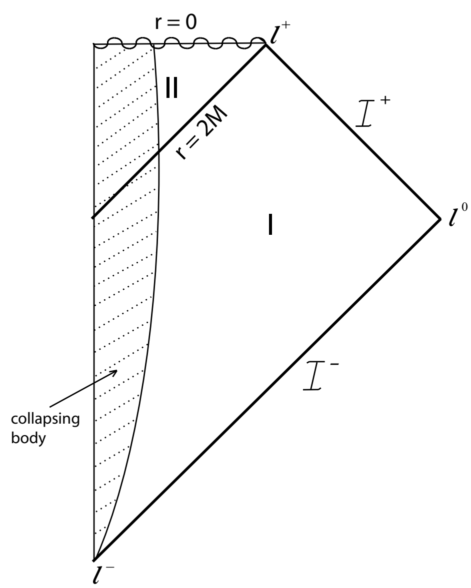

Figure 1.4: Penrose diagram of a spherically collapsing body.

Finally, the Penrose diagram of the Kerr metric is shown in Fig. 1.5, the curve γ describes the path we discussed in the previous section. As you see the surface at $r = r_+$ serves as an event horizon that shields off the singularity for region I.

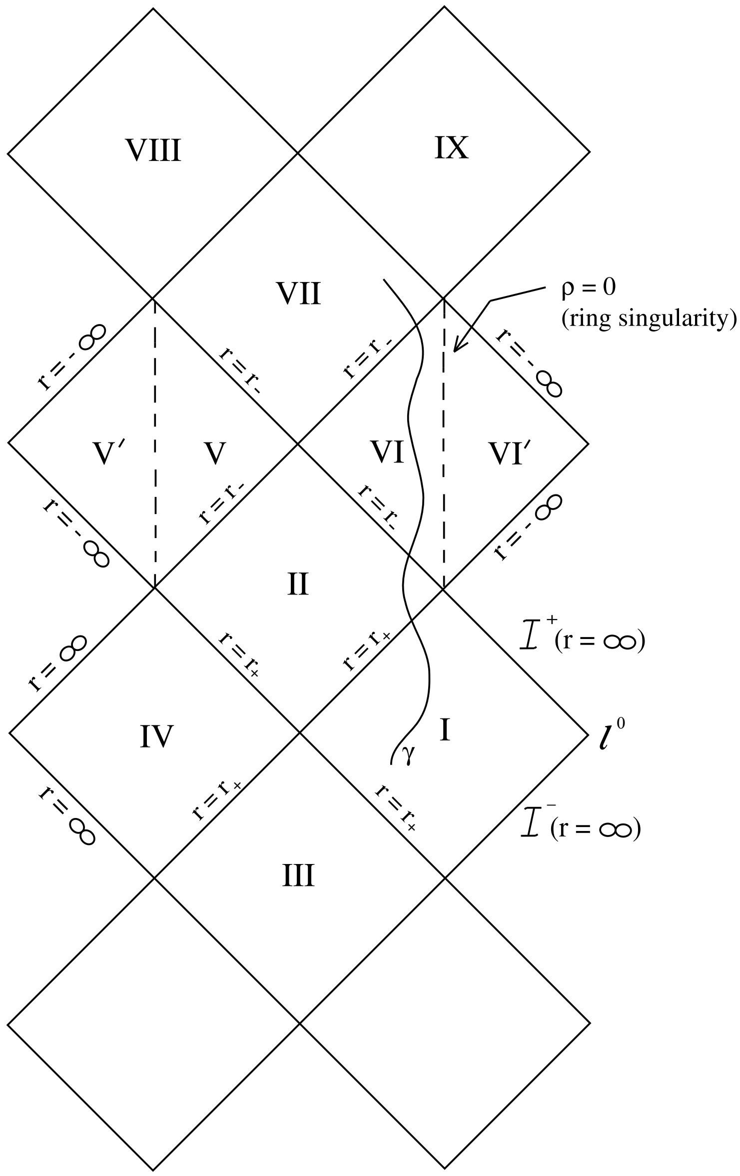

Figure 1.5: Penrose diagram of the extended Kerr metric.

Chapter 2

Black Holes and Thermodynamics

2.1 Area Theorem

Classically, the definition of a black hole is that it is an object with a gravitational attraction from which not even light can escape. Once a particle, massive or massless, passes the event horizon there is no way back. Intuitively, one easily draws the conclusion that therefore the mass of a black hole can only increase. The area of a black hole is defined as the surface that forms the event horizon, so for a schwarzschild black hole of mass M the area is simply

$$A = 4\pi r^2 = 16\pi M^2. \tag{2.1}$$

Since we just concluded that $\delta M \geq 0$ it follows directly from (2.1) that $\delta A \geq 0$. So this intuitive, but naive reasoning led us to the conclusion that the area of a black hole can never decrease in size, this is known as the area theorem.

A rigorous proof of the area theorem can be found for example in [2], here I will try to give you a physical picture of this very mathematical proof, which should help the interested reader understand the full proof.

First we introduce the expansion θ of a null geodesic γ as

$$\theta = D_\alpha \xi^\alpha, \tag{2.2}$$

where D_α is the covariant derivative and ξ^α is the tangent vector of γ. The physical interpretation of the expansion of a bundle of null geodesics is that it is a measure of the focusing of these null geodesics by the spacetime. When θ is positive the light rays are spread out by gravity, whereas when it is negative light rays are focussed towards each other. When the derivative of (2.2) is taken one obtains the focusing equation

$$\xi^\alpha D_\alpha \theta = \frac{d\theta}{d\tau} = -\frac{1}{2}\theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - R_{\alpha\beta}\xi^\alpha\xi^\beta, \tag{2.3}$$

where $R_{\alpha\beta}$ is the Ricci tensor and $\sigma_{\alpha\beta} \equiv D_{(\alpha}\xi_{\beta)}$ is the shear and $\omega_{\alpha\beta} \equiv D_{[\alpha}\xi_{\beta]}$ is the twist of the null geodesics. The shear of a bundle of light rays can be imagined as neighboring light rays experiencing a translation with respect to each other. To understand the meaning of twist imagine a cylinder composed of fibers running in the axial direction indicating different light rays. Then, imagine holding the cylinder in both hands and giving it a twist by rotating one hand in the direction orthogonal to the cylinder axis while holding the other hand still, now the fibers have a twist with respect to one another. When we demand that the null geodesics are orthogonal to the three dimensional hypersurfaces of equal time, it can be shown that their twist is zero. Without proving this it can be understood qualitatively when we make a comparison with the three dimensional picture we have of twist. Here the axis of the cylinder corresponds to the time direction and cross-sections orthogonal to this axis correspond to the hypersurfaces. So demanding that the fibers are orthogonal to these cross-sections results directly in the fibers having no twist with respect to one another. Thus, we have that for hypersurface orthogonal geodesics, the third term in (2.3) vanishes.

The last term of the focusing equation is negative since, according to Einstein's equation

$$R_{\alpha\beta}\xi^\alpha\xi^\beta = 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta \geq 0,$$

where we used that for null geodesics $g_{\alpha\beta}\xi^\alpha\xi^\beta = 0$ and the last step follows from the positive energy condition. Finally, the second term is manifestly non-positive, since it is quadratic. This enables us to obtain the following important inequality from (2.3)

$$\frac{d\theta}{d\tau} + \frac{1}{2}\theta^2 \leq 0 \quad \rightarrow \quad \frac{1}{\theta} \geq \frac{1}{\theta_0} + \frac{1}{2}\tau, \tag{2.4}$$

with θ_0 the value of θ at $\tau = 0$. Once light rays start being converged with some value $\theta_0 < 0$, (2.4) tells us that $\theta \to -\infty$ along those light rays within the affine length $\tau \leq 2/|\theta_0|$. This result is one of the cornerstones of the singularity theorem. We shall not discuss the details of the theorem here explicitly, but it says that spacetimes, satisfying a number of conditions (all satisfied by the Schwarzschild and Kerr metric) and containing a trapped surface, i.e. a surface for which all future directed null geodesics orthogonal to it have a negative expansion θ, will have a singularity.

From the Penrose diagram of the Schwarzschild metric (Fig. 1.4), we see that once light rays enter region II they will inevitably hit the singularity. It can be proven that in spacetimes containing a black hole region B (such as region II in the Schwarzschild metric) every trapped surface T is a subset of the black hole region B; $T \subset B$.

The proof of the area theorem consists of two steps, the first is to prove that the null geodesic generators of the event horizon have $\theta \geq 0$, the second is to prove that this leads to $\delta A \geq 0$.

The event horizon H is the boundary of the black hole region B. Suppose there is a point $p \in H$ for which $\theta < 0$. Let Σ be a surface through p that is intersected only once by every causal (timelike and null) geodesic with no end point (it reaches either $\imath^+$ or $\mathcal{I}^+$). Then $\mathcal{H} = H \cap \Sigma$ is a two surface (note that this is the area of the black hole) and $p \in \mathcal{H}$. Since $\theta < 0$ at p, there is a neighborhood of p for which the expansion is also negative. Now we call the intersection of the surface Σ with this neighborhood K (see Fig. 2.1), so $K \subset \Sigma$. But this leads to a contradiction as follows. Since $\theta < 0$ on K, K is a trapped surface, and as we saw above it follows that $K \subset B$. And thus all the causal geodesics that intersect K have end points at the singularity, so $K \cap \Sigma = \emptyset$. So we must draw the conclusion that $\theta \geq 0$ everywhere on H.

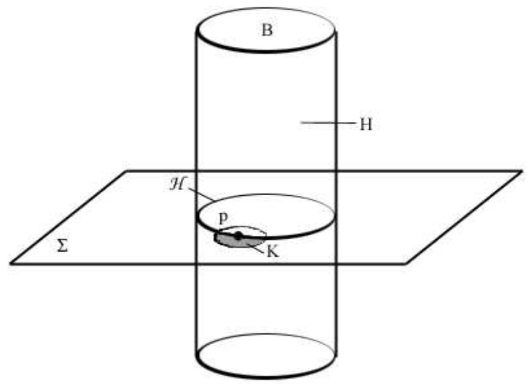

Figure 2.1: Intersection of the event horizon with a Cauchy surface.

This enables us to proof that the area of a black hole can never decrease as follows. Let Σ_1 be a Cauchy surface and $\mathcal{H}_1 = H \cap \Sigma_1$ and let Σ_2 be another Cauchy surface, such that every causal geodesic that passes through Σ_1 will pass through Σ_2 at a later time and $\mathcal{H}_2 = H \cap \Sigma_2$ (see Fig 2.2). Now, through each $p \in \mathcal{H}_1$ passes a null geodesic γ that will intersect Σ_2 at $q \in \mathcal{H}_2$, so there is a map from $\mathcal{H}_1$ into $\mathcal{H}_2$. Finally, since $\theta \geq 0$, the portion of the area of $\mathcal{H}_2$ given by the image of $\mathcal{H}_1$ under this map must be at least as large as the area of $\mathcal{H}_1$. Since new black holes may form between Σ_1 and Σ_2, the area of $\mathcal{H}_2$ may even be larger. So we obtain that the area of a black hole cannot decrease.

2.2 Penrose Process

In 1969 Penrose did the surprising discovery that energy can be extracted from rotating black holes. In section 1.3 we have seen that rotating black holes have an ergosphere. This is a region were it is impossible for massive particles to stand still, but they are not trapped by the black hole yet, they can still escape to infinity. In addition, we found that particles can have negative energy inside the ergosphere, because the Killing vector field k_μ (such that $k^\mu \partial_\mu = \partial_t$) becomes spacelike. So for a massive particle with timelike four-momentum p^μ, the energy inside the ergosphere is

$$E = -k_\mu p^\mu < 0.$$

Can energy be extracted from the black hole by throwing in particles with negative energy? The answer is yes!

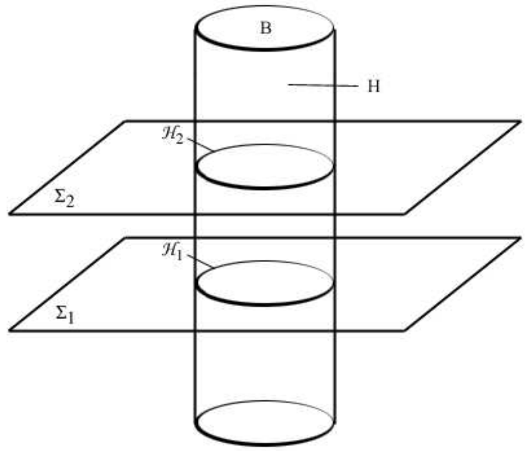

Figure 2.2: Every causal geodesic that intersects Σ_1 will also intersect Σ_2.

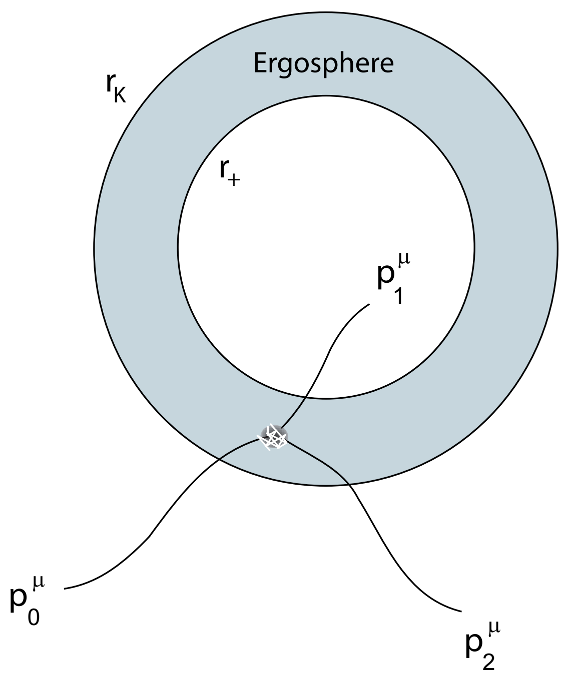

Figure 2.3: Top view of a spinning black hole. A particle with momentum p_0^μ falls towards it and splits into two parts inside the ergosphere.

Let us make this a bit more precise. Consider a particle with positive energy $E_0 = -k_\mu p_0^\mu$ at a large distance from the black hole, we drop this particle into the black hole along a geodesic (its energy-momentum p^μ is conserved). Furthermore, we prepare the particle such that it splits into two parts when it finds itself inside the ergosphere (see Fig. 2.3). The energy E_1 of the part that continues towards the black hole and is finally absorbed, is arranged to be negative from the point of view of an external observer. The other part travels back across the Killing horizon to infinity along a geodesic and is arranged to have a positive energy E_2 outside the ergosphere. Local energy-momentum conservation tells us that

$$p_0^\mu = p_1^\mu + p_2^\mu$$

and contraction with k_μ gives

$$E_0 = E_1 + E_2.$$

Since E_1 is negative, we find $E_2 > E_0$. So we have extracted an energy $|E_1|$ from the rotating black hole.

There is a limit to the amount of energy that can be extracted from a rotating black hole. This limit comes from the fact that particles carrying a negative energy also carry a negative angular momentum, i.e. they move in the direction opposite to the rotation of the black hole. This will eventually cause the black hole to stop rotating. As a consequence the ergosphere will vanish and energy can no longer be extracted from the black hole.

Let us see how we arrive at the relation between a particles energy and its angular momentum. First, we introduce locally non-rotating observers, i.e. they have zero angular momentum, as the closest analog to inertial observers. So, for such observers we may write[1]

$$\begin{aligned} L &= g_{\mu\nu} n^\mu \dot{x}^\nu \\ &= g_{\phi t}\frac{dt}{d\lambda} + g_{\phi\phi}\frac{d\phi}{d\lambda} \\ &= 0. \end{aligned} \tag{2.5}$$

Their coordinate angular velocity is thus

$$\Omega \equiv \frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{a(r^2+a^2-\Delta)}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta},$$

where we inserted

$$\begin{aligned} g_{t\phi} &= -\frac{2Mr}{\rho^2} a \sin^2\theta \\ &= -\frac{r^2+a^2-\Delta}{r^2+a^2(1-\sin^2\theta)} a\sin^2\theta \quad \text{and} \\ g_{\phi\phi} &= (r^2+a^2)\sin^2\theta + \frac{2Mr}{\rho^2}a^2\sin^4\theta, \end{aligned}$$

[1]Remember that we defined $n = n^\mu \partial_\mu = \partial_\phi$ in section 1.3.

which can easily be obtained from the Kerr metric (1.17). For such an observer with zero angular momentum sitting at the event horizon, i.e. $r = r_+$ and $\Delta = 0$, the coordinate angular velocity is

$$\Omega_H = \frac{a}{r_+^2 + a^2}, \tag{2.6}$$

in other words Ω_H is the minimum angular velocity an observer can have at the event horizon, so Ω_H can be naturally defined as the velocity of the event horizon itself. The null vector tangent to the event horizon is then not just k^μ but[2]

$$\chi^\mu = k^\mu + \Omega_H n^\mu.$$

This vector is timelike for $r > r_+$ and spacelike for $r < r_+$, so outside the outer event horizon $\chi^\mu p_\mu < 0$ for every massive particle (note that this includes also the particles that have negative energy inside the ergosphere!), from this it follows that

$$\chi^\mu p_\mu = k^\mu p_\mu + \Omega_H n^\mu p_\mu = -E + \Omega_H L < 0. \tag{2.7}$$

Thus we find that for the particle with negative energy E_2 inside the ergosphere

$$L_2 < E_2/\Omega_H < 0, \tag{2.8}$$

since Ω_H is positive. So the particle moves against the rotation direction of the black hole. Once it is absorbed by the black hole its energy causes a reduction in the mass M of the black hole with an amount E_2 and analogously the black hole's angular momentum J is reduced by an amount L_2. Inserting this into (2.8) gives

$$\delta J < \delta M/\Omega_H.$$

It is interesting to rewrite this as

$$\delta M_{\text{irr}} > 0, \tag{2.9}$$

with the irreducible mass defined as

$$M_{\text{irr}}^2 = \frac{1}{2}\Big(M^2 + \sqrt{M^4 - J^2}\Big). \tag{2.10}$$

A straightforward (but tedious) calculation shows that the variation of (2.10) gives (2.9). For a black hole with zero angular moment we see that its irreducible mass equals its mass, so as we expected its mass cannot be reduced. On the other hand, from a black hole with initial mass M_0 and initial angular momentum J_0 one can maximally extract an energy $M_0 - M_{\text{irr}}(M_0, J_0)$ (in this case the Penrose process should be maximally efficient and the particles that are thrown into the black hole should move along null geodesics to obtain equality in (2.7)). By the time this energy is extracted the black hole's angular momentum will have vanished.

[2] For a proof see [3]

Finally, it is nice to note that (2.9) is nothing more than the area theorem! The area[3] of the Kerr black hole is

$$\begin{aligned} A &= \int_{r=r_+} \sqrt{g_{\phi\phi}g_{\theta\theta}}\,d\phi d\theta \\ &= 4\pi(r_+^2 + a^2), \end{aligned}$$

inserting $r_+ = M + \sqrt{M^2 - a^2}$ gives

$$\begin{aligned} A &= 4\pi(2M^2 + 2M\sqrt{M^2 - a^2}) \\ &= 8\pi(M^2 + \sqrt{M^4 - M^2a^2}) \\ &= 16\pi M_{\text{irr}}^2, \end{aligned} \tag{2.11}$$

where we used $J = Ma$ going from the second to the last line. So the area theorem now leads directly to $\delta M_{\text{irr}} \geq 0$.

2.3 Analogy with Thermodynamics

The previous two paragraphs learned us that a general feature of black hole transformations is that the area of a black hole cannot decrease and moreover tends to increase. In the first section we discussed the general proof of this fact, known as the area theorem. In the second section we discussed the specific case of energy extraction from a black hole by means of the so-called Penrose process, which led to the same conclusion. You might wonder why we discussed the Penrose process if we were able to understand the general proof, the reason is to support the view, first proposed by Beckenstein [4] in 1973, that there exists an analogy between black hole mechanics and thermodynamics. It is probably hard to convince you of the value of this analogy if we proposed it merely because the area theorem resembles the second law of thermodynamics. Namely, the area theorem states that in any physically allowed process the total area of all black holes in the universe cannot decrease. And the second law of thermodynamics states that the total entropy of all matter in the universe cannot decrease.

We elaborated on the Penrose process because it provides three more arguments to take this analogy seriously and investigate it more thoroughly. In the first place, it tells us that an increase in the black hole area is accompanied by an increase in the irreducible mass of the black hole. This means that when the area increases the amount of energy that can be extracted from the black hole and converted into work decreases and transforms into irreducible mass. This suggests that we can regard the irreducible mass as an analog of the degraded energy of a thermodynamic system. The degradation of energy is a characteristic of irreversible processes in thermodynamics (see [5]), degraded energy is energy that can no longer be converted into work. So now we can add to the apparent analogy between area and entropy the analogy between irreducible mass and degraded energy, or even so between black hole mass and system energy.

[3]The area is obtained from the integral $A = \int d\Phi d\Theta$, with $d\Phi = \sqrt{(ds)^2}\Big|_{r,\theta,t=0} = \sqrt{g_{\phi\phi}}d\phi$ and similarly $d\Theta = \sqrt{g_{\theta\theta}}d\theta$.

The second argument is that when two black holes are combined and they merge, this process can provide energy even when it was impossible to extract energy from the two initial black holes separately. Just as two thermodynamic systems in equilibrium can perform work when they are combined. This can be seen as follows, when the two black holes with initial masses M_1 and M_2 merge to form a black hole of mass M, the only restriction on this process is that the area of the newly formed black hole is greater than the sum of the areas of the initial black holes. Since (2.11) tells us that $A \propto M^2$, this restriction implies for the masses that

$$M^2 \geq M_1^2 + M_2^2,$$

so when M lies in the range $M_1^2 + M_2^2 \leq M^2 < M_1^2 + M_2^2 + 2M_1M_2$ this means that $M < M_1 + M_2$ so that the energy that this process can generate is

$$E = M_1 + M_2 - M > 0.$$

Finally, we can derive an analog of the first law of thermodynamics in black hole mechanics using the results of the last section. Combining (2.11) with the definition of the irreducible mass (2.10) gives

$$A = 8\pi\Big(M^2 + \sqrt{M^4 - J^2}\Big).$$

Varying this and solving it for δM gives

$$\delta M = \frac{\sqrt{M^2 - a^2}}{8\pi(2M^2 + 2M\sqrt{M^2 - a^2})}\delta A + \frac{a}{2M^2 + 2M\sqrt{M^2 - a^2}}\delta J. \tag{2.12}$$

This can be simplified by defining the so-called surface gravity κ

$$\kappa \equiv \frac{\sqrt{M^2 - a^2}}{2M^2 + 2M\sqrt{M^2 - a^2}}, \tag{2.13}$$

and rewriting expression (2.6) for Ω_H as

$$\Omega_H = \frac{a}{2M^2 + 2M\sqrt{M^2 - a^2}}.$$

Now, (2.12) can be recast as

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J. \tag{2.14}$$

This equation bears a remarkable resemblance to the first laws of thermodynamics

$$\delta U = T\delta S + \text{work terms}, \tag{2.15}$$

especially when you realize that we had already identified M and A as analogs of U and S respectively.

You may object that we have derived (2.14) from the expression for the area of a specific case, namely the Kerr black hole and thus it can hardly be called a law, but in [6] you can find a proof based on general arguments that holds for any black hole.

The zeroth law of thermodynamics states that a system in thermal equilibrium has a constant temperature. Comparison of the first law of thermodynamics and (2.14) suggests that this means that κ should be constant for a black hole in equilibrium, i.e. a stationary black hole. Since a stationary black hole has constant M and a, the expression for κ (2.13) tells us directly that this is the case. Again we refer to [6] for a proof.

Finally, the third law of thermodynamics (it is impossible to achieve $T = 0$) implies that it should be impossible to achieve $\kappa = 0$. But in the case that $M = a$ the surface gravity does become zero. Remember that in section 1.3 we mentioned this case, but concluded that is was highly unstable. Moreover, we also discussed the case for which $M < a$, but quickly got rid of it, since it contains a naked singularity and would thus be unphysical. On the other hand, κ becomes imaginairy is this case, which also seems quite unphysical. So although, there is as yet no proof of a third law of black hole mechanics it does seem to have something to do with cosmic censorship; the statement that every singularity in spacetime should be shielded from our view by an event horizon.

2.4 Information Theory

In the previous section we found an analogy between black hole mechanics and thermodynamics merely based on a resemblance in the appearance of the equations. In this section we will discuss the possibility of a profound meaning of this resemblance. For this purpose we shall first discuss entropy from the point of view of information theory (we follow the line of reasoning of [4]).

Consider a system that can be in a number of different states and we know that upon measurement it will be found in the n^{th} state with a probability p_n. Its entropy, the measure of one's uncertainty or lack of information about the actual state the system is in, is then given by

$$S = \Sigma_n p_n \ln p_n .$$

We see that once we know exactly in which state the system is, i.e. all p_n are zero except one which equals unity, the entropy is zero. Furthermore, it can be shown that obtaining information about the system, which imposes constraints on the p_n, always leads to a decrease in entropy. As an example, suppose you obtain the information that the system is actually not in state i, then p_i is zero, so the entropy decreases. So there is a direct relation between information and entropy which can formally be expressed as

$$\Delta I = -\Delta S,$$

so an increase in information ΔI corresponds to a decrease in entropy ΔS.

These equations for I and S hold for a wide range of systems whose state is not exactly known. We can look at a black hole as such a system. An observer in the exterior of a black hole can completely describe it by its mass, angular moment and charge. However, black holes can form in a number of different ways, so black holes characterized by the same values for M, J and Q may have different histories. The fact that an external observer has no information whatsoever about these histories, constitutes a lack of information about the black hole state. Consequently, a black hole has an entropy S_{bh}. Note that we derived the fact that black holes may have non-zero entropy from information theory, we ignored the fact that classically black holes have just one internal state.

The second law of thermodynamics, stating that the entropy of a system out of equilibrium increases, can be interpreted as the information of the initial state of the system being washed out by the thermal evolution of the system. Now, let us assume that the black hole entropy is a function of the black hole area, since this seems natural in view of what we found in the previous section. We know that a black hole after its formation is believed to settle down quickly to a state completely determined by three parameters. The loss of information about the initial state suggests that S_{bh} should increase in this process. Because of the area theorem this is what we find, since S_{bh} is a function of A. So in this context the analogy between the second law of thermodynamics and the second law of black hole mechanics seems to be more than just a formal resemblance[4].

Another hint in this direction is that, as we mentioned, an increase in information about a system is accompanied by a decrease in its entropy. Of a black hole, however, it is by definition impossible to obtain information and this is very well represented by the fact that, because of the area theorem, a decrease in S_{bh} is also by definition impossible.

Finally, we want to mention the generalized second law. Although we can decrease the entropy of a system by inquiring information about it, it can be shown that this always leads to an increase in the entropy of the rest of the universe that exceeds the amount of newly obtained information. So the total entropy of the universe never decreases. Now, what happens when some body with an unknown internal state drops into a black hole? This makes the entropy of the visible universe go down, and without a definition of black hole entropy an external observer would be unable to verify that the entropy of the whole universe in fact does not decrease. So the second law must be generalized, such that it says that the entropy of the black hole exterior plus the black hole entropy itself do not decrease. To check this note that when the body drops into the black hole all the information about the body is lost to the external observer. This means that if there was some information available about the body beforehand, this information is now lost to the external observer and added to the black hole entropy together with the original entropy of the body. So the increase in black hole entropy will usually not only equal but actually exceed the decrease in the entropy of the rest of the universe, so

$$\Delta(S_{\mathrm{ext}} + S_{\mathrm{bh}}) > 0.$$

2.5 Inconsistency in Analogy

At the time Beckenstein proposed to take the analogy more seriously based on the arguments we discussed in the previous section, this proposal was swept aside by a crucial inconsistency in the analogy. In thermodynamics systems with a certain temperature are known to radiate with the characteristic Planck spectrum. By definition this radiation is absent for black holes; nothing can

[4]Beckenstein shows in his article [4] that based on arguments from information theory one may even conclude that the black hole entropy is not just a function of the black hole area but is in fact proportional to it.

come out of them, let alone black body radiation! But physicists in favor of the analogy argued that when quantum mechanics was taken into consideration, things might be different.

Steven Hawking was one of the people who was bothered by the superficiality of the arguments in favor of the analogy and the lack of any physical significance (see [7]). He set out to prove that even when quantum mechanics was taken into account there was no such thing as black hole radiation [8]. In the next chapter we will discuss how he approached this and the remarkable results he obtained.

Chapter 3

Hawking Radiation

3.1 Semi-Classical Approach

General relativity tells us that the presence of masses causes spacetime to curve and the metric of a spacetime with a certain energy and mass distribution can be obtained from the Einstein equation. The Schwarzschild metric, for example, describes spacetimes outside spherically symmetric objects centered around $r = 0$, such as the earth. At sufficiently large distances from the center, however, the curvature caused by the massive object sitting at $r = 0$ becomes so weak that is can be neglected. This can be seen by letting $r \to \infty$ in the Schwarzschild metric (1.15), in that case it reduces to the Minkowski metric which describes flat spacetime

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.1}$$

The radius of the earth is proportional to 10^9 times its mass, so at the surface of the earth we find ourselves in a regime where the curvature on small scales is totally negligible. This justifies the fact that in quantum field theory one uses the Minkowski metric instead of the Schwarzschild metric. On earth quantum phenomena are not influenced by gravity, in fact, there seem to be only two cases in which the spacetime curvature is strong enough to seriously influence physics on quantum level. These are the beginning of the universe, where the metric of spacetime is believed to change drastically on very short time scales and in the vicinity of black holes, where the curvature is so high that it cannot be neglected, not even on the scale of particle interactions. This last case is of course the one we are concerned with in this thesis.

In the vicinity of a black hole the description of quantum phenomena requires a full theory of quantum mechanics in which gravity is also incorporated. Unfortunately, as the reader may know, the normal procedure to quantize a field theory does not work in the case of gravity. What makes gravity so different from the other forces of nature is the dual role the metric $g_{\mu\nu}$ plays, on the one hand it is the field that describes the gravitational interaction between particles but on the other hand it is the metric that gives the structure of spacetime. Put differently, one may say that the gravitational field self-interacts, i.e. gravitons exert gravitational forces on each other. So, whereas, for example, photons do not feel the electromagnetic force, gravitons do feel gravity. Roughly speaking it is this major difference that makes it impossible to renormalize gravity, whereas this procedure does work for the other forces of nature. To read more on this subject, see for example [9] or [2].

It may seem that in the absence of a full theory of quantum gravity we are unable to investigate the effect of the presence of a black hole on quantum physics. Fortunately, history provides us with a possibility to circumvent this problem. At the time a full theory of quantum electrodynamics did not yet exist, physicists were able to calculate spontaneous creation of electron-positron pairs by treating their electromagnetic field as a classical background field. In retrospect, we can say that the results obtained in this way are very reliable. This suggests that a similar approximation can be made to obtain a semi-classical description of gravity. More specifically, we could treat gravitation as a classical background field for quantized matter. In terms of formulae this means that the matter fields still solve the usual wave equations, but with the Minkowski metric replaced by the metric that correctly describes the spacetime curvature, i.e. the solution $g_{\mu\nu}$ of the Einstein equation. But now we also have to adjust the Einstein equation, because the energy-momentum tensor of the matter fields is now an operator, while the metric and the Ricci scalar are not. The most natural adjustment is to replace this operator by its expectation value. The Einstein equation then becomes

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi < \hat{T}_{\mu\nu} > . \tag{3.2}$$

Upon more detailed inspection of this formulation of a semi-classical approximation of gravity, we encounter a number of problems. First of all, there seems to be no way around quantizing gravity, which can be seen as follows. Imagine the matter distribution is such that all the matter is found either in region A or in region B. The expectation value of the energy-momentum tensor is then as if half of the matter is in region A and the other half in region B. And the spacetime is curved accordingly. If we now do a measurement and find all the matter to be in region A, this causes the metric to change discontinuously into one that satisfies (3.2) with all the matter in region A. Of course this does not make sense.

A second problem is the so-called backreaction problem. Whenever the energy distribution of the matter fields changes this induces a change in the metric according to (3.2). A change in the metric, however, changes the wave equation which the matter fields must obey, so this change is likely to cause another change in the energy-momentum tensor, and so on. So once something changes you seem to enter a vicious cycle, caused by the coupling between the wave equation and the Einstein equation.

Last but not least, there is a practical obstacle. As mentioned in chapter 1 about general relativity it is very difficult to solve the Einstein equation, and analytic solutions can only be found for very special values of the energy-momentum tensor. In fact, black hole solutions are known only for $T_{\mu\nu} = 0$. So it is practically impossible to use (3.2)! Surprisingly enough, this last problem also suggests what we should do now: instead of replacing the Minkowski metric by the metric that solves (3.2), we should replace it by the Schwarzschild metric that describes the empty spacetime around the black hole. Of course this is yet another approximation, so we would not blame you if you are a bit suspicious of its validity[1]. But this is the approximation Hawking used in 1975 when he set out to prove that black holes do not radiate. The very remarkable result he obtained, namely that black hole do radiate, makes it worthwhile to carry on and discuss his calculation [8] in detail (see also [9] and [10]). We will try to stay close to the notation and arguments used in the original article.

In the next section we will very quickly resume quantum field theory for massless particles, to continue in section 3.3 with a discussion of quantum field theory in a curved spacetime. Then we should be fully equipped to carry out Hawking's calculation, which shall be done in the last two sections. The consequences of the result will be discussed in chapter 4 and in chapter 5 we will return to the discussion of the validity of the semi-classical approximation and also discuss other arguments Hawking uses that might be doubted.

3.2 Quantized Klein-Gordon Field

In this section we will give a very short summary of quantum field theory in flat spacetime. We will restrict ourselves to scalar fields. A single scalar wavefunction $\phi(x)$ obeys the Klein-Gordon equation

$$(\partial^\mu\partial_\mu - m^2)\phi(x) = 0, \tag{3.3}$$

where m is the mass of the particle described by the field and $\partial^\mu\partial_\mu = \eta^{\mu\nu}\partial_\nu\partial_\mu$ with $\eta^{\mu\nu}$ the Minkowski metric[2]. The Klein-Gordon equation can be derived from the following Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\alpha\phi\partial^\alpha\phi + m^2\phi^2) \tag{3.4}$$

using the action principle. From this Lagrangian we see that the field conjugate is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}(x). \tag{3.5}$$

Imposing the usual canonical commutation relations to quantize the field, leads to the following relations

$$\begin{aligned} [\phi(\vec{x},t), \dot{\phi}(\vec{x}',t)] &= \imath\delta(\vec{x}-\vec{x}') \\ [\phi(\vec{x},t), \phi(\vec{x}',t)] &= [\dot{\phi}(\vec{x},t), \dot{\phi}(\vec{x}',t)] = 0. \end{aligned} \tag{3.6}$$

The field can be decomposed in its Fourier components

$$\phi(x) = \Sigma_i[a_i f_i(x) + a_i^\dagger f_i^*(x)], \tag{3.7}$$

where $\{f_i\}$ is a complete orthonormal set of solutions to the Klein-Gordon equation with positive frequencies, i.e.

$$(f_i, f_j) \equiv \imath\int_t d^3x[f_i^*\partial_t f_j - f_j\partial_t f_i^*] = \delta_{ij}, \tag{3.8}$$

[1] An extensive discussion on this subject follows in chapter 5.
[2] Remember that we use the sign convention $(-+++)$ for spacetime metrics.

where t denotes a spacelike hyperplane of simultaneity, and

$$\partial_t f_i = -\imath\omega f_i \quad \text{with } \omega > 0. \tag{3.9}$$

Using (3.6), one easily finds the commutation relations for the operators a_i and $a_i^\dagger$

$$\begin{aligned} [a_i, a_j^\dagger] &= \delta_{ij} \\ [a_i, a_j] &= [a_i^\dagger, a_j^\dagger] = 0. \end{aligned} \tag{3.10}$$

Since these are precisely the commutation relations for the annihilation and creation operators of the harmonic oscillator, we will use the same interpretation for the Klein-Gordon field. We define the vacuum state as

$$a_i|0> = 0 \quad \forall i. \tag{3.11}$$

From this state all the other states that span the Hilbert space can be constructed by acting on it with the creation operators. In this representation the Hilbert space is called a Fock space. The physical significance of the vacuum state becomes clear when we define the Hamiltonian[3]

$$H = \Sigma_i a_i^\dagger a_i \omega \tag{3.12}$$

and calculate its expectation value for the vacuum state

$$< 0|H|0> = < 0|\Sigma_i a_i^\dagger a_i \omega|0> = 0.$$

This tells us that the vacuum state is the state of lowest energy. Since we would like this to correspond to a state without any particles in it, we define the number operator N_i and the total number operator N as

$$N_i = a_i^\dagger a_i \quad \text{and} \tag{3.13}$$

$$N = \Sigma_i a_i^\dagger a_i. \tag{3.14}$$

3.3 Klein-Gordon Field in Curved Spacetime

In curved spacetime the covariant form of the Lagrangian for the Klein-Gordon field is

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2]. \tag{3.15}$$

[3]This expression for the Hamiltonian can be obtained by integrating over the Hamiltonian density T_{00} and normal ordering the result. The energy-momentum tensor can be obtained by variation of the action with respect to the metric

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g(x)}}\frac{\delta S}{\delta g^{\mu\nu}(x)}.$$

For the Klein-Gordon action this leads to

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{2}m^2\phi^2\eta_{\mu\nu}.$$

This yields the Klein-Gordon equation for curved spacetime

$$\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi(x)) - m^2\phi(x) = 0, \tag{3.16}$$

which can be shown to equal:

$$(D^\mu\partial_\mu - m^2)\phi(x) = 0, \tag{3.17}$$

where D_μ is the covariant derivative (1.13).

In the previous section we saw that when we wrote the field on a basis of eigenfunctions of ∂_t, the interpretation of the operators a_i and $a_i^\dagger$ as annihilation and creation operators followed in a very natural way. And, thus, we were able to construct a Fock space and a state of lowest energy. Note that it is essential that ∂_t is a Killing vector in Minkowski spacetime, i.e. $\partial_t\eta^{\mu\nu} = 0$, because this enabled us to define the positive frequency condition (3.9) globally. From this followed globally defined creation and annihilation operators and a global vacuum state, which are all coordinate independent. This is very important since it guarantees us that physics is independent of the coordinates we choose, which is the cornerstone of general relativity. In a general curved spacetime $g^{\mu\nu}$ there is no indication at all that there exists a Killing vector, which means that we have no way to choose a natural basis for the field. In other words, it is impossible to define a global positive frequency condition in a general curved spacetime. This ambiguity in the choice of a basis gives rise to difficulties with defining a vacuum state and the interpretation of the concept of a particle. We can, however, generalize the Hermitian inner product (3.8) to curved spacetime

$$(\psi_i, \psi_j) = \int_\Sigma d\Sigma^\mu \sqrt{g_\Sigma}[\psi_i^* D_\mu\psi_j - \psi_j D_\mu\psi_i^*], \tag{3.18}$$

where Σ is a Cauchy surface and $d\Sigma_\mu = n_\mu d\Sigma$ with n_μ a vector normal to this surface pointing in the positive time direction. We can also introduce a complex basis $\{\psi_i\}$ of solutions to the Klein-Gordon equation such that

$$\begin{aligned}(\psi_i, \psi_j) &= -(\psi_i^*, \psi_j^*) = \delta_{ij}\\ (\psi_i^*, \psi_j) &= (\psi_i, \psi_j^*) = 0.\end{aligned} \tag{3.19}$$

Since (3.19) does not uniquely define the basis, we can also choose an other basis $\{\psi_i'\}$ that satisfies (3.19). Since $\{\psi_i\}$ and $\{\psi_i'\}$ are both orthogonal bases that span the entire Hilbert space, we can write

$$\psi_i' = \Sigma_j[A_{ij}\psi_j + B_{ij}\psi_j^*]. \tag{3.20}$$

When we insert this into (3.19) we find the following relations for the transformation matrices A and B

$$\begin{aligned}AA^\dagger - BB^\dagger &= 1,\\ AB^T - BA^T &= 0.\end{aligned} \tag{3.21}$$

To obtain the inverse of (3.20) we write

$$\begin{aligned}\psi_i' &= \Sigma_j[A_{ij}\psi_j + B_{ij}\psi_j^*]\\ &= \Sigma_j[A_{ij}\Sigma_k(A'_{jk}\psi'_k + B'_{jk}\psi'^*_k) + B_{ij}\Sigma_k(A'^*_{jk}\psi'^*_k + B'^*_{jk}\psi'_k)]\\ &= \Sigma_{j,k}[(A_{ij}A'_{jk} + B_{ij}B'^*_{jk})\psi'_k + (A_{ij}B'_{jk} + B_{ij}A'^*_{jk})\psi'^*_k].\end{aligned} \tag{3.22}$$

From this it follows that

$$\begin{aligned} AA' + BB'^* &= 1, \\ AB' + BA'^* &= 0. \end{aligned}$$

Comparing with (3.21) shows us that these are uniquely satisfied when $A' = A^\dagger$ and $B' = -B^T$. So now we find two additional equations for the matrices A and B, when we demand that A' and B' also satisfy (3.21), namely

$$\begin{aligned} A^\dagger A - B^T B^* &= 1, \\ A^\dagger B - B^T A^* &= 0. \end{aligned} \tag{3.23}$$

As was pointed out in the beginning of this section, there is no preferred basis of solutions in a general spacetime because of the lack of a Killing vector. But a stationary spacetime, i.e. $\partial_t g^{\mu\nu} = 0$, does have a Killing vector. So in a stationary spacetime a natural choice for the basis of the field is the set of eigenfunctions $\{f_i\}$ of this Killing vector with positive frequencies

$$\partial_t f_i = -\imath\omega_i f_i \quad \text{with} \quad \omega_i \geq 0. \tag{3.24}$$

In correspondence with (3.7) we write the general real quantum field satisfying (3.17) as

$$\phi(x) = \Sigma_i [a_i f_i(x) + a_i^\dagger f_i^*(x)], \tag{3.25}$$

and from (3.19) it follows again that the operators a_i satisfy the usual commutation relations for creation and annihilation operators, i.e. (3.10). So as in section 3.2 we can define a vacuum state and a number operator.

The spacetime around a black hole is stationary when the black hole is stationary, so in that case we can define a vacuum state and a unique basis for the solution of the Klein-Gordon equation (3.17). However, when a star collapses to form a black hole the spacetime around it is not stationary, so we cannot define a vacuum state valid in the entire spacetime. Nevertheless, we can say something about the vacuum state at early and late times when we notice that the spacetime of a collapsing body can be split into three regions. The first region is the spacetime at early times, before the collapse, in which there is some configuration of the matter that will eventually form the black hole. This first region is stationary, so there is a natural choice for a vacuum state and a number operator. The second region is the spacetime during the collapse, this is clearly non-stationary. When the event horizon forms the spacetime is known to settle down fast to a stationary state again, which is the third region. Although the first and third region are both stationary, they are clearly not equal so the functions that solve the Klein-Gordon equation in the first region will not be solutions of this equation in the third region. This means that the Fock spaces constructed from the positive frequency eigenfunctions of the first and the third region respectively will be different. In the next section we will discuss this more explicitly and see how this can lead to particle creation.

3.4 Particle Creation in Gravitational Collapse

We know from quantum mechanics in flat spacetime that when a wave of positive frequency e^{ikx} hits a potential barrier, it will be partially reflected and partially transmitted to become of the form $Re^{-ikx}+Te^{ikx}$. As we will see in this section, this reasoning can be extended to curved spacetime in the following way: If a spacetime consists of an initial stationary region I, a non stationary region II and a final stationary region III, then region II can be compared with the potential barrier in the example above and so we should expect a state containing only positive frequencies in the first region to develop into a state containing both positive as well as negative frequencies in the third region. In particular, this means that the vacuum state defined as containing no particles in the first region actually does contain particles in the third region.

Let us for simplicity consider a massless[4] scalar field Φ satisfying the Klein-Gordon equation, i.e. $D^\mu\partial_\mu\Phi = 0$, and investigate its properties in the spacetime of a collapsing body. As was mentioned in the former section, the regions of spacetime before and after the collapse of a star into a black hole are stationary. So before the collapse Φ can be expressed on a complete orthonormal basis $\{f_i\}$ containing only positive frequencies *with respect to the Killing vector at past null infinity* $\mathcal{I}^-$

$$\Phi(x) = \Sigma_j[a_j f_j(x) + a_j^\dagger f_j^*(x)]. \tag{3.26}$$

From section 1.3 we know that in gravitational collapse an event horizon is formed, so all the modes of the field must either pass the event horizon or escape to infinity. Thus at late times the field Φ is completely determined by its data on the event horizon $\mathcal{H}$ and future null infinity $\mathcal{I}^+$

$$\Phi(x) = \Sigma_i[b_i p_i(x) + b_i^\dagger p_i^*(x) + c_i q_i(x) + c_i^\dagger q_i^*(x)], \tag{3.27}$$

where the set $\{p_i\}$ is chosen such that it contains only outgoing modes (that escape to $\mathcal{I}^+$) with positive frequencies *with respect to* $\mathcal{I}^+$, whereas the set $\{q_i\}$ contains only ingoing modes (that disappear into the black hole). It is not clear with respect to what the positive frequency condition on the $\{q_i\}$ should be taken. The timelike Killing vector changes from timelike to spacelike at the horizon, which means that the positive frequency modes outside the black hole are negative frequency modes inside the black hole. This makes it impossible to define positive frequency uniquely for the $\{q_i\}$. Fortunately, the results do not depend on this ambiguity in the $\{q_i\}$. This will be shown below.

Let us first explore how the bases of the initial and final wavefunction are related. The fact that $\{f_i\}$, $\{p_i\}$ and $\{q_i\}$ all satisfy the orthonormality condition (3.19), enables us to write $\{p_i\}$ and $\{q_i\}$ as linear combinations of $\{f_i\}$ and $\{f_i^*\}$

$$\begin{aligned} p_i &= \Sigma_j[A_{ij} f_j + B_{ij} f_j^*], \\ q_i &= \Sigma_j[C_{ij} f_j + D_{ij} f_j^*]. \end{aligned} \tag{3.28}$$

The matrices obey the conditions (3.21) and (3.23). To obtain the relations between the different annihilation and creation operators we substitute (3.28) into (3.27) which gives us the following expression for the annihilation operator at $\mathcal{I}_-$

$$a_j = \Sigma_i(A_{ij} b_i + B_{ij}^* b_i^\dagger + C_{ij} c_i + D_{ij}^* c_i^\dagger). \tag{3.29}$$

[4]See [8] for this calculation including massive and charged scalar fields.

To obtain b_i and c_i in terms of a_i and $a_i^\dagger$ we write $b_i = \Sigma_k[A'_{ik}a_k + B'_{ik}a_k^\dagger]$ and $c_i = \Sigma_k[C'_{ik}a_k + D'_{ik}a_k^\dagger]$ and plug this into (3.29). This can be worked out just as we did in (3.22) and if we then use the relations (3.21) and (3.23) for the transformation matrices, we find the following relations

$$\begin{aligned} b_j &= \Sigma_i[A^*_{ji}a_i - B^*_{ji}a_i^\dagger], \\ c_j &= \Sigma_i[C^*_{ji}a_i - D^*_{ji}a_i^\dagger]. \end{aligned} \tag{3.30}$$

These are called Bogoliubov transformations. Note that imposing the usual commutation relations for annihilation and creation operators on b_i, $b_i^\dagger$ and c_i, $c_i^\dagger$, yields again the conditions (3.21) and (3.23) for the transformation matrices. In other words, these are automatically obeyed.

Measurements at future null infinity are not affected by the fact that we did not impose the positive frequency condition on the $\{q_i\}$. We see this when we write a final state as

$$|\text{final} >= \Sigma_{i,j}\big[\lambda_{ij}|\chi_i >_{\mathcal{I}_+} |\eta_j >_{\mathcal{H}}\big]. \tag{3.31}$$

From which it follows that observables $\mathcal{O}$ at $\mathcal{I}_+$, which act only on the states $|\chi_i >_{\mathcal{I}_+}$, give

$$\begin{aligned} < \text{final}|\mathcal{O}_{\mathcal{I}_+}|\text{final} > &= \Sigma_{i,j}\big[\lambda^*_{ij} < \chi_i|_{\mathcal{I}_+} < \eta_j|_{\mathcal{H}}\big]\mathcal{O}_{\mathcal{I}_+}\Sigma_{k,l}\big[\lambda_{kl}|\chi_k >_{\mathcal{I}_+} |\eta_l >_{\mathcal{H}}\big] \\ &= \Sigma_{i,j,k}\big[\lambda^*_{ij}\lambda_{kj} < \chi_i|_{\mathcal{I}_+}\mathcal{O}_{\mathcal{I}_+}|\chi_k >_{\mathcal{I}_+}\big] \\ &= \mathrm{Tr}(\rho\mathcal{O}_{\mathcal{I}_+}) \end{aligned} \tag{3.32}$$

with ρ the density matrix

$$\rho_{ik} = \Sigma_j\big[\lambda^*_{ij}\lambda_{kj}|\chi_k >_{\mathcal{I}_+} < \chi_i|_{\mathcal{I}_+}\big]. \tag{3.33}$$

So all we need to do now is prove that ρ does not depend on our choice for the $\{q_i\}$. When we change the definition of positive frequencies at the horizon, this means that we do a Bogoliubov transformation on the creation and annihilation operators associated with the states $|\eta_j >_{\mathcal{H}}$. But in the last section we made sure that these transformations leave the orthonormality conditions (3.19) invariant, from which it follows that the states transform as $|\eta_j >= \Sigma_k U_{jk}|\tilde{\eta}_k >$ with U unitary, i.e. $UU^\dagger = 1$. So the final state (3.31) becomes

$$|\text{final} >= \Sigma_{i,k}\big[\tilde{\lambda}_{ik}|\chi_i >_{\mathcal{I}_+} |\tilde{\eta}_k >_{\mathcal{H}}\big],$$

with $\tilde{\lambda} = \lambda U$. And the density matrix becomes

$$\rho_{ik} = \Sigma_j\big[\tilde{\lambda}^*_{ij}\tilde{\lambda}_{kj}|\chi_k >_{\mathcal{I}_+} < \chi_i|_{\mathcal{I}_+}\big],$$

but writing this out in terms of the original coefficients λ shows us that the transformation leaves the density matrix unchanged

$$\begin{aligned} \Sigma_j\tilde{\lambda}^*_{ij}\tilde{\lambda}_{kj} &= \tilde{\lambda}\tilde{\lambda}^\dagger \\ &= (\lambda U)(\lambda U)^\dagger = \lambda UU^\dagger\lambda^\dagger \\ &= \lambda\lambda^\dagger. \end{aligned}$$

So observables at future null infinity are indeed independent of ambiguities in the definition of the $\{q_i\}$.

Now, we can finally show that particles are created in the process of gravitational collapse. Let us start with a vacuum state on past null infinity, i.e. $a_i|0_- >= 0$ for all i and then measure the number of particles in the j-th mode in this state at future null infinity

$$
\begin{aligned}
< 0_-|N(\mathcal{I}_+)_j|0_- > &= < 0_-|b_j^\dagger b_j|0_- > \\
&= < 0_-| - \Sigma_{i,k}[B_{ji}a_i - A_{ji}a_i^\dagger][A_{jk}^* a_k - B_{jk}^* a_k^\dagger]|0_- > \\
&= \Sigma_{i,k} B_{ji} B_{jk}^* < 0_-|a_i a_k^\dagger|0_- > \\
&= \Sigma_{i,k} B_{ji} B_{jk}^* < 0_-|\delta_{ki} - a_k^\dagger a_i|0_- > \\
&= \Sigma_i B_{ji} B_{ij}^\dagger \\
&= \Sigma_i |B_{ji}|^2. \qquad (3.34)
\end{aligned}
$$

Since B, in general, will not be zero, we find that the state containing no particles before the collapse does contain particles afterwards. The energy to create these particles can only come from the gravitational field. So we conclude that in the process of gravitational collapse particles are being created by the interaction of the field Φ with the curvature of spacetime. Note that this can happen in any spacetime with a non-stationary region, not only in gravitational collapse.

3.5 Hawking Radiation

During the process of gravitational collapse particles are created which escape to $\mathcal{I}_+$. In this section we will show that the number of particles created in the collapse is bounded only by the amount of energy containded in the gravitational field and that the character of the particle flux is thermal. To see this we will calculate an asymptotic form of the Bogoliubov coefficients to obtain a specific expression for the particle flux at future null infinity. This calculation will be easier when we decompose the solutions of the wave equation into their Fourier components with respect to advanced time for ingoing solutions and with respect to retarded time for outgoing solutions. General solutions of the massless scalar wave equation are (see Appendix A)

$$r^{-1} R_{\omega l}(r) Y_{lm}(\theta, \phi) e^{-\imath\omega t} \qquad (3.35)$$

with Y_{lm} the spherical harmonics and $R_{\omega l}(r)$ obeying the differential equation

$$\frac{d^2 R_{\omega l}}{dr^{*2}} + \{\omega^2 - [l(l+1)r^{-2} + 2Mr^{-3}][1 - 2Mr^{-1}]\} R_{\omega l} = 0, \qquad (3.36)$$

with $r^* = r + 2M \ln|\frac{r}{2M} - 1|$. When $r \to \infty$, i.e. on $\mathcal{I}_-$ and $\mathcal{I}_+$, (3.36) reduces to

$$\frac{d^2 R_{\omega l}}{dr^{*2}} + \omega^2 R_{\omega l} = 0 \qquad (3.37)$$

$$\longrightarrow R_{\omega l} \sim e^{\pm\imath\omega r^*}. \qquad (3.38)$$

This enables us to write solutions on $\mathcal{I}_-$ and $\mathcal{I}_+$ as

$$r^{-1} Y_{lm}(\theta, \phi) e^{-\imath\omega t \pm \imath\omega r^*},$$

and when we use advanced and retarded time, i.e. $v = t + r^*$ and $u = t - r^*$, we obtain purely ingoing and outgoing solutions

$$f_\omega \sim (r\sqrt{4\pi\omega})^{-1} Y_{lm}(\theta,\phi) e^{-\imath\omega v}, \tag{3.39}$$
$$p_\omega \sim (r\sqrt{4\pi\omega})^{-1} Y_{lm}(\theta,\phi) e^{-\imath\omega u}. \tag{3.40}$$

Note that, unless we put the collapsing body into a box or an otherwise confined region, ω is a continuous parameter and $\{f_\omega\}$ and $\{p_\omega\}$ obey continuous normalization conditions

$$(p_\omega, p_{\omega'}) = (f_\omega, f_{\omega'}) = \delta(\omega - \omega').$$

This condition gives us the normalization factor $(4\pi\omega)^{-1/2}$, which can be seen by working out

$$(f_{\omega lm}, f_{\omega' l' m'}) = \int_\Sigma d\Sigma^\mu \sqrt{g_\Sigma}[f^*_{\omega lm}\partial_\mu f_{\omega' l' m'} - f_{\omega' l' m'}\partial_\mu f^*_{\omega lm}].$$

To do so, note that the f_ω of (3.39) solve the Klein-Gordon equation at past null infinity $\mathcal{I}_-$, where $v \simeq t + r$ and the metric is given by (1.15) so that $\sqrt{g_\Sigma} = (1-2M/r)^{-1/2} r^2 \sin\theta$ and $d\Sigma^\mu = dr d\phi d\theta \sqrt{-(1-2M/r)}$. Furthermore, we know that the spherical harmonics are normalized in the following way

$$\int_0^{2\pi}\int_0^\pi d\phi d\theta \sin\theta Y^*_{l'm'}(\theta,\phi) Y_{lm}(\theta,\phi) = \delta_{ll'}\delta_{mm'}.$$

With these ingredients we find

$$\begin{aligned}
(f_{\omega lm}, f_{\omega' l' m'}) &= \imath \int_r \int_0^{2\pi}\int_0^\pi dr d\phi d\theta \big[r^2 \sin\theta \frac{1}{r^2}\frac{1}{4\pi\sqrt{\omega\omega'}} Y^*_{l'm'} Y_{lm} \\
&\quad [e^{\imath\omega(t+r)}\partial_t e^{-\imath\omega'(t+r)} - e^{-\imath\omega'(t+r)}\partial_t e^{\imath\omega(t+r)}]\big] \\
&= \imath\delta_{ll'}\delta_{mm'}\int_r dr \frac{1}{4\pi\sqrt{\omega\omega'}}(-\imath\omega' - \imath\omega) e^{-\imath(\omega'-\omega)t} e^{-\imath(\omega'-\omega)r} \\
&= \delta_{ll'}\delta_{mm'}\frac{2\omega}{4\pi\omega} 2\pi\delta(\omega' - \omega) \\
&= \delta_{ll'}\delta_{mm'}\delta(\omega' - \omega),
\end{aligned}$$

where we used $\int dx e^{-\imath\omega x} = 2\pi\delta(\omega)$. The same obviously holds for (3.40).

To find the Bogoliubov coefficients we want to know what the wavefunction at past null infinity looks like that gives rise to a positive frequency, outgoing wavefunction at future null infinity, i.e. (3.40). To obtain the initial wavefunction we will trace (3.40) back in time. Since the retarded time u goes to infinity near the event horizon, we can use the geometrical optics approximation (see Appendix B) which says that a particle's world line is a null ray of constant phase ωu in good approximation. Let us trace the final wavefuntion back along one of the last null rays that came from past null infinity, traveled through the collapsing body and escaped to infinity. Let us call this null ray α. It will have a distance $-\epsilon$ to the null ray that generated the event horizon[5], which we will

[5] The null ray that generates the event horizon is the last to leave $\mathcal{I}_-$ and not to be captured by the black hole, so it will not hit the singularity, but neither will it be able to escape to $\mathcal{I}_+$. It is bound to stay at the fixed distance $r = 2M$ forever.

call γ (see Fig. 3.1). Here the distance is measured along a future directed null vector n^a, which is such that $n^a l_a = -1$, when l_a is a null vector tangent to γ. On the other hand, we see from the picture that n_a is tangent to ingoing null rays near the event horizon. It is useful to write the metric in terms of Kruskal coordinates (1.18), since in this form it does not become singular at $r = 2M$. When we set $d\Omega$ to zero, which can be done on grounds of spherical symmetry, the metric reads

$$ds^2 = \frac{2M}{r} e^{-r/2M} d\tilde{u} d\tilde{v}.$$

From this we see that the condition for a null ray: $\frac{ds^2}{d\lambda^2} = 0$ with λ the affine parameter, yields $\frac{d\tilde{u}}{d\lambda} = 0$ or $\frac{d\tilde{v}}{d\lambda} = 0$. So for ingoing null rays the affine parameter is proportional to $\tilde{u}$, i.e.

$$\lambda = C\tilde{u} = -4MCe^{-u/4M},$$

where C has no physical significance and as we will see it drops out in the final result. When we choose the affine parameter to be zero on the event horizon, i.e. where the ingoing null rays intersect γ, it follows that $\lambda = -\epsilon$ when the null rays intersect α

$$\lambda = -\epsilon \quad \rightarrow \quad u = 4M(\ln 4MC - \ln\epsilon). \tag{3.41}$$

So we have found the retarded time in terms of ϵ for the null ray α along which we are tracing back the wavefunction.

We can also express ϵ in terms of the advanced time v, when we continue to trace back the null ray past the endpoint of the event horizon and out to past null infinity. It will have left $\mathcal{I}_-$ just before v_0, which we mark as the advanced time at which the null ray that generated the event horizon left $\mathcal{I}_-$. And since we are using the geometrical optics approximation the distance between these rays will still be $-\epsilon$, where n_a is now tangent to $\mathcal{I}_-$. In this section of spacetime we should write the Schwarzschild metric in terms of the advanced and retarded time

$$ds^2 = -(1 - \frac{2M}{r}) du dv. \tag{3.42}$$

From this it follows that the affine parameter of $\mathcal{I}_-$ is proportional to v, so $-\epsilon = D(v - v_0)$ (again the constant D has no physical significance and will eventually drop out). When we put this into (3.41), we find an expression for u in terms of v. So the wavefunction that gives rise to a positive frequency, outgoing wavefunction at future null infinity has the following form in terms of the advanced time v

$$\begin{aligned} p_\omega(v) &\sim (\sqrt{4\pi\omega})^{-1} \exp(-\imath\omega 4M[\ln 4MC - \ln D(v_0 - v)]) \quad \text{for } v < v_0 \\ p_\omega(v) &= 0 \quad \text{for } v > v_0. \end{aligned} \tag{3.43}$$

In order to find the Bogoliubov coefficients, we need to decompose this result into positive and negative frequency modes with respect to $\mathcal{I}_-$. In other words, we need to write it on the basis $\{f_\omega\}$. This can be done by taking the Fourier transform of $p_\omega(v)$

$$\tilde{p}_\omega(\omega') = \frac{1}{2\pi} \int dv [p_\omega(v) e^{\imath\omega' v}]. \tag{3.44}$$

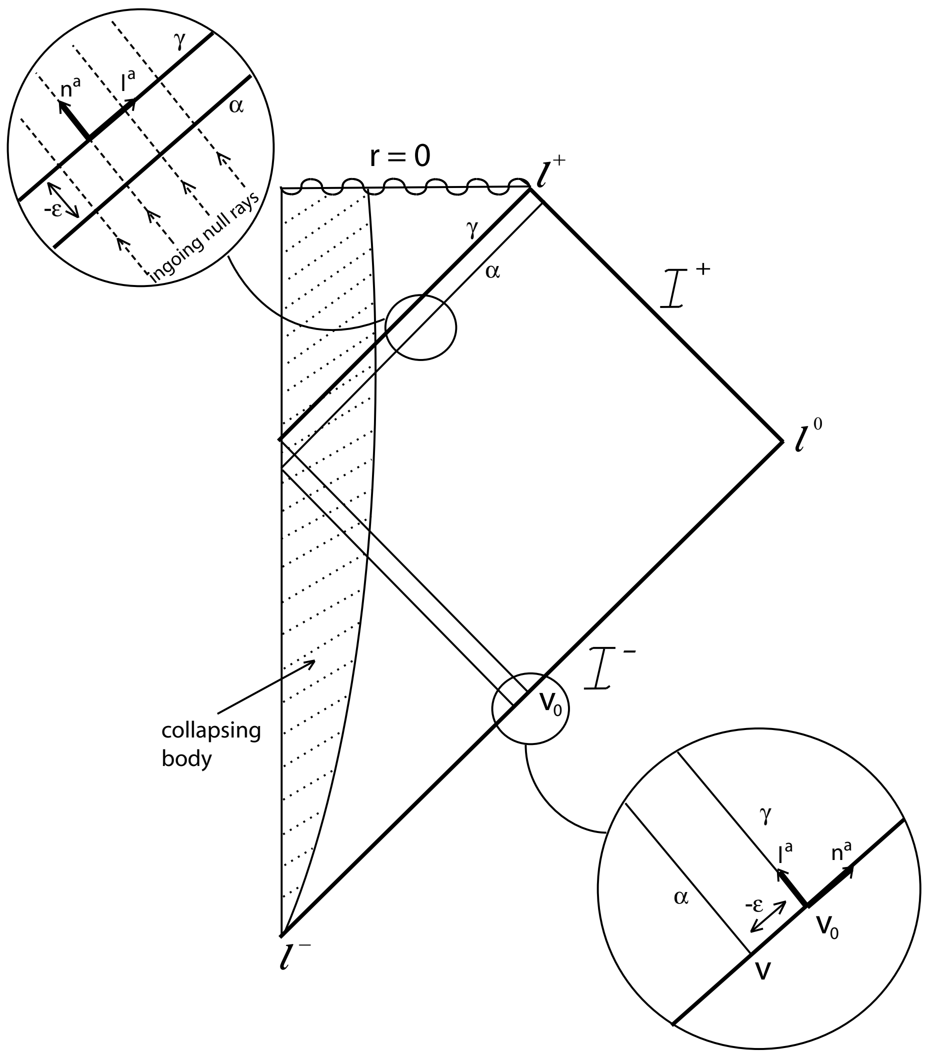

Figure 3.1: We trace back the null ray α that is at a constant distance $-\epsilon$ to the horizon generating null geodesic γ. The null vector l^a is tangent to γ.

Because then we find

$$p_\omega(v) \quad = \quad \int_0^\infty d\omega'[\sqrt{\pi\omega'}\tilde{p}_\omega(\omega')f_{\omega'}(v) + \sqrt{\pi\omega'}\tilde{p}_\omega(-\omega')f^*_{\omega'}(v)],$$

as is readily verified by plugging in (3.44) and (3.39)

$$\begin{aligned} p_\omega(v) &= \int_0^\infty d\omega'\Big[\sqrt{\pi\omega'}\frac{1}{2\pi}\int dv'[p_\omega(v')e^{\imath\omega' v'}]\frac{1}{\sqrt{4\pi\omega'}}e^{-\imath\omega' v} \\ &+ \sqrt{\pi\omega'}\frac{1}{2\pi}\int dv'[p_\omega(v')e^{-\imath\omega' v'}]\frac{1}{\sqrt{4\pi\omega'}}e^{\imath\omega' v}\Big] \\ &= \int dv' p_\omega(v')\int_0^\infty \Big[\frac{d\omega'}{2\pi}\frac{1}{2}e^{-\imath\omega'(v-v')} + \frac{1}{2}e^{-\imath\omega'(v'-v)}\Big] \\ &= \int dv' p_\omega(v')\Big[\frac{1}{2}\delta(v-v') + \frac{1}{2}\delta(v'-v)\Big] \\ &= p_\omega(v) \end{aligned}$$

On the other hand we know that

$$p_\omega \quad = \quad \int_0^\infty d\omega'[A_{\omega\omega'}f_{\omega'} + B_{\omega\omega'}f^*_{\omega'}],$$

so the Bogoliubov coefficients are just the Fourier transforms of (3.43) multiplied by $\sqrt{\pi\omega'}$

$$\begin{aligned} A_{\omega\omega'} &= \sqrt{\pi\omega'}\tilde{p}_\omega(\omega'), \\ B_{\omega\omega'} &= \sqrt{\pi\omega'}\tilde{p}_\omega(-\omega'). \end{aligned} \tag{3.45}$$

Before we continue this calculation it is important to note two things. First of all, we used the geometrical optics approximation which required that $u \gg 1$, so this result is valid only when we stay near the event horizon. But the particles that left past null infinity at advanced times long before v_0 travel through the collapsing body long before the event horizon forms and will be far away from the black hole at the moment it collapses so we may assume that their contribution to the Bogoliubov coefficients is trivial. Secondly, in the geometrical optics approximation one assumes that the wavefunction can be traced back along a null ray, this is only the case if the wavefunction is well enough localized, so we actually should have used a wave packet sharply peaked around a certain frequency. It can be shown that this leads to the same results (see [8]). Moreover, it can be shown that the particle flux is independent of the details of the collapse, by showing that for wave packets that reach $\mathcal{I}_+$, the Bogoliubov coefficients are indeed, as we have assumed, governed by their asymptotic forms (see also [8]).

Let us now calculate the Fourier transform of (3.43):

$$\begin{aligned} \tilde{p}_\omega(\omega') &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\imath\omega' v}p_\omega(v)dv \\ &\sim \frac{1}{4\pi\sqrt{\pi\omega}}\int_{-\infty}^{v_0} \exp(\imath\omega' v - \imath\omega 4M[\ln\alpha - \ln(v_0 - v)])dv \end{aligned} \tag{3.46}$$

$$\text{with} \qquad \alpha = \frac{4MC}{D}$$

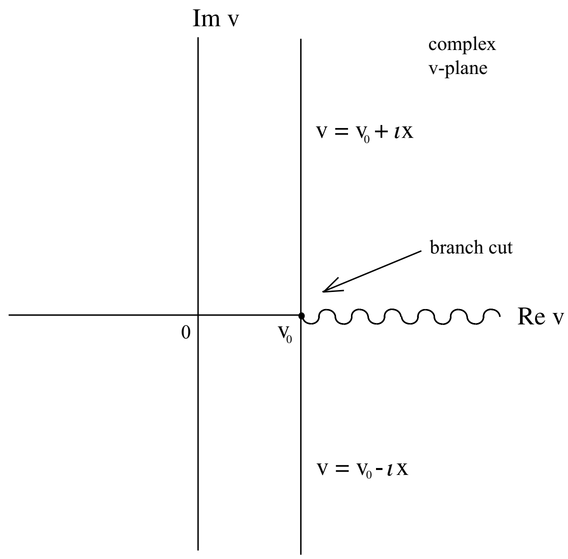

Figure 3.2: The integrand of (3.46) is proportional to $e^{\imath\omega' v+\imath\ln(v_0-v)} = e^{\imath\omega'(y+\imath x)+\imath\ln(v_0-(y+\imath x))}$. There is a branch cut at $y = v_0$, since the integrand vanishes for $y > v_0$. Since there are no singular points the residue theorem tells us that $\oint dve^{\imath\omega' v+\imath\ln(v_0-v)} = 0$. To determine how the contour should be closed we must investigate the asymptotic behavior of the integrand. In the limit that y goes to minus infinity the integrand becomes an extremely high oscillating function. The integral over such a function vanishes. For $\omega' > 0$ the integrand vanishes when x becomes infinite, so we should close the contour in the upper half of the plane to get a zero contribution of the part of the contour that lies at infinity. The same reasoning learns us that for $\omega' < 0$ the contour should be closed in the lower half of the plane.

We can solve this by contour integration in the complex v-plane (see Fig. 3.2), we have to pay attention on how to close the contour, though. For $\omega' > 0$ the contour has to be closed in the upper half of the complex plane, for $\omega' < 0$ in the lower half, because then the contribution of the part of the contour that lies at infinity vanishes, what remains is the integration along the line $v = v_0 \pm \imath x$. Thus we get for $\omega' > 0$

$$\begin{aligned}
\tilde{p}_\omega(\omega') &\sim \frac{\imath}{4\pi\sqrt{\pi\omega}}\int_0^\infty \exp(-\omega' x + \imath\omega' v_0 - \imath\omega 4M[\ln\alpha - \ln(xe^{-\imath\pi/2})])dx \\
&= \frac{\imath}{4\pi\sqrt{\pi\omega}} e^{2M\omega\pi} e^{\imath\omega' v_0}\alpha^{-\imath\omega 4M}\int_0^\infty e^{-\omega' x}x^{\imath\omega 4M}dx \\
&= \frac{\imath}{4\pi\sqrt{\pi\omega}} e^{2M\omega\pi} e^{\imath\omega' v_0}\alpha^{-\imath\omega 4M}\Gamma(1+\imath\omega 4M)(\omega')^{-1-\imath\omega 4M}, \qquad (3.47)
\end{aligned}$$

where the Gamma function is defined as

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt.$$

For $\omega' < 0$ we get

$$\begin{aligned}
\tilde{p}_\omega(\omega') &\sim -\frac{\imath}{4\pi\sqrt{\pi\omega}}\int_0^\infty \exp(\omega' x + \imath\omega' v_0 - \imath\omega 4M[\ln\alpha - \ln(xe^{\imath\pi/2})])dx \\
&= -\frac{\imath}{4\pi\sqrt{\pi\omega}} e^{-2M\omega\pi} e^{\imath\omega' v_0}\alpha^{-\imath\omega 4M}\int_0^\infty e^{\omega' x}x^{\imath\omega 4M}dx. \qquad (3.48)
\end{aligned}$$

When we compare this with the second line of (3.47) we see that

$$\tilde{p}_\omega(-\omega') = -e^{-4M\omega\pi}e^{-2\imath\omega' v_0}\tilde{p}_\omega(\omega') \quad \text{for } \omega' > 0. \qquad (3.49)$$

So we finally obtain the asymptotic Bogoliubov coefficients by combining (3.45), (3.47) and (3.49)

$$\begin{aligned}
A_{\omega\omega'} &= \sqrt{\pi\omega'}\tilde{p}_\omega(\omega') \\
&= \frac{\imath}{4\pi\sqrt{\omega\omega'}} e^{2M\omega\pi} e^{\imath\omega' v_0}\alpha^{-\imath\omega 4M}\Gamma(1+\imath\omega 4M)(\omega')^{-\imath\omega 4M}, \qquad (3.50) \\
B_{\omega\omega'} &= -e^{-4M\omega\pi}e^{-2\imath\omega' v_0}A_{\omega\omega'}. \qquad (3.51)
\end{aligned}$$

Now that we have obtained specific expressions for the Bogoliubov coefficients we can work out (3.34). This tells us what the effect of the collapse is on the initial vacuum state of the field Φ. The fact that we start with a vacuum state means that we assume that there are no particles present initially. Of course the matter that will form the black hole is present, but since this will collapse it has to be localized in the central region of spacetime and its energy is accounted for in the metric. So it seems reasonable to assume that there is vacuum in the asymptotic region. The continuous form of (3.34) is

$$\int_0^\infty |B_{\omega\omega'}|^2 \, d\omega',$$

but since $B_{\omega\omega'}$ goes as $(\omega')^{-1/2}$ for large values of ω' this integral diverges. This means that over all time the collapsing body produces an infinite number of particles in each mode (i.e. for each value of ω), this divergence is connected to the fact that we did not impose the restriction that there is a finite amount of energy present initially, namely the mass of the black hole. To be able to say more about the character of the outcoming particles we should calculate the number of particles per frequency interval $d\omega$ per unit time by constructing finite wavepackets. This is what Hawking does in [8], but we shall adopt a more simple method here, that you will encounter in most of the literature (see for example [9]). Instead of constructing a wavepacket we will discretize the modes by simply confining the system to a box with periodic boundary conditions. For the discrete Bogoliubov coefficients we already found the relation

$$\Sigma_{\omega'}|A_{\omega\omega'}|^2 - |B_{\omega\omega'}|^2 = 1.$$

In addition to this we now also found

$$\Sigma_{\omega'}|B_{\omega\omega'}|^2 = e^{-8M\omega\pi}\Sigma_{\omega'}|A_{\omega\omega'}|^2,$$

(see 3.51), together they give

$$\Sigma_{\omega'}|A_{\omega\omega'}|^2 - |B_{\omega\omega'}|^2 = \left(e^{8M\omega\pi} - 1\right)\Sigma_{\omega'}|B_{\omega\omega'}|^2 = 1.$$

So that the particle number per mode is

$$\begin{aligned} < N_\omega >_{\mathcal{I}_+} &= \Sigma_{\omega'}|B_{\omega\omega'}|^2 \\ &= \left(e^{8M\omega\pi} - 1\right)^{-1}. \end{aligned}$$

The particle flux Φ through a sphere of radius R is then[6]

$$\Phi = \frac{Rd\omega < N >}{2\pi R} = \frac{d\omega}{2\pi}\left(e^{8M\omega\pi} - 1\right)^{-1}. \tag{3.52}$$

From thermodynamics we know that black body radiation is characterized by the Planck spectrum[7]

$$\frac{1}{e^{\beta E} - 1} \quad \text{with} \quad \beta = T^{-1} \quad \text{and} \quad E = \omega,$$

comparing this with (3.52) we can conclude that the black hole radiates at a finite temperature, appropriately named the Hawking temperature, given by

$$T_H = \frac{1}{8M\pi}. \tag{3.53}$$

Now that a black hole appears to have a temperature, we may identify the first law of thermodynamics (2.15) with its analog in black hole mechanics (2.14). For a Schwarzschild black hole one obtains $\kappa = 1/4M$ from (2.13) by setting a to zero. Inserting this in (2.14) gives

$$\begin{aligned} \delta M &= \frac{1}{4(8M\pi)}\delta A \\ &= \frac{1}{4}T_H\delta A. \end{aligned}$$

[6]The wavefunction modes are of the form $\Phi \sim e^{-\imath\omega t + \imath\omega r*}Y_{lm}(\theta,\phi)$. For fixed l and m the periodic boundary condition, i.e. $\Phi(0) = \Phi(R)$, gives $\omega = 2\pi n/R$. So the density of states inside the sphere is $dn = \frac{Rd\omega}{2\pi}$ for l,m fixed. Furthermore, it takes a particle a time R to reach the surface of the sphere, since $c = 1$.

[7]Remember that we work in units where $\hbar = 1$.

Comparison with (2.15) learns that the black hole entropy scales with its area as

$$S = \frac{1}{4}A. \tag{3.54}$$

The proportionality of black hole entropy and area was also anticipated in section 2.4 based on arguments from information theory.

So we find that a black hole emits particles at a steady flux which has a thermal character. This fact will play a crucial role in the next chapter where we will explain the information loss paradox.

But before we go there we should make a final comment. The outcoming particles carry away energy from the black hole so its mass will go down. We see from (3.53) that the temperature is inversely proportional to the mass. Initially the temperature will be low so the radiation will not be strong. In this stage the mass of the black hole will decrease very slowly and can be regarded constant, as we did in our calculation. However, as the black hole loses mass its temperature goes up so that it will radiate stronger. Eventually the radiation will become so strong that the mass can no longer be regarded constant. It can be shown that this will happen only at the final stage of evaporation when the black hole has already radiated away most of its mass (see [8]). There are different scenarios for what will happen at this final stage, they will be discussed in chapter 5.

Chapter 4

Information Loss Paradox

4.1 The Paradox

Let us reconsider for a moment what we found in the last chapter. We started with the question 'What is the effect of the gravitational field of a body that collapses to form a black hole on quantum phenomena?'. Answering this question requires a full theory of quantum gravity, but in the lack of such a theory we used the next best thing: a semi-classical approximation. In this approximation the gravitational field is treated as a classical background in which the quantized matter fields evolve.

Upon inspection we found that for a quantum field theory in curved spacetime there is a difficulty in defining a global vacuum state for the quantized field. This is related to the fact that a curved spacetime in general does not have a timelike Killing vector. As a consequence hereof there is an ambiguity in the choice of a basis for the wavefunction, which results in an ambiguity in the definition of the vacuum state. The different bases are related by the so called Bogoliubov transformations. One can, however, define a vacuum state in a stationary spacetime. The metric of a stationary spacetime is independent of time, so it clearly has a Killing vector.

The spacetime around a collapsing body consists of three regions: 1) a stationary and asymptotically flat region at past timelike infinity, 2) a non-stationary region and 3) another stationary region consisting of a singularity, an event horizon and again an asymptotically flat region at future timelike infinity. In the first and last region we were able to define vacuum states, because they are stationary. However, when we examined the relation between the two vacuum states in terms of the Bogoliubov coefficients we found that they do not correspond. In other words the empty state in the initial region of spacetime is likely to contain particles in the final region. Thus a non-stationary region of spacetime can cause particle creation.

In the last section (3.5) we explicitly calculated the Bogoliubov coefficients for a massless scalar field that evolves in the spacetime of a collapsing body. What we found is that, independent of the details of the collapse, the black hole will radiate a steady flux of particles with a Planckian distribution over the modes. The energy of these particles can have no other origin than the gravitational field. This implies that the radiation carries away energy from the black hole to infinity and we can only conclude that the black hole must lose mass and eventually evaporate.

In the context of chapter 2 this is a very nice result since it establishes that a black hole behaves as a body with a temperature that tends to equilibrium by emitting radiation. This was exactly the missing link in the analogy between black hole mechanics and thermodynamics. On the other hand, it is at least remarkable that the so-called Hawking radiation totally undermines the classical definition of a black hole which states that nothing can escape from its gravitational attraction!

In this chapter we will thoroughly investigate the implications of Hawking's calculation and explain the infamous information loss paradox, but before diving into formulas and calculations, let us give you a qualitative idea of how the paradox comes about. For this purpose let us first try to understand the implications of a singularity. In 1965 Roger Penrose [11] proved, using the fact that gravity is always attractive and the way light cones behave in a curved spacetime (see also section 2.1), that in the classical theory of gravitation the collapse of a bulk of matter unavoidably results in a singularity. A singularity is a region that cannot be described with any of the tools we have to describe nature, so the laws of physics break down at such a point in spacetime. Since we have no means to describe a singularity, we have no idea of what can come out of it. This means that in a spacetime that contains a singularity we would in principle be unable to predict the future.

This is of course a very disturbing fact and many attempts have been made to solve this problem. One of these attempts is called the "cosmic censorship" hypothesis. This says that any singularity should be hidden from our view by an event horizon, because then whatever comes out of the singularity will never be able to interfere with our future[1]. Singularities that are not sealed off by event horizons are called naked singularities. The "big bang" is generally believed to be a naked singularity, but a black hole is not since it has an event horizon.

The event horizon causes observers outside the black hole to have only limited knowledge about the internal state of the black hole. He can only measure its mass, angular momentum and charge, this is called the "no hair" theorem. So any other kind of information about a black hole, such as the kind of matter it is composed of, what kind of object it was before it collapsed, etcetera is lost to an observer outside the black hole. It is believed, however, that this information is somehow stored inside the black hole, so that it is not actually lost to the universe as a whole. This is in accordance with the quantum mechanical law that the universe evolves unitarily in time, since this means that the initial state can always be reconstructed, so information may not be lost.

Now that we have established in the previous section that black holes radiate and evaporate, we must conclude that the radiation must return all the information about the black hole's history to us. Because only in that case black hole formation and evaporation is a unitary process and thus allowed by quantum mechanics. The radiation, however, is completely uncorrelated, as we will show in the following sections, which means that it does not contain any information whatsoever. So after complete evaporation of the black hole, the

[1] This clearly is the most cowardice solution one may think of and it is not quite satisfactory, because the fact that we do not understand the physics of a singularity remains. But fact is that all kinds of calculations on the formation of black holes have so far always produced an event horizon.

only information we have left is the black hole's mass, angular momentum and charge, all the other information is lost. This constitutes the information loss paradox.

So the paradox comes down to the following: From a thermodynamical point of view Hawking radiation is the desired result, but from a quantum mechanical point of view it cannot be good, since it violates unitarity. Is there any way we can reconcile these two very contradictory conclusions?

In the following sections we will show exactly how Hawking radiation violates unitarity. Then in the next chapter we will discuss the validity of Hawking's calculation in detail. In this chapter we will also address the question just posed. Finally, in chapter 6, we will comment shortly on the many different attempts that have been made to resolve the paradox, before we turn our attention to one of these attempts in specific on which we will elaborate in the last chapter.

4.2 The Thermal Density Matrix

In section 3.4 we already introduced the density matrix by its definition

$$< \mathcal{O} >= \mathrm{Tr}(\rho \mathcal{O}), \tag{4.1}$$

with $\mathcal{O}$ an observable. For a pure quantum state $|\Psi >= \Sigma_i \alpha_i |\psi_i >$ the density matrix is simply

$$\rho_p = |\Psi >< \Psi|, \tag{4.2}$$

as is readily checked with 4.1. The density matrix is particularly useful when describing systems at a finite temperature. Following the principles of statistical mechanics we put the system in contact with a heat bath and assign classical probabilities p_A to the quantum mechanical states $|\Psi_A >$. This leads to the thermal density matrix

$$\rho_{th} = \Sigma_A p_A |\Psi_A >< \Psi_A|.$$

For a system at temperature T the probabilities are proportional to the Boltzmann factors $e^{-\beta E_A}$, where $\beta = T^{-1}$, so that

$$\rho_{th} = \frac{\Sigma_A e^{-\beta E_A} |\Psi_A >< \Psi_A|}{\Sigma_A e^{-\beta E_A}}, \tag{4.3}$$

And the expectation value of an observable $\mathcal{O}$ then reads

$$< \mathcal{O} >_{th} = \mathrm{Tr}(\rho_{th} \mathcal{O}) = \frac{\Sigma_A e^{-\beta E_A} < \Psi_A|\mathcal{O}|\Psi_A >}{\Sigma_A e^{-\beta E_A}},$$

as we would expect in statistical mechanics.

In the preceding chapter we found that a black hole emits particles with a thermal spectrum, in other words it evaporates. When the evaporation is complete, the black hole will have disappeared and all that remains is thermal radiation. So the final state is thermal, which means that at late times the thermal density matrix should describe the system correctly. To check more thoroughly that the final state is thermal, one can calculate $< N_j^2 >$, $< N_j^3 >$, etcetera, using ρ_{th} as well as the Bogoliubov coefficients and check that both yield the same results[2]. This is done in [12]. One can also check that all the off-diagonal elements are zero, as is the case for ρ_{th}, by calculating the expectation values of operators like $b_j b_j$ and $b_j^\dagger b_l$ with $j \neq l$. It can be shown that they are all zero (see [12]).

Let us return to the issue of information loss. When the black hole has completely evaporated, the energy of the final state will equal the initial mass of the black hole by conservation of energy. From this we can deduce that every possible configuration of the final state has the same probability in the following manner. Suppose the final state is $|\Psi_A>$, then $E_A = M_0$, so

$$p_A = \frac{e^{-\beta_H E_A}}{\Sigma_A e^{-\beta_H E_A}} = e^{-\beta_H M_0}/Z,$$

where β_H is the temperature of the black hole at the time of emission and $Z = \Sigma_A e^{-\beta_H E_A}$ is the partition function, which is independent of the configuration of the final state. So we have found that the p_A are equal for all A. The fact that every possible configuration is equally probable tells us that it is impossible to reconstruct the initial state (assuming the initial state was pure), since the final state does not carry any information. This explains intuitively that information is lost in black hole evaporation.

There is a subtlety in the above reasoning though; it seems as though we took the temperature of the black hole to be constant. This is of course not the case, because for E_A to equal M_0 the temperature will increase from $\hbar/8\pi M_0$ to infinity. What exactly happens at the final stage of evaporation is unclear (and different scenarios will be discussed later), but it does not seem unreasonable to assume that the black hole radiates away most of its mass very slowly (since the temperature starts low), so that its mass can be assumed constant during most of the evaporation process. However, in this case the issue is not so much whether or not the temperature can be regarded constant, but more if the temperature evolution is more or less the same for different final configurations. There is little reason to assume that this is not the case, so it seems legitimate to say that β_H is independent of the state A.

4.3 Evolution of a Pure State into a Thermal State

In this section we will show that the evolution of a pure state into a thermal state, as we believe happens in the process of black hole formation and evaporation, is not a unitary transformation and thus in contradiction with quantum mechanics.

Let us first refresh our memory: why do transformations need to be unitary in quantum mechanics? Let $\mathcal{H}$ be the infinite dimensional Hilbert space spanned by the orthonormal states $|\Psi_i>$, which are solutions of the wave equation. Let $U : \mathcal{H} \to \mathcal{H}$ be a matrix that transforms the state $|\Psi_A>$ into the state $|\Psi_B>$, then since $<\Psi_i|\Psi_j> = \delta_{ij}$

$$1 = <\Psi_A|\Psi_A> = <\Psi_B|U^\dagger U|\Psi_B> \quad \Rightarrow \quad U^\dagger U = \mathbf{1},$$

[2]For example, $<N_j^2> = \mathrm{Tr}(\rho N_j^2)$ but also $<N_j^2> = <0_-|b_j^\dagger b_j b_j^\dagger b_j|0_->$ which can be expressed in terms of the Bogoliubov coefficients by using 3.30 as we did in 3.34.

in other words, U is unitary[3].

A pure state is described by the pure density matrix (4.2), which is a projector since it projects the whole Hilbert space $\mathcal{H}$ onto one state, so dim(Im ρ_p)$= 1$. Whereas the thermal density matrix has

$$\dim(\mathrm{Im}\rho_{th}) = \dim(\mathcal{H}) = \infty, \tag{4.4}$$

as can be seen directly from its form in (4.3). In quantum mechanics, however, the density matrix that describes a system that started out in a pure state has to obey:

$$\rho' = U\rho_p U^\dagger,$$

so

$$\dim(\mathrm{Im}\rho') = \dim(\mathrm{Im}U\rho_p U^\dagger) = \dim(\mathrm{Im}\rho_p) = 1,$$

since U is invertible. When we compare this with (4.4) we must conclude that

$$\rho_{th} \neq U\rho_p U^\dagger,$$

so a pure state cannot evolve into a thermal state in a unitary way.

This conclusion explains the term paradox. We started with the semi-classical approximation, in which gravitation is incorporated in quantum mechanics by treating it as a classical background field for quantized matter fields. The result is that the matter fields evolve in a non-unitary way. This is in contradiction with the semi-classical approximation since the matter fields should obey the laws of quantum mechanics in this approximation. So the calculation seems to be wrong. On the other hand, regarding the analogy between thermodynamics and black hole mechanics, we seem to have found exactly the missing link, namely the fact that a black hole emits black body radiation.

Nowadays the result that black holes radiate is generally accepted and most physicists believe that in a full theory of black hole dynamics there will be no violation of unitarity and thus no information is lost. But over the past thirty years it seems that nobody has come up with a satisfactory answer to the question what goes wrong in Hawking's calculation, why does it yield two so very contradictory results - one that seems to be dramatically wrong from a quantum mechanical point of view and one that seems to be to good to be true from a thermodynamic point of view.

In the following chapter we will discuss the validity of Hawking's calculation. After that a number of modern points of view on this subject will be discussed in chapter 6.

[3]For the following it is actually enough that U is invertible, so dim(Ker U)$= 0$.

Chapter 5

Discussion of the Validity of Hawking's Calculation

At this point it should be clear to the reader what the information loss paradox is and how it comes about. To understand how the paradox may be resolved it is natural to start by taking the calculation of chapter 3 under the loop. In this chapter we will discuss the many assumptions and approximations that are made in the calculation step by step. Evidently, we will not find a real flaw, otherwise we would have solved the paradox, but we will be able to point out at least two assumptions that may not be fully correct.

5.1 Details

In chapter 3 we have left out a couple of details to be able to keep track of the general reasoning. In this section we will discuss some subtleties that may have bothered the reader, but do not significantly influence the results. At least that is what we shall try to convince the reader of.

5.1.1 Backscattering

In the original article Hawking also takes into account that the particles emitted by the black hole can scatter against the gravitational field surrounding the black hole. This scattering can sent them back across the event horizon, so actually only a fraction of the emitted particles will really escape from the black hole and reach infinity. Suppose a fraction A_ω of the particles emitted with frequency ω reaches infinity, whereas the fraction $(1 - A_\omega)$ of these particles gets scattered back into the black hole. Then the total number of particles radiated away with this frequency is

$$< N_\omega >_{\mathcal{I}_+} \quad = \quad A_\omega \left(e^{8M\omega\pi} - 1 \right)^{-1}.$$

On the other hand, however, for particles that come from past null infinity the same applies. They also partially scatter off the gravitational field outside the black hole. For particles that come from $\mathcal{I}_-$ with a frequency ω the fraction that does not get scattered and thus reaches the black hole is again A_ω. So the ratio of absorption and emission by the black hole is still that of a black body, even when we take the backscattering into account.

5.1.2 Initial Vacuum State

Hawking assumes that the initial state of the massless scalar field is the vacuum state. Strictly speaking the initial state is not completely empty since the matter that will form the black hole is present. But one can assume that this matter is initially localized within a region of radius R, where of course R is greater than the Schwarzschild radius but also much smaller than infinity. Apart from the fact that this is a necessary condition for the matter to collapse it also explains why starting with a vacuum state at spatial infinity is a legitimate choice.

Even if there would be some massless scalar particles present initially, it is highly unlikely that they will influence the process. Either they will travel towards the black hole, pass by it and continue towards infinity. Or they will fall into the black hole where they will just add a tiny bit to the black hole's mass. In either case they do not alter the character of the Hawking radiation in any significant way.

5.1.3 Details of Collapse

To calculate the Bogoliubov coefficients explicitly we needed solutions of the Klein-Gordon equation in the initially and finally stationary regions. These solutions we could only find if we were at sufficiently large r (see (3.36)). Furthermore, it is widely believed that spherically symmetric black holes settle down quite quickly to a stationary state[1]. In that case the spacetime around it can be described by the black hole metric, which is entirely determined by the black hole's mass, angular momentum and charge. All the details of the collapse have been washed out. So what we have done to calculate the Bogoliubov coefficients is typical for scattering processes; we sent in a plane wave from far away and then waited a sufficiently long time before we looked at the wave again. The effect of any finite number of particles that might have been generated by the explicit details of the collapse has been washed out by that time. In addition to this, Hawking proves [8] that the result is also valid if the collapse is not spherically symmetric.

5.1.4 Geometrical Optics Approximation

The geometrical optics approximation (see appendix B) used to relate the plane wave solutions of the Klein-Gordon equation at future null infinity to the plane wave solutions of this equation at past null infinity is valid only for quickly oscillating waves. More precisely the wavelength of the wave has to be negligible compared to the typical radius of curvature of the spacetime, i.e. $\lambda \ll \mathcal{R}$. We shall see that this is the case for the wave we traced back.

First we note that the plane wave solutions of the Klein-Gordon equation are valid only far away from the black hole and long after the collapse (the black hole must have settled down to a stationary state). This means that our observer has to be in the region where $r, t \to \infty$. From the picture (Fig. 5.1)

[1] This is known as the Carter-Israel conjecture (see [13], [14]).

we see that the part of the wavefunction that this observer sees, comes from a very small region on $\mathcal{I}_-$, just before the event horizon generator leaves $\mathcal{I}_-$. The event horizon generator is the null geodesic that does not hit the singularity nor does it reach $\mathcal{I}_+$, it is captured by the black hole and bound to stay at $r = 2M$ for ever.

We have traced the wavefunction back in time so let us first discuss the first part of this narrow strip. Here the retarded time u goes to infinity and this corresponds to a diverging physical frequency. This can be seen from the Schwarzschild metric, since a propertime interval is given by $\Delta\tau = -\sqrt{1 - 2M/r}\Delta t$, which goes to zero as r goes to the Schwarzschild radius. So near the horizon the period of the wave becomes extremely short, which corresponds to an extremely high frequency. So in the first part of the strip the wavelength is extremely short and the geometrical optics approximation is valid.

The second part of the strip lies in the region of spacetime before the collapse. So as we are playing back time, the wave distances itself from the object before collapse and thus its frequency goes down again. However, this time the redshift is clearly less than the blueshift it suffered from the object after collapse[2]. So the physical frequency will go down a little, but as it does, the curvature also goes down. So effectively we still have $\lambda \ll \mathcal{R}$ in the second part if the strip. Consequently, the geometrical optics approximation is legitimate for the entire strip, i.e. for the part of the wavefunction we considered in the calculation.

5.1.5 More General Circumstances

To make the calculation as simple as possible, we carried it out only for a massless scalar field and a chargeless black hole with zero angular momentum. Hawking discusses generalizations of the calculation in the original article. He states that the result remains unchanged for massless fields with integer spin, such as the electromagnetic and linearized gravitational field. So the black hole will also radiate photons and gravitons with a thermal spectrum. For massless fermions however, their anticommuting nature causes the bosonic Bogoliubov matrix relation

$$AA^\dagger - BB^\dagger = 1$$

to change the fermionic relation

$$AA^\dagger + BB^\dagger = 1.$$

It can now easily be derived that massless fermions are radiated with the spectrum that is in accordance with the Fermi-Dirac statistics that applies to them, namely

$$\left(e^{8M\omega\pi} + 1\right)^{-1}.$$

To generalize the calculation to massive fields is a little more tricky since they do not reach null infinity. Hawking argues that this problem can be overcome and shows that massive particles also get emitted with a thermal spectrum.

[2]This effect is discussed in more detail in subsection 5.2.2.

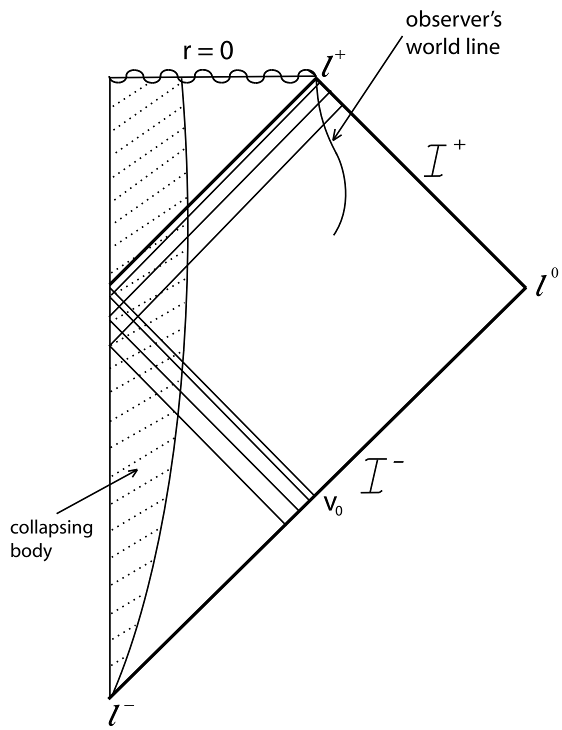

Figure 5.1: An observer at late times only sees modes with large values of the retarded time u. These are the modes for which the geometrical optics approximation is valid.

The only difference now is that the production of massive particles requires more energy and thus will only occur at high enough temperatures (see next subsection).

Finally, one can generalize to a rotating black hole with non-zero charge. In this case one should replace the Schwarzschild metric by the Kerr-Newman metric. It can be shown that for a black hole with angular momentum J and charge Q the radiation spectrum is

$$\left(e^{8M\pi(\omega-l\Omega-e\Phi)}-1\right)^{-1}.$$

where l and e are the angular momentum and charge of the emitted (bosonic[3]) particles respectively and Ω is the angular frequency of the black hole and Φ is its electrostatic potential. The above result tells us that particles with angular momentum in the same direction and charge with the same sign as the black hole are emitted at a higher rate, so they do not only carry away the black hole's mass, but also its angular momentum and charge.

5.1.6 Final Stage

We found that black holes radiate at a temperature inversely proportional to their mass. A black hole will only radiate effectively and loose energy when it has a temperature higher than its surrounding. As we know the universe has a temperature of 3 K originating from the microcosmic background radiation. For a black hole to have a temperature higher than 3 K it has to be lighter than 10^{26} grams[4] (a solar mass is of the order of 10^{33} g)!

Once a black hole starts radiating it will loose mass and as a result its temperature will increase. Initially, the black hole will radiate weakly, which only causes minor changes in its mass and temperature. Since this process develops so gradually we can justify the fact that we kept the mass constant throughout the calculation. Only at the very end of the evaporation will the mass have reduced sufficiently for the production of massive particles. This can be seen as follows. The probability that a particle with energy E is generated is given by $P(E) \propto e^{-E/(k_BT)}$. So the probability that the black hole will create a massive particle of mass m becomes significant when $E/(k_BT) = mc^2/(k_BT_H)$ becomes of order one, i.e. $T_H \sim \frac{k_B}{mc^2}$. So the black hole will start radiating particles of m when the black hole mass has reduced to

$$M_{BH} = \frac{\hbar c^3}{8\pi G T_H k_B} \sim \frac{\hbar c^3}{8\pi G k_B}\frac{k_B}{mc^2} \approx 1.06 \times 10^{22}(mc^2)^{-1} eV g.$$

The lightest particles in the Standard Model are electron-neutrinos, their mass is around 1 eV, so they can be produced by a black hole of a mass of 10^{22}

[3]For fermionic particles one finds

$$\left(e^{8M\pi(\omega-l\Omega-e\Phi)}+1\right)^{-1}.$$

[4]To verify this note that the full expression for the Hawking temperature is given by

$$T_H = \frac{\hbar c^3}{8\pi G M_{BH} k_B} \approx 1.23 \times 10^{26} M_{BH}^{-1}\ gK.$$

grams. Electrons, which have a mass of approximately $0.51\times10^6 eV$, and protons ($m_p \sim 2000 m_e$) are generated by black holes with a mass of the order of 10^{16} grams and 10^{13} grams respectively.

We would like to know at which stage in the evaporation process the approximation that the black hole mass is constant becomes dubious. The rate at which the black hole mass changes is given by the radiation flux out of the black hole times the black hole area

$$\frac{dM_{BH}}{dt} \sim \Phi_{\text{rad}} A_{BH} \sim M_{BH}^{-4} M_{BH}^2,$$

where we used the Stefan-Boltzmann radiation law, i.e. $\Phi_{\text{rad}} = \sigma T^4$ with σ the Stefan-Boltzmann constant, the fact that $T_H \propto M_{BH}^{-1}$ and finally $A_{BH} = 16\pi M_{BH}^2$. From this expression we can see that the change in the black hole mass becomes significant when it becomes of order 1. That is, since we work in unit where $c = \hbar = G = 1$, when M_{BH} becomes of the order of the Planck mass $m_P = \sqrt{\hbar c/G} \approx 10^{-5} g$.

We have no means to calculate what happens to the black hole at this very final stage. However, Hawking claims [8] that the black hole will explode, that it will blast away its remaining energy on an extremely short time scale and thus the black hole will simply disappear. In this scenario information is lost since this remaining energy is insufficient to restore the information that was lost behind the horizon [12].

Another scenario is that there may remain a black hole remnant of some kind that might contain the presumably lost information. But it seems impossible to construct such a remnant without encountering serious difficulties. For discussions of the specific problems one encounters when constructing remnants we refer to [15] and [16].

5.1.7 Violation of the Area Theorem

We have established that in the evolution of a quantized field in a curved non-stationary background metric particles are created. The energy necessary to create these particles can have no other origin than coming from the metric itself. Extracting energy from the metric is accompanied by a decrease in its curvature. In the case of a black hole generating a Schwarzschild metric a reduction in the metric curvature corresponds to a decrease in mass of the black hole.

From chapter 2 we know that a decrease in the mass of a black hole results in a decrease of its area, since

$$\delta M = \frac{\kappa}{8\pi}\delta A.$$

This may seem to be in contradiction with the area theorem and thus with the second law of thermodynamics, but remember that at the end of section 2.4 we anticipated this and concluded that the second law had to be generalized. The generalized second law states that the total entropy of the universe plus all black hole entropy can never decrease. This law is not violated in the process of black hole evaporation, since the decrease in black hole entropy is compensated by the outgoing radiation that causes a maximal increase in the entropy of the rest of the universe.

5.2 Validity of the Semi-Classical Approximation

To consider the validity of the approximation used in the lack of a full theory of quantum gravity let us first restate how this approximation was realized. The gravitational field was treated classically, i.e. it is a solution of the Einstein equation. The matter field was treated quantum mechanically in this classical background field by replacing the flat Minkowski metric in the wave equation by the curved metric that solves the Einstein equation. A problem that immediately arose is the so-called backreaction problem. Once the curvature of spacetime generates particle creation, these newly created particles change (the expectation value of) the energy-momentum tensor. But a change in the energy-momentum tensor causes a change in the spacetime curvature through the Einstein equation, and so on. In addition, there is the problem that solving the Einstein equation analytically becomes practically impossible as soon as the energy-momentum tensor is non-zero.

For the specific case of a collapsing body, however, we argued that initially the matter was well enough localized around $r = 0$, such that the energy-momentum tensor is zero outside a certain radius $R \ll \infty$. And after the collapse all the matter sits at the singularity, so the energy density tensor is now zero everywhere except at $r = 0$. So the initial and final stationary spacetimes could be found analytically from the Einstein equation.

To overcome the backreaction problem one has to be able to argue that it is so small that it can be ignored. We will discuss what the conditions are in which this neglect can be justified and investigate whether these conditions were met in Hawking's calculation.

5.2.1 Backreaction of High Spacetime Curvature

In section 3.3 we established that for a stationary spacetime one can globally define a vacuum state. In the asymptotically flat region of this stationary spacetime there is no ambiguity in the particle number because one can distinguish precisely between all the modes $\{f_i\}$ that form the basis for the wavefunction

$$\phi(x) = \Sigma_i[a_i f_i(x) + a_i^\dagger f_i^*(x)].$$

For a general point in this curved spacetime this is not the case however. This is illustrated in Fig. 5.2. The observer at p can distinguish between the mode f_i with characteristic frequency ω and its complex conjugate f_i^* with frequency $-\omega$ with very high precision when ω is big compared to the local curvature (the indeterminacy can be shown to be exponentially small). So for modes with high frequencies compared to the local spacetime curvature one can accurately distinguish between the annihilation and creation operators, a_i and $a_i^\dagger$, and consequently there is virtually no indeterminacy in the particle number. However, when the characteristic frequency becomes of the order of the local curvature the ambiguity between f_i and its complex conjugate is almost complete. This ambiguity results in a complete ambiguity between the annihilation and creation operator of this mode and consequently there is an uncertainty of $\pm\frac{1}{2}$ in the number operator $a_i^\dagger a_i$ for this mode.

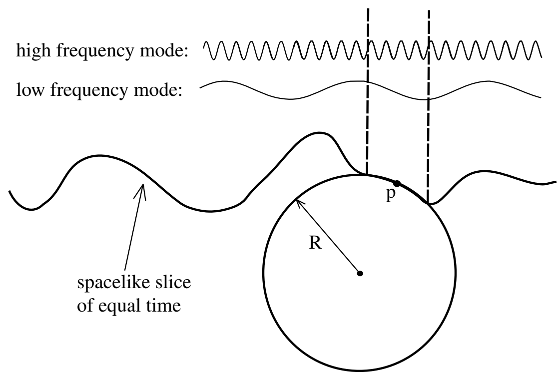

Figure 5.2: The observer at p can distinguish accurately between high frequency modes and their complex conjugates. For low frequency modes, however, the distinction becomes unmeasurable and the ambiguity complete.

A local indeterminacy in particle number will cause an indeterminacy in the local energy-momentum tensor, this in turn will cause an uncertainty in the spacetime curvature via the Einstein equation. Thus the semi-classical approximation will breakdown when the uncertainty in the curvature becomes of the order of the local curvature itself. It can be shown that for a local curvature B the uncertainty in the curvature is of order B^2. This means that the approximation will breakdown when the curvature becomes of order one, however, this corresponds to a local coordinate radius R (see Fig. 5.2) of the order of the Planck length![5] For the spacetime of a black hole such high curvatures will occur only very near the singularity. Since this region is hidden from our view by the event horizon any particle creation caused by these curvatures will not effect us.

5.2.2 Backreaction of High Energy Particles

In the previous section we found that the backreaction of particle creation caused by high spacetime curvatures can be neglected as long as the coordinate radius stays well above the Planck length. On the other hand, there is the backreaction caused by high energy modes of the wavefunction.

Naively one would say that the semi-classical approximation breaks down when there is a significant occupancy of modes with very high frequency. Because then the expectation value of the energy-momentum tensor of the field will become so large that its backreaction on the metric can no longer be ignored. There is a subtlety here, however, because in a free field theory, i.e. where particles do not interact, such a diverging energy can be taken care of by renormalization. It is only because the particles are not free, since they exert

[5] The curvature of spacetime is given by the Riemann tensor, which is the unique tensor that can be constructed from second order derivatives of the metric. Clearly, since the metric is dimensionless and the derivatives are with respect to the spacetime coordinates, the curvature of spacetime has dimensions one over length squared. When the curvature is of order one, so is the coordinate radius. Order one in our units means order Planck length in SI units, that is $L_P = \sqrt{hG/c^3} \approx 10^{-35}m$.

a gravitational attraction on each other, that the semi-classical approximation breaks down for high energy particles. Because in the approximation this interaction is not accounted for and for high energy particles it can not be ignored. This was first pointed out by 't Hooft [17].

Now let us investigate if the condition that there should be no high energy particles, is met in Hawking's calculation. Remember that near the horizon the wavefunction reaches an infinite number of cycles (see Fig.5.1), which corresponds to a diverging physical frequency. This enabled us to construct wave packets whose trajectories in spacetime could be approximated by null geodesics. But since these wavepackets are constructed of diverging frequency modes they will have an extremely high energy. This makes sense, because for a particle to travel through a collapsing body just before the horizon forms and nevertheless be able to escape to infinity it must have an extremely high energy. In view of what we just discussed, however, we must conclude that these high energy particles cause the semi-classical approximation to breakdown near the event horizon.

This argument against the validity of Hawking's calculation is often referred to as backreaction due to the infrared divergence. This can be understood as follows. When a plane wave of a certain frequency approaches a heavy object it will suffer a blue-shift. On the other hand, when it distances itself from the object after passing through it, it will suffer a redshift. If the object is static and thus the background metric is unchanged while the wave passes by, the redshift will exactly cancel the blueshift. For a collapsing body, however, the time scale on which the metric changes is of the same order as the time it takes the wave to travel through it. In that case, the redshift is much larger than the blueshift. This redshift emerges in our calculation of the particle creation in equation (3.43). Pay attention, however, because in the calculation we traced the waves back in time so the redshift also occurs in this direction. Namely, null rays with constant phase u traced back to $\mathcal{I}_-$ pile up densely along the horizon generating null ray for which $v = v_0$, as we see in Fig. 5.1. The redshift is only evident from the picture when you realize that the pile up at $\mathcal{I}_+$ is an artifact of the coordinates (u goes to infinity at the horizon), whereas the pile up at $\mathcal{I}_-$ is indeed a physical blueshift. The infinite number of cycles of the wavefunction thus corresponds to an infinite blueshift. So we would call this an ultraviolet divergence, however, since we did everything in reverse, convention has it that this is called the infrared divergence.

So what we have found in this section is that in Hawking's calculation the condition of sufficiently low spacetime curvatures seems to be met. However, the condition that there should be no high energy particles seems to be violated near event horizon. In this region the matterfield can longer be regarded a free field. Thus, ingoing and outcoming particles, which have a very high energy near the event horizon, will interact with each other and it does not seem unreasonable to this that these interaction will induce long range correlation in the Hawking radiation.

5.3 Are The Bases of $\mathcal{H}$ and $\mathcal{I}_+$ Independent?

Finally, one can doubt an assumption Hawking makes at the very start of his calculation. Namely, he says that the wavefunction is completely determined by its data on the event horizon and future null infinity. And thus writes it as

$$|\Phi >= \Sigma_{i,j} \left[\lambda_{ij} |\chi_i >_{\mathcal{I}_+} |\eta_j >_{\mathcal{H}} \right], \tag{5.1}$$

with $|\chi_i >_{\mathcal{I}_+}$ a complete orthogonal set at future null infinity and $|\eta_j >_{\mathcal{H}}$ a complete orthogonal set at the event horizon. In doing so he assumes that these orthogonal sets are completely independent. In general this is a valid assumption, but in view of the non-locality of the process of particle creation one might fear that in this case they are not independent. Maybe the part of the wave that escapes to infinity picks up information from behind the event horizon by some non-local process. These correlations will cause the spectrum of the radiation to be not exactly thermal and thus unitarity can be restored. This view is advocated in [18]. The argument goes as follows. In a quantum field theory on a fixed background locality is expressed by the fact that the commutator of two observables at any two spacelike separated points x and y vanishes,

$$[\mathcal{O}_i(x), \mathcal{O}_j(y)] = 0. \tag{5.2}$$

However, in a full theory of quantum gravity this is no longer the case. For example, the creation of a high-energetic particle at the point x will cause the geometry of the background to change, which in turn may very well influence observations made at the point y. So then the locality condition (5.2) breaks down. In their article Giddings and Lippert define the so-called locality bound below which the locality condition breaks down. They arrive at a specific expression for this bound, given by

$$(x - y)^2 < |p| \tag{5.3}$$

with p the momentum in the center of mass frame (i.e. for a particle at x with momentum p and a particle at y with momentum $-p$), by noticing that in such a configuration a black hole would form and thus the locality condition is definitely violated.

Now let us see why the validity of the decomposition of the full Hilbert space into a product of two independent Hilbert spaces depends on the validity of the locality condition. Consider a spacelike slice S that cuts across the horizon, such that part of the points that lie on it are inside the black hole and part of them are outside the black hole. Clearly, points x that lie inside the black hole and points y that lie outside the black hole are spacelike separated. Now only if field operators acting at the points x and y commute, it is possible to write a state on the slice S as a state on a product of two independent Hilbert spaces, one inside the black hole and one outside the black hole. So the validity of (5.1) hinges on the validity of the locality condition.

To show that in black hole formation and evaporation the locality condition is violated, we consider a spacelike slice S that cuts across the Hawking radiation outside the black hole as well as the infalling matter inside the black hole. It has been shown [19] that information will escape the black hole at a relatively late time, so in order to compare information contained inside the black hole to that in the Hawking radiation the slice S must be highly deformed. On such an extreme slice the locality bound is unknown, so what we need to do is trace back the Hawking radiation to a point y' near the event horizon. On the new spacelike slice S' that cuts through a point x inside the black hole and the point y' the locality bound is just (5.3). In [18] it is shown that this bound is grossly exceeded and thus the locality condition is violated. Qualitatively this can be understood by remembering that tracing back the Hawking radiation is accompanied by a huge blueshift, so the relative momentum that occurs in the locality bound diverges. For further details we refer to the original article [18].

5.4 Conclusion

Although, Hawking makes quite a number of assumptions and approximations in his calculation, the validity of only two of his many steps can really be doubted. Namely, the validity of the semi-classical approximation near the horizon and the assumption that the wavefunction can be written as a tensor product of states. It is sometimes said that these arguments are actually one and the same. Both objections indeed seem to rely on the view that near the horizon quantum gravitational effects will start to play an important role and that this renders the assumption, that ingoing and outcoming particles are independent, unreliable. So these objections tell us that if we want to solve the information loss paradox we shall have to leave the arena of the semi-classical approximation.

In the next chapter we will shortly discuss some of most well known attempts at solving the paradox. We shall see that nowadays everyone seems to agree on the fact that all processes in nature are unitary, even the evaporation of a black hole. The most widely accepted view on the paradox is then that a black hole does radiate as a body of temperature $T_H = \frac{\hbar}{8M\pi}$, so a black hole with its surrounding is a thermodynamic system semi-classically. But the spectrum of the radiation is not exactly thermal once quantum corrections are taken into account. These corrections are believed to restore the unitarity of the process of black hole formation and evaporation. So the laws of quantum mechanics are not violated.

Chapter 6

Attempts at Resolving the Paradox

Over the past three decades many attempts have been made to find a solution to the paradox. Although a satisfactory solution to the paradox has not yet been found, the issue of information loss seems to have been settled in favor of unitarity. In this chapter we will summarize the attempts that in our opinion are the most interesting, because they have contributed most to the current state of affairs.

We will start by discussing Hawking's initial proposal for a resolution, in which he assumes information is lost. Then we shall discuss two new concepts that resulted from the study of black hole evaporation assuming information is not lost. They are called black hole complementarity and holography. In section 6.4 we shall sketch the AdS/CFT correspondence. This correspondence may be seen as a realization of the holographic principle in string theory. Together with the fact that quantum mechanics is not at variance with the physical world, AdS/CFT contributed strongly to convincing the grand part of the physics community that all physical processes are unitary, even black hole evaporation. Recently, Hawking announced that he also was convinced and admitted to have been wrong in proposing a modification of quantum mechanics. We will shortly discuss his present point of view in the last section of this chapter.

6.1 Hawking's Solution Part 1: Superscattering Operator

Hawking was the first to publish an article that dealt with the consequences of black hole evaporation [12]. He accepted his result that apparently pure states can evolve into mixed states when the effect of gravity on quantized fields is taken into account. Since this is in contradiction with quantum mechanics he proposed a modification of this theory. We will shortly discuss this modification, but before doing so we should emphasize that with this modification Hawking accepted a breakdown of predictability caused by the non-unitarity of the evolution of pure states into mixed states. A full theory of quantum gravity would contain yet another uncertainty above Heisenberg's uncertainty principle.

Einstein once said about this uncertainty principle "God does not play dice". Hawking's solution to the paradox was as he put it: "God not only plays dice, He sometimes throws them where they cannot be seen".

To allow quantum states to evolve from pure into mixed states, Hawking introduced the superscattering operator that maps density matrices to each other. Thus for a system that is described in an initial stage by the density matrix ρ_{ab} and in a final stage by the density matrix $\tilde{\rho}_{cd}$ the superscattering operator $\mathcal{S}$ maps ρ_{ab} to $\tilde{\rho}_{cd}$

$$\tilde{\rho}_{cd} = \mathcal{S}_{cd}{}^{ab}\rho_{ab}. \tag{6.1}$$

When both the initial and the final density matrices describe pure systems, i.e. they are of the form

$$\rho_p = |\Psi><\Psi|,$$

then the superscattering operator can be expressed in terms of the quantum mechanical scattering operator. The scattering operator S maps the initial state to the final state and is unitary so it maps pure states to pure states (see section 4.3). For a unitary process the superscattering operator can be shown to be:

$$\mathcal{S}_{cdab} = \frac{1}{2}(S_{ca}S^{-1}_{bd} + S^{-1}_{ad}S_{cb}).$$

However, if the initial state is such that it has a significant probability to form a black hole, the final state will be thermal and thus the evolution is not unitary. For such processes there is no scattering matrix, so the superscattering operator cannot be expressed in the above form. Hawking proposed that non-unitary processes are nevertheless allowed and their evolution is given by (6.1).

Since we do not have a complete theory that unifies all four forces of nature, we do not know whether or not such a theory would have a unitary time evolution operator. What we do know however is that experiments have as yet never been in contradiction with quantum mechanics. All processes we have been able to measure so far can be described by a scattering operator, so the full theory should at least reduce to a unitary theory in the limit that gravitation is weak.

A problem now immediately arises with Hawking's modified theory, which can be understood as follows. It is believed that on the Planck scale gravitational collapse of massive objects occurs frequently. In Hawking's modified quantum theory this would lead to a non-unitary time evolution at all scales. After all, in his theory the formation of Planck size black holes would effect our universe in such a way that pure states will evolve into mixed states, which makes his theory behave non-unitarily in the quantum mechanical limit. As we discussed above this cannot be reconciled with what we see.

In a very readable and, I might even say, entertaining article [20] Page comments on the effects of the introduction of a superscattering operator. He proves that if the superscattering operator is to be CPT invariant, then it maps pure initial density matrices into pure final density matrices. He suggests a number of possibilities for further investigation of the evolution described by a superscattering operator. He concludes his article with saying that "maybe if God throws dice where they cannot be seen, they cannot affect us".

Further difficulties that Hawking's proposal encounters are discussed in [21], [22] and [23].

6.2 Holographic Principle

The holographic principle was introduced by 't Hooft in 1993 [24]. To understand this idea we should go back to the analogy between thermodynamics and black hole mechanics for a moment. First it was found that the area of a black hole behaves in the same manner as entropy. Clues from information theory indicated that a black hole might even have an entropy that scales with its surface area. Finally, Hawking's discovery that black holes radiate, which indicates that they have a temperature, was even more evidence that the analogy meant more than just a similarity in formula's.

However, when a black hole is regarded as a hot body that tends to an equilibrium state with its surrounding, then it has a remarkable feature. Namely, its entropy scales with its area, whereas the entropy of a thermodynamic system scales with its volume. In thermodynamics entropy is an extensive quantity. This can be easily understood from its definition as the logarithm of the number of accessible states. Consider a system of volume V that is build up from V cubes of unit volume. Each cube has a certain number of accessible states. For example, consider a system that is allowed to have the value 0 or 1 in each cube, in that case the number of accessible states is 2. The total number of states for such a system is then simply:

$$Z = 2^V,$$

and thus the entropy scales with the volume of the system:

$$S = \ln Z = V \ln 2.$$

In [24] the fact that black hole entropy scales with the black hole area is investigated and a remarkable conclusion is reached. 't Hooft considers a system that has two accessible states for each unit volume, just as the one described above. The system is confined to a volume $V = 4/3\pi(d/2)^3$ and has an energy smaller than $d/4$, such that it does not collapse to form a black hole. He then shows that the entropy of this system is small compared to that of a black hole if the volume is sufficiently large. Furthermore, he shows that the entropy is maximal when the volume is filled with one black hole of the largest possible size, i.e. with radius $d/2$. The entropy is then $s = 4\pi(d/2)^2 = A/4$, where A is the surface area. This is exactly the result we obtained in chapter 3.5 (see (3.54))!

For a quantum field theory that is build on two assumptions: (1) at Planckian distance scales it has discrete degrees of freedom and (2) the evolution of the field theory must be reversible in time, one seems to do an enormous over counting of the number of accessible states or degrees of freedom. This is because in such a theory the degrees of freedom are thought to scale with the volume. 't Hooft explains this over-counting as a result of the fact that almost all the states of the field have such a high energy that they would collapse to form black holes before they can influence the future of the system. He concludes that to describe what happens inside a volume it is enough to know the degrees of freedom on the surface of this volume!

The comparison with a hologram is obvious. A two-dimensional hologram of a three-dimensional object is made by shining a laser beam onto the object and letting the reflected beam interfere with an unperturbed laser beam. The interference pattern registered on a photographic plate is the two-dimensional hologram and the three-dimensional object can be recovered by shining with an unperturbed laser beam at the plate. So it appears that all the information necessary to describe the three-dimensional object can be stored on a two dimensional surface. This comparison can be extended to explain how sharp we see our universe. The information we can store on the photographic plate is limited by the resolution of our instruments, similarly the information one can store on the surface area of a system is limited by the finiteness of the Planck length. The blurring caused by this limitation is so small that in practice we perceive our universe very sharply.

The holographic principle states that a field theory on a two-dimensional closed surface suffices to describe all process that take place in the three-dimensional volume within this surface. If one would be able to find this field theory it would provide a solution to the information loss paradox. For example, if this theory is found to be unitary, it is settled that pure states evolve into pure states, while at the same time the black hole behaves as a hot body.

6.3 Black Hole Complementarity

Also in 1993 Susskind, Thorlacius and Uglum [25] published an article in which they proposed an original view on black holes in order to reconcile quantum mechanics and black hole evaporation. They argue that an outside observer, that stays at a sufficiently large distance from a black hole, should be able to describe black hole evaporation using a semi-classical approximation. Furthermore, they assume that the semi-classical theory has a unitary time evolution, so there is no information loss in the process of black hole formation and evaporation to an outside observer.

For an observer to stay outside the black hole he has to have a constant acceleration away from the black hole horizon. It is calculated that such a Rindler observer experiences a bath of thermalized particles and that the temperature of this bath goes up as he gets closer to the horizon. In fact, to a Rindler observer the region near the horizon seems to be very hot, so his semi-classical theory will breakdown there. To avoid this, the concept of a stretched horizon is introduced. The stretched horizon is located at a certain distance from the horizon. This distance is determined by the energy scale up to which the observer can justify the use of his theory. This energy scale is called the cut-off. The observer cannot penetrate beyond the stretched horizon, so for his observations to be consistent this surface should behave as a hot membrane.

The stretched horizon can be shown to have no observer independent existence. This, together with the fact that an outside observer does not see any information loss, led Susskind et al. to formulate the principle of black hole complementarity which, if correct[1], solves the information loss paradox. To state the principle we quote from [26]:

- "From the point of view of an external observer, the stretched horizon exists and is a collection of quantum mechanical, microscopic degrees of freedom which can absorb, store, thermalize and emit any quantum mechanical information which falls into the black hole.

[1] Some serious arguments against the principle are discussed in [26].

- "A freely falling observer will not detect the stretched horizon, nor will he experience any other local signal when he crosses the horizon[2]."

Complementarity implies that the information carried by the matter that falls into a black hole is thermalized by the stretched horizon and re-emitted in the Hawking radiation. The radiation contains the information of the infalling matter in long-distance correlations, this means that to recover exactly what fell into the black hole by studying the outcoming radiation one has to measure for an infinitely long time. In practice this is of course impossible. But this is no different in everyday processes: try to recover the text written on a piece of paper that you just burned by looking at the ashes and the heat that was released.

It was soon realized that the principle of black hole complementarity and the holographic principle are closely related. The stretched horizon that absorbs and re-emits all the information that falls onto it is closely analogous to a surface that encodes all the information contained in the volume it encloses. Shortly after the formulation of these principles, Susskind and Uglum [26] suggested that string theory was a possible candidate to realize a holographic description of black holes.

6.4 String Theory

By now it should have become clear to the reader that solving the information loss paradox requires a theory that is valid in a regime where both gravity and quantum phenomena play a role, i.e. a full theory that describes all four forces of nature quantum mechanically. In the absence of such a theory we were only able to indicate where the validity of Hawking's calculation breaks down and speculate about whether or not the full theory would be unitary. In short, without a theory of quantum gravity we are left to vague statements and speculations. Since this, of course, is rather unsatisfying we chose to study the most promising candidate of a theory of quantum gravity in the remainder of this thesis, namely string theory. Not only is string theory a quantized theory that contains gravity, but also more specific clues are present that indicate how string theory might solve the information loss paradox.

Before we start discussing these clues, it must be emphasized that we make some huge leaps through discoveries in string theory over the past 10 years and the reader is by no means expected to really understand the arguments. Rather we hope to convince the reader that our choice to concentrate on string theory is based on solid grounds. Furthermore, we must stress that although our arguments are based on string theory, no real knowledge about this theory is required to understand the following chapters.

As we discussed previously, the main problem of incorporating gravity into quantum field theory is that it gives rise to a non-renormalizable theory. In the low energy limit field interactions can be expanded in a perturbation series, but in the case of gravity higher order corrections give diverging contributions. These divergences can be suppressed when gravity interactions are smeared out over spacetime (see Fig 6.1). String theory is the only known theory that

[2] In [26] the authors have restricted themselves to big black holes for which the spacetime curvature is low, such that tidal forces are very small.

smears out these divergences in a consistent way, i.e. respecting causality and general coordinate invariance. This is a popular argument why string theory can describe all four forces consistently.

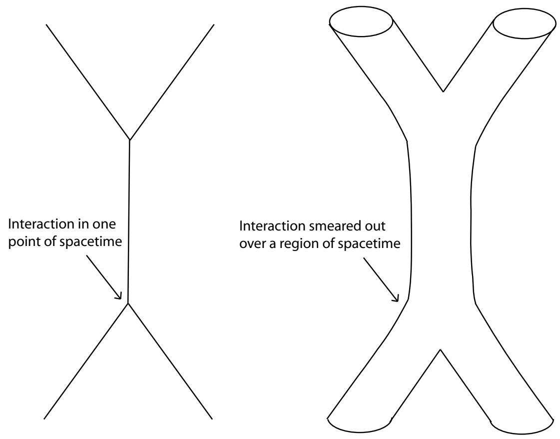

Figure 6.1: A Feynman diagram in quantum field theory and in string theory.

Another convincing feature of string theory is that in order to be consistent it must contain exactly one massless spin-2 particle. In other words, it has a unique candidate for the graviton. In a low energy limit one can calculate tree-diagrams for these gravitons and show that they correspond to the interactions one finds in the classical limit. I.e. the four-point interactions of $h_{\mu\nu}$, which gives the deviation of the curved metric $g_{\mu\nu}$ from the Minkowski metric $\eta_{\mu\nu}$, are equal to those of the graviton. Finally, there is a conceptually difficult method using β-functions[3] that enables one to recover the Einstein equation from string theory.

So far for the arguments in favor of string theory as a candidate for the unifying theory. We will now turn our discussion to the relation of string theory with the information loss paradox. In the beginning of this chapter we anticipated the importance of the AdS/CFT correspondence. This correspondence relates string theory in an Anti-de Sitter background of d-dimensions to a conformal field theory on the $(d-1)$-dimensional boundary. It was derived[4] by Maldacena in 1997 [27]. The fact that a theory that lives on the boundary of a spacetime can describe all that happens in the bulk strongly reminds us of the holographic principle. And not only does string theory exhibit a holographic character, it also tells us that, if correct, nature has a unitary time evolution and information is not lost in any regime. This is because the conformal field theory that effectively describes all physics in the bulk is unitary.

But this is not all, there is more evidence that string theory can solve the paradox. To understand this evidence we will discuss a problem that is closely

[3] β-Functions tell you how your physical quantities depend on the energy. It can be shown that setting these functions to zero, i.e. demanding that string theory is scale independent, gives the Einstein equation.

[4] Unfortunately, it is very difficult to test this correspondence directly. This has to do with the fact that in both theories one can only calculate quantities perturbatively and the low-energy limit in the one theory corresponds to the high energy limit of the other.

related to information loss and has to do with black hole entropy. The statistical mechanical interpretation of the entropy of a thermodynamical system is that it measures the number of microscopic states of the system. Classically, however, a black hole, is empty except for the pointlike singularity at its center. So it seems to have only one internal state and one would accordingly expect it to have zero entropy. If the identification of the black hole entropy with its area is correct, which implies that they have non-zero entropy, we would want string theory to tell us that black holes do have microstates. In the classical limit these microstates should reduce to our classical picture of a black hole consisting of just empty space except for the singularity.

In 1996 Strominger and Vafa [28] published an article in which they reported that using new techniques in string theory they were able to count the internal states of a specific kind of five-dimensional black hole. When they compared what they found to the area of the black hole they retrieved the relation $S_{\mathrm{BH}} = A/4$. This relation was by then known as the Bekenstein-Hawking area-entropy relation. Moreover, their article was shortly followed by a number of publications ([29], [30], [31]) in which also the Hawking radiation rate and Hawking temperature of this particular type of black hole (called near-extremal black hole) were correctly found from the string theory description of this black hole[5]. Although all these results were obtained for a type of black hole that has little to do with the physical Schwarzschild black hole, they are very promising. And what is more, they give way to an approach for solving the paradox that finally goes beyond speculation.

We would like to conclude this section by restating that we believe that

- Resolving the information loss paradox requires a theory of quantum gravity and string theory exhibits a number of features that makes it a very good candidate for such a theory.

- The fact that actual calculations can be done, makes this approach all the more attractive to pursue in the quest for a resolution of the paradox.

Before we get to work, we will shortly comment on Hawking's present opinion about the paradox in the next section.

6.5 Hawking's Solution Part 2: Path-Integral Approach

In the first section of this chapter we discussed a modification of quantum mechanics proposed by Hawking in order to solve the paradox. The modification was such that processes could be non-unitary and thus information could be lost. For more than two decades Hawking has strongly believed that information was lost in black hole formation and evaporation.

Somewhere in the past years, however, he started doubting that information could really be lost. And, finally, in the summer of 2004 he announced he had solved the paradox in favor of unitarity! In England Hawking is a celebrity and the announcement caught a lot of attention from the press. His talk, at the 17th

[5]In the next chapter we attempt to give the reader an idea of what this string theory description of black holes is.

International Conference on General Relativity and Gravitation held in Dublin, attracted an enormous crowd (for a nice account of this event and a transcript of the talk see [32]).

We will be very short on the content of the talk, because up till now no article that provides the calculations necessary to proof his statements has appeared[6]. Schematically he suggests the following. Black hole formation and evaporation can be thought of as a scattering process. Prepare a certain configuration such that it is very likely to form a black hole, then sit at infinity and send in a bunch of particles and wait a very long time and see what comes out. So far this may sound very familiar to you and you would probably guess that what will come out is Hawking radiation. At this point, however quantum mechanics comes into play and it tells you that since you are at infinity you can never be sure that a black hole really formed, no matter how certain it was in the classical theory. This is the fact that provides a way to maintain unitarity.

Processes as the one described above can be calculated by doing the Feynman path-integral over all histories. Unfortunately, how to do this is not at all trivial, but usually one starts with selecting a couple of histories that one assumes to give the most interesting results. From our previous discussion we assume these to be the geometries in which a black hole forms and the ones in which no black hole forms, the so-called trivial geometries. Now Hawking argues that the integral over the trivial geometries gives a unitary result, whereas the black hole geometries do not. But when one adds the two contributions the final result is unitary, this has to do with the fact that correlation functions decay exponentially in a black hole background. Therefore the final state in the black hole metric is independent of the initial state and accordingly does not contribute to the transition amplitude.

In the physics society Hawking has received a lot of criticism, because he violated an unwritten rule: never declare you have solved something, before you have actually done the calculations. Besides this there are a number of physicists who believe that Hawking's solution is not original. In the 2001 article [34] Maldacena reports a calculation that seems to be exactly the one that Hawking proposes. Hawking argues that Maldacena did not draw the right conclusions in that article, but many seem to doubt whether Hawking's conclusions are legitimate. To proof this he (or better: his students) will have to provide us with a calculation.

[6] Just before this thesis was finished, Hawking put an article [33] on the web following his annoucement of having solved the paradox. The article does not contain any information that was not already provided at his talk. In the acknowledgements he mentions that his student C. Galfard is still working on a proof to support his claims.

Chapter 7

Resolving the Paradox with String Theory?

In the previous chapter we introduced string theory as a consistent theory of quantum gravity that is successful in explaining black hole entropy microscopically, albeit only for a special class of black holes. Exactly how black holes are constructed in string theory and how their microscopic states are counted is way beyond the scope of this thesis. We will, however, try to give the reader a qualitative idea of what stringy black holes are and how black hole microstates can be identified in string theory.

In very rough lines, what we shall do in this chapter is compare the classical picture of a black hole to a special kind of stringy black hole proposed by Mathur and try to see whether this stringy black hole, that does have internal states as opposed to the classical black hole, is able to send out information about its internal states along with the Hawking radiation. This test is of course crucial if we want these stringy black holes to be able to solve the information loss paradox.

We must warn the reader that the stringy black hole lives in ten dimensions and in the classical limit it reduces to an exotic type of black hole that has little to do with the physical Schwarzschild black hole. Nevertheless, the outcome of this test can give us insight in whether or not the approach of Mathur is likely to be successful in solving the paradox in the long run.

7.1 Constructing Black Holes with Strings

To get an idea of how black holes arise in string theory, we must introduce a couple of concepts. First of all, there are the building blocks of string theory which are strings and D-branes. Strings are one-dimensional extended objects which can either be closed or open, i.e. their endpoints can be joined together to form a closed loop or not. For open strings their endpoints can either move freely in spacetime (Neumann boundary condition) or be fixed in spacetime (Dirichlet boundary condition). Strings live in a d dimensional spacetime[1] and

[1] There are five consistent string theories that live in 10 dimensions. They can all be related to one another by dualities (of which we will come to speak shortly) and since they can all be obtained in suitable limits from eleven dimensions, it is believed that they are all different

Dp-branes are usually defined as $(p+1)$ dimensional objects on which open strings with Dirichlet boundary conditions end. Since there is a momentum flow from the open string with fixed endpoints to the Dp-brane, the Dp-brane is itself a dynamical object.

Obviously, the strings can vibrate and when they are quantized, different vibration modes can be identified as being different particles. For example, one of the massless modes of the closed string can be identified as the graviton, similarly the photon is the massless vibration mode of the open string with free endpoints. If string theory is to be the unified theory of quantum mechanics and gravity, then all the Standard Model particles must arise from the string spectrum. Furthermore, the string length has to be of the order of the fundamental Planck length. In turn the string tension is inversely proportional to the string length squared, consequently massive string modes are of the order of the Planck mass and this energy scale is way out of reach experimentally. This means that all the Standard Model particles must in fact be contained in the massless string spectrum. You may worry that this seems unreasonable, since most known particles are massive, but compared to the Planck mass these masses are very small, so to first order they are zero and small symmetry-breaking effects could make them non-zero.

For this low energy regime, where only the massless fields play a role, there is an effective action that describes a supergravity theory. Essentially, this effective theory is obtained by describing a single string or brane in a background of the fields that arise from the massless string spectrum. The strength with which strings couple to each other is given by the string coupling constant g_s, on the other hand the D-brane tension is inversely proportional to the string coupling constant so branes will never arise in a perturbative theory for small g_s. This is why D-branes are called non-perturbative objects.

This is a good point to say some words about another important concept in string theory, namely dualities. Dualities are believed to be exact symmetries of string theory that are spontaneously broken. Let us first discuss T-duality. As we mentioned above the strings usually live in 10 dimensions, but this is of course not the dimensionality of the spacetime we experience. This should be solved by compactifying 6 of the 10 dimensions such that the volume of the compactified manifold is much smaller than the length scales that are currently experimentally accessible. T-duality is the duality between a theory compactified on a circle with radius R and a theory compactified on a circle with radius $1/R$. S-duality, on the other hand, is the duality between the strong coupling regime of one theory and the weak coupling regime of another theory. This is because it sends the coupling constant g_s to $1/g_s$.

Why these dualities are important in our discussion of black holes in string theory, is because there are black hole solutions of supergravity that arise from certain fundamental string configurations as well as ones that arise from certain D-brane configuration and these different solutions can be related by performing a number of S- and T-dualities.

At this point we have introduced the concepts one must know to appreciate what the geometries we wish to study have to do with strings and branes. In the following section we focus on the particular brane configuration that gives rise to the black hole geometry we are interested in. And we shall try to make it

limits of one and the same 11 dimensional theory, mysteriously called M-theory.

plausible that one can derive a classical black hole geometry for this system as well as a less naive picture that consists of a collection of geometries that may be identified with black hole microstates. After that we will end this -perhaps a bit superficial- introduction and we will state the geometries we have taken from string theory and start studying them thoroughly.

7.2 The D1-D5 System

The geometry we wish to investigate in this chapter is generated by the so-called D1-D5 system. This system is composed of n_p D1-branes that are wrapped around the y direction which is compactified on a circle of radius R. In addition, there are four more dimensions compactified on a 4-manifold M (which we shall take to be a 4-torus). And there are n_1 D5-branes wrapped around all five compactified directions. This picture is probably quite hard to visualize so now we invoke the S- and T-dualities to relate the D1-D5 system to the F-P system[2]. The F-P system is just a single string (called F) that is wrapped n_1 times around the compactified y direction and carries n_p units of momentum charge P in this same direction. To obtain this configuration the string has to be stretched long enough to wrap n_1 times around a circle of radius R. Since the string has a certain tension it gains energy when it is stretched. So you can imagine that a string, stretched very long and rolled up again, may have enough energy to cause the background in which it is embedded to curve highly. In fact, it is possible to give the string so much energy in this way that it generates a singularity! The supergravity solution that is produced by this very heavy string is asymptotically flat and has a singularity at its center ($r = 0$)[3]. But Mathur [35] claims that this geometry corresponds to the F-P system only naively, because it is a solution of the low energy supergravity equations away from $r = 0$. He claims that the singularity at $r = 0$ is not really allowed by the full string theory. He illustrates this as follows. The string that is wrapped along the y direction carries momentum. This momentum has to be bound to the y direction if the state is to be a bound state. So the momentum takes the form of traveling waves along the string. As a consequence of these transverse vibrations the string moves away from its center in the directions transverse to the y direction. In other words, upon looking closely one sees that the string is not confined to $r = 0$, this is an indication that the singular supergravity solution may not be correct. If one takes the transverse vibration modes of the string into account when going to the supergravity theory, one finds a different geometry for each vibration mode. Although, these geometries are still singular, it can be shown [36] that the singularities miraculously disappear upon going to the dual D1-D5 system.

So Mathur argues that the D1-D5 system generates a collection of geometries that all look very similar to the naive geometry, the only difference is that in each geometry the singularity is replaced by a smooth 'cap'. At large distances from the center all these smooth geometries look the same, but they all have a slightly different cap at their center. Mathur interprets these different caps are the different quantum states or microstates of the black hole. And it can be

[2] We just state that these two systems are dual without explaining it, because that requires a much deeper understanding of string theory.

[3] What the intermediate region looks like will be discussed in the next section.

shown [37] that counting these microstates gives you an entropy that corresponds to the macroscopic entropy that comes from the horizon area[4].

To summarize, Mathur claims that there is a classical geometry that corresponds to the D1-D5 system and a collection of geometries that should be interpreted as the microstates of the quantum mechanical version of the black hole corresponding to the D1-D5 system. He calls the quantum mechanical black hole a fuzzball, since up to the horizon it looks like a classical black hole, but inside the horizon it looks like a fuzzball, because of all the different microstates (see Fig. 7.1). His work is original, not because he was able to find the entropy by a count of microstates (that was already done for a similar stringy system by Strominger and Vafa [28]), nor because the found all these different geometries corresponding to one stringy system (that was done by Lunin, Maldacena and Maoz [36]), but because he interpreted these different geometries as the microstates of the black hole.

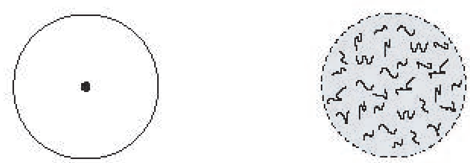

Figure 7.1: On the left a schematic picture of a classical black hole with, except for the singularity, empty space inside the horizon. On the right the fuzzball, the proposed quantum mechanical picture of a black hole. The microstates extend right up to the horizon.

In the next section we will study the classical black hole geometry and a generic case of the quantum mechanical black hole geometry, namely the geometry that corresponds to the simplest vibration mode of the F-P system. from now on we shall refer to these two geometries as the CBH (classical black hole) geometry and the QBH (quantum mechanical black hole) geometries respectively. In section 7.4 we will propose a calculation that allows us to compare the two geometries. Specifically, we are interested to see if the QBH geometries exhibit properties that may lead to non-local corrections to the classical limit, i.e. the CBH geometry.

There is a number of reasons why one expects that solving the paradox requires non-locality. The most naive way to put this is that non-local effects near the event horizon would allow the Hawking radiation to carry away information

[4]There are quite a number of subtleties here. Namely, the singular geometry that is generated here is the one of a massless (!) black hole. This follows from the fact that the horizon sits at $r = 0$, so the horizon area which is proportional to the mass squared, is zero. From this in turn we must conclude that the macroscopic entropy is zero! Whereas a counting of the different vibration modes of the string gives rise to a non-zero microscopic entropy. As a solution for this discrepancy between the macroscopic and microscopic entropy Mathur proposes to identify the horizon as the boundary of the region where the non-naive geometries differ from each other, i.e. the boundary of the caps. And in fact, it can be shown that this identification gives a macroscopic entropy that corresponds well with the microscopic entropy [37].

from the region inside the black hole horizon to infinity and thus restore the information that would otherwise be lost in the process of black hole evaporation. Another argument comes from holography. Remember that according to the AdS/CFT correspondence one can describe everything that happens in a $(d+1)$-dimensional volume with a conformal field theory that lives in d dimensions, which is local and unitary. The fact that the theory on the boundary is local actually implies that the theory in the bulk can not be local. The example we discussed before about entropy illustrates this. In a local theory the entropy of a system scales with the volume of that system, however, holography tells us that it scales with the surface area of the system. This is clearly a non-local feature, since it requires that all the information about the internal system is stored on its surface. So according to the AdS/CFT correspondence we should expect that quantum gravity is a non-local theory[5]. An elaborate discussion of this argument can be found in [38].

7.3 Properties of The Black Hole Geometries

In this section we shall study the CBH and the QBH geometries, in particular we shall give a short introduction to Anti-de-Sitter spacetime (AdS for short) and BTZ black holes (for a nice review see [39]).

7.3.1 Anti-de-Sitter spacetime and the BTZ black hole

AdS is a solution to the Einstein equation with a negative cosmological constant. It is a hyperbolic spacetime, which means that it has a constant negative curvature. Just like a 3-sphere (S^3) can be embedded in $\mathbb{R}^4$, one can embed AdS_3 in $\mathbb{R}^{(2,2)}$. So the metric of this four-dimensional flat spacetime with signature $(--++)$ is[6]

$$ds^2 = -dU^2 - dV^2 + dX^2 + dY^2, \tag{7.1}$$

and AdS_3 is defined by the hypersurface

$$-U^2 - V^2 + X^2 + Y^2 = -l^2.$$

If we choose the hyperbolic coordinates χ, ϕ and t/l such that

$$-V^2 + X^2 = -l^2 \cosh^2\chi \quad \text{and} \quad -U^2 + Y^2 = l^2 \sinh^2\chi$$

and $V = l\cosh\chi\cosh\phi$, $X = l\cosh\chi\sinh\phi$, $U = l\sinh\chi\sinh(t/l)$ and $Y = l\sinh\chi\cosh(t/l)$, the metric for AdS_3 that follows from (7.1) is

$$ds^2 = -\sinh^2\chi dt^2 + l^2(d\chi^2 + \cosh^2\chi d\phi^2).$$

With a final coordinate transformation defined by

$$r = l\cosh\chi$$

[5]Note that this means that if you believe in holography it is actually a miracle that in a classical limit gravity behaves like a local theory!

[6]In this spacetime there are closed timelike curves, but we shall assume that this periodicity in the timelike direction has been removed by going to the universal covering space. Both AdS spacetimes, with or without periodicity in the timelike direction, are solutions to the Einstein equation with a negative cosmological constant $\Lambda = -1/l^2$.

we find the metric of AdS_3 in "Schwarzschild" coordinates

$$ds^2 = -\Big(\frac{r^2}{l^2} - 1\Big)dt^2 + \Big(\frac{r^2}{l^2} - 1\Big)^{-1} dr^2 + r^2 d\phi^2. \tag{7.2}$$

Notice that ϕ, t and χ range from minus infinity to plus infinity and consequently we have $0 < r < \infty$. There is a coordinate singularity at $r = l$, but this is merely an artifact of the coordinates, since AdS is a smooth spacetime with no singularities. As in Euclidean space it can be proven that in AdS spacetime two geodesics that intersect each other, do so in only one point. On the other hand, two geodesics that do not intersect have a minimum distance between them (as opposed to non-intersecting geodesics in Euclidean space which are always equidistant). This minimum distance is such that there is a unique geodesic segment of minimal length that connects the two geodesics at right angles.

The fact that we chose to write the metric for AdS_3 in "Schwarschild" coordinates is of course no coincidence. It is now straightforward to obtain the simplest BTZ metric[7], namely by making the following identification: $\phi \to \phi + 2\pi$. This BTZ metric describes a 2+1 dimensional non-rotating black hole. The minimum distance between the two identified geodesics occurs at $r = l$ and is $2\pi l$. This is referred to as the horizon "area", since it is the minimum distance around the black hole. Clearly, this horizon size changes when we choose a different identification for ϕ. The standard form of the BTZ metric is obtained by first taking a more general period for ϕ, namely $2\pi a$, and then redefining the coordinates so that ϕ has its usual period

$$\phi \to a\phi \quad r \to r/a \quad t \to t/a.$$

The BTZ metric then reads

$$ds^2 = -\Big(\frac{r^2}{l^2} - m\Big)dt^2 + \Big(\frac{r^2}{l^2} - m\Big)^{-1} dr^2 + r^2 d\phi^2, \tag{7.3}$$

where ϕ has period 2π and $m = 1/a^2$. The minimal length around the black hole is now $2\pi l\sqrt{m}$ and the dimensionless quantity m is called the mass parameter[8].

The BTZ metric is asymptotically AdS. This can be seen by writing the AdS metric in different coordinates. Take the $\mathbb{R}^{(2,2)}$ coordinates to be

$$\begin{aligned} V &= l\cosh\chi \sin\frac{t}{l} \\ U &= -l\cosh\chi\cos\frac{t}{l} \\ X &= l\sinh\chi\sin\theta \\ Y &= -l\sinh\chi\cos\theta, \end{aligned}$$

this leads to the metric

$$ds^2 = -\cosh^2\chi dt^2 + l^2(d\chi^2 + \sinh^2\chi d\theta^2),$$

[7]This three-dimensional black hole was first found by Bañados, Teitelboim and Zanelli ([40] and [41]), hence the name BTZ black hole.

[8]To construct a rotating BTZ black hole one has to apply another identification simultaneously with the identification $\phi \to \phi + 2\pi$, namely $t \to t + \frac{\pi J}{m}$. In this way one obtains a rotating black hole with mass $M = m + \frac{J^2}{4ml^2}$ and angular momentum J. For details see [39].

where χ and t are given the full range, but θ is now periodic: $0 < \theta < 2\pi$. Finally, upon substituting $r = l \sinh \chi$, we find

$$ds^2 = -\Big(\frac{r^2}{l^2} + 1\Big)dt^2 + \Big(\frac{r^2}{l^2} + 1\Big)^{-1} dr^2 + r^2 d\theta^2. \tag{7.4}$$

If we now compare this with (7.3) we see that they become equal for $r \to \infty$. So the BTZ metric is indeed asymptotically AdS. Furthermore, we see that for $m = -1$ the BTZ metric is just AdS.

A special type of BTZ black hole is the $m = 0$ case. This is the one we are particularly interested in, because this is the type of black hole that is generated by the D1-D5 system in the classical limit. Since the horizon size of a BTZ black hole is given by $2\pi l\sqrt{m}$, we find that the $m = 0$ BTZ black hole has zero horizon size. It is not really clear if one should consider the $m = 0$ BTZ black hole as a black hole with a horizon at $r = 0$ or a singularity at $r = 0$ or both[9]. For our purposes it suffices to acknowledge that it is a black hole.

7.3.2 The Classical Black Hole Geometry

The low energy supergravity solution that corresponds to the D1-D5 system away from $r = 0$ is the CBH geometry. Remember that the branes are wrapped around compactified dimensions. The y-direction is compactified on a circle of radius R and the four z_a-directions are compactified on a four-torus[10] T^4. The CBH geometry is given by the following metric

$$\begin{aligned} ds^2 &= \frac{1}{h}(-dt^2 + dy^2) + h\sum_{i=0}^{3} dx_i dx_i \\ &+ \frac{\sqrt{1+\frac{Q_1}{r^2}}}{\sqrt{1+\frac{Q_5}{r^2}}}\sum_{a=0}^{3} dz_a dz_a, \end{aligned} \tag{7.5}$$

with

$$h = \Big[\Big(1 + \frac{Q_1}{r^2}\Big)\Big(1 + \frac{Q_5}{r^2}\Big)\Big]^{1/2}, \tag{7.6}$$

where $r = \sum x_i^2$ and Q_1 and Q_5 proportional to the number of D1- and D5-branes respectively[11]. The coordinate y is periodic, it runs from zero to $2\pi R$.

[9]We have to be a bit careful with the word singularity here. There is a coordinate singularity for $r = 0$, but as we know that does not yet make it a true singularity. In fact it can be shown that the spacetime is smooth at $r = 0$ and it can be continued to negative r^2. However, in that case ϕ becomes timelike and since ϕ is periodic, this means that there are closed timelike curves in the negative r^2 region. Since closed timelike curves are unphysical, the common practice is to remove them from the spacetime. Hence we shall restrict the metric to $r > 0$ and call the removed point $r = 0$ a singularity.

[10]This torus will play a role of little importance for reasons that will become clear in the next section.

[11]As we said in the previous section, the metric was first obtained for the F-P system and then by dualities transformed into this metric for the D1-D5 system. The dual F-P system consists of n_1 strings (F) and the string carries n_p units of momentum (P). The duality transformation can be represented schematically as follows P(n_p) F(n_1) $\leftrightarrow$ D1(n_p) D5(n_1), so $Q_1 \propto n_p$ and $Q_5 \propto n_1$. Both Q_1 and Q_5 have units [length]^2.

When we introduce polar coordinates for the x_i-directions, the metric becomes

$$\begin{aligned} ds^2 &= \frac{1}{h}(-dt^2 + dy^2) + hdr^2 \\ &+ hr^2\Big(d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2\Big) \\ &+ \frac{\sqrt{1+\frac{Q_1}{r^2}}}{\sqrt{1+\frac{Q_5}{r^2}}}\sum_{a=0}^{3} dz_a dz_a, \end{aligned} \tag{7.7}$$

where $r > 0$, θ runs from zero to π and ϕ and ψ have period 2π, so the second line of (7.7) is just a 3-sphere. Finally, t runs from minus to plus infinity.

To understand better what kind of spacetime we have at hand let us zoom in on two regions of this spacetime. First, we let r go to infinity. From (7.6) we see that in that limit $h \to 1$, so (7.7) reduces to

$$\begin{aligned} ds^2 &\approx -dt^2 + dy^2 + dr^2 \\ &+ r^2\Big(d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2\Big) \\ &+ \sum_{a=0}^{3} dz_a dz_a, \end{aligned} \tag{7.8}$$

which is just flat (ten-dimensional) spacetime (with 4 compactified dimensions and a circle). Note that in this limit the radius of the 3-sphere goes to infinity as r^2. On the other hand, when we take the limit $r^2 \ll Q_1Q_5$, we find that $h \approx \frac{\sqrt{Q_1Q_5}}{r^2}$ and the factor in front of the four-torus becomes approximately $\sqrt{Q_1/Q_5}$. So in this limit the metric reduces to

$$\begin{aligned} ds^2 &\approx \frac{r^2}{\sqrt{Q_1Q_5}}(-dt^2 + dy^2) + \frac{\sqrt{Q_1Q_5}}{r^2}dr^2 \\ &+ \sqrt{Q_1Q_5}\Big(d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2\Big) \\ &+ \sqrt{\frac{Q_1}{Q_5}}\sum_{a=0}^{3} dz_a dz_a. \end{aligned} \tag{7.9}$$

First of all, we see that the 3-sphere becomes of a fixed size (as does the 4-torus). Furthermore, the first line is proportional to the metric of the $m = 0$ BTZ black hole. This can be seen as follows

$$\begin{aligned} &\frac{r^2}{\sqrt{Q_1Q_5}}(-dt^2 + dy^2) + \frac{\sqrt{Q_1Q_5}}{r^2}dr^2 \\ = &\sqrt{Q_1Q_5}\Big(\frac{r^2}{Q_1Q_5}(-dt^2 + dy^2) + \frac{dr^2}{r^2}\Big) \\ = &\sqrt{Q_1Q_5}\Big(-r'^2dt'^2 + r^2dy'^2 + \frac{dr'^2}{r'^2}\Big), \end{aligned}$$

with $t = Rt'$, $r = r'/(R\sqrt{Q_1Q_5})$ and $y = Ry'$, so $0 < y' < 2\pi$. So this part of the metric is proportional to the $m = 0$ BTZ metric, as can be seen by setting $l = 1$ and $m = 0$ in (7.3).

So the CBH geometry consists of an asymptotically flat region, an inner region for which the 3-sphere has a fixed size, this is called the throat and finally a horizon/singularity at its center ($r = 0$). Fig. 7.2 pictures the geometry schematically.

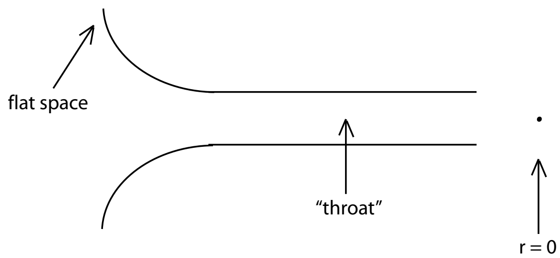

Figure 7.2: A schematic picture of the CBH geometry, with an asymptotically flat region, a throat and a singularity at $r = 0$.

7.3.3 The Quantum Mechanical Black Hole Geometries

In this section we shall study the geometry of one of the microstates of the fuzzball. As we mentioned in the previous section, the geometries of the different microstates are obtained from different vibration profiles of the F-P system. The geometries that correspond to the D1-D5 system are then obtained after performing a number of dualities. The microstate we shall study corresponds to a very simple vibration profile of the F string. Although, we will not derive the geometry from this vibration profile, but merely state the result, we shall for completeness give the explicit profile that corresponds to the metric we shall study. The string is restricted to bend only in the 4 noncompact directions x_i. So the displacement profile of the string is given by a four-vector $\vec{F}(v)$, where $v = t - y$. For a string with all its energy in the lowest harmonic the vibration profile is

$$F_1 = \hat{a}\cos\omega v, \quad F_2 = \hat{a}\sin\omega v, \quad F_3 = F_4 = 0,$$

where $\hat{a}$ is a constant and $\omega = (n_1 R')^{-1}$, with R' the radius of the compactified direction y before dualities and n_1 the number of strings in the system. After dualities the geometry that corresponds to this profile is found to be (see [42])

$$\begin{aligned} ds^2 &= \frac{1}{h}(-dt^2 + dy^2) + hf\Big(d\theta^2 + \frac{dr^2}{r^2 + a^2}\Big) \\ &- \frac{2a\sqrt{Q_1 Q_5}}{hf}(\cos^2\theta d\psi dy + \sin^2\theta d\phi dt) \\ &+ h\Big[\Big(r^2 + \frac{a^2 Q_1 Q_5 \cos^2\theta}{h^2 f^2}\Big)\cos^2\theta d\psi^2 \\ &\quad +\Big(r^2 + a^2 - \frac{a^2 Q_1 Q_5 \sin^2\theta}{h^2 f^2}\Big)\sin^2\theta d\phi^2\Big] \\ &+ \frac{\sqrt{Q_1 + f}}{\sqrt{Q_5 + f}}\sum_{a=0}^{3} dz_a dz_a, \end{aligned} \tag{7.10}$$

with h given in (7.6), $f = r^2 + a^2\cos^2\theta$ and a proportional to $\hat{a}$. The constants Q_1 and Q_5 are the same as in the previous subsection. Finally, it can be shown that $a = \sqrt{Q_1 Q_5}/R$ ([35]), so as expected a has units [length]. From now on we shall call this geometry the SQBH geometry, where the S stands for simple, since it was derived for a very simple vibration profile.

Just as we did for the CBH geometry, we shall zoom in on some particular regions of this spacetime to get a better understanding of its properties. In this case, however, we have three regions of interest instead of two (this is of no great surprise, when you realize that we have introduced a second lengthscale a beside the original one: $(Q_1Q_5)^{1/4}$).

First we investigate the asymptotic region: $r \to \infty$. In that case we find that h approximates unity again and $f \approx r^2$ and we can neglect all $1/f$ terms. Finally, also the term in front of the four-torus approaches unity in this limit. So we find that the metric again has a flat asymptotic region

$$\begin{aligned} ds^2 &\approx -dt^2 + dy^2 + r^2\Big(d\theta^2 + \frac{dr^2}{r^2}\Big) \\ &+ \Big[r^2\cos^2\theta d\psi^2 + r^2\sin^2\theta d\phi^2\Big] \\ &+ \sum_{a=0}^{3} dz_a dz_a, \end{aligned} \tag{7.11}$$

and again the radius of the 3-sphere goes to infinity in this region (compare with (7.8)).

The inner region, i.e. $r \ll (Q_1Q_5)^{1/4}$, can be split into two regions, one with $(Q_1Q_5)^{1/4} \gg r \gg a$ which is called the near horizon region and the other with $r \ll a$ which is called the cap. Let us first go to the inner region, by taking $r \ll (Q_1Q_5)^{1/4}$. In this limit we find that $h \approx \frac{\sqrt{Q_1Q_5}}{f}$, so the metric becomes

$$\begin{aligned} ds^2 &\approx \frac{f}{\sqrt{Q_1Q_5}}(-dt^2 + dy^2) + \sqrt{Q_1Q_5}\Big(d\theta^2 + \frac{dr^2}{r^2+a^2}\Big) \\ &- 2a(\cos^2\theta d\psi dy + \sin^2\theta d\phi dt) \\ &+ \frac{\sqrt{Q_1Q_5}}{f}\Big[\Big(r^2 + a^2\cos^2\theta\Big)\cos^2\theta d\psi^2 + \Big(r^2 + a^2 - a^2\sin^2\theta\Big)\sin^2\theta d\phi^2\Big] \\ &+ \sqrt{\frac{Q_1}{Q_5}}\sum_{a=0}^{3} dz_a dz_a. \end{aligned}$$

The third line is actually just $\sqrt{Q_1Q_5}(\cos^2\theta d\psi^2 + \sin^2\theta d\phi^2)$, as can be seen by taking a factor $f = r^2 + a^2\cos^2\theta$ out of the big square brackets, which cancels with the $1/f$ in front of it. Furthermore, we would like to get rid of the crossterms in the second line. To do so we collect all the terms containing dt, dy, $d\phi$ and/or $d\psi$ and massage this a little:

$$\begin{aligned} & \frac{f}{\sqrt{Q_1Q_5}}(-dt^2 + dy^2) - 2a(\cos^2\theta d\psi dy + \sin^2\theta d\phi dt) \\ +\ & \sqrt{Q_1Q_5}\Big[\cos^2\theta d\psi^2 + \sin^2\theta d\phi^2\Big] \\ =\ & -(r^2+a^2)\frac{dt^2}{\sqrt{Q_1Q_5}} + r^2\frac{dy^2}{\sqrt{Q_1Q_5}} + a^2\sin^2\theta\frac{dt^2}{\sqrt{Q_1Q_5}} + a^2\cos^2\theta\frac{dy^2}{\sqrt{Q_1Q_5}} \end{aligned}$$

$$
\begin{aligned}
&- \quad 2a(\cos^2\theta d\psi dy + \sin^2\theta d\phi dt) + \sqrt{Q_1Q_5}\Big[\cos^2\theta d\psi^2 + \sin^2\theta d\phi^2\Big] \\
&= \quad -(r^2+a^2)\frac{dt^2}{\sqrt{Q_1Q_5}} + r^2\frac{dy^2}{\sqrt{Q_1Q_5}} \\
&+ \quad \sqrt{Q_1Q_5}\Big[\cos^2\theta\Big(d\psi - \frac{ady}{\sqrt{Q_1Q_5}}\Big)^2 + \sin^2\theta\Big(d\phi - \frac{adt}{\sqrt{Q_1Q_5}}\Big)^2\Big].
\end{aligned}
$$

To get a better idea of what the inner region looks like, we need to transform to new angular coordinates

$$\psi' = \psi - \frac{ady}{\sqrt{Q_1Q_5}} \qquad \phi' = \phi - \frac{adt}{\sqrt{Q_1Q_5}}.$$

Notice that on the old coordinates we had the identifications

$$(\psi, y) \sim (\psi + 2\pi, y) \sim (\psi, y + 2\pi R),$$

whereas on the new coordinates they have become

$$(\psi', y) \sim (\psi' + 2\pi, y) \sim (\psi' - \frac{a2\pi R}{\sqrt{Q_1Q_5}} = \psi' - 2\pi, y + 2\pi R),$$

where we used $a = \sqrt{Q_1Q_5}/R$. In these new coordinates the metric in the inner region becomes

$$
\begin{aligned}
ds^2 \quad &\approx \quad -(r^2+a^2)\frac{dt^2}{\sqrt{Q_1Q_5}} + r^2\frac{dy^2}{\sqrt{Q_1Q_5}} + \sqrt{Q_1Q_5}\frac{dr^2}{r^2+a^2} \\
&+ \quad \sqrt{Q_1Q_5}\Big[d\theta^2 + \cos^2\theta d\psi'^2 + \sin^2\theta d\phi'^2\Big] \\
&+ \quad \sqrt{\frac{Q_1}{Q_5}}\sum_{a=0}^{3} dz_a dz_a.
\end{aligned}
\tag{7.12}
$$

The first line is actually just AdS_3, this can be seen as follows

$$
\begin{aligned}
& -(r^2+a^2)\frac{dt^2}{\sqrt{Q_1Q_5}} + r^2\frac{dy^2}{\sqrt{Q_1Q_5}} + \sqrt{Q_1Q_5}\frac{dr^2}{r^2+a^2} \\
= \quad & \sqrt{Q_1Q_5}\Big(-(r'^2+1)\frac{a^2dt^2}{Q_1Q_5} + r'^2\frac{a^2dy^2}{Q_1Q_5} + \frac{dr'^2}{r'^2+1}\Big) \\
= \quad & \sqrt{Q_1Q_5}\Big(-(r'^2+1)dt'^2 + r'^2dy'^2 + \frac{dr'^2}{r'^2+1}\Big),
\end{aligned}
$$

where $r' = r/a$, $t' = ta/\sqrt{Q_1Q_5} = t/R$ and $y' = ya/\sqrt{Q_1Q_5} = y/R$, so the period of the new coordinate y' is again 2π. This is indeed AdS_3 (see (7.4) with $l = 1$).

So the inner region is $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ and the 3-sphere and the 4-torus have a fixed size in this region. Just as the CBH geometry, the SQBH geometry consists of an asymptotically flat region and then for smaller r is has a throat with local geometry $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$. In fact, the near horizon region, i.e. $a \ll r \ll (Q_1Q_5)^{1/4}$, is exactly the same as the inner region of the CBH geometry. This can be seen by setting a to zero in (7.12) and comparing this with (7.9). This is no surprise because the CBH geometry is exactly the SQBH geometry with a set to zero (see (7.10))!

Clearly, the difference between the two geometries becomes manifest when r becomes of the order of a. And by taking the limit $r \ll a$ in (7.12) it is easily seen that the metric becomes

$$\begin{aligned} ds^2 &\approx -a^2 \frac{dt^2}{\sqrt{Q_1 Q_5}} + r^2 \frac{dy^2}{\sqrt{Q_1 Q_5}} + \sqrt{Q_1 Q_5} \frac{dr^2}{a^2} \\ &+ \sqrt{Q_1 Q_5} \Big[d\theta^2 + \cos^2\theta d\psi'^2 + \sin^2\theta d\phi'^2 \Big] \\ &+ \sqrt{\frac{Q_1}{Q_5}} \sum_{a=0}^{3} dz_a dz_a, \end{aligned} \tag{7.13}$$

which is smooth, so there is no singularity at $r = 0$. This region is called the cap. For different vibration profiles of the string one finds similar asymptotic and near horizon regions, but different caps ([35]).

So we have found that the SQBH geometry looks very much like the CBH geometry for large enough values of r. The asymptotic region is flat. When moving towards the center there seems to be an $m = 0$ BTZ black hole with a throat at the center of the geometry. But upon moving even closer one finds that there is no horizon nor singularity at the center, instead the geometry is smooth at $r = 0$. It is often said that 'the throat is sealed off with a smooth cap' (see Fig. 7.3).

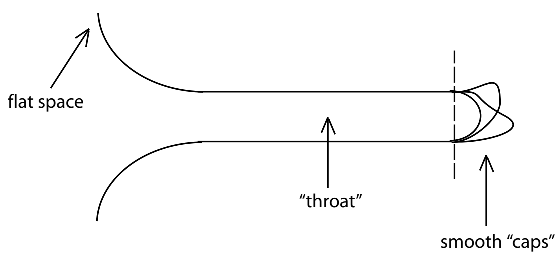

Figure 7.3: A schematic picture of the SQBH geometry, with an asymptotically flat region and a throat that is sealed off with a smooth cap, so there is no singularity at $r = 0$.

7.4 Subjecting The Fuzzball Proposal to a Test

We have seen that there is problem that is closely related to the information loss paradox, which has to do with the black hole entropy. There are two ways to calculate the entropy of a system. In one approach it is derived from macroscopic properties of the system and in the other it is derived from the number of microscopic states which the system has access to. Either way the answer should of course be the same. For classical black holes, however, the entropy derived macroscopically is non-zero, whereas the microscopic approach yields zero entropy.

Already in 1996 Strominger and Vafa discovered that string theory provided an answer to this problem. Roughly speaking, the answer is that black holes do have microstates, but they can only be seen when black holes are discussed in a full theory of quantum gravity. In the fuzzball proposal of Mathur it is made explicit what these black hole microstates are. They correspond to different vibration modes of the string system.

If the fuzzball proposal is to solve the information loss paradox, it is not enough that it has microstates, in addition it has to be able to communicate information about these microstates to the region outside the black hole. For a classical black hole the regions inside and outside the horizon are causally disconnected, this means that there is no correlation between the value of a matterfield at a position X inside the horizon and its value at a position Y outside the horizon. So the correlation function (also called Green's function or propagator) of the field between positions X and Y vanishes. What we would thus like to see is that the fuzzball picture yields corrections to the correlation function such that the black hole can sent out information about its microstates. This is the test we propose to do to find out whether or not the fuzzball is not only a solution to the entropy problem but maybe also to the information loss paradox.

Ideally we would calculate the correlation function of a matterfield[12] in the CBH geometry as well as in the SQBH geometry and compare the results. Unfortunately, the differential equation we run into in this calculation cannot be solved analytically. In fact, it turned out that finding the correlation function for the CBH geometry was already too difficult a task. What we did instead was solving the differential equation for the matterfield in both geometries and then, using the fact that the SQBH metric reduces to the CBH metric for $a \to 0$, writing the matterfield in the SQBH geometry as the matterfield in the CBH geometry plus corrections. Finally, from this we try to extract information about the difference between the correlation functions in the different metrics.

The critical reader may have noted something funky about what we just proposed. We want to write the wavefunction that lives in one spacetime as the wavefunction that lives in *another* spacetime plus corrections. The wavefunctions are functions of the coordinates of *different* spacetimes, so comparing the wavefunctions is a bit like comparing apples to oranges. However, we justify this comparison, because the spacetimes are very similar and in the limit of $a \to 0$ they are equal. So by setting a to zero we can check that the coordinates we choose in the SQBH spacetime reduce to the same coordinates in the CBH spacetime. This enables us to be quite confident in saying where a point X in one spacetime would be in the other spacetime.

In the following sections we shall start by solving the wave equation for a massless scalar field in the CBH geometry. Then we shall use perturbation theory to find the solution of the wave equation in the SQBH geometry to first order in a. After we normalize both these solutions we shall concentrate on the correlation function. In the final section we present our results and propose a number of ideas for successive research.

7.5 Solving The Wave Equation

We have already discussed the Klein-Gordon equation in a curved spacetime in section 3.3. The wave equation for the massless scalar field Φ in a curved

[12] We shall restrict ourselves to the massless scalarfield.

spacetime with metric $g_{\mu\nu}$ reads

$$\Box\Phi \equiv \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}g^{\mu\nu}\partial_\mu\Phi) = 0.$$

To solve this equation we shall have to plug in the metrics for the D1-D5 system given in the previous sections. As you can expect this is going to be a pretty messy calculation, so we would like to simplify things as much as possible. We want to compare two geometries, so we are particularly interested in their differences. We noted that the SQBH and the CBH geometry are equal for a zero. So clearly the non-local effects, if there are any, will come about when a becomes non-zero. Now we note that for both the CBH metric as well as the SQBH metric the four-torus looks exactly the same, whether a is zero or not. This suggests that the four-torus cannot be the source of any differences in locality between the two geometries. So to make things easier we shall just discard the whole four-torus and compare two six-dimensional geometries instead. For the CBH metric the six-dimensional part of interest is

$$\begin{aligned} ds^2 &= \frac{1}{h}(-dt^2 + dy^2) + hdr^2 \\ &+ hr^2\Big(d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2\Big), \end{aligned} \tag{7.14}$$

and for the SQBH metric this is

$$\begin{aligned} ds^2 &= \frac{1}{h}(-dt^2 + dy^2) + hf\Big(d\theta^2 + \frac{dr^2}{r^2 + a^2}\Big) \\ &- \frac{2a\sqrt{Q_1 Q_5}}{hf}(\cos^2\theta d\psi dy + \sin^2\theta d\phi dt) \\ &+ h\Big[\Big(r^2 + \frac{a^2 Q_1 Q_5 \cos^2\theta}{h^2 f^2}\Big)\cos^2\theta d\psi^2 \\ &\quad + \Big(r^2 + a^2 - \frac{a^2 Q_1 Q_5 \sin^2\theta}{h^2 f^2}\Big)\sin^2\theta d\phi^2\Big]. \end{aligned} \tag{7.15}$$

7.5.1 The Klein-Gordon Equation in The CBH Geometry

The CBH metric is diagonal, so its inverse is trivially found. This also means that the wave equation consists of six terms, each of them containing derivatives with respect to one variable only. Finally, we note that the metric is a function of r and θ. With these observations we can write the Klein-Gordon equation as

$$\Box\Phi = \Box_{r,\theta}\Phi + \Box_{t,\phi,y,\psi}\Phi = 0,$$

with

$$\begin{aligned} \Box_{r,\theta}\Phi &= \frac{1}{\sqrt{-g}}\partial_r(\sqrt{-g}g^{rr}\partial_r\Phi) + \frac{1}{\sqrt{-g}}\partial_\theta(\sqrt{-g}g^{\theta\theta}\partial_\theta\Phi) \\ \Box_{t,y,\phi,\psi}\Phi &= g^{tt}\partial_t^2\Phi + g^{yy}\partial_y^2\Phi + g^{\phi\phi}\partial_\phi^2\Phi + g^{\psi\psi}\partial_\psi^2\Phi. \end{aligned}$$

The fact that the CBH metric has four Killing vectors ∂_t, ∂_y, ∂_ϕ and ∂_ψ, i.e. the metric does not depend on t, y, ϕ and ψ (so $\partial_t g_{\mu\nu} = 0$, etcetera), suggests us to write the wavefunction as $\Phi = e^{-\imath\omega t + \imath\lambda y + \imath\nu\psi + \imath\mu\phi}\Psi(r,\theta)$. Physically this can be understood as follows. When the metric is independent of a certain coordinate, there is no curvature in that direction, thus a particle moving in this spacetime will feel no force in that direction and moves freely. Its wavefunction is then just a plane wave with a constant frequency. This frequency is the conserved charge associated to the Killing vector (see section 1.2).

Upon writing the solution like an eigenfunction of the Killing vectors the differential equation for Φ reduces to a differential equation for Ψ depending only on two variables. When we plug in this expression for Φ and the components of the inverse of the CBH metric (7.14), we find that

$$\begin{aligned}\Box_{t,y,\phi,\psi}\Phi &= (hr^2)^{-1}\Big\{\omega^2\Big[\frac{Q_1Q_5}{r^2}+Q_1+Q_5+r^2\Big]\\ &- \lambda^2\Big[\frac{Q_1Q_5}{r^2}+Q_1+Q_5+r^2\Big]-\frac{\mu^2}{\sin^2\theta}-\frac{\nu^2}{\cos^2\theta}\Big\}\Phi.\end{aligned}$$

And when we use that the determinant of the metric is $g=-h^2r^6\sin^2\theta\cos^2\theta$, we find that the remaining part of the wave equation reads

$$\Box_{r,\theta}\Phi = (hr^2)^{-1}\Big\{\frac{1}{r}\partial_r(r^3\partial_r\Phi)+\frac{1}{\sin 2\theta}\partial_\theta(\sin 2\theta\partial_\theta\Phi)\Big\}.$$

After dividing out the factor $(hr^2)^{-1}$ and the exponential part of the wavefunction, we can use the separation of variables trick again (we write $\Psi(r,\theta)=H(r)\Theta(\theta)$) to obtain the following radial and angular equations

$$\frac{1}{r}\partial_r\Big(r^3\partial_r H(r)\Big)+\Big\{(\omega^2-\lambda^2)\Big[\frac{Q_1Q_5}{r^2}+Q_1+Q_5+r^2\Big]-\Lambda\Big\}H(r)=0 \quad (7.16)$$

and

$$\frac{1}{\sin 2\theta}\partial_\theta\Big(\sin 2\theta\partial_\theta\Theta(\theta)\Big)-\Big\{\frac{\mu^2}{\sin^2\theta}+\frac{\nu^2}{\cos^2\theta}\Big\}\Theta(\theta)=-\Lambda\Theta(\theta), \quad (7.17)$$

where Λ is the 'separation constant'.

The Angular Equation

The angular equation is just the Laplace equation on S^3. Its eigenvalues are known to be $k(k+2)$. So for $\Lambda=k(k+2)$ the solutions of (7.17) are

$$\Theta_{k\mu\nu}(\theta)=\cos^{|\nu|}\theta\sin^{|\mu|}\theta P_c^{(|\mu|,|\nu|)}(\cos 2\theta), \quad (7.18)$$

where $P_c^{(|\mu|,|\nu|)}$ is the Jacobi polynomial

$$P_c^{(|\mu|,|\nu|)}(u)=\frac{1}{2^c}\Sigma_{i=0}^c\left(\begin{array}{c}|\mu|+c\\ i\end{array}\right)\left(\begin{array}{c}|\nu|+c\\ c-i\end{array}\right)(u+1)^i(u-1)^{d-i},$$

and

$$c=\frac{k-(|\mu|+|\nu|)}{2}$$

and k, μ and ν are integers, $|\mu|+|\nu|\le k$ and $\mu+\nu\equiv k(\text{mod } 2)$.

The Radial Equation

The remaining task is to solve the radial equation (7.16). Unfortunately, we can not find the exact solution to this equation. However, we can solve it in the limit that $r^2 \ll \sqrt{Q_1Q_5}$. We found in the previous sections that the CBH and the SQBH geometry look very similar for large values of r. And, as we said, we are interested in the differences between the geometries. So we can argue again that we do not loose any crucial information when we restrict ourselves to the inner region of the spacetimes.

In this region, i.e. for $r^2 \ll \sqrt{Q_1Q_5}$, the radial equation (7.16) reduces to

$$\frac{1}{r}\partial_r\Big(r^3\partial_r H(r)\Big) + \Big\{(\omega^2-\lambda^2)\frac{Q_1Q_5}{r^2} - \Lambda\Big\}H(r) = 0.$$

With the substitution $z = 1/r$ this can be rewritten as

$$z^3\partial_z\Big(\frac{1}{z}\partial_z H(z)\Big) + \Big\{(\omega^2-\lambda^2)z^2(Q_1Q_5) - \Lambda\Big\}H(z) = 0.$$

Now put $H(z) = zG(z)$ to obtain for $G(z)$

$$\begin{aligned} z^2\partial_z^2 G(z) + z\partial_z G(z) + (\omega^2-\lambda^2)Q_1Q_5 z^2 G(z) \\ -(\Lambda+1)G(z) = 0. \end{aligned}$$

This is Bessel's differential equation and its solutions are[13]

$$G(z) = C_1 J_\sigma(\sqrt{Q_1Q_5(\omega^2-\lambda^2)}z) + C_2 Y_\sigma(\sqrt{Q_1Q_5(\omega^2-\lambda^2)}z),$$

with

$$\sigma = \sqrt{\Lambda+1} = k+1,$$

since $\Lambda = k(k+2)$. So the solution to the radial equation is

$$H(r) = \frac{1}{r}\Big[C_1 J_\sigma\Big(\frac{\sqrt{Q_1Q_5(\omega^2-\lambda^2)}}{r}\Big) + C_2 Y_\sigma\Big(\frac{\sqrt{Q_1Q_5(\omega^2-\lambda^2)}}{r}\Big)\Big]. \quad (7.19)$$

So finally we found the eigenmode solutions to the massless Klein-Gordon equation in the CBH metric for $r \ll \sqrt{Q_1Q_5}$ to be

$$\begin{aligned} \Phi_{\omega,\lambda,\mu,\nu,k} &\propto e^{-\imath\omega t + \imath\lambda y + \imath\mu\phi + \imath\nu\psi} \\ &\times \frac{1}{r}\Big[C_1 J_\sigma\Big(\frac{\sqrt{Q_1Q_5(\omega^2-\lambda^2)}}{r}\Big) + C_2 Y_\sigma\Big(\frac{\sqrt{Q_1Q_5(\omega^2-\lambda^2)}}{r}\Big)\Big] \\ &\times \cos^{|\nu|}\theta\,\sin^{|\mu|}\theta\, P_c^{(|\mu|,|\nu|)}(\cos 2\theta). \end{aligned} \quad (7.20)$$

[13]More generally the Bessel functions $Z_\nu(x)$ solve the differential equation

$$x^2\partial_x^2 Z_\nu(x) + x\partial_x Z_\nu(x) + (x^2-\nu^2)Z_\nu(x) = 0.$$

and $J_\nu(x)$ is the Bessel function of the first kind, given explicitly by

$$J_\nu(x) = \frac{x^\nu}{2^\nu}\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{2^{2k}k!\Gamma(\nu+k+1)} \quad [|\arg x| < \pi],$$

since r is real, so is $z = 1/r$ and thus its argument is zero. The general solution to the Bessel equation is of the form $c_1 J_\nu(x) + c_2 J_{-\nu}(x)$. From the expression for J_ν we see that, for ν a natural number n, $J_{-n}(x) = (-1)^n J_n(x)$, so they are no longer linearly independent. In that case the solution is of the form $c_1 J_n(x) + c_2 Y_n(x)$, where $Y_n(x)$ is the Bessel function of the second kind. For more on Bessel functions see [43].

with k a non-negative integer, $|\mu|+|\nu| \leq k$ and $\mu+\nu \equiv k (\mathrm{mod}\ 2)$. Furthermore, C_1 and C_2 are constants and

$$\begin{aligned} c &= \frac{k-(|\mu|+|\nu|)}{2}, \\ \sigma &= k+1. \end{aligned}$$

Finally, remember that the y coordinate is periodic; $y \sim y + 2\pi R$. When we impose the periodic boundary condition on the wavefunction, $\Phi(y=0) = \Phi(y = 2\pi R)$, this implies that λ is quantized

$$\begin{aligned} 1 &= e^{2\pi\imath R\lambda} \\ \Rightarrow 2\pi\imath R\lambda &= n\pi \quad \text{with } n \text{ integer} \\ \Rightarrow \lambda &= \frac{n}{2R}. \end{aligned} \tag{7.21}$$

7.5.2 The Klein-Gordon Equation in The SQBH Geometry

The procedure to solve the Klein-Gordon equation in the SQBH metric is exactly analogous to the procedure we followed in the previous section. The determinant and non-zero components of the inverse of the SQBH metric (7.15) are[14]

$$\begin{aligned} \sqrt{-g} &= hfr\sin\theta\cos\theta \\ g^{\theta\theta} &= g_{\theta\theta}^{-1} = (hf)^{-1} \\ g^{rr} &= g_{rr}^{-1} = (hf)^{-1}(r^2+a^2) \\ g^{tt} &= \frac{-f^2h^2(r^2+a^2)+a^2Q_1Q_5\sin^2\theta}{f^2h(r^2+a^2)} \\ g^{t\phi} &= g^{\phi t} = -\frac{a\sqrt{Q_1Q_5}}{fh(r^2+a^2)} \\ g^{\phi\phi} &= (h(r^2+a^2)\sin^2\theta)^{-1} \\ g^{yy} &= h + \frac{a^2Q_1Q_5\cos^2\theta}{f^2hr^2} \\ g^{y\psi} &= g^{\psi y} = \frac{a\sqrt{Q_1Q_5}}{fhr^2} \\ g^{\psi\psi} &= (hr^2\cos^2\theta)^{-1}. \end{aligned}$$

Again by inserting $\Phi = e^{-\imath\omega t+\imath\lambda y+\imath\nu\psi+\imath\mu\phi}\Psi(r,\theta)$ we find

$$\begin{aligned} \Box_{r,\theta}\Phi &= (hf)^{-1}\Big\{\frac{1}{r}\partial_r(r(r^2+a^2)\partial_r\Phi) + \frac{1}{\sin 2\theta}\partial_\theta(\sin 2\theta\partial_\theta\Phi)\Big\} \\ \Box_{t,\phi}\Phi &= (hf)^{-1}\Big\{\omega^2\Big[\frac{Q_1Q_5}{r^2+a^2} + Q_1 + Q_5 + r^2\Big] - \frac{2a\mu\omega\sqrt{Q_1Q_5}}{r^2+a^2} + \frac{a^2\mu^2}{r^2+a^2} \\ &+ \omega^2a^2\cos^2\theta - \frac{\mu^2}{\sin^2\theta}\Big\}\Phi \end{aligned}$$

[14]Note that we chose to use the metric in these coordinates instead of the twisted coordinates (7.12). Clearly, for the result this makes no difference, however, if you choose the twisted coordinates you will have to twist them back in order to compare the solution in the SQBH metric to the solution we found in the CBH metric.

$$\begin{aligned}\Box_{y,\psi}\Phi &= (hf)^{-1}\Big\{-\lambda^2\Big[\frac{Q_1Q_5}{r^2}+Q_1+Q_5+r^2\Big]-\frac{2a\nu\lambda\sqrt{Q_1Q_5}}{r^2}-\frac{a^2\nu^2}{r^2}\\ &- \lambda^2a^2\cos^2\theta-\frac{\nu^2}{\cos^2\theta}\Big\}\Phi\end{aligned}$$

Two separate differential equations for the variables r and θ can now be obtained by writing $\Psi = H(r)\Theta(\theta)$. We find them to be

$$\begin{aligned}&\frac{1}{r}\partial_r\Big(r(r^2+a^2)\partial_r H(r)\Big)+\Big\{\omega^2\Big[\frac{Q_1Q_5}{r^2+a^2}+Q_1+Q_5+r^2\Big]\\ &- \frac{2a\mu\omega\sqrt{Q_1Q_5}}{r^2+a^2}+\frac{a^2\mu^2}{r^2+a^2}-\lambda^2\Big[\frac{Q_1Q_5}{r^2}+Q_1+Q_5+r^2\Big]\\ &- \frac{2a\nu\lambda\sqrt{Q_1Q_5}}{r^2}-\frac{a^2\nu^2}{r^2}\Big\}H(r)-\Lambda H(r)=0 \qquad (7.22)\end{aligned}$$

and

$$\frac{1}{\sin 2\theta}\partial_\theta\Big(\sin 2\theta\partial_\theta\Theta(\theta)\Big)+\Big\{(\omega^2-\lambda^2)a^2\cos^2\theta-\frac{\mu^2}{\sin^2\theta}-\frac{\nu^2}{\cos^2\theta}\Big\}\Theta(\theta) = -\Lambda\Theta(\theta). \qquad (7.23)$$

The Angular Equation

We shall start with the angular equation again. This time, however, we will not be able to find an exact solution to the differential equation. As expected the equation reduces to the Laplace equation on S^3 when a is set to zero. So to zeroth order in a the solution to this equation was already found in the previous section, see (7.18). We can find the first order correction to this solution using perturbation theory. This is done in appendix D and we find that the angular part of the wave function to first order in $\epsilon = a^2(\omega^2-\lambda^2)$ is

$$\begin{aligned}\Theta(\theta) &= \cos^\nu\theta\sin^\mu\theta P_c^{(\mu,\nu)}(\cos 2\theta)\\ &+a^2(\omega^2-\lambda^2)\cos^\nu\theta\sin^\mu\theta[AP_{c+1}^{(\mu,\nu)}(\cos 2\theta)+BP_{c-1}^{(\mu,\nu)}(\cos 2\theta)],\end{aligned}$$

with

$$\begin{aligned}A &= \frac{(c+1)(c+\mu+\nu+1)}{4(2c+\mu+\nu+1)(2c+\mu+\nu+2)^2}\\ B &= \frac{(c+\mu)(c+\nu)}{4(2c+\mu+\nu+1)(2c+\mu+\nu)^2}\end{aligned}$$

and the corresponding eigenvalues are

$$\Lambda = k(k+2)+a^2(\omega^2-\lambda^2)\Big[\frac{(\mu^2-\nu^2)}{2(2c+\mu+\nu+1)(2c+\mu+\nu)}-\frac{1}{2}\Big].$$

The Radial Equation

To solve the radial part of the differential equation we proceed as in the previous section, i.e. we restrict ourselves to the region of spacetime where $r^2 \ll \sqrt{Q_1Q_5}$. So (7.22) reduces to

$$\begin{aligned}&\frac{1}{r}\partial_r\Big(r(r^2+a^2)\partial_r H(r)\Big)+\Big\{\omega^2\frac{Q_1Q_5}{r^2+a^2}\\ &- \frac{2a\mu\omega\sqrt{Q_1Q_5}}{r^2+a^2}+\frac{a^2\mu^2}{r^2+a^2}-\lambda^2\frac{Q_1Q_5}{r^2}\\ &- \frac{2a\nu\lambda\sqrt{Q_1Q_5}}{r^2}-\frac{a^2\nu^2}{r^2}\Big\}H(r)-\Lambda H(r)=0\end{aligned}$$

We can recast this as

$$\begin{aligned}&\frac{1}{r}\partial_r\Big(r(r^2+a^2)\partial_r H(r)\Big)\\+\quad&\Big\{\frac{(\omega\sqrt{Q_1Q_5}-a\mu)^2}{r^2+a^2}-\frac{(\lambda\sqrt{Q_1Q_5}+a\nu)^2}{r^2}\Big\}H(r)-\Lambda H(r)=0\end{aligned} \quad (7.24)$$

and then we change to $x=r^2$ to find

$$\begin{aligned}&4\partial_x\Big(x(x+a^2)\partial_x H\Big)\\+\quad&\Big(\frac{(\omega\sqrt{Q_1Q_5}-a\mu)^2}{x+a^2}-\frac{(\lambda\sqrt{Q_1Q_5}+a\nu)^2}{x}\Big)H-\Lambda H=0.\end{aligned}$$

When we substitute $H(x)=x^\alpha(x+a^2)^\beta\tilde{H}(x)$, with $\alpha=(\sqrt{Q_1Q_5}\lambda+a\nu)/(2a)$ and $\beta=(\sqrt{Q_1Q_5}\omega-a\mu)/(2a)$, the remaining differential equation for $\tilde{H}(x)$ is a hypergeometric equation:

$$\begin{aligned}4x(x+a^2)\partial_x^2\tilde{H}+4[2x(\beta+\alpha+1)+a^2(1+2\alpha)]\partial_x\tilde{H}&\\+[4(\beta+\alpha)(\beta+\alpha+1)-\Lambda]\tilde{H}=0.&\end{aligned}$$

This equation is solved by $\tilde{H}(x)=F(d,e,f,-x/a^2)$, with[15]

$$\begin{aligned}d&=\alpha+\beta+1/2+1/2\sqrt{1+\Lambda}\\e&=\alpha+\beta+1/2-1/2\sqrt{1+\Lambda}\\f&=1+2\alpha.\end{aligned}$$

When we construct the solution to the Klein-Gordon equation in the SQBH metric from all the intermediate results, we find that the eigenmodes of the wavefunction for small a and in the region where $r^2\ll\sqrt{Q_1Q_5}$ are the following

$$\begin{aligned}\Phi_{\omega,\lambda,\mu,\nu,k}\;&\propto\;e^{-\imath\omega t+\imath\lambda y+\imath\mu\phi+\imath\nu\psi}\cos^{|\nu|}\theta\sin^{|\mu|}\theta\\&\times\;\Big[P_c^{(|\mu|,|\nu|)}(\cos 2\theta)\\&\quad+a^2(\omega^2-\lambda^2)[AP_{c+1}^{(|\mu|,|\nu|)}(\cos 2\theta)+BP_{c-1}^{(|\mu|,|\nu|)}(\cos 2\theta)]\Big]\\&\times\;r^{2\alpha}(r^2+a^2)^\beta F(d,e,f,-(r/a)^2),\end{aligned}\quad (7.25)$$

with

$$\begin{aligned}d&=\alpha+\beta+1/2+1/2\sqrt{1+\Lambda}\\e&=\alpha+\beta+1/2-1/2\sqrt{1+\Lambda}\\f&=1+2\alpha\\\alpha&=(\sqrt{Q_1Q_5}\lambda+a\nu)/(2a)\end{aligned}$$

[15] The hypergeometric function $F(a,b,c,z)$ (also written as ${}_2F_1(a,b;c;z)$) solves the equation

$$-z(z+1)\partial_z^2F(a,b,c,z)+[z(a+b+1)-c]\partial_zF(a,b,c,z)+abF(a,b,c,z)=0.$$

and is given by

$$F(a,b,c,x)\equiv\Sigma_{i=0}^{\infty}\frac{\Gamma(a+i)\Gamma(b+i)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+i)}\frac{x^i}{i!}.$$

$$\begin{aligned}\beta &= (\sqrt{Q_1Q_5}\omega - a\mu)/(2a)\\ A &= \frac{(c+1)(c+\mu+\nu+1)}{4(2c+\mu+\nu+1)(2c+\mu+\nu+2)^2}\\ B &= \frac{(c+\mu)(c+\nu)}{4(2c+\mu+\nu+1)(2c+\mu+\nu)^2}\end{aligned}$$

and

$$\Lambda = k(k+2) + a^2(\omega^2-\lambda^2)\Big[\frac{(\mu^2-\nu^2)}{2(2c+\mu+\nu+1)(2c+\mu+\nu)} - \frac{1}{2}\Big], \tag{7.26}$$

where k is a non-negative integer and $|\mu|+|\nu| \leq k$ and $\mu+\nu \equiv k(\text{mod } 2)$. Finally, λ is quantized as before (see (7.21)).

7.6 The Propagator

In section 7.4 we argued that to test the metrics found for the quantum mechanical black hole, or actually just a generic example of such a metric, namely the SQBH metric, we should calculate the fieldpropagator in this metric. The propagator or Green's function for an arbitrary linear differential operator $\hat{D}$ in six dimensions is defined by

$$\hat{D}G(X,Y) = \delta(X-Y), \tag{7.27}$$

where X and Y are six dimensional position vectors. Suppose $\phi_n(X)$ are the eigenfunctions of the differential operator and their eigenvalues are E_n, so $\hat{D}\phi_n(X) = E_n\phi_n(X)$, then an explicit expression for the propagator can often be found in terms of this basis of eigenfuncions. The procedure is the following. Write the Green's function on the basis of eigenfunctions

$$G(X,Y) = \sum_{n=0}^{\infty} a_n(Y)\phi_n(X), \tag{7.28}$$

as well as the delta function

$$\delta(X-Y) = \sum_{n=0}^{\infty} b_n(Y)\phi_n(X). \tag{7.29}$$

To find the coefficients b_n, multiply both sides of the expression for the delta function by $\phi_m^*(Y)$ and integrate over X

$$\begin{aligned}\int \phi_m^*(Y)\delta(X-Y)dX &= \sum_{n=0}^{\infty}\int \phi_m^*(Y)b_n(Y)\phi_n(X)dX\\ \Rightarrow \quad \phi_m^*(Y) &= \sum_{n=0}^{\infty} b_n(Y)\delta_{mn} = b_m(Y),\end{aligned}$$

so plugging this result back into (7.29), we obtain

$$\delta(X-Y) = \sum_{n=0}^{\infty} \phi_n^*(Y)\phi_n(X). \tag{7.30}$$

We can now obtain the coefficients a_n by plugging (7.28) and (7.30) into (7.27)

$$\begin{aligned}
\hat{D}\sum_{n=0}^{\infty} a_n(Y)\phi_n(X) &= \sum_{n=0}^{\infty} \phi_n^*(Y)\phi_n(X) \\
\to \sum_{n=0}^{\infty} a_n(Y)[\hat{D}\phi_n(X)] &= \sum_{n=0}^{\infty} \phi_n^*(Y)\phi_n(X) \\
\to \sum_{n=0}^{\infty} a_n(Y)[E_n\phi_n(X)] &= \sum_{n=0}^{\infty} \phi_n^*(Y)\phi_n(X) \\
\to a_n(Y) &= \frac{\phi_n^*(Y)}{E_n},
\end{aligned}$$

so the Green's function reads

$$G(X,Y) = \sum_{n=0}^{\infty} \frac{\phi_n(X)\phi_n^*(Y)}{E_n}. \tag{7.31}$$

In the previous sections we have found the eigenfunctions of the differential operator for the Klein-Gordon field for both metrics, so in theory we have all the ingredients to calculate the propagators for the massless scalar fields and compare them. However, as you can easily imagine from looking at the expressions for the wavefunctions, it is impossible to perform the mode sum in (7.31) analytically[16].

Another option for finding the propagator in the CBH metric would be by performing the following steps

- Remember that the CBH metric is the product of the metric of a 3-sphere and the metric of a BTZ black hole with zero mass parameter. We shall write this as BTZ$\times$S^3. Due to this fact we may write the eigenfunctions in this metric as $\phi(X) = \phi_{BTZ}(X_{BTZ})\phi_S(X_S)$ and consequently the propagator can be obtained from the propagator in the BTZ metric and the eigenmodes in the S^3 metric (see [44]):

$$G(X,Y) = \sum_I G_I(X_{BTZ}, Y_{BTZ})\phi_S^I(X_S)\phi_S^{*I}(Y_S),$$

  where the sum is over the multiindex $I = k, \mu, \nu$.

- The eigenmodes of the Klein-Gordon equation on S^3 are spherical harmonics [44].

- The propagator in the BTZ metric can be obtained from the propagator in the AdS$_3$ metric using the imaging technique [45].

Unfortunately, apart from many subtleties[17] that will arise when one really tries to carry out these steps, one runs into an infinite sum when calculating the

[16] It would be very interesting to perform the mode sum computationally. We did not look into this possibility further, since writing a program that performs this calculation can be the subject of a thesis in its own right.

[17] Such as boundary conditions in AdS$_3$ that prevent causality violations caused by lightrays that travel to infinity and back in a finite amount of time.

propagator in the BTZ metric [45] that is possibly worse than the mode sum in (7.31), which we are trying to circumvent with this approach.

It seems that at this point we must conclude that we shall not be able to find explicit expressions for the propagators, at least not analytically. Nevertheless, we will not yet give up our aim to test Mathur's fuzzball proposal. Instead we suggest the following. Suppose we can find a way to express the wavefunction in the SQBH metric, $\Phi_n^Q(X)$, for small a as

$$\Phi_n^Q(X) \approx \Phi_n^C(X) + \delta\Phi_n(X), \tag{7.32}$$

where $\Phi_n^C(X)$ is the wavefunction in the CBH metric and $\delta\Phi_n(X)$ is the leading correction there to. Then it follows from the fact that the eigenfunctions form a complete basis that we can write

$$\delta\Phi_n = \Sigma_m \tilde{G}_{nm}\Phi_m^C, \tag{7.33}$$

the symbol we used for the coefficient-matrix is suggestive, since it has something to do with the Fourier transform of the correction on the propagator in the CBH metric. This can be seen as follows

$$\begin{aligned}
G_Q(X,Y) &= \Sigma_n \frac{\Phi_n^Q(X)(\Phi_n^Q(Y))^*}{E_n} \\
&\approx \Sigma_n \frac{(\Phi_n^C + \delta\Phi_n)(\Phi_n^C + \delta\Phi_n)^*}{E_n} \\
&= \Sigma_{n,m} \frac{(\Phi_n^C + \tilde{G}_{nm}\Phi_m^C)(\Phi_n^C + \tilde{G}_{nm}\Phi_m^C)^*}{E_n} \\
&\approx G_C(X,Y) + \Sigma_{n,m} \frac{\Phi_n^C(\tilde{G}_{nm}\Phi_m^C)^* + \tilde{G}_{nm}\Phi_m^C(\Phi_n^C)^*}{E_n}.
\end{aligned} \tag{7.34}$$

where $G_Q(X,Y)$ and $G_C(X,Y)$ are the propagator in the SQBH metric and in the CBH metric respectively. From this expression for the propagator in the SQBH metric we see that it is going to differ significantly from the propagator in the CBH metric when $\tilde{G}_{ik}$ becomes of order one. So by calculating this function we may be able to find out for which modes the propagators become significantly different, in other words for which frequencies of the wavefunction there may be non-local effects.

To find $\tilde{G}$ we take the innerproduct of the (normalized!) CBH modes with (7.33)

$$\begin{aligned}
(\Phi_k^C, \delta\Phi_n) &= (\Phi_k^C, \Sigma_m \tilde{G}_{nm}\Phi_m^C) \\
&= \Sigma_m \tilde{G}_{nm}\delta_{km} \\
&= \tilde{G}_{nk}.
\end{aligned} \tag{7.35}$$

The innerproduct is a generalization of the innerproduct in four-dimensional curved spacetimes given in (3.18) to a six-dimensional spacetime

$$(\Phi_i, \Phi_j) = \int_\Sigma d\Sigma^\mu \sqrt{g_\Sigma}[\Phi_i^* \partial_\mu \Phi_j - \Phi_j \partial_\mu \Phi_i^*], \tag{7.36}$$

where Σ is a Cauchy surface and $d\Sigma^\mu = n^\mu d\Sigma$ with n^μ a six-dimensional vector normal to this surface pointing in the positive time direction.

In the following subsections we shall consecutively normalize the eigenmodes of the Klein-Gordon equation in the CBH metric and expand the SQBH eigenmodes in a Taylor series in a to obtain an expression of the form (7.32).

7.6.1 Normalizing The Eigenmodes

In this subsection we shall normalize the eigenmodes (7.20) of the Klein-Gordon equation in the CBH metric such that

$$\begin{aligned}(\Phi_i, \Phi_j) &= -(\Phi_i^*, \Phi_j^*) = \delta_{ij} \\ (\Phi_i^*, \Phi_j) &= (\Phi_i, \Phi_j^*) = 0.\end{aligned} \tag{7.37}$$

The innerproduct is given in (7.36). Since n^μ a unit vector pointing in the positive time direction, we have $n^t = \imath\sqrt{h}$ and $n^i = 0$ for $i = y, r, \theta, \psi, \phi$. So $d\Sigma^\mu \partial_\mu = \imath\sqrt{h} drdyd\phi d\psi d\theta\partial_t$, furthermore we have $g_\Sigma = hr^6 \sin^2\theta\cos^2\theta$ and for $r^2 \ll \sqrt{Q_1 Q_5}$ we may use the approximation $h \approx \sqrt{Q_1Q_5}/r^2$. When we use the general argument that the innerproduct should be independent of the Cauchy surface one chooses to evaluate it on, in other words it should be independent of time ($\partial_t(\Phi_i, \Phi_j) = 0$), we find for the CBH eigenmodes, whose time-dependent part is $e^{-\imath\omega t}$, that

$$\begin{aligned}\partial_t(\Phi_i, \Phi_j) &= \int_\Sigma d\Sigma^\mu \sqrt{g_\Sigma}[\partial_t(\Phi_i^*\partial_\mu\Phi_j) - \partial_t(\Phi_j\partial_\mu\Phi_i^*)] \\ &= (\imath\omega - \imath\omega')(\Phi_i, \Phi_j) \\ &= 0 \quad \rightarrow \quad \omega = \omega'.\end{aligned} \tag{7.38}$$

Note that we have chosen to indicate the quantum numbers of Φ_j with a prime ($\omega_j = \omega'$, $\lambda_j = \lambda'$, etcetera) and those of Φ_i without a prime ($\omega_i = \omega$, etcetera). This is just to minimize the number of indices.

So to normalize the modes we have to evaluate the following integral

$$\begin{aligned}(\Phi_i, \Phi_j) &= \int_\Sigma \imath hr^3 \sin\theta\cos\theta\delta(\omega' - \omega)[\Phi_i^*\partial_t\Phi_j - \Phi_j\partial_t\Phi_i^*]drdyd\phi d\psi d\theta \\ &= \int_\Sigma \imath hr^3 \sin\theta\cos\theta\delta(\omega' - \omega)(-\imath\omega' - \imath\omega)\Phi_j\Phi_i^* drdyd\phi d\psi d\theta.\end{aligned} \tag{7.39}$$

The integrals over ψ, ϕ and y are easily obtained

$$\int_0^{2\pi R} dy \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi e^{\imath\lambda' y + \imath\mu'\phi + \imath\nu'\psi} e^{-\imath\lambda y - \imath\mu\phi - \imath\nu\psi} = (2\pi)^3 R\delta_{\lambda'\lambda}\delta_{\mu'\mu}\delta_{\nu'\nu}.$$

The integral over θ now reads

$$\int d\theta \cos\theta\sin\theta\cos^{2|\nu|}\theta\sin^{2|\mu|}\theta P_{c'}^{(|\mu|,|\nu|)}(\cos 2\theta)P_c^{(|\mu|,|\nu|)}(\cos 2\theta).$$

When we make the substitution $z = \cos 2\theta$ we can rewrite this integral into a standard form. Note that $\cos\theta\sin\theta d\theta = 1/2\sin 2\theta d\theta = -1/4dz$ and $\cos^2\theta = (1+z)/2$ and $\sin^2\theta = (1-z)/2$. So the integral reads

$$\begin{aligned}& -\frac{1}{4}2^{-|\nu|-|\mu|}\int dz(1+z)^{|\nu|}(1-z)^{|\mu|}P_{c'}^{(|\mu|,|\nu|)}(z)P_c^{(|\mu|,|\nu|)}(z) \\ =& -\frac{1}{4}2^{-|\nu|-|\mu|}\delta_{c'c}\frac{2^{|\nu|+|\mu|+1}\Gamma(|\mu|+c+1)\Gamma(|\nu|+c+1)}{c!\Gamma(|\mu|+|\nu|+2c+1)\Gamma(|\mu|+|\nu|+c+1)} \\ \equiv& -N_\theta(\mu,\nu,k)\delta_{k'k}.\end{aligned}$$

The solution to the integral was found in [43]. Note that since we already found delta functions for μ and ν, the delta function for c could be replaced by $\delta_{k'k}$.

The remaining integral over r is

$$\int dr\sqrt{Q_1Q_5}r\frac{1}{r^2}\Big[C_1J_\sigma(\alpha/r)+C_2Y_\sigma(\alpha/r)\Big]\Big[C_1J_{\sigma'}(\alpha'/r)+C_2Y_{\sigma'}(\alpha'/r)\Big]$$

where we have defined $\alpha = \sqrt{Q_1Q_5(\omega^2-\lambda^2)}$ and inserted the approximate value for h. This can be simplified using the fact that $\alpha=\alpha'$, since we already found delta functions for both ω and λ and even more so, since we just found $k=k'$, we also have $\sigma=\sigma'$ (remember that $\sigma=k+1$). Finally, we note that Y_σ diverges for $r\to 0$, so it is not square integrable. It follows that we must set C_2 to zero. So we should evaluate

$$\int dr\sqrt{Q_1Q_5}\frac{1}{r}\Big[J_\sigma(\alpha/r)\Big]^2$$

Which gives, as can be found in [43], after substituting $t=1/r$

$$-\int dt\frac{1}{t}J_\sigma(\alpha t)J_\sigma(\alpha t) = -\frac{1}{2\sigma}. \tag{7.40}$$

We can plug all these results into (7.39) to obtain

$$\begin{aligned}(\Phi_i,\Phi_j) &= \imath(-2\imath\omega)(2\pi)^3R\delta_{\lambda'\lambda}\delta_{\mu'\mu}\delta_{\nu'\nu}\delta_{k'k}\delta(\omega-\omega')\\ &\quad\times N_\theta(\mu,\nu,k)C(\sigma)\\ &\equiv N_C^{-2}\delta_{\lambda'\lambda}\delta_{\mu'\mu}\delta_{\nu'\nu}\delta_{k'k}\delta(\omega-\omega')\end{aligned} \tag{7.41}$$

with

$$C(\sigma)=\sqrt{Q_1Q_5}\frac{C_1^2}{2\sigma}$$

and

$$N_\theta(\mu,\nu,k)=\frac{\Gamma(|\mu|+c+1)\Gamma(|\nu|+c+1)}{2c!\Gamma(|\mu|+|\nu|+2c+1)\Gamma(|\mu|+|\nu|+c+1)}.$$

So the normalization constant is given by

$$N_C(\sigma,\mu,\nu,k)=\big(2\omega(2\pi)^3RN_\theta(\mu,\nu,k)C(\sigma)\big)^{-1/2}. \tag{7.42}$$

7.6.2 Expanding The Eigenmode in a Taylor Series

In this subsection we shall write the wavefunction in the SQBH metric as the wavefunction in the CBH metric plus corrections, $\Phi^Q(X)\approx\Phi^C(X)+\delta\Phi(X)$. When we compare the two expressions for the wavefunctions (7.25) and (7.20), we see that it is clear that the first reduces to the latter upon setting a to zero except for the radial part of the function. So what remains to be done is express the hypergeometric function in terms of Bessel functions. This can be done (see [46]), but there is an easier way. We can simply use perturbation theory again to find that the radial part of the wavefunction in the SQBH metric to first order in a reads (see appendix E)

$$H(r)\approx\frac{1}{r}C_1\Big[J_{\sigma_0}(\sqrt{\alpha_0}/r)+a\frac{\alpha_1}{4\sqrt{\alpha_0}r}(J_{\sigma_0-1}(\sqrt{\alpha_0}/r)-J_{\sigma_0+1}(\sqrt{\alpha_0}/r))\Big]$$

with

$$\begin{aligned}\alpha_0 &\equiv Q_1Q_5(\omega^2-\lambda^2)\\ \alpha_1 &\equiv -\sqrt{Q_1Q_5}(\omega\mu+\lambda\nu)\\ \sigma_0 &= k+1.\end{aligned}$$

Now let us define

$$\begin{aligned}F_i^0(t,y,\psi,\phi) &= e^{-\imath\omega_i t+\imath\lambda_i y+\imath\mu_i\phi+\imath\nu_i\psi}\\ \Theta_i^0(\theta) &= \cos^{\nu_i}\theta\sin^{\mu_i}\theta P_{c_i}^{\mu_i,\nu_i}(\cos 2\theta)\\ H_i^0(z) &= zC_1J_{k+1}(\sqrt{\alpha_0}z).\end{aligned}$$

The normalized solution to the Klein-Gordon equation in the CBH metric then reads

$$\Phi_i^C = N_{C_i}F_i^0(t,y,\psi,\phi)\Theta_i^0(\theta)H_i^0(z),$$

with N_C given in (7.42). The normalized solution to the Klein-Gordon equation in the SQBH metric, on the other hand is

$$\Phi_i^Q = N_{Q_i}F_i^0(t,y,\psi,\phi)[\Theta_i^0(\theta)+a^2\Theta_i^1(\theta)][H_i^0(z)+aH_i^1(z)],$$

where

$$\begin{aligned}\Theta_i^1(\theta) &= \cos^{\nu_i}\theta\sin^{\mu_i}\theta(\omega^2-\lambda^2)\Big[AP_{c_i+1}^{\mu_i,\nu_i}(\cos 2\theta)+BP_{c_i-1}^{\mu_i,\nu_i}(\cos 2\theta)\Big]\\ H_i^1(z) &= z^2\frac{\alpha_1}{4\sqrt{\alpha_0}}C_1[J_k(\sqrt{\alpha_0}z)-J_{k+2}(\sqrt{\alpha_0}z)]\end{aligned}$$

From the normalization condition on the SQBH modes we find that $N_{Q_i}\approx N_{C_i}$ to first order in a (see appendix E). So finally the correction to first order in a on the CBH modes is found to be

$$\begin{aligned}\delta\Phi_i &\equiv N_{Q_i}\Psi_i^Q - N_{C_i}\Psi_i^C\\ &= aN_{C_i}F_i^0(t,y,\psi,\phi)\Theta_i^0(\theta)H_i^1(z)+\mathcal{O}(a^2).\end{aligned}$$

7.6.3 Calculating $\tilde{G}_{ik}$

In the previous two subsections we have found all the ingredients we need to calculate $\tilde{G}_{ik}$. From (7.35) we see that we have to evaluate the following inner-product

$$\begin{aligned}\tilde{G}_{ik} &= (\Phi_k^C,\delta\Phi_i)\\ &= (N_{C_k}F_k^0\Theta_k^0H_k^0, aN_{C_i}F_i^0\Theta_i^0H_i^1).\end{aligned}$$

The integrals over y, ψ, ϕ and θ are just as in section 7.6.1 and are simply delta functions for their respective quantumnumbers, i.e. λ, ν, μ and k. The integral over r causes a bit more trouble. It is given by

$$\begin{aligned}I_r &\equiv -\int\frac{dz}{z^3}H_k^0H_i^1\\ &= -\int dz\,J_{k+1}(\sqrt{\alpha_0}z)\Big[J_k(\sqrt{\alpha_0'}z)-J_{k+2}(\sqrt{\alpha_0'}z)\Big],\end{aligned}$$

where we have used $z = 1/r$ and thus $hr^3 \approx \sqrt{Q_1 Q_5} r dr \propto -z^{-3} dz$. And we have used the fact that the integral over θ gave a deltafunction for k to set $k_k = k_i \equiv k$. The solution to this integral can be found in [43] and is given by

$$I_r = \begin{cases} \frac{1}{\sqrt{\alpha_0}} \left(\frac{\alpha_0'}{\alpha_0}\right)^{k/2} & \text{for } \alpha_0 > \alpha_0' \\ 0 & \text{for } \alpha_0 = \alpha_0' \\ -\frac{1}{\sqrt{\alpha_0'}} \left(\frac{\alpha_0}{\alpha_0'}\right)^{k/2+1/2} & \text{for } \alpha_0' > \alpha_0 \end{cases} .$$

Plots of this function for various values of the parameters are given in Fig. 7.4.

As you can see this function vanishes almost everywhere, except for a narrow strip where α_0 and α_0' approach each other. As k increases the strip becomes more narrow and as α_0 and α_0' increase the maximum value of the function decreases. Most important, however, is that even its maximum value is really quite small. And in addition to that remember that this term is suppressed by a factor a in the expression for the propagator in the SQBH metric (7.34). So it seems that we have found that the propagator in the SQBH metric equals the propagator in the CBH metric to first order in a. A false conclusion would be, however, that there are no correlations in the SQBH metric that would appear non-local in the CBH metric. The result is probably due to our approach as can be seen as follows. If there are non-local effects in a certain spacetime this means that a particle traveling through spacetime that finds itself at a position X can be affected by things that occur at a causally disconnected position Y. In terms of the Lagrangian this means that there should be interaction terms that connect the positions X and Y, in other words there should be non-local terms in the Lagrangian. In our approach we wrote the SQBH wavefunction as the CBH wavefunction plus corrections, $\Phi_Q(X) = \Phi_C(X) + \delta\Phi(X)$, clearly this is a local correction. Consequently, this local correction cannot introduce non-local terms in the Lagrangian.

The appropriate conclusion is thus that possible non-local effects in the CBH metric caused by corrections from the SQBH metric will not be manifested in the function $\tilde{G}$.

7.7 Comparison of The Wavefunctions

In the subsections 7.5.1 and 7.5.2 we have found the wavefunctions of a massless particle in the inner region of the CBH metric and SQBH metric respectively. In this section we will compare the radial parts of these wavefunctions. To simplify things we will set the quantum numbers λ, k, μ and ν, corresponding to the compactified directions y, θ, ϕ and ψ respectively, to zero. The respective radial parts of the wavefunction are then given by

$$H_{CBH}(r) \propto \frac{1}{r} J_1\left(\frac{\sqrt{Q_1 Q_5}\omega}{r}\right) \tag{7.43}$$

and

$$H_{SQBH}(r) \propto (r^2 + a^2)^\beta F(\beta + 1, \beta - 1, 1, -(r/a)^2), \tag{7.44}$$

with $\beta = \omega\sqrt{Q_1 Q_5}/(2a)$.

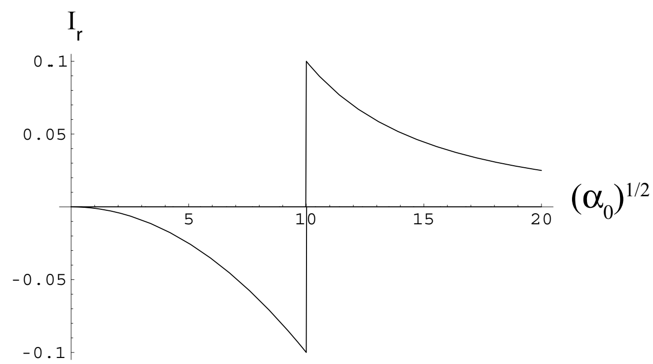

Figure 7.4: The plot shows I_r as a function of $\sqrt{\alpha_0}$, for $\sqrt{\alpha_0'} = 10$ and $k = 1$.

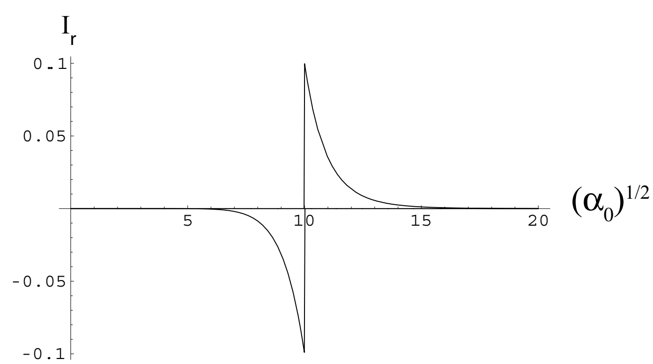

Figure 7.5: The plot shows I_r as a function of $\sqrt{\alpha_0}$, for $\sqrt{\alpha_0'} = 10$ and $k = 10$.

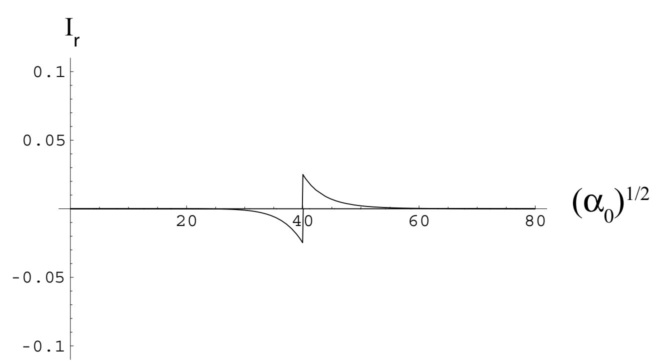

Figure 7.6: The plot shows I_r as a function of $\sqrt{\alpha_0}$, for $\sqrt{\alpha_0'} = 40$ and $k = 10$.

The periodicity of H_{CBH} blows up as r approaches zero. This can be interpreted as the particle getting caught in the throat and bouncing back and forth inside the throat an infinite number of times before hitting the singularity. The hypergeometric function H_{SQBH} that solves the radial part of the differential equation in the SQBH metric exhibits quite a different behavior in this respect. The number of times the particle bounces back and forth inside the throat is finite, thus it can reemerge from the throat in a finite amount of time.

Furthermore, we have seen that the solution in the SQBH metric reduces to the solution in the CBH metric for a sufficiently small. From figures 7.7 and 7.8 we see that their behavior is indeed very similar for sufficiently large values of r. From the plots we see that the approximation breaks down just after the wavefunction crosses zero for the second time (counting in the direction of decreasing r). However, it is very difficult to determine at what distance this is in terms of the characteristic lengths of the system. This is because the behavior of H_{SQBH} depends much stronger on the relation between a and $\omega\sqrt{Q_1 Q_5}$ than on the relation between r and these parameters.

We find that for increasingly large values of $\beta = \omega\sqrt{Q_1 Q_5}/(2a)$, the number of oscillations in the region of small r also increases. So for fixed values of the characteristic lengths of the system, i.e. a and $\sqrt{Q_1 Q_5}$, this seems to imply that as the energy or frequency ω becomes larger, and thus the wavelength shorter, the number of times that the particle, described by the wavefunction, bounces back and forth inside the throat, before it reaches the cap and gets reflected, also increases. On the other hand, the resemblance between H_{CBH} and H_{SQBH} completely disappears when β becomes of order one. Or equivalently, when ω becomes of the order $a/\sqrt{Q_1 Q_5}$. This can be explained as follows. Remember that first of all we restricted ourselves to the inner regions of spacetimes by setting $r^2 \ll \sqrt{Q_1 Q_5}$. In addition we found that H_{SQBH} approximates[18] H_{CBH} for small values of a, i.e. for $a \ll r$. So the breakdown of the resemblance is simply due to the fact that the wavelength of a wave with frequency $\sim a/\sqrt{Q_1 Q_5}$ is $\sim \sqrt{Q_1 Q_5}/a$, which exceeds the size of the region in which we found the resemblance, i.e. $a^2 \ll r^2 \ll \sqrt{Q_1 Q_5}$.

Finally, we see from the plots of H_{CBH} and H_{SQBH} that our results do not seem to contradict Mathur's idea that a black hole geometry is in fact a superposition of various geometries like the SQBH geometry. Remember that the SQBH geometry is the supergravity solution corresponding to one possible vibration mode of the fundamental string. Different vibration modes give rise to geometries similar to the SQBH geometry, but each with a slightly different cap. The different caps correspond to different microstates of the black hole. Since the SQBH geometry is a generic case we would expect that also the wavefunction we found in this background can be considered generic. When we compare the plots in Figs. 7.7 and 7.8 it does not seem unreasonable that a superposition of functions like the radial part of the wavefunction in the SQBH metric will look similar to the the radial part of the wavefunction in the CBH metric.

[18] We found this by solving the radial wave equation for small values of a using perturbation theory. However, one can also find this directly from the expression for H_{SQBH} by letting a go to zero. This procedure can be found in [46].

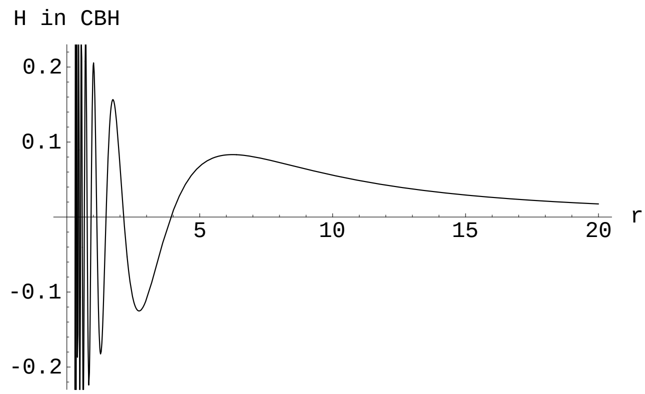

Figure 7.7: Radial part of the wavefunction in the CBH metric. On the vertical axis its amplitude, on the horizontal axis the distance to the singularity at $r = 0$.

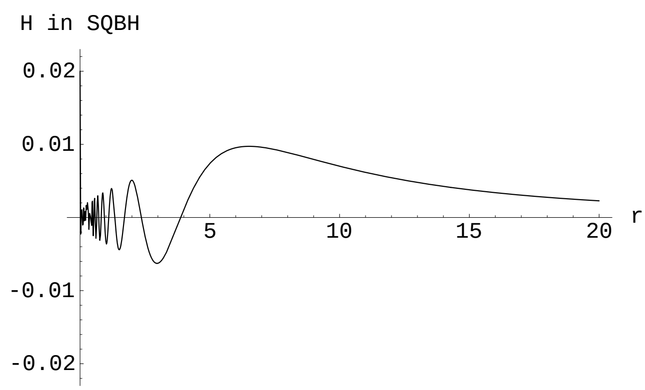

Figure 7.8: Radial part of the wavefunction in the SQBH metric. On the vertical axis its amplitude, on the horizontal axis the distance to the origin $r = 0$. We have set $\omega\sqrt{Q_1 Q_5} \gg a$.

7.8 Discussion

In this chapter we have studied a candidate solution to the information loss paradox put forward by Mathur. We motivated our choice of this particular proposal with two reasons. Firstly, because we argued that solving the paradox required a theory of quantum gravity. String theory, that underlies Mathur's work, is such a theory, and in addition to that it exhibits a holographic feature that seems to be a characteristic of black holes. The other aspect of Mathur's proposal that made it attractive to us is that it provides the possibility to do actual calculations.

After this chapter we are confident to say that the fuzzball proposal has indeed proven to be a fruitful source of lengthy calculations. Our efforts led to nice expressions of the solutions to the Klein-Gordon equation in both the CBH and the SQBH metrics. We normalized the CBH solution and expanded the SQBH solution in a Taylor series for small values of a, the characteristic lengthscale on which the two geometries differ. Not surprisingly, the SQBH solution reduces to the CBH solution to zeroth order in a.

At this point we seemed fully equipped to calculate the propagators in both metrics, however, we ran into quite a number of difficulties. In retrospect, the choice we made at that point, namely to calculate the first order correction to the CBH propagator, was unfortunate. It failed to provide us any information about possible non-local effects in the SQBH metric and this was most probably caused by the fact that we had already filtered out these effects by writing the SQBH solution as the CBH solution plus corrections.

Nevertheless, we learned from our mistakes and would not want to miss the opportunity to propose some alternative methods to obtain the answer to the question "Does the fuzzball have the property required to restore information, namely non-locality?". One option is to stick to comparing the propagators in the different metrics. In that case, one should either turn to computational methods, or invest more time in studying the differential equations to obtain an analytical expression for the propagator. Either way it seems wise to use the twisted coordinates for the SQBH metric (see (7.12)), since then the calculation is the same in both metrics. One only has to be a bit careful when transforming back to the original coordinates.

A different option is to study geodesics instead of propagators. Locality and causality are closely related concepts and it would be a very good indication of non-locality if one could find a geodesic in the SQBH metric that connects two points that are spacelike separated, i.e. causally disconnected, in the CBH metric. Since calculating propagators has proven to be quite difficult in these metrics, this seems a nice alternative.

Finally, we made a very handwaving observation regarding Mathur's idea that a superposition of SQBH-like metrics will reduce to the CBH metric. It would be nice to make this observation more precise, since as we said Mathur argues that the metrics, which correspond to the different vibration modes of the string, reduce to the CBH metric upon 'coarse-graining' over these metrics. What he means by coarse-graining is keeping only the part of the geometries outside the horizon, where the horizon is identified as the place where the caps start, i.e. where the geometries start to differ. This is similar to the procedure for going from statistical physics to thermodynamics where the differences between the microstates of the system are ignored and the system is described by a few macroscopic parameters that the microstates have in common. We find this explanation of the 'coarse-graining' procedure a bit unsatisfying, since it will trivially work in this case, because the metrics already equal the CBH metric except for the cap, but it does not explain at all how the singularity arises in this 'coarse-graining' procedure. It may be that the inspection of a superposition of functions like H_{SQBH} will gives some more insight in this matter.

Conclusion and Outlook

The first goal we set ourselves at the beginning of this thesis was to give an account of the information loss paradox accessible to a fourth year undergraduate student in physics. Obviously, the reader is in a better position than we are to judge whether or not we have achieved this goal. Nevertheless, we would like to note the following. Since the paradox is widely studied, many introductions on the subject of various levels of complexity are available. During our studies we encountered many an introduction that does not really mention the analogy between thermodynamics and black hole mechanics. They merely point out that the area of a black hole cannot decrease which is reminiscent of the second law of thermodynamics for entropy. In our opinion it is crucial to know that the analogy extends much further in order to fully appreciate the paradox. Hawking's calculation provides the missing link in the analogy, namely that black holes have a temperature just as thermodynamic systems. If this were not the case it is unlikely that anyone would have believed the calculation, since its implication, information loss, is in contradiction with the very fundaments the calculation was built on.

It is very difficult to get a clear overview of Hawking's calculation because of the many steps and approximations he makes. Clearly, this also makes it difficult to point out where the calculation may be wrong. After a thorough study of the validity of the many steps in the calculation, we concluded that the semi-classical approximation most probably breaks down near the event horizon due to the diverging redshift photons suffer in this region.

The breakdown of the semi-classical approximation already suggests that information is not lost in black hole evaporation. Namely, Hawking radiation is *exactly* thermal in the semi-classical approximation, which leaves the possibility that in reality the radiation does carry information through long range correlations. In that case information is not lost, it is just very hard to restore, but this is no different in the case of a burned piece of paper that used to have text on it. Most physicists always believed that information did not really get lost in black hole evaporation and the discovery of the AdS/CFT correspondence probably took their last doubts away. In short, this is because 't Hooft argued five years before this discovery that black holes seem to tell us that nature is in principle holographic. That is, all the information about a bulk can be stored on its boundary. And this is exactly what the AdS/CFT correspondence says, namely everything that happens in the bulk can be described by a theory that lives on its boundary. And what is more, it tells us that the theory that lives on the boundary is a unitary theory, which means that information does not get lost.

In the final chapter we studied the fuzzball proposal, an idea of Mathur that seems to offer a solution to the black hole entropy problem. This problem is closely related to the information loss paradox, so an obvious question to ask, is if this idea also has the potential to solve this problem. In an attempt to answer this question we decided to try and calculate the propagator of a field in a generic fuzzball metric and compare it to a propagator of a field in a classical black hole metric. This proved to be a particularly difficult task and although we did find the wavefunctions in the different metrics we failed to find the matching propagators. Since, in essence, all that you need to find the propagator is the wavefunction, one could pursue our line of research by resorting to computational methods. But we tend to believe that studying geodesics in the different metrics instead of propagators is more promising if one wants to find non-local effects.

Although, we could not answer the question as to whether or not the fuzzball proposal is likely to offer a solution to the information loss paradox based upon actual results, we will not refrain from commenting on the subject anyway. First of all, we believe the fuzzball proposal does not really solve the paradox as long as there are no four dimensional fuzzballs. On the other hand, the paradox seems to indicate that black holes have microstates and in the framework of string theory it seems very plausible that these microstates arise from string vibrations. So as long as there is no evidence that says that four dimensional fuzzballs cannot exist, we think it is a neat idea that is worthy of further investigation. The suggestions we make for successive research can be crucial in determining its potential for solving the paradox. They can make or break the fuzzball proposal, so we strongly recommend their pursuit.

Appendix A

General Solutions of the Klein-Gordon Equation in the Schwarzschild Metric

The Klein-Gordon equation for massless fields in curved spacetime reads

$$D^{\mu}\partial_{\mu}\Phi(x) = 0. \tag{A.1}$$

Equivalently,

$$g^{\mu\nu}[\partial_{\mu}\partial_{\nu} - \Gamma^{\alpha}_{\mu\nu}\partial_{\alpha}]\Phi = 0, \tag{A.2}$$

where $\Gamma^{\alpha}_{\mu\nu}$ is the Christoffel-symbol. Since, for the Schwarzschild solution, the metric is diagonal (see (1.15)), we only need the non-zero components of the Christoffel-symbol with $\mu = \nu$. They are

$$\begin{aligned} \Gamma^1_{00} &= \frac{M}{r^3}(r-2M) \qquad \Gamma^1_{11} = -\tfrac{M}{r(r-2M)} \qquad \Gamma^1_{22} = -(r-2M) \\ \Gamma^1_{33} &= -(r-2M)\sin^2\theta \qquad\qquad \Gamma^2_{33} = -\sin\theta\cos\theta. \end{aligned} \tag{A.3}$$

And the non-zero components of the inverse of the metric are

$$g^{00} = -(1-\frac{2M}{r})^{-1} \quad g^{11} = (1-\frac{2M}{r}) \quad g^{22} = r^{-2} \quad g^{33} = (r\sin\theta)^{-2}. \tag{A.4}$$

So we can write (A.2) as

$$\begin{aligned} \Big[-(1-\frac{2M}{r})^{-1}\partial_t^2 + (1-\frac{2M}{r})\partial_r^2 + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 \\ -\frac{2M-2r}{r^2}\partial_r + \frac{\cos\theta}{r^2\sin\theta}\partial_\theta\Big]\Phi = 0. \end{aligned} \tag{A.5}$$

Because of the spherical symmetry of this problem we can write $\Phi = U(r,t)Y(\theta,\phi)$, which enables us to split (A.5) into two parts, one depending only on r and t and the other only on θ and ϕ, so each part should be constant.

We will choose this constant such that $Y(\theta,\phi)$ are the spherical harmonics. The two equations obtained are

$$\Big[\partial_\theta^2 + \frac{1}{\sin^2\theta}\partial_\phi^2 + \frac{\cos\theta}{\sin\theta}\partial_\theta\Big]Y(\theta,\phi) = l(l+1)Y(\theta,\phi) \tag{A.6}$$

$$\Big[-(1-\frac{2M}{r})^{-1}\partial_t^2 + (1-\frac{2M}{r})\partial_r^2 - \frac{2M-2r}{r^2}\partial_r\Big]U(r,t) = \frac{l(l+1)}{r^2}U(r,t). \tag{A.7}$$

Since we also have translational symmetry, we can write $U(r,t) = T(t)u(r)$ and split (A.7) again which gives rise to

$$\partial_t^2 T(t) = -\omega^2 T(t) \tag{A.8}$$

and

$$\Big[(1-\frac{2M}{r})^2\partial_r^2 - (1-\frac{2M}{r})\frac{2M-2r}{r^2}\partial_r\Big]u(r) = \Big[(1-\frac{2M}{r})\frac{l(l+1)}{r^2} - \omega^2\Big]u(r).$$

So $T(t) \sim e^{-\imath\omega t}$, when we throw away the solutions moving in the negative timelike direction. To simplify the second equation we write $u(r) = r^{-1}R(r)$ and obtain

$$\Big[(1-\frac{2M}{r})^2\partial_r^2 + (1-\frac{2M}{r})\frac{2M}{r^2}\partial_r\Big]R(r) = \Big[(1-\frac{2M}{r})\big[\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\big] - \omega^2\Big]R(r).$$

Finally when we use $r^* = r + 2M\ln|\frac{r}{2M}-1|$, we can write $\partial_r = (1-2M/r)^{-1}\partial_{r^*}$ and $\partial_r^2 = (1-2M/r)^{-2}\partial_{r^*}^2 - (1-2M/r)^{-2}(2M/r^2)\partial_{r^*}$. So the equation for the radial part of the wave function becomes

$$\partial_{r^*}^2 R(r) + \{\omega^2 - [\frac{l(l+1)}{r^2} + \frac{2M}{r^3}][1-\frac{2M}{r}]\}R(r) = 0. \tag{A.9}$$

Appendix B

Geometrical Optics Approximation

Intuitively it makes sense that quantized matter fields should describe particles moving along geodesics in the classical limit, otherwise general relativity would not be the classical limit of quantum theory. But what exactly are the conditions for the fields to be correctly described by the classical limit? And how do we obtain the classical limit from quantum theory? These questions will be addressed in this appendix.

First of all, note that the particles that succeed at traveling through the collapsing body just before the horizon forms must have extremely high kinetical energy and will approximately travel at the speed of light and their mass will be negligible. In other words, these particle are quantum theoretically described by massless quantized fields and classically they move along null geodesics.

Let us state the fundamental laws of geometrical optics in curved spacetime: (1) massless particles move along null geodesics; and (2) the amplitude is governed by an adiabatic invariant which, in quantum language, states that the number of quanta is conserved. In general there is a third law that says: (3) the polarization vector is perpendicular to the rays and is parallel transported along the rays, but the waves we consider do not have a polarization since we restrict ourselves to massless scalar fields. For the derivation of this last law we refer to [47], where the calculation is done for the electromagnetic vector potential.

Now we will define three typical lengths, onto which we have to impose certain conditions to be able to work in the classical limit. These lengths are: (1) the wavelength λ of the waves as measured in Riemann normal coordinates; (2) the typical length $\mathcal{L}$ over which the amplitude, wavelength and polarization of the waves vary; and (3) the typical radius of curvature $\mathcal{R}$ of the spacetime through which the waves travel.

The geometrical optics approximation, i.e. the classical limit, is valid whenever the wavelength of the waves is very short compared to the other two typical lengths

$$\lambda \ll \mathcal{L} \quad \text{and} \quad \lambda \ll \mathcal{R}, \tag{B.1}$$

in other words, the waves should locally look like plane waves traveling through flat spacetime.

This answers our first question about the conditions our field has to satisfy in order to obey the laws of geometrical optics. In the following we will use (B.1) to obtain the laws of geometrical optics from the quantum theoretical wave equation, which is in our case the massless Klein-Gordon equation.

As we saw in section 3.5 we can write the outgoing waves (see (3.40)) as

$$\Phi = \Re[A(r,\theta,\phi)e^{-\imath\omega u}] = \Re[A(x^\mu)e^{-\imath\theta}], \tag{B.2}$$

where $\theta \equiv \omega u$ is the phase and $A(x^\mu)$ is the amplitude of the wave. The condition that the amplitude of the wave should vary only over distances much larger than the wavelength, tells us that the amplitude can be written as a series in λ

$$A(x^\mu) = a(x^\mu) + b(x^\mu) + c(x^\mu) + \ldots \quad \text{with} \quad a \propto \lambda^0,\ b \propto \lambda^1,\ c \propto \lambda^2,\ \ldots$$

Furthermore, we know that $\theta \sim 1/\lambda$, since ω is inversely proportional to the wavelength. If we now define the dimensionless parameter

$$\lambda/L \quad \text{with} \quad L = min(\mathcal{L},\mathcal{R}),$$

we can keep track of the order of expansion by introducing the dummy expansion parameter ϵ, which has eventual value unity. So any term with a factor ϵ^n in front of it varies as $(\lambda/L)^n$. When the conditions for the classical limit, i.e. (B.1), are satisfied, λ/L will approach zero, so higher order terms in ϵ will vanish. When we use this parameter we obtain the following form for (B.2)

$$\Phi = \Re[(a(x^\mu) + \epsilon b(x^\mu) + \epsilon^2 c(x^\mu) + \ldots)e^{-\imath\theta/\epsilon}]. \tag{B.3}$$

With this we find

$$\begin{aligned}\partial^\alpha\Phi = \Re\big([\partial^\alpha(a(x^\mu) + \epsilon b(x^\mu) + \epsilon^2 c(x^\mu) + \ldots)\\ -\frac{\imath}{\epsilon}(a(x^\mu) + \epsilon b(x^\mu) + \epsilon^2 c(x^\mu) + \ldots)\partial^\alpha\theta]e^{-\imath\theta/\epsilon}\big),\end{aligned}$$

So the Klein-Gordon equation, i.e. $D_\alpha\partial^\alpha\Phi = 0$, gives us

$$\begin{aligned} D_\alpha\partial^\alpha\Phi &= \Re\big([D_\alpha\partial^\alpha(a(x^\mu) + \epsilon b(x^\mu) + \ldots) - \frac{\imath}{\epsilon}(a(x^\mu) + \epsilon b(x^\mu) + \ldots)D_\alpha\partial^\alpha\theta \\ &- 2\frac{\imath}{\epsilon}\partial^\alpha\theta\partial_\alpha(a(x^\mu) + \epsilon b(x^\mu) + \ldots) \\ &- \frac{1}{\epsilon^2}(a(x^\mu) + \epsilon b(x^\mu) + \ldots)\partial^\alpha\theta\partial_\alpha\theta]e^{-\imath\theta/\epsilon}\big) \\ &= 0. \end{aligned} \tag{B.4}$$

From the leading term ($\propto \epsilon^{-2}$) we obtain

$$\partial^\alpha\theta\partial_\alpha\theta = 0, \tag{B.5}$$

To obtain the classical laws in their simplest form we define the wave vector as $k^\mu = \partial^\mu\theta$. From this it follows that the phase is of the form $\theta = k_\alpha x^\alpha +$ constant. When we write down the corresponding wave

$$\Phi \sim \Re[e^{-\imath(\vec{k}\cdot\vec{x} - k^0 t)}],$$

we see that k^0 is the angular frequency and that $\vec{k}$ points along the direction of propagation of the wave for each surface of constant phase. At each point in spacetime there is a wave vector and when they are glued together end-to-end they construct a family of curves $\mathcal{P}(\sigma)$ with tangent vector k^μ, which are the light rays since those are defined to be the curves normal to the surfaces of constant phase. This suggests that we write the wave vector as

$$k^\alpha = \frac{dx^\alpha}{d\sigma}. \tag{B.6}$$

It is now simple to obtain from (B.5) that massless particles move along null geodesics. First we note that

$$k^\mu k_\mu = g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = 0,$$

which is exactly the condition for a geodesic to be null. Secondly, we find also from (B.5) that

$$0 = D_\mu(k^\alpha k_\alpha) = 2k^\alpha D_\mu k_\alpha,$$

but since $D_\mu k_\alpha = D_\mu \partial_\alpha \theta = D_\mu D_\alpha \theta = D_\alpha D_\mu \theta = D_\alpha k_\mu$, this equals

$$\begin{aligned}
0 &= k^\alpha D_\alpha k_\mu \\
&= \frac{dx^\alpha}{d\sigma} D_\alpha \frac{dx_\mu}{d\sigma} \\
&= \frac{dx^\alpha}{d\sigma}\left[\partial_\alpha \frac{dx_\mu}{d\sigma} - \Gamma^\beta_{\alpha\mu}\frac{dx_\beta}{d\sigma}\right] \\
&= \frac{dx^\alpha}{d\sigma}\left[\frac{d\sigma}{dx^\alpha}\frac{d}{d\sigma}\frac{dx_\mu}{d\sigma} - \Gamma^\beta_{\alpha\mu}\frac{dx_\beta}{d\sigma}\right] \\
&= \frac{d^2 x_\mu}{d\sigma^2} - \Gamma^\beta_{\mu\alpha}\frac{dx^\alpha}{d\sigma}\frac{dx_\beta}{d\sigma},
\end{aligned}$$

which is exactly the geodesic equation! Furthermore, we can now identify σ as the affine parameter.

We have now derived the first law of geometrical optics, which states that massless particles move along null geodesics. The third law can be derived by looking at the term proportional to ϵ^{-1} in (B.4). For the details we refer to [47].

Appendix C

Derivation of Hawking Temperature using Path Integrals

The method described in this appendix is not meant to give further insight in the physics behind the calculation of Hawking radiation, it is rather meant as a tool to find the expression for the Hawking temperature in a quick and simple way [48].

Path integrals are only defined in Euclidean spacetime, so we will use a sequence of substitutions to obtain a suitable expression of a Euclidean metric from the Schwarzschild metric (1.15)

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

We can simplify this by choosing to work in the equatorial plane, i.e. $\theta = \pi/2$. We then define $\tau = \imath t$, to obtain the Euclidean signature

$$ds^2 = (1 - \frac{2M}{r})d\tau^2 + (1 - \frac{2M}{r})^{-1}dr^2 = \frac{r - 2M}{2M}d\tau^2 + \frac{2M}{r - 2M}dr^2.$$

Near but outside the horizon we have $r = 2M + \rho$ with $\rho > 0$, substituting this in the equation above gives us

$$ds^2 = \frac{\rho}{2M}d\tau^2 + \frac{2M}{\rho}d\rho^2. \tag{C.1}$$

If we finally set $\rho = y^2$, giving $d\rho^2 = 4\rho dy^2$ and substitute this into (C.1), we get

$$ds^2 = \frac{y^2}{2M}d\tau^2 + 8Mdy^2 = 8M(dy^2 + y^2\frac{d\tau^2}{16M^2}).$$

Now compare this last equation with $dr^2 + r^2d\phi^2$, which is a well defined description of a smooth Euclidean space if $0 \leq \phi \leq 2\pi$, we then see that this condition gives us a periodic time condition

$$\frac{d\tau^2}{16M^2} = d\big(\frac{\tau}{4M}\big)^2 \quad \rightarrow \quad 0 \leq \big(\frac{\tau}{4M}\big)^2 \leq 2\pi \quad \rightarrow \quad 0 \leq \tau \leq 8\pi M.$$

Path integrals with periodic time are equivalent to systems with finite temperature $T = \hbar/\tau_{\max}$, so this final argument leads to precisely the Hawking temperature

$$T_H = \frac{\hbar}{8M\pi}.$$

Appendix D

The Angular Equation in The SQBH Geometry For Small a

In this appendix we shall calculate the solution to the angular differential equation in the SQBH metric to first order in the small parameter $\epsilon = a^2(\omega^2 - \lambda^2)$. Let us restate the differential equation (7.23)

$$\frac{1}{\sin 2\theta}\partial_\theta\Big(\sin 2\theta\partial_\theta\Theta(\theta)\Big) + \Big\{(\omega^2 - \lambda^2)a^2\cos^2\theta - \frac{\mu^2}{\sin^2\theta} - \frac{\nu^2}{\cos^2\theta}\Big\}\Theta(\theta) = -\Lambda\Theta(\theta). \quad \text{(D.1)}$$

This differential equation in θ reduces to its analog in the CBH metric (7.17) if $a\omega,\ a\lambda \to 0$. So for small a the solution is approximately (7.18) with the eigenvalues Λ approximated by

$$\Lambda = l(l+2) + O((a\omega)^2) + O((a\lambda)^2). \quad \text{(D.2)}$$

The leading correction to these eigenvalues (for solutions with $\mu = \nu = 0$) is calculated in the appendix of [46] and is found to be

$$\delta\Lambda = -\frac{\epsilon}{8} = -\frac{a^2}{8}(\omega^2 - \lambda^2). \quad \text{(D.3)}$$

Using perturbation theory we can also find a first order correction to the eigenstates $\Theta(\theta)$ (we shall start with $\mu = \nu = 0$ and generalize later). This goes as follows, we introduce the differential operator $\hat{D}$

$$\hat{D} \equiv \frac{1}{\sin 2\theta}\partial_\theta\Big(\sin 2\theta\partial_\theta\Big) + (\omega^2 - \lambda^2)a^2\cos^2\theta,$$

or with the substitution $z = \cos 2\theta$

$$\hat{D} = 4\partial_z\Big((1 - z^2)\partial_z\Big) + (\omega^2 - \lambda^2)a^2\frac{1+z}{2}.$$

For $a\omega,\ a\lambda \to 0$ the second part of the differential operator is small compared to the first part. Using the small parameter ϵ we can write $\hat{D} = \hat{D}_0 + \epsilon\hat{D}_1$,

$\Theta = \Theta_0 + \epsilon\Theta_1$ and $\Lambda = \Lambda_0 + \epsilon\Lambda_1$. With this notation the differential equation reads

$$(\hat{D}_0 + \epsilon\hat{D}_1)(\Theta_0 + \epsilon\Theta_1) = -(\Lambda_0 + \epsilon\Lambda_1)(\Theta_0 + \epsilon\Theta_1),$$

dropping terms of order ϵ^2 we find

$$\hat{D}_0\Theta_0 + \epsilon(\hat{D}_0\Theta_1 + \hat{D}_1\Theta_0) = -\Lambda_0\Theta_0 - \epsilon(\Lambda_0\Theta_1 + \Lambda_1\Theta_0). \tag{D.4}$$

The leading part of this equation is just (7.17) with $\mu = \nu = 0$ and is solved by the Legendre polynomials $P_n(z)$ for $\Lambda_0 = 4n(n+1)$,[1] so $\Theta_0(z) = P_n(z)$. Furthermore, we already know that $\Lambda_1 = -1/8$ from (D.3). When we put all this into (D.4), we can reduce the equation for Θ_1 to

$$D_0\Theta_1 + \frac{1}{2}zP_n(z) = -4n(n+1)\Theta_1.$$

The solution can be found to be

$$\Theta_1(z) = \frac{1}{4(2n+1)}(P_{n+1}(z) - P_{n-1}(z)).$$

To obtain Λ_1 and $\Theta_1(z)$ for general $\mu, \nu \geq 0$, we first rewrite (D.1) by substituting $\Theta(\theta) = \cos^\nu\theta \sin^\mu\theta T(\theta)$ and $z = \cos 2\theta$, this gives

$$\begin{aligned}\Big(-\frac{1}{4}(\mu+\nu)^2 - \frac{1}{2}(\mu+\nu)\Big)T(z) + (\nu - \mu - (\mu+\nu+2)z)\partial_z T(z)& \\ +(1-z^2)\partial_z^2 T(z) + \frac{a^2}{8}(\omega^2 - \lambda^2)(1+z)T(z)& \\ = -\frac{\Lambda}{4}T(z).&\end{aligned}$$

As before we write $\hat{D} = \hat{D}_0 + \epsilon\hat{D}_1$, $T = T_0 + \epsilon T_1$ and $\Lambda = \Lambda_0 + \epsilon\Lambda_1$. In this case we have

$$\begin{aligned}\hat{D}_0 &= -\frac{1}{4}(\mu+\nu)^2 - \frac{1}{2}(\mu+\nu) + (\nu - \mu - (\mu+\nu+2)z)\partial_z + (1-z^2)\partial_z^2 \\ \hat{D}_1 &= \frac{1+z}{8} \\ \Lambda_0 &= k(k+2) \\ T_0(z) &= P_c^{(\mu,\nu)}(z) \quad \text{with} \quad c = \frac{k-(\mu+\nu)}{2}.\end{aligned}$$

To find T_1 and Λ_1 we have to solve

$$\hat{D}_0 T_1 + \hat{D}_1 T_0 = -\frac{\Lambda_0}{4}T_1 - \frac{\Lambda_1}{4}T_0, \tag{D.5}$$

or equivalently

$$\begin{aligned}\frac{z+1}{8}P_c^{(\mu,\nu)}(z) + c(c+\mu+\nu+1)T_1 + (\nu - \mu - (\mu+\nu+2)z)\partial_z T_1& \\ +(1-z^2)\partial_z^2 T_1 = -\frac{\Lambda_1}{4}P_c^{(\mu,\nu)}(z),& \qquad \text{(D.6)}\end{aligned}$$

[1] Comparing with (D.2) tells us that these eigenvalues correspond to even values of l only ($l = 2n$). The eigenvalues for l odd can be found by considering the case in which $(\mu - \nu)$ is an odd integer.

where we inserted $T_0(z) = P_c^{(\mu,\nu)}(z)$ and $\Lambda_0 = k(k+2) = (2c+\mu+\nu)(2c+\mu+\nu+2)$. Comparison with the case where $\mu = \nu = 0$ suggests us that $T_1(z) = AP_{c+1}^{(\mu,\nu)}(z) + BP_{c-1}^{(\mu,\nu)}(z)$ may very well solve this equation. And some algebra shows that indeed it does. Start by noticing that[2]

$$
\begin{aligned}
&(\nu - \mu - (\mu+\nu+2)z)\partial_z(AP_{c+1}^{(\mu,\nu)}(z) + BP_{c-1}^{(\mu,\nu)}(z)) \\
&\qquad +(1-z^2)\partial_z^2(AP_{c+1}^{(\mu,\nu)}(z) + BP_{c-1}^{(\mu,\nu)}(z)) \\
&= -A(c+1)(c+\mu+\nu+2)P_{c+1}^{(\mu,\nu)}(z) - Bc(c+\mu+\nu)P_{c-1}^{(\mu,\nu)}(z).
\end{aligned}
$$

Plugging this into (D.6) gives

$$
\begin{aligned}
&\frac{z+1}{8}P_c^{(\mu,\nu)}(z) + c(c+\mu+\nu+1)(AP_{c+1}^{(\mu,\nu)}(z) + BP_{c-1}^{(\mu,\nu)}(z)) \\
&-A(c+1)(c+\mu+\nu+2)P_{c+1}^{(\mu,\nu)}(z) - Bc(c+\mu+\nu)P_{c-1}^{(\mu,\nu)}(z) \\
&\qquad = -\frac{\Lambda_1}{4}P_c^{(\mu,\nu)}(z),
\end{aligned}
$$

rearranging this gives

$$
\begin{aligned}
zP_c^{(\mu,\nu)}(z) &= 8(2c+\mu+\nu+2)AP_{c+1}^{(\mu,\nu)}(z) - 8(2c+\mu+\nu)BP_{c-1}^{(\mu,\nu)}(z) \\
&- (2\Lambda_1+1)P_c^{(\mu,\nu)}(z).
\end{aligned}
$$

This can be solved for A, B and Λ_1 by using the relation

$$
\begin{aligned}
&(2c+\mu+\nu+1)(2c+\mu+\nu)(2c+\mu+\nu+2)zP_c^{(\mu,\nu)}(z) \\
&\qquad +(2c+\mu+\nu+1)(\mu^2-\nu^2)P_c^{(\mu,\nu)}(z) \\
&= 2(c+1)(c+\mu+\nu+1)(2c+\mu+\nu)P_{c+1}^{(\mu,\nu)}(z) \\
&\qquad +2(c+\mu)(c+\nu)(2c+\mu+\nu+2)P_{c-1}^{(\mu,\nu)}(z).
\end{aligned}
$$

We then obtain

$$
\begin{aligned}
A &= \frac{(c+1)(c+\mu+\nu+1)}{4(2c+\mu+\nu+1)(2c+\mu+\nu+2)^2} \\
B &= \frac{(c+\mu)(c+\nu)}{4(2c+\mu+\nu+1)(2c+\mu+\nu)^2} \\
\Lambda_1 &= \frac{(\mu^2-\nu^2)}{2(2c+\mu+\nu+1)(2c+\mu+\nu)} - \frac{1}{2}.
\end{aligned}
$$

It can easily be checked that this is in accordance with our previous results by putting $\mu = \nu = 0$ (note that you obtain $\Lambda_1 = -1/2$, instead of the expected $-1/8$, but this is only a matter of definition which you can see upon comparing (D.4) and (D.5)). So to first order in $\epsilon = a^2(\omega^2 - \lambda^2)$ we found

$$
\begin{aligned}
\Theta(\theta) &= \cos^\nu\theta \sin^\mu\theta P_c^{(\mu,\nu)}(\cos 2\theta) \\
&+a^2(\omega^2-\lambda^2)\cos^\nu\theta\sin^\mu\theta[AP_{c+1}^{(\mu,\nu)}(\cos 2\theta) + BP_{c-1}^{(\mu,\nu)}(\cos 2\theta)]
\end{aligned}
$$

and

$$
\Lambda = k(k+2) + a^2(\omega^2-\lambda^2)\Big[\frac{(\mu^2-\nu^2)}{2(2c+\mu+\nu+1)(2c+\mu+\nu)} - \frac{1}{2}\Big].
$$

[2] See for example [43].

Appendix E

The Radial Equation in The SQBH Geometry For Small a

To find the radial solution in the SQBH metric to first order in a we do perturbation theory, just as we did for the angular differential equation. We start from (7.24)

$$\begin{aligned}
&\frac{1}{r}\partial_r\Big(r(r^2+a^2)\partial_r H(r)\Big)+\Big\{(\omega^2-\lambda^2)(Q_1+Q_5+r^2)\\
+\quad&\frac{(\omega\sqrt{Q_1Q_5}-a\mu)^2}{r^2+a^2}-\frac{(\lambda\sqrt{Q_1Q_5}+a\nu)^2}{r^2}\Big\}H(r)-\Lambda H(r)=0
\end{aligned}$$

and recast the differential operator to be of the form $\hat{D}_0+a\hat{D}_1$, where $\hat{D}_0$ is the radial part of the differential operator in the CBH metric. This can be done by doing a number of substitutions and then neglect all terms of order a^2 and higher. Start by substituting $z=1/r$ and $H(z)=zG(z)$ and dividing by z, this gives

$$\begin{aligned}
z^2\partial_z^2G(z)+z\partial_zG(z)-G(z)+a^2\Big[z^3\partial_z(zG(z))+z^3\partial_z(z^2\partial_zG(z))\Big]&\\
+\Big\{z^2\frac{(\omega\sqrt{Q_1Q_5}-a\mu)^2}{1+z^2a^2}-z^2(\lambda\sqrt{Q_1Q_5}+a\nu)^2\Big\}G(z)&\\
-\Big\{-(\omega^2-\lambda^2)(Q_1+Q_5+z^{-2})+\Lambda\Big\}G(z)=0.&
\end{aligned}$$

As before we can neglect the term $(\omega^2-\lambda^2)(Q_1+Q_5+z^{-2})$ when we restrict ourselves to the inner region of spacetime, i.e. $r^2\ll\sqrt{Q_1Q_5}$. Now we throw away all terms that are of order a^2 or higher. What remains is

$$\begin{aligned}
z^2\partial_z^2G(z)+z\partial_zG(z)&\\
+\Big\{Q_1Q_5(\omega^2-\lambda^2)-a\sqrt{Q_1Q_5}(\omega\mu+\lambda\nu)\Big\}z^2G(z)=(\Lambda+1)G(z).&
\end{aligned}$$

This is again a Bessel's differential equation and the solution is

$$G(z)=C_1J_\sigma(\sqrt{\alpha_0+a\alpha_1}z)+C_2J_{-\sigma}(\sqrt{\alpha_0+a\alpha_1}z),$$

with

$$\begin{aligned}\alpha_0 &= Q_1 Q_5(\omega^2 - \lambda^2)\\ \alpha_1 &= -\sqrt{Q_1 Q_5}(\omega\mu + \lambda\nu)\\ \sigma &= \sqrt{\Lambda + 1}.\end{aligned}$$

We would like to write this as $G(z) = G_0(z) + aG_1(z)$. To do so we first note that

$$\sqrt{\alpha_0 + a\alpha_1}z \approx \sqrt{\alpha_0}(1 + a\frac{\alpha_1}{2\alpha_0})z$$

and since Λ can also be expanded in orders of a (see (7.26)), so can σ

$$\begin{aligned}\sigma &= \sqrt{\Lambda+1} \approx \sqrt{\Lambda_0 + a^2\Lambda_1 + 1}\\ &\approx \sqrt{\Lambda_0+1}(1 + a^2\frac{\Lambda_1}{2(\Lambda_0+1)}) \equiv \sigma_0 + a^2\sigma_1.\end{aligned}$$

Now we can develop the Bessel function in a Taylor series in a, to obtain

$$J_{\sigma_0 + a^2\sigma_1}(\sqrt{\alpha_0 + a\alpha_1}z) \approx J_{\sigma_0}(\sqrt{\alpha_0}z) + a\frac{\alpha_1}{2\sqrt{\alpha_0}}z\partial_x J_{\sigma_0}(x)\Big|_{x=\sqrt{\alpha_0}z}$$
$$+\frac{a^2}{2}\Big(\frac{\alpha_1}{2\sqrt{\alpha_0}}z\Big)^2\partial_x^2 J_{\sigma_0}(x)\Big|_{x=\sqrt{\alpha_0}z} + a^2\sigma_1\partial_\nu J_\nu(\sqrt{\alpha_0}z)\Big|_{\nu=\sigma_0}.$$

From this we see that to first order in a the Bessel function is of integer order, since $\sigma_0 = k+1$ is integer. Finally, using

$$\partial_x J_\nu(x) = \frac{1}{2}(J_{\nu-1}(x) - J_{\nu+1}(x))$$

we find

$$G(z) \approx C_1\Big[J_{k+1}(\sqrt{\alpha_0}z) + a\frac{\alpha_1}{4\sqrt{\alpha_0}}z(J_k(\sqrt{\alpha_0}z) - J_{k+2}(\sqrt{\alpha_0}z))\Big].$$

As you can see we have put C_2 to zero, this is again to make sure that the wavefunction is square integrable.

Finally, we would like to normalize $H(z)$ to first order in a. Upon comparison with the normalization procedure for the modes in the CBH metric, we find that we have to solve (see (7.40))

$$-\sqrt{Q_1Q_5}\int\frac{dz}{z}G^2(z) \approx -\sqrt{Q_1Q_5}\int\frac{dz}{z}C_1^2\Big[J_{k+1}(\sqrt{\alpha_0}z)\Big]^2$$
$$-2\sqrt{Q_1Q_5}a\int dz C_1^2\frac{\alpha_1}{4\sqrt{\alpha_0}}J_{k+1}(\sqrt{\alpha_0}z)[J_k(\sqrt{\alpha_0}z) - J_{k+2}(\sqrt{\alpha_0}z)]$$
$$= -\sqrt{Q_1Q_5}\frac{C_1^2}{2(k+1)},$$

in the last step we have used that

$$\int dz J_\mu(\beta z)[J_{\mu-1}(\beta z) - J_{\mu+1}(\beta z)] = \frac{1}{2\beta} - \frac{1}{2\beta} = 0,$$

as can be found in [43]. So to first order[1] in a the modes in the SQBH metric have the same normalization constants as the modes in the CBH metric.

[1] To second order in a this is not true since there is at least a correction that comes from normalizing the angular modes found in appendix D.

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