D-branes, nontrivial B-fields and K-theory

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Contents

1 Introduction

In this thesis we will study configurations of D-branes with nontrivial B-field. D-branes are objects that appear in string theory, and can be described as dynamical hyper surfaces in space-time on which open strings can end. The open strings give a $U(n)$ gauge field on the brane. Closed strings give rise to a B-field in all of space-time. There is a gauge freedom between the B-field and the gauge field A on the brane. But for nontrivial B-fields, i.e. B fields that can only described locally, the local gauge freedom breaks down. In this case the A-field has to be interpreted as a connection in a twisted vector bundle, in order to make the path integral factors well defined. An explicit calculation of the anomaly cancellation will be done in the last chapter. It is mainly based on [12] and [7].
The first chapter will give an introduction in (super)string theory. The second chapter gives an introduction to D-branes with some emphasis on Sen's construction of tachyon condensation. The third chapter gives some basics of K-theory and its application in string theory. Topological K-theory gives a classification of vector bundles other than that of cohomology, that narrowly resembles possible D-brane charges. The last chapter describes the problems related to D-branes in a background of nontrivial B-fields. The twisted line bundles that are needed to cancel the anomaly of the B-field, give rise to the proposal in [3] that D-brane charges in such a background should be measured in twisted K-theory.

2 Bosonic strings

Describing a one-dimensional object moving through space-time, one needs two variables. A time-like parameter τ, parameterizing the eigentime of the string, and a parameter σ for the different positions on the string. There are two possible topologies: a loop, the *closed string*, and a line-segment with two endpoints, the *open string*. The embedding of the string in space-time, is a map from the parameter-space, called the *worldsheet*, Σ into space-time. We will start very generally with D-space-time dimensions, so $D-1$ space and 1 time dimension.

$$X : \Sigma \to \mathbb{R}^D. \tag{2.1}$$

In the case of a closed string the worldsheet is an infinite cylinder with circumference 2π. For an open string it is the strip $]-\infty,\infty[\times[0,\pi]$. Everything starts, as always, by choosing an appropriate action for the string. By analogy with the relativistic point particle, where the action is the relativistic length of its world line, a natural choice is the *Nambu-Gotu action*, the relativistic area of the worldsheet embedded in space-time.

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_\Sigma d\sigma d\tau\sqrt{|\det\partial_a X^\mu \partial_b X_\mu|}. \tag{2.2}$$

We use the indices $(a, b = 0, 1)$ for the worldsheet parameters $\sigma^0 = \tau$ and $\sigma^1 = \sigma$, and the Greek indices $(\mu, \nu = 0, 1, ...D-1)$ for space-time coordinates. The constant α' is the Regge slope and has units of the space-time length squared. This action has the desired Poincaré and reparameterization invariance, but the square root makes it not very practical. The expression

$$h_{ab} = \partial_a X^\mu \partial_b X_\mu = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \tag{2.3}$$

can be thought of as the space-time metric $\eta_{\mu\nu}$ pulled back to the worldsheet. The action can now be rewritten

$$S_P = -\frac{1}{4\pi\alpha'} \int_\Sigma d\sigma d\tau \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X_\mu \tag{2.4}$$

where h is the determinant of h_{ab} and h^{ab} is the inverse of h_{ab}. Now consider more generally h^{ab} as a symmetric 2-tensor field on the worldsheet, independent of X. The action S_p above with arbitrary fields X^μ and h_{ab} is called the *Polyakov action*. Variation of h^{ab} gives

$$\begin{aligned} \delta S &= -\frac{1}{4\pi\alpha'} \int_\Sigma d\sigma d\tau \sqrt{h} T_{ab} \delta h^{ab} \\ \text{with } T_{ab} &= \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab} h^{cd} \partial_c X^\mu \partial_d X_\mu. \end{aligned} \tag{2.5}$$

Under a reparameterization $\sigma^a \to \sigma^a + \xi^a$ of the world sheet

$$\begin{aligned} X^\mu &\to X^\mu + \xi^a \partial_a X^\mu, \\ S &\to S + \frac{-1}{4\pi\alpha'} \int_\Sigma d\sigma d\tau \sqrt{h}\, T_{ab} (\nabla^a \xi^b + \nabla^b \xi^a). \end{aligned} \tag{2.6}$$

Here ∇ denotes the covariant derivative with respect to h^{ab}. So with h^{ab} transforming as:

$$\begin{aligned} h^{ab} &\to h^{ab} - \nabla^a \xi^b - \nabla^b \xi^a \\ &= h^{ab} + \xi^c \partial_c h^{ab} - h^{ac} \partial_c \xi^b - h^{bc} \partial_c \xi^a, \end{aligned} \tag{2.7}$$

i.e. transforming as a 2-tensor, S is invariant under worldsheet reparameterization just as the Nambu-Gotu action. The Polyakov action however has an extra symmetry. It is invariant under local (on the worldsheet) rescaling of the metric with any positive factor:

$$h^{ab} \to e^{\omega(\sigma,\tau)} h^{ab}. \tag{2.8}$$

This is called *Weyl-rescaling*. The equations of motions with respect to h^{ab} give $T^{ab} = 0$. This condition implies

$$\partial_a X^\mu \partial_b X^\mu = \frac{1}{2} \left(h^{cd} \partial_c X^\mu \partial_d X_\mu \right) h_{ab}. \tag{2.9}$$

which is the same as saying that h_{ab} is a Weyl-rescaling of the pull-back metric $\partial_a X^\mu \partial_b X_\mu$. Thus we see that the Polyakov action leads to the same theory as

the Nambu-Gotu action, only with an extra invariance.
This extra invariance can be used in combination with reparameterization invariance to fix the metric, at least locally, to the standard flat metric $h_{ab} = \eta_{ab}$. Switching to an Euclidean worldsheet action, and using complex coordinates $w = \sigma + \mathrm{i}\tau$ this simplifies the action to

$$S = \frac{1}{2\pi\alpha'} \int_{\Sigma} dw d\bar{w}\ \partial_w X^{\mu} \partial_{\bar{w}} X_{\mu}. \tag{2.10}$$

2.1 Conformal field theory

One might think that fixing the metric leads to the loss of scale invariance. This is not so. Consider reparameterizations, such that the metric changes with a positive factor. By (2.7) these satisfy

$$\nabla^a \xi^b + \nabla^b \xi^a = \omega(\sigma, \tau) h^{ab} \tag{2.11}$$

Because the original Polyakov action (2.4) is invariant under Weyl-rescaling independent of the reparameterization, the action is also invariant if we only reparameterize X keeping the metric fixed. Such a transformation thus only gives a different local rescaling of the pull-back metric. An action invariant under such a transformation, is called *conformally invariant.* In complex coordinates, using

$$\begin{aligned} h^{ab} &= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad v^z = v^1 + iv^2, \quad v^{\bar{z}} = v^1 - iv^2, \\ h_{ab} &= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad v_z = \frac{1}{2}(v^1 - iv^2), \quad v_{\bar{z}} = \frac{1}{2}(v^1 + iv^2) \\ \partial &= \frac{\partial}{\partial z} = \frac{1}{2}(\partial_1 - \mathrm{i}\partial_2), \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2) \end{aligned} \tag{2.12}$$

we get that this transformation is of the form $z \to z + \xi$ with $\bar{\partial}\xi = 0$, or $\bar{z} \to \bar{z} + \bar{\xi}$ with $\partial\bar{\xi} = 0$. So conformal transformations are anti-holomorphic and holomorphic transformations. We can use the conformal invariance, to define new coordinates for the worldsheet, transforming

$$w \to z = e^{-iw} = \exp(-\mathrm{i}\sigma + \tau), \ \bar{w} \to \bar{w} = e^{\mathrm{i}\bar{w}} = \exp(\mathrm{i}\sigma + \tau) \tag{2.13}$$

The closed worldsheet is now mapped to the entire complex plane, with circles around the origin giving the string coordinates at equal time. The origin itself corresponds to $\tau \to -\infty$ and times moves radially outward. The open string is mapped to the upper half-plane.
Under a conformal transformation

$$\delta S \sim \int_{\Sigma} \sqrt{h} (\nabla^a \xi^b + \nabla^b \xi^a) T_{ab} = \int_{\Sigma} \sqrt{h} \omega h^{ab} T_{ab}, \tag{2.14}$$

Therefore in a conformal theory, in which this δS vanishes for any scalar function ω, $h^{ab} T_{ab} = 0$. For the string theory stress tensor in complex coordinates this means $T_{z\bar{z}} = T_{\bar{z}z} = 0$. Furthermore $\nabla^a T_{ab} = 0$, which follows from the equations of motion by (2.6). In complex coordinates:

$$\bar{\partial} T_{zz} = \partial T_{\bar{z}\bar{z}} = 0. \tag{2.15}$$

We define

$$\begin{aligned} T(z) &:= T_{zz}(z) = -\frac{1}{\alpha'}\partial X^\mu \partial X_\mu, \\ \bar{T}(\bar{z}) &:= T_{\bar{z}\bar{z}}(\bar{z}) = -\frac{1}{\alpha'}\bar{\partial} X^\mu \bar{\partial} X_\mu. \end{aligned} \tag{2.16}$$

So far all we have done is solving classical equations of motion. Now let's see what happens when we try to quantize the theory. Expectation values can be calculated by using the path integral method

$$< \mathcal{F}[X] >= \int \mathcal{D}X \exp(-\frac{1}{2\pi\alpha'}\int dz d\bar{z}\ \partial_z X^\mu \partial_{\bar{z}} X_\mu)\mathcal{F}[X] \tag{2.17}$$

where $\mathcal{F}$ is a functional of X, that is a function that depends on all values of X anywhere on the worldsheet. Using the fact that the path integral of a functional derivative is zero

$$0 = \int \mathcal{D}X \frac{\delta \exp(-S[X])}{\delta X_\mu(z,\bar{z})} = \left\langle -\frac{\delta S[X]}{\delta X_\mu(z,\bar{z})} \right\rangle = \frac{1}{\pi\alpha'}\langle \partial\bar{\partial} X^\mu(z,\bar{z})\rangle. \tag{2.18}$$

Adding any operator not depending on the value of X at z gives the same answer. So this must correspond to operator equation in the Hilbert space formalism

$$\partial\bar{\partial}\hat{X}^\mu(z,\bar{z}) = 0. \tag{2.19}$$

This is in line with Ehrenfest's theorem as the classical equation of motion $\partial\bar{\partial}X^\mu = 0$ translates into the corresponding operator equation. From

$$\begin{aligned} 0 &= \int \mathcal{D}X \frac{\delta \exp(-S) X^\nu(z',\bar{z}')}{\delta X_\mu(z,\bar{z})} \\ &= \int \mathcal{D}X \exp(-S)(\eta^{\mu\nu}\delta^2(z-z',\bar{z}-\bar{z}') + \frac{1}{\pi\alpha'}\partial_z\partial_{\bar{z}} X^\mu(z,\bar{z}) X^\nu(z',\bar{z}')) \end{aligned} \tag{2.20}$$

it follows that

$$\frac{1}{\pi\alpha'}\partial_z\partial_{\bar{z}}\langle X^\mu(z,\bar{z})X^\nu(z',\bar{z}')\rangle = -\eta^{\mu\nu}\langle\delta^2(z-z',\bar{z}-\bar{z}')\rangle. \tag{2.21}$$

So the correspondence between classical and quantum mechanics no longer holds when the local operators are evaluated at coincident points. It is therefore useful to introduce the following normal ordering

$$:X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2): = X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2) + \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_1 - z_2|^2. \tag{2.22}$$

The point is that the normal ordered product does satisfy the equation of motion.

$$\partial_1\bar{\partial}_1 :X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2): = 0 \tag{2.23}$$

In the Taylor expansion of $:X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2):$ for z_1 around z_2, all terms $\partial^k\bar{\partial}^l..$ with $k > 0$ and $l > 0$ vanish by (2.23). Thus from (2.22)

$$\begin{aligned} X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2) &= -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_1-z_2|^2 + :X^\mu X^\nu(z_2,\bar{z}_2): \\ +\sum_{k=1}^{\infty}\frac{1}{k!}\left[(z_1-z_2)^k :X^\nu\partial^k X^\mu(z_2,\bar{z}_2): \right. &\left. + (\bar{z}_1-\bar{z}_2)^k :X^\nu\bar{\partial}^k X^\mu(z_2,\bar{z}_2):\right]. \end{aligned} \tag{2.24}$$

This expression is called an *Operator Product Expansion (OPE)*, the expansion of a product of local operators into a sum of local operators evaluated at one point, the coefficients depending on the separation of the local operators in the product. The symbol '$\sim$' will be used, when only the terms with coefficient that are nonsingular for $z_1 \to z_2$ are specified. So

$$\begin{aligned} X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2) &\sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_1-z_2|^2, \\ \partial X^\mu(z_1)\partial X^\nu(z_2) &\sim -\frac{\alpha'}{4(z_1-z_2)^2}. \end{aligned} \tag{2.25}$$

A *primary field* $\phi(z,\bar{z})$ of weight $(h,\bar{h})$ is a field transforming $\phi \to (\partial f)^h(\bar{\partial}\bar{f})^{\bar{h}}\phi$ under $(z,\bar{z}) \to (f(z),\bar{f}(\bar{z}))$. For a holomorphic field $\phi(z)$ of course $\bar{h} = 0$, and the infinitesimal transformation under $z \to f(z) = z + \xi$ reads

$$\delta_\xi\phi(z) = (\xi(z)\partial + h\partial\xi(z))\phi(z). \tag{2.26}$$

The stress tensor is now given by

$$T(z) = -\frac{1}{\alpha'}:\partial X^\mu\partial X_\mu:, \quad \bar{T}(\bar{z}) = -\frac{1}{\alpha'}:\bar{\partial}X^\mu\bar{\partial}X_\mu:. \tag{2.27}$$

The OPE of the stress tensor with a local operator ϕ gives its behaviour under conformal transformations, since the infinitesimal change in ϕ under a conformal reparameterizations $z \to z + \xi$ or $\bar{z} \to \bar{z} + \bar{\xi}$ (conformal implies that ξ is holomorphic, resp. $\bar{\xi}$ is anti-holomorphic) is given by

$$\begin{aligned} \delta_\xi\phi(z,\bar{z}) &= [\oint\frac{dz'}{2\pi\mathrm{i}}\xi(z')T(z'),\phi(z,\bar{z})], \\ \delta_{\bar{\xi}}\phi(z,\bar{z}) &= [\oint\frac{d\bar{z}'}{2\pi\mathrm{i}}\bar{\xi}(\bar{z}')\bar{T}(\bar{z}'),\phi(z,\bar{z})]. \end{aligned} \tag{2.28}$$

Combining this equation with (2.26), calculating the contour integral using Cauchy's theorem, the OPE of T and a holomorphic field ϕ of weight h should be

$$T(z_1)\phi(z_2) \sim \frac{h\phi(z_2)}{(z_1-z_2)^2} + \frac{\partial\phi(z_2)}{z_1-z_2}. \tag{2.29}$$

The stress tensor itself is not a primary field, since one can calculate using Wick's theorem, that its OPE with itself is not of the form (2.29)

$$T(z_1)T(z_2) \sim \frac{c}{2(z_1-z_2)^4} + \frac{2T(z_2)}{(z_1-z_2)^2} + \frac{\partial\phi(z_2)}{z_1-z_2} \tag{2.30}$$

where $c = d+1$ the number of space-time dimensions. In conformal field theory this constant is called the *central charge.* The stress tensor has a Laurent expansion

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad L_n^\dagger = L_{-n} \text{ or equivalently } L_n = \oint \frac{dz}{2\pi \mathrm{i}} z^{n+1} T(z).$$

and similar the operators $\bar{L}$ for $\bar{T}(\bar{z})$. The operators form an algebra. Using (2.30) it is possible to calculate

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\ [\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{\bar{c}}{12}m(m^2-1)\delta_{m+n,0}, \\ [\bar{L}_m, L_n] &= 0. \end{aligned} \tag{2.31}$$

This is a *Virasoro algebra.* The L_n and $\bar{L}_m$ are the generators of the conformal transformations $z \to z + \epsilon z^{n+1}$ and $\bar{z} \to \bar{z} + \epsilon \bar{z}^{m+1}$. Under such transformations a primary field of weight $(h, \bar{h})$ transforms as (cf. (2.26) and (2.28))

$$\begin{aligned} \delta\phi &= [L_n, \phi(z,\bar{z})] = (z^{n+1}\partial + h(n+1)z^n)\phi(z,\bar{z}) && \text{for } z \to z + \epsilon z^{1-n}, \\ \delta\phi &= [\bar{L}_m, \phi(z,\bar{z})] = (\bar{z}^{m+1}\bar{\partial} + \bar{h}(m+1)\bar{z}^m)\phi(z,\bar{z}) && \text{for } \bar{z} \to \bar{z} + \epsilon \bar{z}^{1-m}. \end{aligned} \tag{2.32}$$

Classically $T_{ab} = 0$ had to be imposed after gauge fixing, but by (2.31) there is no nonzero $|\Psi\rangle$, for which $L_n|\Psi\rangle = 0$ for all $n \in \mathbb{Z}$, thus for which $T|\Psi\rangle = 0$. In fact we only have to impose $\langle\Psi|T|\Psi\rangle = 0$. The state for which this is satisfied will be denoted by $|0\rangle$ and $L_n|0\rangle = 0$ for all $n \geq 0$. But then

$$||L_{-1}|0\rangle||^2 = \langle 0|[L_1, L_{-1}]|0\rangle = \langle 0|L_0|0\rangle = 0. \tag{2.33}$$

So then also $L_{-1}|0\rangle = 0$, but this means that this state is invariant under translations in the plane. The operators $L_0, L_{\pm 1}$ can be seen to generate $\mathfrak{sl}(2,\mathbb{C})$. The state $|0\rangle$ is therefore called the sl_2-invariant vacuum. Note that this is the vacuum of the worldsheet-theory, which does not mean a space-time vacuum without strings, but a string worldsheet without any excitations. A state Ψ describing a propagating string is not necessarily translation invariant, as the coordinate $\tau = -\infty$ is fixed at the origin. So in fact it is possible that $\langle\Psi|T\langle\Psi|$ has a pole for $z \to 0$. The worldsheet of a propagating string is a cylinder and the transformation $w = \sigma + \mathrm{i}\tau \mapsto exp(-iw)$ is only conformal between the cylinder and the complex plane without the origin. In fact for the transformation between the cylinder and the plane

$$T_{ww}(w) = (\partial_w z)^2 T_{zz}(z) + \frac{c}{24}. \tag{2.34}$$

With this expression the Hamiltonian H, the generator of translations in the τ-direction, is given by

$$H = \int_0^{2\pi} \frac{d\sigma}{2\pi} T_{\tau\tau} = -\int_0^{2\pi} \frac{d\sigma}{2\pi} T_{ww} + T_{\bar{w}\bar{w}} = L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24}. \tag{2.35}$$

Remember that τ is only a coordinate on the worldsheet, so it has an arbitrary rescaling with respect to the space-time coordinates. The parameterization can be fixed, using for instance light-cone coordinates (see [9] or [18]), but then the X^μ are no longer free to move in the σ and τ-direction. Thus only $D-2$ of the D bosonic fields X^μ remain. For this theory then $c = D-2$. Translation in the τ-direction is now directly related to a translation of the worldsheet-cylinder in the longitudinal direction. Demanding invariance under this translation implies $H|\Psi\rangle = 0$. Because invariance under rotation of the string, i.e. invariance under σ-translation, implies $(L_0 - \bar{L}_0)|\Psi\rangle = 0$ we then have

$$L_0|\Psi\rangle = \bar{L}_0|\Psi\rangle = \frac{D-2}{24}|\Psi\rangle. \tag{2.36}$$

This condition which is essential in the calculation of the string spectrum can be derived in various ways. Frequently used are the *light-cone approach* and *old covariant quantization*. These methods however do not make clear how the classical condition $T_ab = 0$ breaks down after quantization. A more abstract, but also more convincing method, uses so called Fadeev Popov ghosts and BRST-quantization. For a good introduction in BRST-quantization of the bosonic string see [18] vol. I. We will not treat this here, as it would take too much space in this already rather lengthy introduction. We will however discuss the ghost fields and some of their properties as they are essential in understanding some subjects later on (for instance the vertex operators in superstring theory see chapter 3).

2.2 Fadeev Popov ghosts

Remember that the stress tensor of the Polyakov action vanishes by the equation of motion of h_{ab}:

$$\langle T_{ab}(\sigma,\tau)\rangle = \left\langle \frac{\delta S_P[X,h]}{\delta h_{ab}(\sigma,\tau)} \right\rangle = \int \mathcal{D}X\mathcal{D}h \frac{\delta \exp(-S_P[X,h])}{\delta h_{ab}(\sigma,\tau)} = 0 \tag{2.37}$$

The Polyakov action is invariant under reparameterization and local rescalings of the metric, Weyl rescaling. The path integral diverges as we integrate over gauge equivalent configurations giving the same action. Choosing one representation in every gauge class $[(X,h)]$ by choosing one fixed metric $\hat{h}$, there are two possible problems. It is possible that one gauge class contains more than one pair $(X,\hat{h})$. This happens if there are reparameterizations which do not change the metric, so $\delta h^{ab} = \nabla^a\xi^b + \nabla^b\xi^a = 0$. These are called killing vectors. For instance the sphere has six such reparameterizations. Another problem arises if not every gauge class contains a pair $(X,\hat{h})$. Then there are different metrics

which cannot be gauged into each other. The space of these gauge inequivalent metrics is called the moduli space.
By choosing one fixed metric $\hat{h}$ and omitting the integral $\mathcal{D}h$, the equation of motion $T_{ab} = 0$ above no longer follows. The variation of the metric in the form of a reparameterization, so $\delta h^{ab} = \nabla^a \xi^b + \nabla^b \xi^a$, is lost. Indeed $T_{ab}(\nabla^a \xi^b + \nabla^b \xi^a)$ need not vanish, as the gauge fixed action is no longer reparameterization invariant. It only vanishes for conformal reparameterizations $\nabla^a \xi^b + \nabla^b \xi^a = \omega h^{ab}$, which corresponds to the original Weyl invariance.
There is however a way to preserve this variation in the gauge fixing. If there are no moduli, by a reparameterization the diagonal elements of the metric can be send to zero. Then we have a Weyl rescaling of the standard flat metric

$$h_{ab} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{2.38}$$

So for every metric h there is a reparameterization Φ such that $\Phi(h) = \omega \cdot \hat{h}$ with ω a scalar function.
For the fixing of the metric we need a functional equivalent of the Dirac delta distribution. This $\delta[h]$ is defined such that

$$\int \mathcal{D}h\ \mathcal{F}[h]\delta[h - \hat{h}] = \mathcal{F}[\hat{h}] \tag{2.39}$$

for any functional $\mathcal{F}[h]$ depending on the metric. To fix at $\Phi(h) = \omega \cdot \hat{h}$ we have to insert $\delta[\omega \cdot \hat{h} - \Phi(h)]$. It can be decomposed in delta functionals of its entries.

$$\delta[\omega \cdot \hat{h} - \Phi(h)] = \delta[2\omega - \Phi(h)^{z\bar{z}}]\delta[\Phi(h)^{zz}]\delta[\Phi(h)^{\bar{z}\bar{z}}]. \tag{2.40}$$

The Dirac delta distribution has Fourier decomposition $\delta(x) = \int dk \exp(ikx)$. Equivalently for the delta functionals we have

$$\begin{aligned} \delta[\Phi(h)^{zz}] &= \int \mathcal{D}B_{zz} \exp\left(\mathrm{i} \int d^2 z B_{zz}(z,\bar{z})\Phi(h)^{zz}(z,\bar{z})\right), \\ \delta[\Phi(h)^{\bar{z}\bar{z}}] &= \int \mathcal{D}B_{\bar{z}\bar{z}} \exp\left(\mathrm{i} \int d^2 z B_{\bar{z}\bar{z}}(z,\bar{z})\Phi(h)^{\bar{z}\bar{z}}(z,\bar{z})\right). \end{aligned} \tag{2.41}$$

For a bijective map $f : \mathbb{R}^n \to \mathbb{R}^n$ we have the following identity with δ^n the higher dimensional delta distribution.

$$\int d^n x \delta^n(f(x)) \left|\det \frac{\partial f}{\partial x}\right| = 1. \tag{2.42}$$

A similar identity holds for the delta functionals.

$$\begin{aligned} 1 &= \int \mathcal{D}\Phi \mathcal{D}\omega\ \delta[\omega\hat{h} - \Phi(h)] \cdot \left|\det \frac{\delta(\omega\hat{h} - \Phi(h))}{\delta(\omega, \Phi)}\right| \\ &= \int \mathcal{D}\Phi \mathcal{D}\omega\ \delta[\omega\hat{h} - \Phi(h)] \cdot \left|\det \frac{\delta 2\nabla^z \xi^z}{\delta \xi^z}\right| \left|\det \frac{\delta 2\nabla^{\bar{z}} \xi^{\bar{z}}}{\delta \xi^{\bar{z}}}\right|. \end{aligned} \tag{2.43}$$

The last line is derived using the fact that the only contribution comes from Φ, ω such that $\Phi(h) = \omega \cdot \hat{h}$. Then a variation $\delta\Phi = \xi$ gives $\delta\Phi(h)^{zz} = 2\nabla^z \xi^z$. The variation of ω gives an infinite factor in the determinant, but using regularization techniques, it can be shown that this does not give a problem. If the moduli space is non trivial, this expression should include an integral over this space. The integral over Φ should be over all reparameterizations modulo the killing transformations, otherwise the determinant is zero. The determinants can be evaluated using anti-commuting fields $b_{zz}, b_{\bar{z}\bar{z}}, c^z$ and $c^{\bar{z}}$.

$$\begin{aligned} \left|\det \frac{\delta 2\nabla^z \xi^z}{\delta \xi^z}\right| &= \int \mathcal{D}b_{zz}\mathcal{D}c^z \exp\left(-\frac{1}{2\pi}\int d^z\ b_{zz}(z,\bar{z})\nabla^z c^z(z,\bar{z})\right), \\ \left|\det \frac{\delta 2\nabla^{\bar{z}} \xi^{\bar{z}}}{\delta \xi^{\bar{z}}}\right| &= \int \mathcal{D}b_{\bar{z}\bar{z}}\mathcal{D}c^{\bar{z}} \exp\left(-\frac{1}{2\pi}\int d^2 z\ b_{\bar{z}\bar{z}}(z,\bar{z})\nabla^{\bar{z}} c^{\bar{z}}(z,\bar{z})\right). \end{aligned} \tag{2.44}$$

The factor $-1/(2\pi)$ arises from rescaling the fields, and is chosen for convenience. Putting this all together, we get

$$\begin{aligned} 1 &= \int \mathcal{D}\Phi\mathcal{D}\omega\mathcal{D}B\mathcal{D}b\mathcal{D}c\ \delta[\omega\hat{h} - \Phi(h)] \cdot \exp(-S_{gf}[B,\omega,\Phi(h)] - S_{gh}[b,c]), \\ S_{gf}[B,\omega,\Phi(h)] &= \frac{\mathrm{i}}{2\pi}\int d^2 z \left[B_{zz}(z,\bar{z})\Phi(h)^{zz}(z,\bar{z}) + B_{\bar{z}\bar{z}}(z,\bar{z})\Phi(h)^{\bar{z}\bar{z}}(z,\bar{z})\right], \\ S_{gh}[b,c] &= \frac{1}{2\pi}\int d^2 z \left[b_{zz}(z,\bar{z})\nabla^z c^z(z,\bar{z}) + b_{\bar{z}\bar{z}}(z,\bar{z})\nabla^{\bar{z}} c^{\bar{z}}(z,\bar{z})\right]. \end{aligned} \tag{2.45}$$

Now everything comes down to exploiting all gauge invariances. Add (2.45) to the Polyakov path integral, the non-gauge fixed path integral with Polyakov action, and put the integrals $\mathcal{D}\Phi\mathcal{D}\omega$ in front. Write $S_P[\Phi(X),\Phi(h)] = S_P[X,h]$. After the integration over $B_{zz}, B_{\bar{z}\bar{z}}$ and h, $\Phi(h)$ changes into $\omega\hat{h}$, but it only appears in S_P which is Weyl-invariant. The $\Phi(X)$ can be changed back into X by a change of variables in X. Then none of the terms depend on ω or Φ anymore, so their integrals just give the volume of the total gauge-group and can be omitted. So in total

$$\int \mathcal{D}X\mathcal{D}b\mathcal{D}c\mathcal{D}\tilde{b}\mathcal{D}\tilde{c}\ \exp\left(-\frac{1}{2\pi\alpha'}\int d^2 z\partial X\bar{\partial}X - \frac{1}{2\pi}\int d^2 z b\bar{\partial}c + \tilde{b}\partial\tilde{c}.\right) \tag{2.46}$$

Here $b, \tilde{b}, c$ and $\tilde{c}$ are short for $b_{zz}, b_{\bar{z}\bar{z}}, c^z$ and $c^{\bar{z}}$. The terms $\nabla^z c^z$ and $\nabla^{\bar{z}} c^{\bar{z}}$ are translated in $\bar{\partial} c^z$ and $\partial c^{\bar{z}}$, as the ∇'s depend on $\Phi(h)$ and only contributions with $\Phi(h) = \omega\hat{h}$ have to be considered. In that case the only non vanishing terms of the Christoffel symbol are Γ^z_{zz} and $\Gamma^{\bar{z}}_{\bar{z}\bar{z}}$ and these do not appear in $\nabla^z c^z$ and $\nabla^{\bar{z}} c^{\bar{z}}$. If there are moduli there should be integrals over the moduli space, and the covariant derivatives should remain in the ghost action.
The ghost fields also form a conformal field theory. The $b(z), c(z)$ are holomorphic fields of weight 2 and -1 and $\tilde{b}(\bar{z}), \tilde{c}(\bar{z})$ are anti-holomorphic of weight 2 and -1. The b, c-fields and the $\tilde{b}, \tilde{c}$-fields can be treated separately. Because the b, c-theory will occur in other cases, we will treat it here more generally.

2.3 First order langrangians in 2d CFT

Let b and c be fields of weight λ and $1-\lambda$ and

$$S = \frac{1}{2\pi}\int d^2 z b\bar{\partial} c. \tag{2.47}$$

In this paragraph we will give the main properties of this theory. Not all derivations are given in detail. More information can be found in [8]. We will consider both anti-commuting ($\epsilon = 1$) as commuting ($\epsilon = -1$) b and c. By a calculation analogous to (2.21)

$$\langle b(z)c(w)\rangle = \epsilon(z-w)^{-1} \tag{2.48}$$

Expanding the fields gives

$$\begin{array}{rll} b(z) & = \sum_n b_n z^{-n-\lambda}, & b_n^\dagger = \epsilon b_{-n}, \\ c(z) & = \sum_n c_n z^{-n-(1-\lambda)}, & c_n^\dagger = c_{-n}, \\ c_m b_n + \epsilon b_n c_m & = \delta_{m,-n}. & \end{array} \tag{2.49}$$

The stress tensor can be found to be

$$\begin{aligned} T(z) &= -\lambda{:}b(z)\partial c(z){:} + (1-\lambda){:}(\partial b(z))c(z){:} \\ T(z)T(w) &\sim \frac{\epsilon(1-3Q^2)}{2(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w), \\ Q &= \epsilon(1-2\lambda). \end{aligned} \tag{2.50}$$

The normal ordering is defined by subtracting the pole $\epsilon(z-w)^{-1}$ in (2.48) for $z \to w$. The central charge can be read of to be $c = \epsilon(1-3Q^2)$. So for the $\lambda = 2$-reparameterization ghosts $Q = -3$ and $c = -26$. Writing out

$$L_k = \sum_n (k\lambda + n) b_{-n} c_{n+k} \text{ for } k \neq 0 \tag{2.51}$$

(normal ordering has no influence for $k \neq 0$) shows that states $|\Psi\rangle$ that satisfy

$$b_{-n}|\Psi\rangle \neq 0 \text{ for some } n \text{ implies } c_{n+k}|\Psi\rangle = 0 \text{ for all } k > 0 \tag{2.52}$$

are states for which $L_k|\Psi\rangle = 0$ for all $k > 0$. If there is no n for which $b_{-n}|\Psi\rangle = 0$. Ignoring some possibilities we will therefore consider states that satisfy the stronger condition

$$b_{-n}\langle\Psi| = 0 \text{ for all } n < N \text{ and } c_n|\Psi\rangle = 0 \text{ for all } n > N. \tag{2.53}$$

for some $N \in \mathbb{Z}$. These states will be labeled $|q\rangle$

$$\begin{aligned} b_{-n}|q\rangle &= 0, \quad n < -\epsilon q + \lambda, \\ c_n|q\rangle &= 0, \quad n \geq -\epsilon q + \lambda. \end{aligned} \tag{2.54}$$

A useful tool is the number current

$$j(z) = -{:}b(z)c(z){:} = \sum_n j_n z^{-n-1}. \tag{2.55}$$

The normal ordering was defined subtracting the pole $\epsilon(z-w)^{-1}, z \to w$ of (2.48). But this calculation is only valid if there are no boundary conditions at $z = 0$. This situation, which is invariant under translation of the origin, therefore corresponds to the incoming state $|0\rangle$, as it is the only state with no poles for $z \to 0$ when acting with $b(z)$ or $c(z)$. The normal ordering of $j(z)$ exactly cancels the pole $\epsilon(z-w)^{-1}, z \to w$. Therefore $j(0)|0\rangle = j_0|0\rangle = 0$. Using the definition of $|0\rangle$ we can relate the normal ordering to a creation/annihilation ordering such that $j_0|0\rangle = 0$. This gives

$$j_0 = \epsilon \sum_{n=-\infty}^{\lambda-1} c_n b_{-n} - \sum_{n=\lambda}^{\infty} b_{-n} c_n. \tag{2.56}$$

Then $j_0|q\rangle = q|q\rangle$. We can apply this to L_0 as well

$$L_0 = -\epsilon \sum_{n=-\infty}^{\lambda-1} n c_n b_{-n} + \sum_{n=\lambda}^{\infty} n b_{-n} c_n. \tag{2.57}$$

and calculate

$$L_0|q\rangle = \epsilon \frac{1}{2} q(q+Q)|q\rangle. \tag{2.58}$$

Both $|0\rangle$ and $|-Q\rangle$ have $L_0 = 0$, but only $|0\rangle$ has $L_{-1} = 0$ (use (2.51)), so $|0\rangle$ is the sl_2-invariant vacuum. Defining $\langle q|$ such that $\langle q|j_0^\dagger = \langle q|q$ results in $\langle q|c_n^\dagger = 0$ if and only if $c_n|q\rangle = 0$, $\langle q|(b_{-n})^\dagger = 0$ if and only if $b_{-n}|q\rangle = 0$, and $\langle 0|L_k^\dagger = \langle 0|L_{-k} = 0$ for $k \geq -1$. However

$$\begin{aligned} j_0^\dagger &= \sum_{n=-\infty}^{\lambda-1} b_n c_{-n} - \epsilon \sum_{n=\lambda}^{\infty} c_{-n} b_n \\ &= -\epsilon \sum_{n=-\infty}^{-\lambda} c_n b_{-n} + \sum_{n=-(\lambda-1)}^{\infty} b_{-n} c_n \end{aligned} \tag{2.59}$$

and thus $j_0^\dagger = -j_0 - Q$. Then

$$\langle p|j_0|q\rangle = q\langle p|q\rangle = \epsilon(-p-Q)\langle p|q\rangle. \tag{2.60}$$

So the hermitian conjugate of $|q\rangle$ is the bra $\langle -q-Q|$ and not $|q\rangle$. Let us see how this is related to the path integral calculation. We take as an example the sphere and focus on the anti-commuting b, c-theory. Consider complex coordinates z on the entire sphere without the 'south pole', with $z = 0$ at the 'north pole' and $z \to \infty$ corresponding to the 'south pole'. The coordinates $z' = 1/z$ can be used for the entire sphere without the north-pole, with $z' = 0$ at the south pole and the limit $z' \to \infty$ for the north pole. As in the path-integral both $z = 0$ and $z' = 0$ are not fixed. The path integral calculation corresponds to a calculation with the translation invariant incoming state $|0\rangle$ and outgoing state $\langle 0|$. Suppose $\xi \sim z^{-n-(1-\lambda)}$ is a weight $(1-\lambda)$ field. Then

$$\xi'(z') = \left(\frac{\partial z}{\partial z'}\right)^{1-\lambda} \xi(z) \sim z'^{n-(1-\lambda)} \tag{2.61}$$

So for ξ to be holomorphic on the entire sphere: $(1-\lambda) \leq n \leq -(1-\lambda)$, such that there are no poles for $z \to 0$ and $z' \to 0$. If such fields exist then there is a symmetry of the action $c \to c + \eta\xi$, with η an anti-commuting constant, because $\bar{\partial}\eta\xi = 0$. If in the path integral we integrate over these superfluous fields - this is an integration over anti-commuting variables which are not in the integrand- the path integral vanishes just as $\langle 0|0\rangle$ vanishes. The vanishing of $\langle 0|0\rangle$ can be remedied by including all c_n with $(1-\lambda) \leq n \leq -(1-\lambda)$ between $\langle 0|$ and $|0\rangle$. The b_{-n} and c_n can namely be used to switch between the different vacua. Using (2.54) with $\epsilon = 1$, it is readily checked that

$$\begin{aligned} c_{-q+\lambda-1}|q\rangle &\cong |q+1\rangle, \\ b_{-(-q+\lambda)}|q\rangle &\cong |q-1\rangle, \end{aligned} \tag{2.62}$$

because the states on the left-hand side have the same defining properties of (2.54). For instance if $\lambda = 2$ there are vector (thus weight $-1 = (1-\lambda)$) fields $1, z, z^2$ and we should insert the operators c_{-1}, c_0 and c_1. Then $c_{-1}c_0c_1|0\rangle = |3\rangle$. We can also let these operators act on the right-hand side giving $\langle 0|c_{-1}c_0c_1 = \langle -3|$. So depending on how much of them we shift to the right $\langle 0|..|0\rangle$ turns into $\langle -3|..|0\rangle, \langle -2|..|1\rangle, \langle -1|..|2\rangle$ or $\langle 0|..|3\rangle$. All of these do not vanish because of (2.60). Using

$$\partial^n c(0)|0\rangle = c_{1-n}|0\rangle \tag{2.63}$$

instead of inserting c_{-1}, c_0 and c_1, we could also insert $\partial^2 c(0), \partial c(0)$ and $c(0)$ on the right hand-side. In the path integral, this insertion cancels the integration over $c(0), \partial c(0)$ and $\partial^2 c(0)$. Thus these three values are fixed, and this exactly fixes the gauge freedom of the killing vector fields $1, z, z^2$.
In general if there are m different killing vectors by inserting the operators $c(z_1), ..c(z_m)$ for different z_i by fixing the $c(z_i)$ we fix the gauge freedom. On the torus with $\lambda = 2$ there is only one holomorphic vector field, but now there is also a weight 2 field with $\bar{\partial}\omega = 0$, which gives a gauge freedom $b \to b + \omega$. So we have to fix the gauge by inserting both a b_{-2} and a c_1. In general by the Riemann-Roch theorem we have

$$(\#\text{number of zero modes of } c - \#\text{number of zero modes of } b) = Q(1-g)$$

with g the genus of the Riemann surface.
For the commuting b, c-theory there are two differences. The integration over gauge equivalent field configurations, does not make the path integral vanish but diverge. Furthermore the operators b_{-n} and c_n can no longer be used to switch between the different $|q\rangle$, because in deriving (2.62) we used $c_n^2 \neq 0$ and $b_{-n}^2 \neq 0$. In fact in this case the $|q\rangle$'s correspond to inequivalent representations of the b, c-algebra. There is however a way to construct an operator that switches between the different representations by rewriting the fields. Replace the action with

$$S = -\int d^2z \frac{1}{2}\epsilon\partial\phi(z)\bar{\partial}\phi + \frac{1}{8}Q\sqrt{h}R\phi. \tag{2.64}$$

and calculate the OPE:

$$\phi(z)\phi(w) \sim \epsilon \ln(z-w). \tag{2.65}$$

This rewriting is also useful for both the commuting ($\epsilon = -1$) as the anti-commuting case ($\epsilon = 1$). First for $\epsilon = 1$ the following OPE-identities hold.

$$
\begin{aligned}
\exp(-\phi(z))\exp(\phi(w) &\sim \frac{\epsilon}{z-w}, \\
\exp(-\phi(z))\exp(-\phi(w)) &= \mathcal{O}(z-w), \\
\exp(\phi(z))\exp(\phi(w)) &= \mathcal{O}(z-w),
\end{aligned}
$$

These are the same as $b(z)c(w), b(z)b(w)$ and $c(z)c(w)$ of the anti-commuting b, c-theory. The weight of the operator $\exp(q\phi)$ can be calculated to be $\frac{1}{2}\epsilon q(q + Q)$. So for the anti-commuting theory we can make the identification

$$
b(z) \cong \exp(-\phi(z)), \quad c(z) \cong \exp(\phi(z)). \tag{2.66}
$$

This process of writing the b, c-theory as a bosonic ϕ-theory is called *bosonizing*. For the commuting theory ($\epsilon = -1$) the weight of $\exp(\phi(z))$ is $-\lambda$. Therefore we make the identification

$$
b(z) = \exp(-\phi(z))\partial\xi(z), c(z) = \exp(\phi(z))\eta(z), \tag{2.67}
$$

with η and ξ an auxiliary b, c-theory with weights 1 and 0. They must be anti-commuting to give the right OPE:

$$
b(z)c(w) \cong \exp(-\phi(z))\partial\xi(z)\exp(\phi(w))\eta(w) \sim \frac{1}{w-z}. \tag{2.68}
$$

By similar OPE calculations (again for details see [8]), the number current can be identified with $\epsilon\partial\phi$. In particular

$$
[j_0, \exp(q\phi(z))] = q\exp(q\phi(z)). \tag{2.69}
$$

So the operator $\exp(q\phi(z))$ shifts the vacuum state $|q'\rangle$, which is a j_0-eigenstate with eigenvalue q', to $|q' + q\rangle$.

2.4 Gauge fixing details

The zero modes of c for the reparameterization ghost system ($\lambda = 2$) correspond to the vector fields that leave the metric invariant (killing vectors). After (2.45) we noticed that these should be divided out of the $\mathcal{D}\Phi$-integration the same way as they are divided out of $\mathcal{D}c$. This means that these killing reparameterizations still form a gauge invariance of the action $S = \int d^2z\partial X\bar{\partial}X$, as this reparameterization of X is not fixed by the insertion of the δ's.
The zero modes of b correspond to the moduli. The moduli are namely variations of the metric, thus weight 2 fields, that are orthogonal to reparameterizations and Weyl rescalings. Starting from a Weyl rescaling of the flat metric, any variation of $h_{z\bar{z}} = h_{\bar{z}z}$ is a Weyl rescaling. Thus the moduli are variations of h_{zz} and $h_{\bar{z}\bar{z}}$, such that δh_{zz} is orthogonal to $\bar{\partial}\xi^z$, thus $\bar{\partial}\delta h_{zz} = 0$ and likewise for $h_{\bar{z}\bar{z}}$. Which are thus indeed the zero modes of b_{zz} and $b_{\bar{z}\bar{z}}$.

All gauge fixing problems can now be resolved by an extension of the gauge fixing term. Let κ be the number of killing vectors and μ the dimension of the moduli space, so $\kappa - \mu = Q(1-g)$, and let $t^i, 1 \leq i \leq \mu$ parameterize the moduli space. The variation in the metric is denoted by $t^i \delta_i h$. This is the first order variation of the metric by the moduli, which is what we need for the Jacobian. The gauge fixing term $\delta(\Phi(h) - \omega \hat{h})$ has to be extended to $\delta(\Phi(h) + t^i \delta_i h - \omega \hat{h})$. For the identity in (2.45) still to hold the ghost action has to extended to

$$S_{g,ext}[b, c, \tau] = \int d^2 z \; b_{zz}(\nabla^z c^z + \tau^i \delta_i h^{zz}) + cc. \tag{2.70}$$

where τ is the anti-commuting ghost corresponding to t. To fix reparameterizations in μ points we have to add $\delta(\Phi(z_1))\delta(\Phi(z_2))..\delta(\Phi(z_\kappa))$. To insert these in the path integral we have to compensate with ghost terms $\exp(\zeta^j_\alpha c^\alpha(z_j)), 1 \leq j \leq \kappa$. The ghosts τ and ζ can be integrated out immediately to give

$$\begin{aligned} &\int \mathcal{D}b\mathcal{D}c \; d^\kappa \zeta \; d^\mu \tau \exp(-S_{g,ext}[b,c,\tau]) \prod_{j=1}^{\kappa} \exp(\zeta^j_\alpha c^\alpha(z_j)) \\ = &\int \mathcal{D}b\mathcal{D}c \exp(-S_g[b,c]) \prod_{i=1}^{\mu} (\int d^2 z \; b_{zz} \delta^i h^{zz} \times cc.) \prod_{j=1}^{\kappa} c^z(z_j) c^{\bar{z}}(\bar{z}_j). \end{aligned} \tag{2.71}$$

The extra b and c terms are precisely those that are needed to gauge fix the ghost system. Following the same procedure as before replace $S_p[X, h]$ by $S_p[\Phi(X), \Phi(h)]$. After the integration over h this becomes $S_p[\Phi(X), \omega\hat{h} - t_i \delta^i h]$. A change of variables $X \to \Phi(X)$ makes this $S_p[X, \omega \hat{h} - t_i \delta^i h]$. If there are other operators $\mathcal{F}[X]$ in the path integral depending on X, we have to replace $\mathcal{F}[X]$ by $\mathcal{F}[\Phi^{-1}(X)]$. The full expression becomes

$$\begin{aligned} &\int \mathcal{D}\Phi \mathcal{D}\omega \mathcal{D}X d^\mu t \exp(-S_p[X, \omega\hat{h} - t_i \delta^i h]) \mathcal{F}[\Phi^{-1}(X)] \\ &\qquad \times \prod_{j=1}^{\mu} \delta(\Phi(z_j)) \times \text{ghost term}(2.71). \end{aligned} \tag{2.72}$$

The physical requirements for $\mathcal{F}[X]$-term is investigated in the next-paragraph.

2.5 Vertex Operators

The Hilbert space is a direct product of $\mathcal{H}^X$ of the bosonic field theory and $\mathcal{H}^g$ of the ghosts. The stress tensor is the sum

$$T(z) = T^m(z) + T^g(z) = -\frac{1}{\alpha'} {:}\partial X(z) \partial X(z){:} + {:}c(z)\partial b(z) + 2(\partial c(z)) b(z){:}$$

of matter (X) and ghost (b, c) fields (and similar for the stress tensor $\bar{T}$ of the anti-holomorphic fields). In order to restore the full reparameterization invariance of the Polyakov path integral, the condition $\langle T_{ab} \rangle = 0$ has to be imposed in the gauge fixed situation. Because the gauge fixed theory is conformally invariant $\langle T^a_a \rangle = 0$ should follow from the equations of motions, as the trace of the stress tensor corresponds to the conformal reparameterizations. This is however not true for every gauge choice. The metric was fixed to an arbitrary Weyl rescaling of the standard flat metric: $h_{ab}(\sigma, \tau) = \omega(\sigma, \tau)\hat{h}_{ab}(\sigma, \tau)$. It appears that after quantization the Weyl rescaling invariance and therefore the conformal invariance can be broken. In fact

$$\langle T^a_a \rangle = -\frac{c}{12}R \tag{2.73}$$

where R is the worldsheet curvature of the metric h_{ab}. (In this calculation one assumes that $c = \tilde{c}$, see [18] §3.4). The vanishing of this Weyl anomaly is required in order to prevent that different gauge choices are inequivalent and lead to the loss of covariance or unitarity of the theory. Therefore the total central charge $c = c^X + c^g = D - 26$ has to vanish. The bosonic string theory can only be quantized without problems in $D = 26$ dimensions.
The non-conformal reparameterization invariance has to be preserved in the gauge fixed theory by imposing $T_{zz} = T_{\bar{z}\bar{z}} = 0$. This means that physical states should satisfy $L_n|\mathsf{Phys}\rangle = \bar{L}_n|\mathsf{Phys}\rangle = 0$ for $n \geq 0$. The state $|\mathsf{Phys}\rangle$ is the direct product of $|\mathsf{Matter}\rangle \in \mathcal{H}^X$ and a ghost vacuum state $|q\rangle \in \mathcal{H}^g$. In order for $L_0|\mathsf{Phys}\rangle = (L_0^X + L_0^g)|\mathsf{Phys}\rangle$ to give zero, the L_0 eigenvalues of $|\mathsf{Matter}\rangle$ and $|q\rangle$ should be zero together. Later we will argue that the right ghost vacuum is $c_1|0\rangle$. As $L_0^g c_1|0\rangle = -|\rangle 0$, we get the condition $L_0|\mathsf{Matter}\rangle = |\mathsf{Matter}\rangle$. With $D = 26$ this is the same as in (2.36).
Now let us address the question of which operators $\mathcal{F}[X]$ we can add to the path integral. As the path-integral over the X-fields with no additional operators is invariant under a reparameterization of X only, in particular translation corresponding to L_{-1}, this describes the vacuum of the matter fields. So we should insert an operator that maps the vacuum to a physical state with $L_0|\mathsf{Matter}\rangle = |\mathsf{Matter}\rangle$. This should be done for both the incoming as the outgoing state. Any additional operators should be such that they do not break the full reparameterization invariance of the total matter+ghost-system, i.e. the condition

$$\langle T_{ab} \rangle = \langle T^m_{ab} + T^g_{ab} \rangle = 0. \tag{2.74}$$

Primary fields $\phi(z, \bar{z})$ with weight $(h, \tilde{h}) = (1, 1)$ have the special property that (cf.(2.32))

$$\begin{aligned} [L_n, \phi(z, \bar{z})] &= \partial(z^{n+1}\phi(z, \bar{z})), \\ [\bar{L}_n, \phi(z, \bar{z})] &= \bar{\partial}(\bar{z}^{n+1}\phi(z, \bar{z})). \end{aligned} \tag{2.75}$$

These expressions will vanish when integrated over z and $\bar{z}$. So inserting integrals $\int d^2z$ of weight $(1, 1)$ operators in the path integral does not break the total reparameterization invariance. From the previous paragraph it becomes clear that the leftover killing reparameterizations, are fixed by adding delta functions. They will cancel some of the integrals $\int d^2z$ replace them by $c(z)\tilde{c}(\bar{z})$ for some point z. Any additional operators keep their integral. These integrals of weight $(1,1)$-fields, are invariant under reparameterization by

$$\int d^2z\phi(\Phi(z,\bar{z})) = \int d^2z\partial\Phi(z,\bar{z})\bar{\partial}\Phi(z,\bar{z})\phi(\Phi(z,\bar{z})) = \int d^2z'\phi(z',\bar{z}') \quad (2.76)$$

with $(z',\bar{z}') = \Phi(z,\bar{z})$. This means that all dependence on Φ in (2.72) is gone. We have managed to completely separate the reparameterization freedom in the Polyakov path integral. The integrals over Φ and ω can now simply left out, which is equivalent to dividing by the volume of the gauge freedom.
Zero loop n-point diagrams in particle physics, correspond to sphere-like diagrams with no holes (handles) in string theory. We have to choose three points to fix the killing reparameterizations. In describing a propagating string, i.e. a cylinder diagram, at least the points $z = 0$ and $z = \infty$ have to be fixed. This still leaves one holomorphic and one anti-holomorphic killing field. This corresponds to the fact that on an infinite cylinder the choice of the origin, the $(\sigma,\tau) = (0,0)$, is arbitrary. It is natural to fix the point $z = \exp(\tau - i\sigma) = 1$. The left-moving and right-moving ghost amplitudes are now of the form

$$\begin{aligned} \langle 0|c^\dagger(\infty)c(1)c(0)|0\rangle &= \langle 0|c_{-1}c_0c_1|0\rangle, \\ \langle 0|\tilde{c}^\dagger(\infty)\tilde{c}(1)\tilde{c}(0)|0\rangle &= \langle 0|\tilde{c}_{-1}\tilde{c}_0\tilde{c}_1|0\rangle. \end{aligned} \quad (2.77)$$

The total interaction with additional operators is now of the form

$$\begin{aligned} \int dz_1 dz_2..dz_r \langle 0|c_{-1}\tilde{c}_{-1}\phi_f^\dagger(\infty,\infty)\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)..& \\ ...\phi_r(z_r,\bar{z}_r)c_0\tilde{c}_0\phi_0(1,1)\phi_i(0,0)c_1\tilde{c}_1|0\rangle.& \end{aligned} \quad (2.78)$$

The initial and final state of this interaction can be recognized to be

$$\begin{gathered} \phi_i(0,0)c_1\tilde{c}_1|0\rangle, \\ \langle 0|c_{-1}\tilde{c}_{-1}\phi_f^\dagger(\infty,\infty). \end{gathered} \quad (2.79)$$

The states with ghost vacuum $c_1\tilde{c}_1|0\rangle$ and matter state of the form $|\mathsf{Matter}\rangle = \phi(0,0)|0\rangle$ with ϕ a weight $(1,1)$ primary field, are therefore the physical states. They describe the different free strings in the theory. The additional operators can be thought of as interactions with other strings. They are therefore called *vertex operators*, as they can be seen as the branching off of strings from the worldsheet. In fact we will see that they can also be interpreted as interactions due to the presence of background fields.

2.6 The closed string spectrum

In the following we will construct all primary operators of weight $(1,1)$ and thereby the full spectrum of the closed bosonic string theory. After quantization we should have

$$[X_\mu(\sigma,\tau), P_\nu(\sigma',\tau)] = \mathrm{i}\eta_{\mu\nu}\delta(\sigma-\sigma'). \quad (2.80)$$

One can check, using $\partial_\tau = L_0 + \bar{L}_0$ so $P^\mu = \frac{i}{\alpha'}[L_0 + \bar{L}_0, X^\mu]$, that this also follows from the path integral method with (2.21). Then

$$[P_\mu, :\!\exp(ik_\mu X^\mu)\!:] = k_\mu :\!\exp(ik_\mu X^\mu)\!:. \tag{2.81}$$

The operator $:\!\exp(ik_\mu X^\mu)\!:$ is a primary field of weight $(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4})$. So the first vertex operator we find, ignoring the integral $\int d^2 z$, is $:\!\exp(ik_\mu X^\mu(z,\bar{z})\!:$ with $k^2 = \frac{4}{\alpha'}$. With the notation $|k_\mu; 0\rangle = :\!\exp(ik_\mu X^\mu)\!: c_1|0\rangle$ we have $P_\mu|k_\mu;0\rangle = k_\mu|k_\mu;0\rangle$. So the state $|k_\mu;0\rangle$ with $k^2 = -\frac{4}{\alpha'}$ is physical. It has negative mass $M^2 = -k^2$ and it is called the *tachyon* state.
The other vertex operators are of the form

$$\begin{aligned}&\mathcal{V}(k_\mu, \{N_m\}, \{\tilde{N}_m\})(z,\bar{z}) = \\ &\quad :\prod_{m=1}^{\infty} (\partial^m X^\mu(z))^{N_m} \prod_{m=1}^{\infty} (\bar{\partial}^m X^\mu(\bar{z}))^{\tilde{N}_m} \exp(ik_\mu X^\mu(z,\bar{z})):.\end{aligned} \tag{2.82}$$

With $N = \sum_m mN_m, \tilde{N} = \sum_m m\tilde{N}_m$ these operators have weight $(N + \frac{\alpha' k^2}{4}, \tilde{N} + \frac{\alpha' k^2}{4})$. In general however they are not primary fields. For $\mathcal{V}^{\mu\nu} = \partial X^\mu \bar{\partial} X^\nu e^{ik_\lambda X^\lambda}$ the OPE

$$\begin{aligned}T(z)\mathcal{V}^{\mu\nu}(w) \sim\ & \frac{1}{(z-w)^3} :\!(k^\nu \partial X^\mu(w) + k^\mu \bar{\partial} X^\nu(w)) e^{ik_\lambda X^\lambda(w)}\!: + \\ & \left[\frac{\frac{\alpha' k^2}{4} + 1}{(z-w)^2} + \frac{1}{z-w}\partial_w\right] :\!\partial X^\mu(w) \bar{\partial} X^\nu(w) e^{ik_\lambda X^\lambda(w)}\!:\end{aligned} \tag{2.83}$$

shows (cf. (2.29)) that only the operator $a_{\mu\nu}\mathcal{V}^{\mu\nu}$ with

$$k^\mu a_{\mu\nu} = k^\nu a_{\mu\nu} = 0, \quad k^2 = 0, \tag{2.84}$$

is a primary field of weight 1. These conditions are called *physical state conditions* as they decide which states of the form $a_{\mu\nu}\mathcal{V}^{\mu\nu}|0\rangle$ are physical. The poles $(z-w)^{-2}, (z-w)^{-3}, ..$ correspond with the operators $L_0, L_1, ...$ One can show that the leading singularity in the OPE of T and an operator of the form (2.82) is of order $N+2$. There are therefore only a finite number $(N+1)$ of physical state conditions. The operators of the first mass levels are

$$\begin{array}{ll}
M^2 = -k^2 = -\frac{4}{\alpha'}, N = \tilde{N} = 0: & :\!\exp(ik_\mu X^\mu)\!: \\
M^2 = 0, N = \tilde{N} = 1: & a_{\mu\nu} :\!\partial X^\mu \bar{\partial} X^\nu \exp(ik_\mu X^\mu)\!: \\
M^2 = -k^2 = \frac{4}{\alpha'}, N = \tilde{N} = 2: & p_{\mu\nu} :\!\partial^2 X^\mu \bar{\partial}^2 X^\nu \exp(ik_\mu X^\mu)\!: \\
& q_{\lambda\mu\nu} :\!\partial X^\lambda \partial X^\mu \bar{\partial}^2 X^\nu \exp(ik_\mu X^\mu)\!: \\
& r_{\lambda\mu\nu} :\!\partial^2 X^\lambda \bar{\partial} X^\mu \bar{\partial} X^\nu \exp(ik_\mu X^\mu)\!: \\
& s_{\kappa\lambda\mu\nu} :\!\partial X^\kappa \partial X^\lambda \bar{\partial} X^\mu \bar{\partial} X^\nu \exp(ik_\mu X^\mu)\!:
\end{array}$$

etc...

each with N physical state conditions plus the condition $k^2 = \frac{4}{\alpha'} - N$. Later we will see that the massless modes $a_{\mu\nu}$ can be interpreted as the graviton field.

2.7 Open string spectrum

Until now we have considered only conformal field theory on the entire complex plane, so $\sigma = \arg z$ is a periodic coordinate running from 0 to 2π. For the open string we let σ run from 0 to π, so we only use the upper halve plane. Moreover we need to apply some boundary conditions for $\sigma = 0$ and $\sigma = \pi$. The action in σ and τ coordinates

$$S[X] = \frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma (\partial_\tau X(\tau,\sigma))^2 + (\partial_\sigma X(\tau,\sigma))^2 \tag{2.85}$$

after a variation of $X^\mu \to X^\mu + \delta X^\mu$ changes with

$$\begin{aligned}\delta S[X] =& \frac{-1}{2\pi\alpha'} \int d\tau \int_0^\pi d\sigma (\partial_\tau^2 X^\mu(\tau,\sigma) + \partial_\sigma^2 X_\mu(\tau,\sigma))\delta X^\mu(\tau,\sigma) \\ &+ \frac{1}{2\pi\alpha'} \int d\tau [\delta X_\mu(\tau,0)\partial_\sigma X^\mu(\tau,0) - \delta X_\mu(\tau,\pi)\partial_\sigma X^\mu(\tau,\pi)].\end{aligned} \tag{2.86}$$

From the first term we get the equation of motion $\partial_\tau^2 X^\mu + \partial_\sigma^2 X^\mu = \partial\bar\partial X^\mu = 0$. For the last term to vanish, we have two possibilities

$$\begin{aligned} &\partial_\sigma X^\mu(\tau,0) = \partial_\sigma X^\mu(\tau,\pi) = 0 && \text{Neumann boundary conditions,} \\ &X^\mu(\tau,0) = X_0^\mu,\ X^\mu(\tau,\pi) = X_1^\mu && \text{Dirichlet boundary conditions.}\end{aligned} \tag{2.87}$$

This last possibility, where we keep the endpoints of the string fixed, will be used for D-branes in chapter 4. The equation of motion says that X^μ is a harmonic function, therefore it can be written as a sum $X^\mu(z,\bar z) = X_L^\mu(z) + X_R^\mu(\bar z)$. We already saw this for the closed string, where the fields $\partial X^\mu(z) = \partial X_L^\mu(z)$ and $\bar\partial X^\mu(\bar z) = \bar\partial X_R^\mu(\bar z)$ and the corresponding stress tensors $T(z)$ and $\bar T(\bar z)$ could be treated completely separately. They have a Laurent expansion

$$\partial X^\mu(z) = -\mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu z^n,\ \bar\partial X^\mu(\bar z) = -\mathrm{i}\sqrt{\frac{\alpha'}{2}} \sum_n \tilde\alpha_n^\mu \bar z^n. \tag{2.88}$$

In the case of open strings the boundary conditions imply $\partial X^\mu = \pm\bar\partial X^\mu$ for $\sigma = 0,\pi$ (+ for Neumann and $-$ for Dirichlet). This gives $\alpha_n = \pm\tilde\alpha_n$. Extend $\partial X^\mu(z)$ to the entire complex plane using the same expansion. Now $\bar\partial X(\bar z)$ with $\operatorname{Im}\bar z \geq 0$ thus $0 \leq \sigma \leq \pi$, can be expressed in the values of $\partial X^\mu(z)$ on the lower half plane, by $\bar\partial X^\mu(\bar z) = \pm\partial X^\mu(z)$ for $\operatorname{Im}\bar z > 0$. So for open strings we can treat ∂X^μ as a holomorphic field on the entire complex plane and simply ignore the anti-holomorphic fields, as they are directly related to the holomorphic.
For the stress tensor we only consider $T(z) = T^X(z) + T^g(z)$, and for the ghosts only b and c. The vertex operators have weight $h = 1$ and are of the form

$$\mathcal{V}(k_\mu, \{N\})(z,\bar z) = :\prod_{m=0}^\infty (\partial^{m+1} X^\mu(z))^{N_m} \exp(ik_\mu X^\mu):. \tag{2.89}$$

Because of the boundary conditions the variation of $X(z,\bar z)$ in (2.21) also changes $X(\bar z, z)$. Therefore the normal ordering changes to

$$:X^\mu(z_1,\bar z_1)X^\nu(z_2,\bar z_2): = X^\mu(z_1,\bar z_1)X^\nu(z_2,\bar z_2) + \alpha'\eta^{\mu\nu}\ln|z_1 - z_2|^2 \tag{2.90}$$

and the weight of the operator $:\exp(ik_\mu X^\mu):$ becomes $\frac{1}{2}\alpha' k^2$. The physical state conditions are $M^2 = -k^2 = \frac{2}{\alpha'}(N-1)$ and N other conditions concerning the orientation with respect to k^μ. There is again a tachyon state with $M^2 = -\frac{2}{\alpha'}$ and a massless state with vertex operator

$$A_\mu :\partial X^\mu \exp(ik_\mu X^\mu): \tag{2.91}$$

and physical state conditions

$$k^2 = k^\mu A_\mu = 0. \tag{2.92}$$

This describes the $D-2$ states of a massless vector particle. The states are labeled by $|A_\nu; k_\mu\rangle := A_\nu :\partial X^\mu \exp(ik_\mu X^\mu): c_1|0\rangle$ and carry an index μ. A special case is when $A_\mu = ck_\mu$ with c a constant. Then we can write $|ck_\mu; k_\mu\rangle = 2cL_{-1}|0; k_\mu\rangle$. Any physical state which can be written as $L_{-k}|\Psi\rangle$ with Ψ some arbitrary state, is physically trivial because all correlations with other physical states $\langle \mathrm{Phys}|L_{-k}|\Psi\rangle$ vanish. So in the physical spectrum $|A_\nu; k_\mu\rangle \sim |A'_\nu; k_\mu\rangle$ if $A_\nu - A'_\nu$ is a multiple of k^ν. Thus we see that massless mode of the open string spectrum has the same gauge freedom as the vector potential of Maxwell theory. In this way we can identify this spin-1 state as the photon.

2.8 Strings in backgrounds

Thus far we have dealt with two dimensional worldsheet field theory only. The X^μ have space-time indices, but they just seem to label a number of independent bosonic fields. We have seen that this number should be 26 to cancel the Weyl anomaly. Further we saw that the A^μ that label the photon states, have the same freedom and gauge invariance as a $U(1)$-space-time-gauge field. In the same way we can also show that the massless closed string states have the structure of a free graviton field. Let us now consider the case of a curved space-time by replacing the $\eta^{\mu\nu}$, the standard flat metric, that was implicit in the definition of the Polyakov action, by a more general curved space-time metric $G^{\mu\nu}(X)$

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma \sqrt{h}\; G_{\mu\nu}(X) h^{ab}\partial_a X^\mu \partial_b X^\nu. \tag{2.93}$$

A curved space-time can be seen as a coherent background of gravitons of strings interacting with the propagating string. This can be seen by expanding a close-to-flat metric $G^{\mu\nu}(X) = \eta^{\mu\nu} + \chi^{\mu\nu}(X)$. Then

$$\exp(-S_\sigma) = \exp(-S_p)(1 - \frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma \sqrt{h}\, \chi_{\mu\nu}(X) h^{ab}\partial_a X^\mu \partial_b X^\nu + ...). \tag{2.94}$$

If we expand $\chi(X)$ in it's Fourier modes, we recognize in the first order term an expansion of graviton vertex operators $a_{\mu\nu}\partial X^\mu \partial X^\nu \exp(ik_\mu X^\mu)$. These terms can be interpreted as 1-point interactions of the string with the graviton background, and the higher order terms as higher order interactions. Note that these use only the symmetric part of $a_{\mu\nu}$, and as the trace of the metric is always equal to the number of space-time dimensions D, also $a^\mu_\mu = 0$ in these interactions. By adding an anti-symmetric tensor field $B_{\mu\nu}(X)$ and a scalar field $\Phi(X)$, the *dilaton*, to the background

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma\sqrt{h}\ \left[(G_{\mu\nu}(X)h^{ab} + \mathrm{i}\epsilon^{ab}B_{\mu\nu})\partial_a X^\mu \partial_b X^\nu + \alpha' R\phi(X)\right], \tag{2.95}$$

one uses all massless closed string states. The fact that only vertex operators where added, provided the $G_{\mu\nu}, B_{\mu\nu}$ and ϕ fields only have massless modes, seems enough to maintain the conformal invariance. However as in the case of the flat background, the vanishing of the Weyl anomaly puts an extra constraint on the theory. After an elaborate calculation

$$\begin{aligned}
T^a_a &= -\frac{1}{2}[h^{ab}\beta^G_{\mu\nu} + \mathrm{i}\epsilon^{ab}\beta^B_{\mu\nu}]\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta^\Phi R,\\
\beta^G_{\mu\nu} &= \mathbf{R}_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{1}{4}H_{\mu\kappa\lambda}H^{\kappa\lambda}_\nu + \mathcal{O}(\alpha'),\\
\beta^B_{\mu\nu} &= -\frac{1}{2}\nabla^\lambda H_{\lambda\mu\nu} + (\nabla^\lambda\Phi)H_{\lambda\mu\nu} + \mathcal{O}(\alpha'),\\
\beta^\Phi &= \frac{D-26}{6\alpha'} - \frac{1}{2}\nabla^2\Phi + \nabla_\lambda\Phi\nabla^\lambda\Phi - \frac{1}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + \mathcal{O}(\alpha').
\end{aligned} \tag{2.96}$$

The first R is the worldsheet curvature, while $\mathbf{R}_{\mu\nu}$ is the space-time Ricci tensor. The field H is the field strength of B

$$H_{\lambda\mu\nu} = \partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}. \tag{2.97}$$

So in the language of differential forms where $B_{\mu\nu}$ are the components of a 2-form B, $H_{\lambda\mu\nu}$ are the components of the 3-form H and $H = dB$.
This theory with action S_σ describes the propagation of a string in a target space where the characteristic string length $\sqrt{\alpha'}$ is much smaller than the characteristic radius of the space-time curvature. The higher mass string states, which are of order α', can then be ignored and for the vanishing of the Weyl anomaly we have to impose $\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0$ up to terms of order α'. Looking at (2.96) we see that this means that the metric has to satisfy Einstein's equation coupled to the B and Φ-field. If B and Φ are constant we also recover the condition $D = 26$. The condition $\beta^B_{\mu\nu} = 0$ makes the B field into a sort of gauge field (compare with the Maxwell equation $\nabla^\lambda F_{\lambda\mu} = 0$) with gauge freedom $B \sim B + d\lambda$, where λ is any 1-form.
It is possible to construct an action for the background-fields, for which the equations of motion are exactly the anomaly canceling condition $\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0$. One can then try to quantize this system of background-fields, resulting in a quantized theory of gravity. Of course this system has all the problems of quantized gravity field theory, but now we know that it can be seen as a low energy approximation of a theory that does not have these problems. The

action, called *low energy effective action* is given by

$$S = \frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G} e^{-2\Phi} \left[-\frac{2(D-26)}{3\alpha'} + \mathbf{R} \right. \\ \left. - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4\partial_\mu \Phi \partial^\mu \Phi + \mathcal{O}(\alpha') \right] . \tag{2.98}$$

The constant κ_0 is arbitrary and can be changed by a redefinition of Φ.
The Euler number $\chi = 2 - 2g - b$, with g the genus and b the number of boundaries, of the worldsheet, can be calculated

$$\chi(\Sigma) = \frac{1}{4\pi} \int_\Sigma R. \tag{2.99}$$

Writing the dilaton field $\Phi(X) = \Phi_0 + \Phi'(X)$ as the sum of the vacuum expectation value Φ_0 and Φ', we see that the term

$$\frac{1}{4\pi} \int_\Sigma R\Phi_0 = \chi \Phi_0 \tag{2.100}$$

in the action becomes an overall factor. Summing over all possible topologies, we see that every extra handle on the worldsheet, which decreases χ by 2, gives an extra factor $\exp(-2\Phi_0)$ in the path integral. Thus

$$g_s = \exp(\Phi_0) \tag{2.101}$$

acts a string coupling constant in the theory. Compare the low energy effective action with the usual Einstein-Hilbert action, in which the constant κ in $\sqrt{-G}R/(2\kappa^2)$ is the gravitational constant, which in has the value

$$\kappa = \frac{\sqrt{8\pi G_N}}{\hbar c^5} = 4.106 \times 10^{-19} GeV^{-1}. \tag{2.102}$$

Henceforth one can make the identification

$$\kappa = \kappa_0 e^{\phi_0} = \kappa_0 g_s. \tag{2.103}$$

3 Superstrings

Consistency in string theory is a very important and delicate subject. It leads to very strict constraints on possible theories. For example the bosonic string theory is only consistent in 26 space-time dimensions. There is however still an important problem with this theory. It contains a tachyon state, a particle of negative-mass squared. This means that the vacuum of the theory is not stable. Another problem is that the spectrum of the theory contains no fermions, whereas in nature these particles are clearly observed. The constraints on our string model make it impossible to simply project out the tachyon and add new states to the theory in a consistent way. A more radical step is needed. The

bosonic string theory is basically a 2-dimensional conformal field theory with only bosonic fields. We have seen that the spectrum and gauge symmetries of the worldsheet theory, translate into a spectrum of space-time particles, such as the graviton and the photon, with their characteristic gauge invariances. We will see that if we make the worldsheet theory super-symmetric, that is adding fermions and enlarging the conformal symmetry group to a super-conformal symmetry group, the spectrum of space-time fields will also contain fermions. Moreover there is a natural and, more importantly, consistent way of projecting out a part of the spectrum including the tachyon, after which not only the worldsheet theory but also the theory of the space-time fields is super-symmetric.
An overview of superstring theory, including the most recent developments until 1998, is given in the second volume of [18]. As an introduction however it is rather concise. A better, but older, introduction to the basics of the theory is given in [9] and [8].

3.1 Superconformal field theory

A 2 dimensional conformal field theory can be made into a super-conformal field theory by adding anti-commuting coordinates $\theta, \bar{\theta}$ to the complex coordinates $z, \bar{z}$. Again the theory splits into a holomorphic, depending on (z, θ), and a anti-holomorphic part with $(\bar{z}, \bar{\theta})$. There are complex super derivatives

$$D = \partial_\theta + \theta\partial, \quad \bar{D} = \partial_{\bar{\theta}} + \bar{\theta}\bar{\partial} \tag{3.1}$$

We will focus on the holomorphic part to avoid repetition. Holomorphic fields of weight h can now be extended to super-fields of the form

$$\phi(z, \theta) = \phi_0(z) + \theta\phi_1(z). \tag{3.2}$$

A conformal field theory extended in this way is called a super-conformal field theory. It has an enlarged symmetry group. Next to the usual conformal reparameterizations of the z-coordinates, there are now also reparameterizations involving the anti-commuting coordinates. The super-conformal symmetries are generated by the super stress tensor

$$T(z, \theta) = T_f + \theta T_b(z) = \sum_n z^{-r-\frac{3}{2}} G_r + \theta \sum_n z^{-n-2} L_n \tag{3.3}$$

Like in ordinary conformal field theory, the transformation properties of the fields can be derived by calculating the OPE's of the stress tensor with the fields. Some more details can be found in [8]. We will just give the results for the Laurent components L_n and G_r of the super stress tensor. They form a super Virasoro algebra:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}, \\
[L_m, G_r] &= (\frac{1}{2}m - r)G_{m+r}.
\end{aligned} \tag{3.4}$$

The L_n are the usual generators of the conformal symmetries

$$\begin{aligned}[L_m, \phi_0(z)] &= (z^{m+1}\partial + h(m+1)z^m)\phi_0(z), \\ [L_m, \phi_1(z)] &= (z^{m+1}\partial + (h+\frac{1}{2})(m+1)z^m)\phi_1(z).\end{aligned} \tag{3.5}$$

Note that ϕ_0 has weight $h+\frac{1}{2}$. If h is half integer we will call ϕ_0 the bosonic and ϕ_1 the fermionic component of the super-field, and vice versa for super-fields of integer weight h. The G_r generate the super-symmetries which mixes the bosonic and fermionic part of ϕ. (ϵ is an anti-commuting constant)

$$\begin{aligned}[\epsilon G_r, \phi_0(z)] &= \epsilon z^{r+\frac{1}{2}}\phi_1(z), \\ [\epsilon G_r, \phi_1(z)] &= \epsilon(z^{r+\frac{1}{2}}\partial + 2(r+\frac{1}{2})hz^{r-\frac{1}{2}})\phi_0(z).\end{aligned} \tag{3.6}$$

By going from the coordinates $w = \sigma + \mathrm{i}\tau$ to $z = \exp(-iw)$, the fermionic fields, the components with weight h half integer, transform as $\phi_f(z) = (\partial_w \exp(-iw))^h$ $\phi_f(w)$. The half integer exponent changes single valued fields in double valued fields and vice versa. Both possibilities, periodic and anti-periodic, have to be considered. The Hilbert space splits in two subspaces: The Neveu-Schwarz sector with $\phi_f^{NS}(e^{2\pi \mathrm{i}}z) = \phi_f^{NS}(z)$ and the Ramond sector with $\phi_f^R(e^{2\pi \mathrm{i}}z) = -\phi_f^R(z)$. Notice that this means that ϕ_f is periodic in the R(amond)-sector, and anti-periodic in the NS-sector. For the NS-sector the Laurent expansion of all fermionic fields are in components $\phi_{f,r}$ with $r+\frac{1}{2} \in \mathbb{Z}$. The R-sector has integer indices for the $\phi_{f,r}$.

3.2 Superstring theory, vertex operators and spin fields

The superstring theory is the super-conformal analogue of (2.10)

$$S = \frac{1}{4\pi}\int d^2z d^2\theta \bar{D}\mathbf{X}^\mu D\mathbf{X}_\mu = \frac{1}{4\pi}\int d^2z\ \bar{\partial}X^\mu\partial X_\mu + \psi^\mu\bar{\partial}\psi_\mu + \tilde{\psi}^\mu\partial\tilde{\psi}_\mu, \tag{3.7}$$

with the fields (in units with $\alpha' = 2$)

$$\mathbf{X}^\mu(z,\bar{z},\theta,\bar{\theta}) = X_l^\mu(z) + X_r^\mu(\bar{z}) + \theta\psi^\mu(z) + \bar{\theta}\tilde{\psi}^\mu(\bar{z}), \tag{3.8}$$

and stress tensor

$$T(z,\theta) = -\frac{1}{2}D\mathbf{X}^\mu D^2\mathbf{X}_\mu = -\frac{1}{2}\psi^\mu\partial X_\mu + -\frac{1}{2}\theta(\partial X^\mu\partial X_\mu + \psi^\mu\partial\psi_\mu). \tag{3.9}$$

The central weight of the X fields is again $c = D$ the number of space-time dimensions. For D is even, with the pairs $\psi^{i,+}$ and $\psi^{i,-}$

$$\begin{aligned}\psi^{0,\pm} &= \frac{1}{2}\sqrt{2}(\pm\psi^0 + \psi^1) \\ \psi^{i,\pm} &= \frac{1}{2}\sqrt{2}(\psi^{2i} \pm \psi^{2i+1}) \text{ with } i = 1, 2, ..D/2\end{aligned} \tag{3.10}$$

the action for the ψ^μ's can be written as

$$S = \int d^2z \sum_{i=0}^{D/2} \psi^{i,+} \bar{\partial} \psi^{i,-}. \tag{3.11}$$

So in fact there are $D/2$ b,c-systems with $\lambda = \frac{1}{2}$. The central charge is then $D/2$ (see (2.50)). So the total central charge of the matter fields is $c = \frac{3}{2}D$.
For the construction of the vertex operators we can use the fact that for a weight $h = \frac{1}{2}$ super field $\phi(z) = \phi_f(z) + \theta\phi_b(z)$

$$[\epsilon G_r, \phi(z)] = \epsilon D(z^{r+\frac{1}{2}}\phi(z)). \tag{3.12}$$

After integration of this expression over θ only $\partial(z^{r+\frac{1}{2}}\phi_f)$ remains which vanishes after integration over z. The integral of $[L_n, \phi(z)]$ over θ only gives $[L_n, \phi_b(z)] = \partial(z^{n+1}\phi_b(z))$ because ϕ_b has weight 1 and this also vanishes after integration over z. Including the right-moving fields in this discussion, the integral $\int d^2zd^2\theta$ of a weight $(\frac{1}{2}, \frac{1}{2})$ super field, commutes with the generators of the super-conformal symmetries L_n and G_r. Thus adding such an expression in the path integral does not break these symmetries. Those operators are the super-symmetric version of the vertex operators of bosonic string theory. The mass formula changes to $M^2 = -k^2 = \frac{2}{\alpha'}(N-1)$ as the integrated operators are now of weight $\frac{1}{2}$. There is a tachyon

$$\int d^2zd^2\theta \exp(ik_\mu \mathbf{X}^\mu) = \int d^2z k_\mu \psi^\mu k_\nu \tilde{\psi}^\nu \exp(ik_\mu x^\mu) \tag{3.13}$$

with $k^2 = \frac{2}{\alpha'}$ and the massless mode

$$\int d^2zd^2\theta a_{\mu\nu} D\mathbf{X}^\mu \bar{D}\mathbf{X}^\nu \exp(ik_\kappa \mathbf{X}^\kappa) \tag{3.14}$$

where $a_{\mu\nu}$ satisfies $a_{\mu\nu}k^\mu = a_{\mu\nu}k^\nu = 0$.
Operators involving only super-fields do not switch between the NS and the R-sector, because these do not change the periodicity conditions. For the NS-sector $G_{-1/2}^2 = L_{-1}$. There is a unique state $|0\rangle$, called the super-symmetric vacuum, which is invariant under $G_{\pm 1/2}, L_0$ and $L_{\pm 1}$. In the R-sector there is no state with $L_0 = 0$ since $L_0 = G_0^2 + \frac{c}{24} = G_0^2 + \frac{D}{16}$. The lowest possible L_0 eigenvalue is therefore $\frac{D}{16}$. The state(s) with this L_0-value thus has $G_0 = 0$. To reach it from the vacuum state $|0\rangle$ in the NS-sector, we have to bosonize the ψ

$$\psi^{i,+} \cong e^{\mathrm{i}H^i}, \quad \psi^{i,-} \cong e^{-\mathrm{i}H^i}. \tag{3.15}$$

The zero modes of ψ satisfy

$$\{\psi_0^\mu, \psi_0^\nu\} = 2\eta^{\mu\nu}. \tag{3.16}$$

They are thus a complex representation of the Clifford algebra $Cl_{D-1,1}$ corresponding to the $(-,+,+..)$ metric of D-dimensional space-time(see also appendix A.6. The elements of the representation space are called Dirac spinors.

A basis of this space is given by $s_\alpha = (s_{\alpha,0}, ..s_{\alpha,D/2-1})$ with each $s_{\alpha,i} = \pm\frac{1}{2}$ and $\alpha = 1, ..2^{D/2}$. It is thus a $2^{D/2}$ dimensional space. The operator

$$S_\alpha = \exp(\mathrm{i}\sum_{i=0}^{D/2-1} s_{\alpha,i} H^i), \tag{3.17}$$

creates the state $S_\alpha|0\rangle$. It corresponds to the basis element s_α of the Dirac spinor representation. The $\psi_0^{i,+}$ and $\psi_0^{i,-}$ act on it as raising and lowering operators. For instance with $s_\alpha = (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ and $s_\beta = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$

$$\psi_0^{0,+} S_\alpha|0\rangle = S_\beta|0\rangle, \quad \psi_0^{0,+} S_\beta|0\rangle = 0, \quad \psi_0^{0,-} S_\beta|0\rangle = S_\alpha|0\rangle \tag{3.18}$$

The $2^{D/2}$ by $2^{D/2}$ matrix that corresponds to ψ_0^μ acting on the basis $S_\alpha|0\rangle$ will be denoted by $\Gamma^\mu_{\alpha\beta}$. The states $S_\alpha|0\rangle$ have all the desired properties. Firstly it gives an OPE with ψ^μ (from now on summation over double spinor indices $\alpha, \beta..$ is assumed):

$$\psi^\mu(z) S_\alpha(w) \sim (z-w)^{-1/2} \Gamma^\mu_{\alpha\beta} S_\beta(w) \tag{3.19}$$

Thus S_α transforms the fermionic fields ψ^μ from periodic to anti-periodic fields and we are indeed in the R-sector. Further the weight of the spin-field S_α is $D/16$, giving $S_\alpha|0\rangle$ the lowest possible L_0-eigenvalue $D/16$ in the R-sector.
Consider the state $|u^\alpha\rangle = u^\alpha e^{\mathrm{i}k_\mu X^\mu(0)} S_\alpha(0)|0\rangle = 0$. The condition $G_0|u^\alpha\rangle = 0$ implies

$$k_\mu \Gamma^\mu_{\alpha\beta} u^\beta = 0 \tag{3.20}$$

which is the massless Dirac equation. As the weight of S_α is $D/16$, the condition $L_0|u^\alpha\rangle = D/16|u^\alpha\rangle$ implies that the weight of $e^{\mathrm{i}k_\mu X^\mu}$ is 0, thus $k^2 = 0$. So the lowest energy states of the left-moving R-sector, can be recognized as a massless Dirac spinor in 10 dimensions.
There are $2^{D/2}$ independent states $S_\alpha|0\rangle$, $\alpha = 1, ..2^{D/2}$. The physical state condition, the Dirac equation, relates half of them to the other half. This can be seen by switching to a preferred momentum frame. Take for instance $k_0 = k_1, k_\mu = 0, \mu \notin \{0, 1\}$. Then

$$k_\mu \Gamma^\mu_{\alpha\beta} u^\alpha = (k_0 \psi_0 + k_1 \psi_1)_{\alpha\beta} u^\beta. \tag{3.21}$$

So $\psi^{0,+}|u^\alpha\rangle = 0$ and thus the first spin number of u^α is fixed to $+\frac{1}{2}$, leaving only $2^{D/2-1}$ options. These remaining states form a complex representation of the Clifford algebra Cl_{D-2}, the Clifford algebra corresponding to the standard Euclidean metric in $\mathbb{R}^{D-2}$.

3.3 Super-ghosts

The action (3.7) is in fact a gauge fixed action in which the worldsheet metric and its super-symmetric partner (gravitino) have been fixed. As for the bosonic string theory the fixing in the full super-symmetric Polyakov path integral has to be compensated with a Jacobian, which can be calculated by a path integral of super ghost fields, the super-symmetric version of the reparameterization ghosts b, c. Again focusing on the holomorphic part:

$$\begin{aligned} &C = c + \theta\gamma,\ B = \beta + \theta b, \\ &S_{gh} = \int d^2 z d^2\theta B\bar{D}C = \int d^2 z\ \beta\bar{\partial}\gamma - b\bar{\partial}c. \end{aligned} \tag{3.22}$$

The β and γ ghosts are commuting fields of weight $\frac{3}{2}$ and $-\frac{1}{2}$. This combination of commuting fields with half-integer weight is opposite to the rule for the matter fields. The same is true for the b, c ghosts. In the NS-sector β and γ are periodic, have β_r and γ_r-components with half-integer r. The ghost-vacua $|q\rangle$ have integer q. In the R-sector this is all opposite: anti-periodic fields, integer r and half integer q. As explained in §2.3 the β, γ-theory can be bosonized using a scalar field ϕ and anti-commuting fields η and ξ of weight 1 and 0 and the following identification

$$\beta(z) \cong \exp(-\phi(z))\partial\xi(z),\ \gamma(z) \cong \exp(\phi(z))\eta(z). \tag{3.23}$$

The operator $e^{q\phi}$ of weight $-\frac{1}{2}q(q+Q)$ can then be used to interpolate between the different vacua.

As for the bosonic theory the vanishing of the Weyl anomaly requires the total central charge to be zero. The central charge of the b, c fields is again -26. For the β, γ fields it is 11 (see (2.50)). The total central charge is then $D + D/2 - 26 + 11$. So we need $D = 10$. The superstring theory only works in 10 space-times dimensions.

Again the gauge choice does not entirely fix the symmetries of the action. There are again three holomorphic killing reparameterizations (for the genus 0 calculations) which have to be fixed with three c's and cancel three integrals of the vertex operators. The conformal symmetries have been enlarged to super-conformal symmetries. As a result there are now extra super-conformal killing fields. These give rise to zero modes in the β, γ system. The β, γ-system having weights $(\lambda, 1-\lambda) = (\frac{3}{2}, -\frac{1}{2})$, has $Q = \epsilon(1-2\lambda) = 2$. The gauge freedom in the β, γ system has to be fixed by inserting operators $e^{q\phi}$ that switch between the ghost vacua $|a\rangle$ and $|a+q\rangle$. The gauge freedom in the b, c-system corresponds to a gauge freedom in the matter (X and ψ-fields) system, that is not fixed by choosing a fixed metric. Similarly the gauge freedom in the β, γ-system corresponds to a leftover gauge freedom in the gauge-fixed matter theory. We will not give details of how this leftover gauge-freedom is fixed, but it is very similar to the bosonic string where three of the integrated vertex operators loose their integral $\int d^2 z$, of which one describes the creation at $z = 0$ of the incoming state, one gives the outgoing state and the possible third gives an interaction at some intermediate point. For the superstring we have vertex operators of the form $\int d^2 z d^2\theta \Phi(z, \bar{z}, \theta, \bar{t}heta)$ with $\Phi = (\Phi_f(z) + \theta\Phi_b(z))(\tilde{\Phi}_f(\bar{z}) + \bar{\theta}\tilde{\Phi}_b(\bar{z}))$. Again three of them loose their integral $d^2 z$ if we fix the killing reparameterizations of the matter system. Notice that the integral $d^2 z$ picks out the $\theta\bar{\theta}\Phi_b(z)\tilde{\Phi}_b(\bar{z})$ term. The fixing of the extra super-conformal killing reparameterizations, is done by replacing this term in the vertex operators at $z=0$ and $z=\infty$, with a $\Phi_f(z)\tilde{\Phi}_f(\bar{z})$ term, and placing a $\exp(-\phi(z))$ at those two points which are needed in the $Q=2$ β,γ-system. So in general a physical initial state is of the form

$$\Phi_f(z)\tilde{\Phi}_f(\bar{z})c_1\tilde{c}_1e^{-\phi(0)}e^{-\tilde{\phi}(0)}|0\rangle. \tag{3.24}$$

Notice that, since the weights of Φ_f, c_1 and $e^{-\phi}$ are respectively $\frac{1}{2}, -1$ and $\frac{1}{2}$, this state indeed has $L_0=0$. Focusing entirely on the matter system this gives the condition that $|\mathsf{Matter}\rangle = \Phi_f(z)|0\rangle^m$ has to have $L_0=\frac{1}{2}$. This is the physical state condition for the NS-sector. The initial state made with (3.13) is then ($|0\rangle^m$ denoting the matter vacuum)

$$:\exp(\mathrm{i}k_\mu X^\mu(0)):|0\rangle^m, \qquad \text{with } k^2=\frac{2}{\alpha'} \tag{3.25}$$

and that of the massless states

$$a_{\mu\nu}\psi^\mu(0)\tilde{\psi}^\nu(0):\exp(\mathrm{i}k_\mu X^\mu(0)):|0\rangle^m, \qquad \text{with } k^2=k^\mu a_{\mu\nu}=k^\mu a_{\nu\mu}=0. \tag{3.26}$$

Again we have the spurious state decoupling of states which are of the form $L_{-k}|\Psi\rangle$. For the massless states this gives an equivalence $a_{\mu\nu}\sim a_{\mu\nu}+ck_\mu k_\nu$ for any constant c. This is the same as for the massless modes of the bosonic string. Again it can be interpreted as giving rise to a graviton field inducing a space-time metric $G_{\mu\nu}$, an anti-symmetric 2-form field $B_{\mu\nu}$ and a dilaton field Φ.

In the R-sector the vacua of the β,γ-theory are labeled with half-integer q. The sl_2-invariant vacuum $|0\rangle$ is not in the R-sector but in the NS-sector. To switch from this state to a vacuum in the R-sector, we can simply use $\exp(q\phi(0))$ with half-integer q. To create a physical state in the R-sector from the NS-vacuum $|0\rangle$, the spin field S_α of weight $\frac{D}{16}=\frac{5}{8}$ has to be accompanied by $\exp(-\phi(0)/2)$ of weight $\frac{3}{8}$, mapping ghost-vacuum $|0\rangle$ in NS to $|-\frac{1}{2}\rangle$ in R. The physical ground state of the R-sector including ghosts is then (now focussing on the left-moving sector only)

$$u^\alpha S_\alpha(0)e^{-\phi(0)/2}:e^{ik_\mu X^\mu(0)}:c_1|0\rangle \tag{3.27}$$

with u^α a Dirac spinor. The non-trivial physical state conditions are $k^2=0$ such that $L_0=0$ and $k_\mu\Gamma^\mu_{\alpha\beta}u^\alpha=0$ such that $G_0=0$. Note that this is a state with $G_0^2=L_0=0$, which is possible since the total central charge vanishes.

3.4 GSO-projection

In a Dirac spinor representation the operator

$$\Gamma^{11}=-\mathrm{i}^{D/2}\Gamma^0\Gamma^1..\Gamma^{D-1} \tag{3.28}$$

has the following properties

$$(\Gamma^{11})^2=0, \quad \{\Gamma^{11},\Gamma^\mu\}=0. \tag{3.29}$$

It has eigenvalue $+1$ for spinors s_α with an even number of $s_{\alpha,i} = -\frac{1}{2}$, and eigenvalue -1 for an odd number. Projecting on one of the two eigenspaces gives the two inequivalent Weyl spinor representations, having so called positive and negative chirality. This construction can be generalized to an operator $(-1)^F$ which satisfies

$$\{(-1)^F, \psi_r\} = 0 \tag{3.30}$$

for all r. For the R-ground state this means that it acts as Γ^{11} (identifying $\psi_0^\mu \cong \Gamma^\mu$). In the whole R-sector every mass-level splits in two eigenspaces of $(-1)^F$. So projecting on one of the two eigenspaces we get halve the number of states at each mass level. F is the worldsheet fermion number operator. In the NS-sector it counts the number of ψ's that are needed to create a state from the vacuum. The ψ's switch between mass-levels $M^2 = \frac{n}{\alpha'}$ and mass $M^2 = \frac{n+1/2}{\alpha'}$. So projecting out one of the eigenspaces of $(-1)^F$ gives either the tachyon state and every other mass level $M^2 = \frac{n+1/2}{\alpha'}$ with $n \geq -1$, or we get the massless modes and every mass-level with mass an integer multiple of $\frac{1}{\alpha'}$. We choose of course the last option, keeping the graviton and getting rid of the tachyon. In the R-sector we can choose between of positive and negative chirality.

This projection, called Gliozzi-Scherk-Olive (GSO) projection, is an essential step in obtaining a well defined superstring theory. First of all we get rid of the tachyon, which solves all kinds of stability problems. Furthermore it can be shown that it gives a local field theory on the worldsheet, that is the branch-cuts, the anti-periodicity around some point at the worldsheet created by the fermionic fields, disappear in all amplitudes. The projection is also required in diagrams such as the torus, in which the ψ-fields not have periodicity conditions in two directions. Finally it turns out that the projection results in a spectrum of space-time fields that are super-symmetric.

At the massless level of the left-moving NS-sector we have $D-2 = 8$ independent states (the physical state property $k_\mu a^\mu = 0$ and the gauge freedom $a^\mu \sim a^\mu + k^\mu$ reduces it by 2). This is the same for the R-sector as $2^{D/2} = 32$ becomes 16 after the GSO-projection, and the physical state condition (Dirac equation) leaves 8 independent states. To calculate the number of states at other mass levels we calculate

$$\mathrm{Tr}(q^{\alpha' M^2}) = \mathrm{Tr}(q^{N_b + N_f + a}) \tag{3.31}$$

using $M^2 = \frac{N_b+N_f+a}{\alpha'} = 0$ with

$$N_b = \sum_{n>0} \alpha_{-n}^\mu \alpha_{\mu,n} = \sum_{n>0}\sum_\mu n N_{b,n}^\mu, \quad N_f = \sum_{r\geq 0} \psi_{-r}^\mu \psi_{\mu,r} = \sum_{r>0}\sum_\mu r N_{f,r}^\mu \tag{3.32}$$

where $N_{b,n}^\mu \in \mathbb{Z}_{\geq 0}$ and $N_{f,r}^\mu = 0, 1$ are the bosonic and fermionic occupation numbers. a is a normal ordering constant which is $\frac{1}{2}$ in the NS-sector, and 0 in the R-sector. For the R-sector

$$
\begin{aligned}
\mathrm{Tr}_R(q^{\alpha' M^2}) &= 8 \sum_{\{N^\mu_{B,n},N^\mu_{F,r}\}} q^{\sum_{\mu=0}^{9}(\sum_{n>0} nN^\mu_{B,n}+\sum_{r>0} rN^\mu_{F,r})} \\
&= 8\prod_{n=1}^{\infty}(\sum_{N=0}^{\infty} q^{nN})^8 \prod_{r=1}^{\infty}(1+q^r)^8 \\
&= 8\prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^8
\end{aligned}
\tag{3.33}
$$

The 8 in front are the 8 physical ground states of the R-sector. The fact that the $\sum_\mu = 0^9$ gives an exponent 8 instead of 10, is because we sum over the occupation numbers $\{N^\mu_{B,n}, N^\mu_{F,r}\}$ of the physical spectrum for which every index μ has only 8 out of 10 independent orientations. For the NS-sector we use that $1+(-1)^F = 0$ for the states projected out by GSO. So that we can write

$$
\begin{aligned}
\mathrm{Tr}_{NS}(q^{\alpha' M^2}) &= \frac{1}{2}\mathrm{Tr}[\ q^{N_b+N_f-1/2}(1+(-1)^F)] \\
&= \frac{1}{2\sqrt{q}}\left[\prod_{n=1}^{\infty}\left(\frac{1+q^{n-1/2}}{1-q^n}\right)^8 - \prod_{n=1}^{\infty}\left(\frac{1-q^{n-1/2}}{1-q^n}\right)^8\right].
\end{aligned}
\tag{3.34}
$$

Both expressions can be proven to be equal. From a power expansion the number of states at each mass level can be read of

$$
8 + 128q + 1152q^2 + 7680q^3 + 42112q^4 + 200448q^5 + \mathcal{O}(q^6). \tag{3.35}
$$

This means that at every mass level the left-moving NS-sector contains as many states as the left-moving R-sector. This is a strong evidence for space-time super-symmetry.

3.5 Type II strings

For the closed superstring the left moving (holomorphic) fields are completely independent from the right moving. What we have shown so far is the spectrum of the left moving fields, with at the massless level both in the NS as in the R-sector 8-independent states. The right-moving spectrum is of course completely analogous. The spectrum of the closed superstring is the direct product of the two. However there is a choice we can make here. For GSO-projection in the R-sector we could choose between states with left and right-chirality. Both choices give a similar spectrum. But in the direct product it does matter whether we choose opposite chirality for left- and right-movers or the same chirality. The first theory is called the type IIA superstring; the second type IIB.
The choice between NS and R-sector on the left and on the right, gives 4 sectors of the closed superstring. Each has 8×8 independent physical states at the massless level.
• NS-NS: Completely analogous to the bosonic string (only now in 10 dimensions), the massless level contains a symmetric rank 2 tensor $G_{\mu\nu}$, the graviton,

with 35 states, an anti-symmetric rank 2 tensor (2-form field) $B_{\mu\nu}$ with 28 states and 1 state for a scalar field Φ, the dilaton.
• NS-R: This sector contains a massless spin 1/2 fermion (8 states) with vertex operator

$$u^\alpha \Gamma^\mu_{\alpha\beta} \psi_\mu \tilde{S}^\beta e^{-\tilde{\phi}/2} e^{ik_\mu X^\mu} \tag{3.36}$$

and a massless spin 3/2 fermion (56 states), the gravitino:

$$u^{\alpha\mu} \psi_\mu \tilde{S}_\alpha e^{-\tilde{\phi}/2} e^{ik_\mu X^\mu}, \quad \Gamma^\mu_{\alpha\beta} u_{\alpha\mu} = 0 \tag{3.37}$$

with physical state conditions

$$k^2 = k_\mu \Gamma^\mu_{\alpha\beta} u^\alpha = k_\mu \Gamma^\mu_{\alpha\beta} u^{\alpha\nu} = k_\mu u^{\alpha\mu} = 0. \tag{3.38}$$

• R-NS: Naturally, this sector contains the same states as the NS-R sector. Only in type IIA they have opposite chirality, and in type IIB they are of the same chirality as in the NS-R sector.
• R-R: In the Dirac spinor representation the product

$$u^\alpha v^\beta (\Gamma^{\mu_1} \Gamma^{\mu_2} .. \Gamma^{\mu_n} C)_{\alpha\beta} \tag{3.39}$$

of two spinors u^α and v^β with C the charge conjugation matrix, transforms as an n-tensor. A reordering of the indices only changes sign by $\{\Gamma^\mu, \Gamma^\nu\} = \eta^{\mu\nu}$. Using the notation

$$\Gamma^{[\mu_1} \Gamma^{\mu_2} ... \Gamma^{\mu_n]} \tag{3.40}$$

for the completely anti-symmetrized product, expressions of the form

$$(\Gamma^{[\mu_1} \Gamma^{\mu_2} ... \Gamma^{\mu_n]} C)_{\alpha\beta} u^{\alpha\beta} \tag{3.41}$$

with $1 \leq n \leq D$ form a decomposition in anti-symmetric tensors of an element $u^{\alpha\beta}$ in the direct product representation of Dirac spinors. Thus

$$\mathbf{2^{D/2}_{Dirac}} \otimes \mathbf{2^{D/2}_{Dirac}} = [0] \oplus [1] \oplus ...[D] \tag{3.42}$$

where $[n]$ denotes the space of anti-symmetric n-tensors. The n-tensors are related by the $D-n$-tensors by

$$\Gamma^{[\mu_1} \Gamma^{\mu_2} ... \Gamma^{\mu_n]} = \pm \Gamma^{11} \Gamma^{[\mu_{n+1}} \Gamma^{\mu_{n+2}} ... \Gamma^{\mu_D]} \tag{3.43}$$

So $[n] \cong [D-n]$ by the linear isomorphism Γ^{11}. As we remarked before the R-ground states satisfying the physical state conditions can be seen as a Weyl spinor in 8 dimensions.

$$\begin{aligned} \mathbf{2^4_{Dirac}} \times \mathbf{2^4_{Dirac}} &= [0] + [1] + [2] + [3] + [4] + [5] + [6] + [7] + [8] \\ &= [0]^2 + [1]^2 + [2]^2 + [3]^2 + [4]. \end{aligned} \tag{3.44}$$

The following equation with $D = 2k$

$$(\Gamma^{[\mu_1} \Gamma^{\mu_2} ... \Gamma^{\mu_n]} C)_{\alpha\beta} (\Gamma u)^\alpha v^\beta = (-1)^{k+n} (\Gamma^{[\mu_1} \Gamma^{\mu_2} ... \Gamma^{\mu_n]} C)_{\alpha\beta} u^\alpha (\Gamma^{11} v)^\beta \tag{3.45}$$

shows that if we take the direct product of two Weyl spinors in 8 dimensions with opposite chirality, we only get n-tensors with odd n and the product with equal chirality gives even-n-tensors. So the $\mathbf{2^4_{Dirac}} \times \mathbf{2^4_{Dirac}}$ breaks in four sectors depending on whether we project onto a Weyl spinor of positive $\mathbf{2^3}$ or negative chirality $\mathbf{2^{3'}}$ on the left and on the right.

$$\begin{aligned} \mathbf{2^3} \times \mathbf{2^3} &= [0] + [2] + [4]_+, \\ \mathbf{2^{3'}} \times \mathbf{2^3} &= [1] + [3], \\ \mathbf{2^3} \times \mathbf{2^{3'}} &= [1] + [3], \\ \mathbf{2^{3'}} \times \mathbf{2^{3'}} &= [0] + [2] + [4]_-. \end{aligned} \tag{3.46}$$

The type IIA R-R sector is given by the second or the third possibility, with 8 states corresponding to an 8 dimensional vector, and 56 states corresponding to an 8 dimensional anti-symmetric 3-tensor.
The type IIB R-R sector is given by the first or the fourth possibility, with 1 state for a scalar and 28 for an anti-symmetric 2-tensor in 8 dimensions, plus halve (is 35) of the states of an 8-dimensional anti-symmetric 4-tensor.
Again in 10 dimensions, so ignoring the physical state conditions, the vertex operators are given by

$$G_{\mu_1 \mu_2 .. \mu_n} e^{-\phi/2} e^{-\tilde{\phi}/2} S^\alpha \tilde{S}^\beta (\Gamma^{[\mu_1} \Gamma^{\mu_2} ... \Gamma^{\mu_n]} C)_{\alpha\beta} \tag{3.47}$$

with G a anti-symmetric n-tensor. From (3.45) with $k = D/2 = 5$, we see that we have 10-dimensional even n-tensors in type IIA, and odd n-tensors for type IIB. They are related to the $n-1$-tensors in the 8-dimensional decomposition. The anti-symmetric n-tensors just as the fields in the NS-NS sector, can be interpreted as background anti-symmetric n-tensor fields, i.e. they are n-forms on the space-time manifold. They will be denoted as $G^{(n)}$. So we have

$$\begin{aligned} G^{(2)}, G^{(4)}, G^{(6)}, G^{(8)}, &\quad \text{for type IIA}, \\ G^{(1)}, G^{(3)}, G^{(5)}, G^{(7)}, G^{(9)}, &\quad \text{for type IIB}. \end{aligned} \tag{3.48}$$

The n-form $G^{(n)}$ is directly related to the dual $(10-n)$-form $G^{(10-n)}$ by $G^{(n)} = *G^{(10-n)}$. $G^{(5)}$ in type IIB satisfies a self-duality relation.

3.6 Type I string

For the introduction of open strings, we have to impose boundary conditions. The conditions for X are as in 2.7: $\partial X^\mu(z) = \pm\bar{\partial} X^\mu(\bar{z})$ at $\operatorname{Im} z = 0$ with $+$ for Neumann and $-$ for Dirichlet boundary conditions. Super-symmetry ($\{G_r, \partial X^\mu\} = z^{r+\frac{1}{2}} \psi(z)$ and $\{\bar{G}_r, \bar{\partial} X^\mu\} = \bar{z}^{r+\frac{1}{2}} \tilde{\psi}(\bar{z})$) then implies

$$\begin{aligned} \text{Neumann:} \quad \psi^\mu(z) &= \epsilon \tilde{\psi}^\mu(\bar{z}) \Big|_{\operatorname{Im} z=0}, \\ \text{Dirichlet:} \quad \psi^\mu(z) &= -\epsilon \tilde{\psi}^\mu(\bar{z}) \Big|_{\operatorname{Im} z=0} \end{aligned} \tag{3.49}$$

with $\epsilon = 1$ in the NS-sector and $\epsilon = \mathrm{sgn}(\mathrm{Re}(z))$ for the R-sector. It is again possible to accomplish this by using a doubling trick, where holomorphic fields on the entire complex plane describe the left-moving and the right-moving fields on the upper half plane. So with a holomorphic field $y(z)$ on the entire plane, $\partial X(z) = y(z)$ and $\bar{\partial} X(\bar{z}) = y(\bar{z})$ for $\mathrm{Im}\, z \geq 0$. And with $\Psi(z)$ a holomorphic fermionic field in the NS or in the R-sector describes $\psi^\mu(z) = \Psi^\mu(z)$ and $\tilde{\psi}^\mu(\bar{z}) = \pm\Psi^\mu(\bar{z})$ for $\mathrm{Im}\, z \geq 0$. The spin-fields are given by $S_\alpha(z) = s_\alpha(z)$ and $\tilde{S}_\alpha(\bar{z}) = s_\alpha(\bar{z})$ with $s_\alpha(z)$ a holomorphic spin-field with

$$\Psi^\mu(z) s_\alpha(w) \sim (z-w)^{-1/2} \Gamma^\mu_{\alpha\beta} s^\beta(w). \tag{3.50}$$

The boundary conditions $S_\alpha(z) = S_\alpha(\bar{z})$ at $\mathrm{Im}\, z = 0$ are such that if ψ^μ is a fermion with Neumann conditions in the NS-sector, $\psi^\mu S_\alpha$ is in the R-sector and vice versa. This can only be combined with the boundary conditions for ψ^μ and $\tilde{\psi}^\mu$ if we have Neumann conditions for all μ. How to change some of them to Dirichlet conditions is explained in §4.2.
A consistent string theory with open strings also contains closed strings as a closed string can branch of of any open string diagram. Because of the boundary condition $S_\alpha(z) = \tilde{S}_\alpha(\bar{z})$ at $\mathrm{Im}\, z = 0$, the state $S_\alpha(0)|0\rangle = \tilde{S}_\alpha|0\rangle$, has the chirality for the left- and right-moving fermions. Therefore open superstrings with only Neumann boundary conditions can only be combined with type IIB closed strings. It turns out that this combination is not yet a consistent theory. The problems are solved if we project out all states that are odd under the parity operator which acts as

$$\Omega : X_L(z) \leftrightarrow X_R(\bar{z}). \tag{3.51}$$

The result of such a projection is called an unoriented string theory. Furthermore the gauge fields of the open string must have a $SO(32)$-gauge group. Actually this must be $Spin(32)/mod\mathbb{Z}_2$ with another than the usual $\mathbb{Z}_2$, but we will keep the notation $SO(32)$ as in most of the literature. In the massless NS-NS sector only the graviton $G_{\mu\nu}$ and the dilaton survive the projection, while the 2-form $B_{\mu\nu}$ is projected out. The only surviving states in the massless R-R sector are those of the 3-form $G^{(3)}$. Finally only the linear combination (NS-R)+(R-NS) survives, giving one gravitino and one Weyl spinor. The resulting theory is called the *type I open and closed unoriented superstring theory.*

3.7 Heterotic strings

The heterotic string is a combination of the bosonic string including the b, c ghosts, on the left-moving side, and of type II superstring on the right-moving side. It is again a 10 dimensional string theory. As the X fields on the left has central weight 10 and the b, c fields have -26, we need a compensating theory with weight 16. The simplest possibility is to add 32 left-moving spin 1/2 fields λ^A. The index A is an internal index. The λ^A have a $SO(32)$ symmetry. Another possibility uses the gauge group $E8 \times E8$. We will not give these theories anymore attention than to say that these are the last two of the 5 consistent superstring theories.

3.8 Super-gravity, low-energy effective theory

The spectra of the three superstring theories we have described are all produce super-symmetric space-time theories. It is important to realize that this is a local space-time symmetry. As the commutator of two super-symmetry transformations is a translation, this means that we have local Poincaré-invariance, which is the basis of general relativity. Theories with local super-symmetry are therefore super-gravity theories.

Just as for the bosonic string theory we can look at superstring theory in a nontrivial background and look at what conditions the theory lays upon these space-time fields. In general these equations of motions will contain an infinite number of terms. However if we are merely interested in a 'everyday life range' approximation, we can discard terms of order α' or higher. In this range, in which the 'typical string length' $l \to 0$, so we are again in the range of ordinary particle/field-theories, we ignore all massive fields and even a lot of higher order corrections in the massless fields themselves. Very often the theory of space-time fields that we obtain in this way can be described by an action, which is called a *low-energy effective action*. As the superstring theories were super-symmetric at every mass level, this low-energy theory is in fact a super-gravity theory.

... And even for those two numbers there are only a few possible combinations. There is a unique super-gravity theory in 11 dimensions, which is also the maximum dimension, and all other theories can be derived from it. In 10 dimensions we have already seen what the possible super-gravity theories consist of. There are two $N = 2$ theories, named type IIA and type IIB, which fields are the same as the massless modes of the corresponding string-theory. The $N = 1$ theory can be coupled to a super Yang Mills theory with gauge group $SO(32)$ or $E8 \times E8$. They are equal to the low energy effective theory of type I and heterotic $SO(32)$ string theory and to heterotic $E8 \times E8$ string theory. The fact type I and heterotic $SO(32)$ have the same low-energy approximation, while they are very different theories in general, can be explained by the fact that their low energy effective actions are only equal after changing $\Phi \to -\Phi$ for the dilaton field. As the string coupling $g_s \sim e^{\Phi}$, the weak coupling limit of one corresponds to the strong coupling limit of the other theory.

The weak/strong coupling duality, such dualities are called S-dualities in general, is a first example of a duality in string theory. These dualities play a very important role in string theory. Although it appears that there are five distinct superstring theories, it turns out that there is a web of dualities relating all of them. In fact, just as the different super-gravity theories can be derived by dimensional reduction from super-gravity in 11 dimensions, all superstring-theories appear to be different limits of a single 11 dimensional theory, coined M-theory. This theory is not a superstring theory. It not yet clear how to describe it in a perturbative way, although there is a conjecture describing it in some sort of matrix quantum mechanics.

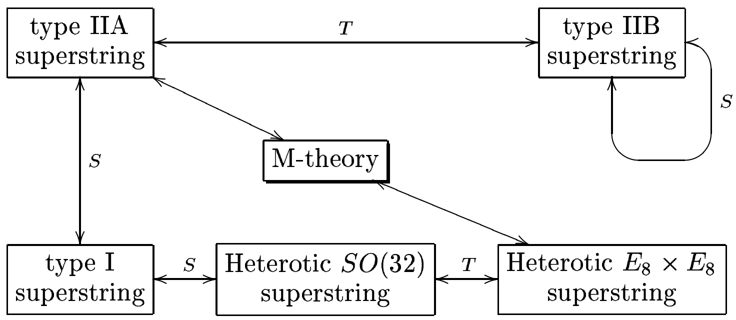

Figure 1: An overview of string dualities.

4 D-branes

The fact that the different superstring theories are so strictly related to each other is only known for a couple of years. This is because before 1995 string theory was only known in its perturbative regime. All calculations were made in an expansion of multi loop diagrams giving higher and higher orders of g_s. This is of course only valid in the weak-coupling limit with $g_s \ll 1$. The situation changed dramatically with the discovery of non-perturbative objects, called D-branes, with contributions of order g_s^{-1}, and the realization of S-dualities interchanging weakly and strongly coupled theories.
The first article giving a complete description of these objects is [20]. A more elementary introduction is given in [19] and [18]. Some technicalities are more clearly explained in [1].

4.1 Compact dimensions and T-duality

So far we have assumed our space-time to have a topology equal to $\mathbb{R}^{9+1}$. In coming to a realistic model of quantum gravity in real life, something has to be done about the discrepancy between the clearly observed 4 space-time dimensions and the 10 dimensions that are required in superstring theory. A way out of this is to consider the leftover dimensions to be compact with very small radii, such that they can only be observed at high energies. Moreover we will see that compactification plays an important role in relating different theories with one another.
We start by considering the ninth dimension to be compact with radius R, i.e. periodic $x^9 \sim x^9 + 2\pi R$. The first consequence is that the space-time momentum in this dimension becomes quantized (in order for the space-time translation operator $\exp(ip^9 d)$ to be single-valued when d is an integer multiple of $2\pi R$). So with mode expansion

$$X^\mu(z, \bar{z}) = x_0^\mu + \sqrt{\frac{\alpha'}{2}} \left(\mathrm{i}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau + (\alpha_0^\mu - \tilde{\alpha}_0^\mu)\sigma + \mathrm{i} \sum_{n \neq 0} \frac{\alpha_n^\mu z^{-n} + \tilde{\alpha}_n^\mu \bar{z}^{-n}}{n} \right)$$

and $P^9(z,\bar z) = \frac{\mathrm{i}}{\alpha'}(z\partial X^9 + \bar z\bar\partial X^9)$, for the total momentum

$$p^9 = \frac{1}{2\pi\mathrm{i}}\int \frac{\mathrm{i}}{\alpha'}(dz\ z\partial X^9 + d\bar z\ \bar z\bar\partial X^9) = \frac{\alpha_0^9 + \tilde\alpha_0^9}{\sqrt{2\alpha'}} = \frac{n}{R}. \tag{4.1}$$

In noncompact dimensions for X to be single valued $\alpha_0 - \tilde\alpha_0 = 0$. In the compact direction however X can make a winding $X^9 \to X^9 + 2\pi wR$ when going around $\sigma \to \sigma + 2\pi$ with $w \in \mathbb{Z}$ the winding number. Thus

$$\alpha_0^9 - \tilde\alpha_0^9 = \sqrt{\frac{2}{\alpha'}}wR \tag{4.2}$$

and so

$$\alpha_0^9 = \frac{1}{\sqrt{2\alpha'}}\left(\alpha'\frac{n}{R} + wR\right), \quad \tilde\alpha_0^9 = \frac{1}{\sqrt{2\alpha'}}\left(\alpha'\frac{n}{R} - wR\right). \tag{4.3}$$

The mass as observed in the $8+1$ noncompact dimensions

$$\begin{aligned} M^2 = -p^\mu p_\mu &= \frac{2}{\alpha'}(\alpha_0^9)^2 + M^2_{osc.} \\ &= \frac{2}{\alpha'}(\tilde\alpha_0^9)^2 + \tilde M^2_{osc.} \end{aligned} \tag{4.4}$$

with a summation μ over the noncompact dimensions 0..8 only and $M^2_{osc.}$ the mass of the left-moving bosonic and fermionic oscillators- $(N-1/2)/2\alpha'$ for NS and $N/2\alpha'$ for R- and $\tilde M^2_{osc.}$ the mass of the right-movers. The different number n states are the Kaluza-Klein modes, while the number w has no counterpart in field theory. As $R \to \infty$ the states with winding number $w > 0$ become infinitely massive, and p^9 becomes continuous again, thus retrieving a noncompact dimension. The limit $R \to 0$ makes all states with $n \neq 0$ infinitely heavy, but as the $w > 0$ become lighter and lighter, a new continuum of states with $n = 0$ and different w appears. So in this limit the compactified dimension does not disappear as in field theory. Instead we get a spectrum which is similar to that of the $R \to \infty$ limit. This is a consequence of the following symmetry. If we interchange

$$R \leftrightarrow \frac{\alpha'}{R}, \quad n \leftrightarrow w, \quad \alpha_0^9 \leftrightarrow \alpha_0^9, \quad \tilde\alpha_0^9 \leftrightarrow -\alpha_0^9 \tag{4.5}$$

we get exactly the same spectrum. In fact in the bosonic string theory if we interchange $\tilde\alpha_m^\mu \leftrightarrow -\tilde\alpha_m^\mu$ for all m with μ the compact direction, we get exactly the same theory. So with $X^\mu(z,\bar z) = X_L^\mu(z) + X_R^\mu(\bar z)$ we change to

$$X'^\mu(z,\bar z) = X_L^\mu(z) - X_R^\mu(\bar z). \tag{4.6}$$

So the physics of a string theory with compact dimension R is equal to that of a theory with compact dimension $R' = \alpha'/R$. This is a remarkable phenomenon, which is a specific string feature. This symmetry, which can be regarded as a space-time parity operation on the right-moving degrees of freedom, is known

as *T-duality* (Target space duality).
In superstring theory, taking again $\mu = 9$ as the compact dimension, by supersymmetry we also have to change

$$\psi'^9(z) = \psi^9(z), \quad \tilde{\psi}'^9(\bar{z}) = -\tilde{\psi}^9(\bar{z}). \tag{4.7}$$

By interchanging $\tilde{\psi}^9(\bar{z}) \leftrightarrow -\tilde{\psi}^9(\bar{z})$, we also interchange one of the pairs $\tilde{\psi}^{\pm,i} \leftrightarrow \psi^{\mp,i}$ in (3.10) thus changing the chirality of the fermions in the R-sector. This means that T-duality maps type IIA superstrings in a target space with a compact dimension of radius R to type IIB with a compact dimension of radius α'/R. The spin-fields transform

$$S'_\alpha(z) = S_\alpha(z), \quad \tilde{S}'_\alpha(\bar{z}) = \Gamma^9\Gamma\tilde{S}_\alpha(\bar{z}) \tag{4.8}$$

in order to preserve the OPE (3.19)

$$\begin{aligned} \tilde{\psi}'^9(\bar{z})\tilde{S}'(\bar{w}) = -\tilde{\psi}^9(\bar{z})\Gamma^9\Gamma\tilde{S}(\bar{w}) &\sim -\Gamma^9\Gamma(\bar{z}-\bar{w})^{-1/2}\Gamma^9\tilde{S}(\bar{w}) \\ &= (z-w)^{-1/2}\Gamma^9\tilde{S}'(\bar{w}) \end{aligned} \tag{4.9}$$

and similar $\tilde{\psi}^\mu\tilde{S}'$ for the other μ, using that $\Gamma^{11}\Gamma^9$ anti-commutes with Γ^9 and commutes with the other Γ's. Applying this to the R-R vertex operators the effect on the fields G is that it removes an $\mu = 9$-index if one is present, and otherwise adds one, in this way transforming the type IIA-fields in type IIB and vice versa.

4.2 Open strings and T-duality

For the open string again the momentum in the compact direction. There is however no such thing as a winding number. So one might wonder what happens if we apply the same T-duality (4.6),(4.7). The answer is that if we start with an open string with Neumann boundary conditions $\partial X^9 = \bar{\partial} X^9$ at $\mathrm{Im}\, z = 0$, we get Dirichlet boundary conditions $\partial X'^9 = -\bar{\partial} X'^9$ (this follows directly from (4.6)). It means that where the string endpoints could move freely first, in the dual theory they are fixed in time: $\partial_\tau X^9(\tau, 0) = \partial_\tau X^9(\tau, \pi) = 0$. So still being able to move freely in the other directions, they are fixed to a 8 dimensional hyperplane. With y its coordinate in the compact dimension, we can write $X'^9(\tau, 0) = X'^9(\tau, \pi) = y$ for all τ. It is now clear what happens to the quantum number n of the momentum. Since the endpoints of the string are fixed in the dual theory, it can have a winding number $X'^9(\tau, 0) - X'^9(\tau, \pi) = 2\pi w R, w \in \mathbb{Z}$. T-duality interchanges $n \leftrightarrow w$.
The hyperplane we just saw is what is called a Dirichlet-brane. A D-brane of $p+1$ space-time dimensions is called a Dp-brane. So our construction is a D8-brane. A theory in which open strings can move freely in the entire target-space of 10 space-time dimensions can be seen as a D9 brane.

Let us first consider multiple D8-branes. There is a compact ninth dimension. Let $y_1, ...y_m$ be the coordinates in this dimension of the m D8-branes. The open strings can stretch between the different branes. The Hilbert space of

Figure 2: Open strings stretching between multiple D-branes at different positions $X'^9 = y_1, y_2, y_3, y_4, ..$

open strings splits in different sectors. A state can be seen as a matrix where the ij-element represents a string with boundary conditions

$$X'^9(\tau,0) = y_i, \quad X'^9(\tau,\pi) = y_j + m2\pi R'. \tag{4.10}$$

The number w in the difference $X'^9(\tau,\pi) - X'^9(\tau,0) = 2\pi R'w$ is no longer integer, rather

$$w = \frac{y_j - y_i}{2\pi R'} + m, \quad m \in \mathbb{Z}. \tag{4.11}$$

In the T-dual picture, this is the 'original' picture with only Neumann boundary conditions and no branes, the momentum (use: $w \leftrightarrow n$ and $R' = \alpha'/R$)

$$p^9 = \frac{w}{R} = \frac{y_j - y_i}{2\pi\alpha'} + \frac{m}{R} \tag{4.12}$$

is no longer an integer multiple of $1/R$. This means that strings from the ij-sector in the dual picture pick up a phase

$$\exp\left(\mathrm{i}\frac{y_j - y_i}{R'}\right) \tag{4.13}$$

under going around the compact dimension. Define the matrix

$$\Lambda = \mathrm{diag}\{e^{-\mathrm{i}y_1/R'}, e^{-\mathrm{i}y_2/R'}, ..e^{-\mathrm{i}y_m/R'}\}. \tag{4.14}$$

A general state of open strings stretching between the different D-branes was a matrix with the ij-th element an open string stretching between the i-th and the j-th brane. In the T-dual picture of this, the matrix Ψ of the T-dual of these states transform

$$\Psi \to \Lambda^{-1}\Psi\Lambda. \tag{4.15}$$

when translated around the compact dimension. In this picture we thus still have different sectors of the Hilbert space. One can interpret this by saying that each endpoint has a label i, called *Chan-Paton factor*, running from 1 to m, attached to it. So the string string from the ij-sector has a label i at its beginning and a label j at its endpoint. One can say that there are m different space-filling D9-branes with open strings stretching between brane i and j.
The phase factor Λ can be seen as a Wilson Line of a gauge field. Consider a background field A^μ on a single D-brane. Its contribution to the string action is

$$\int_{\partial\Sigma} A_\mu(X(\sigma,\tau))\dot{X}^\mu(\sigma,\tau). \tag{4.16}$$

This is simply the photon vertex operator integrated over the worldsheet boundary. It gives a contribution to the momentum operator of a string stretched between the i-th and j-th brane is

$$P^9(\tau,\sigma) = \dot{X}^\mu(\tau,\sigma) + \mathrm{i}\delta(\sigma)A_i^\mu(X^\mu(\tau,0)) - \mathrm{i}\delta(\sigma-\pi)A_j^\mu(X^\mu(\tau,0)). \tag{4.17}$$

So the endpoints of the open string are charged under the gauge field A^μ. Under going around a loop the string picks up a phase

$$\exp(\mathrm{i}\oint A_j - A_i). \tag{4.18}$$

That this is indeed the right interpretation of the phase factor, becomes clear if we compare the vertex operator of the massless photon A^9 with its T-dual in the D8-branes system (in X'-coordinates)

$$\begin{aligned} V &= \partial_\tau X^9 = z\partial X + \bar{z}\bar{\partial}X \\ &= \mathrm{i}\partial_\sigma X'^9 \end{aligned} \tag{4.19}$$

So the massless photon state of A_i^9 corresponds to variation of the endpoint of the coordinate, i.e. a fluctuation of the D-brane itself! Let us look at the mass spectrum of an ij string in the D8-branes system

$$\begin{aligned} M^2 &= (\alpha_0^9 + \tilde{\alpha}_0^9)^2 + M_{osc.}^2 \\ &= \left(\frac{2\pi nR' + y_j - y_i}{2\pi\alpha'}\right)^2 + M_{osc.}^2 \end{aligned} \tag{4.20}$$

In the D8-brane picture $n \in \mathbb{Z}$ is the winding number. If $y_j = y_i$ there is a massless state for $n = 0$. In the NS-sector it is a gauge field A^μ for the noncompact dimensions, and the fluctuation of the D-brane coordinate in the compact direction. In the Neumann-picture this last state is the gauge field A^9 in the compact direction. In this case as well as for $n \neq 0$ the Wilson loop vanishes for $\oint A^9 = 2\pi n$. This makes A^9 a massless gauge field satisfying the Dirac charge quantization condition (this is explained in §4.6). When brane i shifts away from j, the massless mode gains mass and in the Neumann picture we have a non-trivial Wilson loop.

If none of the D-branes coincide, only the strings with begin- and endpoints at the same D-brane have massless modes. If n of the m D-branes coincide the n^2 massless modes can be interpreted as a $U(n)$ gauge field. If all D-branes coincide, in the Neumann picture we have a full $U(m)$ group acting on the Chan-Paton factors. The shifting apart of one of the D-branes results in the breaking down of $U(m)$ to a subgroup $U(m-1) \times U(1)$.

Applying a few T-dualities in different directions, we get open strings with Dirichlet boundaries in those directions, i.e. they are fixed on a lower dimensional hyperplane. For a Dp-brane, we have $p+1$ Neumann and $9-p$ Dirichlet conditions for an open string attached to it. Since T-duality interchanges N and D boundaries, a further T-duality in a direction tangent to a Dp-brane reduces it to a $(p-1)$-brane, while a T-duality in a orthogonal direction turns it into a $(p+1)$-brane. For open strings with only Neumann boundaries we found the condition $S_\alpha(z) = \tilde{S}_\alpha(\bar{z})$ at $\mathrm{Im}\, z = 0$. As the open strings can be patched together to form closed strings, there should be closed strings satisfying this. This requires the left- and right-moving spinors of the R-R-ground state to be of the same chirality. Thus a space filling D9-brane is only consistent in type

type IIA	$p = 0, 2, 4, 6, 8$
type IIB	$p = -1, 1, 3, 5, 7, 9$
type I	$p = 1, 5, 9$

Table 1: The possible Dp-branes in superstring theory

IIB. Actually this is only after the parity projection which makes it a type I theory, but we will come to that. Applying a T-duality in the d-direction, the boundary condition becomes

$$S'_\alpha(z)|_{\mathrm{Im}\, z=0} = \Gamma^d \Gamma^{11} \tilde{S}'_\alpha(\bar{z}) \tag{4.21}$$

This is only possible for opposite chirality R-R-states of the type IIA string. Applying an odd number of T-dualities starting from the D9-brane, gives an even Dp-brane in the type IIA-theory and an even number of T-dualities a odd Dp-brane of type IIB. For the unoriented type I case the boundary condition for the spin-field must be invariant under $S_\alpha \leftrightarrow \tilde{S}_\alpha$ this is only true for $p = 1, 5, 9$. All possibilities are given in table 1. The D1-brane is a string like object, the D0 is a particle and the D-1-brane an instanton.

4.3 The D-brane action

In the last paragraph we saw that the T-dual of a gauge field in a compact direction, is the fluctuation of a D-brane in this direction. Let us consider a Dp-brane with open strings attached to it, giving a massless $U(1)$ vector field on the brane and $9 - p$ scalars describing the fluctuations. They have interaction with the massless closed strings, that can still move through the entire target-space. This can be described as an interaction with background fields. Using coordinates $\xi^a, a = 0, ...p$ for the brane, we can write an low energy effective action for this

$$S_p = -T_p \int d^{p+1}\xi e^{-\Phi} \sqrt{-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}, \tag{4.22}$$

where G_{ab} and B_{ab} are the pull-back of these space-time fields to the brane, and $F_{ab} = \partial_a A_b - \partial_b A_a$ the field strength of A_a. The integral of $\sqrt{-\det G_{ab}}$ is the world volume of the brane, the simplest coordinate invariant action. Note that this an action for the scalars describing the fluctuations as well. The pull-back of G and B are namely given by

$$G_{ab}(\xi) = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu}(X(\xi)), \quad B_{ab}(\xi) = \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} B_{\mu\nu}(X(\xi)). \tag{4.23}$$

with $X(\xi)$ the embedding of the D-brane in the target space. The dependence on F_{ab} can be understood as follows. Consider a Dp-brane and an open string attached to it, starting at $X(\xi_0)$ and ending at $X(\xi_1)$. Locally around $X(\xi_0)$ we can choose space-time coordinates such that the metric $G^{\mu\nu}$ is flat and such that the X^μ-directions for $\mu = 0, ..p$ are parallel to the brane. Consider a constant gauge field F_{12}. It is the field strength of a gauge field A with $A_1 = -X^2 F_{12}$. The endpoints of the string are free to move in the X^1 and X^2 direction. Moving both endpoints in the X^1 direction keeping $X^2(\xi_0)$ and $X^2(\xi_1)$ fixed, after going around the complete X^1-dimension, the string picks up a phase

$$\exp\left(\mathrm{i}2\pi R F_{12}\left[X^2(\xi_1) - X^2(\xi_0)\right]\right). \tag{4.24}$$

Applying a T-duality in the X^1-direction the endpoints are fixed in this direction. The phase factor gets translated in a difference of the X^1-coordinate of the brane at both endpoints.

$$X'^1(\xi_b) - X'^1(\xi_a) = 2\pi\alpha' F_{12}(X^2(\xi_1) - X^2(\xi_0)) \tag{4.25}$$

Choose local coordinates x^a on the brane around ξ_0 such that $X^a(\xi) = X^a(\xi_0) + x^a$ with $a = 0, ..p$. So the point ξ_0 is given by $x^a = 0$. After T-duality $X'^a(x) = X^a(x)$ for $a \neq 1$ and $X'^1(x) = X'^1(0) + 2\pi\alpha' F_{12}(X^2(x) - X^2(0)) = X'^1(0) + 2\pi\alpha' F_{12} x^2$. We can discard the x^1-coordinate. The only thing that changes in G'_{ab} is

$$G'_{22}(x) = \frac{\partial X'^1}{\partial x^2}\frac{\partial X'^1}{\partial x^2} G_{11}(X(x)) + \frac{\partial X'^2}{\partial x^2}\frac{\partial X'^2}{\partial x^2} G_{11}(X(x)) = (1 + (2\pi\alpha' F_{12}))^2 G_{22}(x)$$

so that

$$\det G'_{ab} = (1 + (2\pi\alpha' F_{12})^2)\det G_{ab} = \det(G_{ab} + F_{ab}). \tag{4.26}$$

Finally the dependence on B_{ab} is due to the following gauge freedom of the world sheet action with background fields

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \quad A_\mu \to A_\mu + \Lambda_\mu \tag{4.27}$$

that will be explained in detail in §6.7.

Suppose we have a single compact dimension, the low energy effective action in the noncompact dimensions is determined by integrating $\int 1/(2\kappa_0^2)\sqrt{-G}\mathbf{R}$ out over the compact dimension. This should be invariant under T-duality, thus

$$\frac{1}{2\kappa_0^2} 2\pi R = \frac{1}{2\kappa_0'^2} 2\pi R'. \tag{4.28}$$

This determines the transformation of the string coupling g_s and of Φ:

$$\frac{\kappa_0'^2}{\kappa_0^2} = \frac{g_s^2}{g_s'^2} = \frac{e^{2\Phi}}{e^{2\Phi'}} = \frac{R}{'}R = \frac{\alpha'}{R^2}. \tag{4.29}$$

Consider now a Dp-brane wrapped on a p-dimensional torus in with radii R_i in a flat background $G_{\mu\nu} = \eta_{\mu\nu}, B_{\mu\nu} = F_{\mu\nu} = 0$ and constant dilaton. The action is simply

$$S = T_p e^{-\Phi}\prod_i 2\pi R_i. \tag{4.30}$$

After a T-duality in the compact dimension j, $R_j \to R'_j = \alpha'/R_j$ and $\exp(-\Phi) \to \exp(-\Phi') = \exp(-\Phi)\sqrt{\alpha'}/R_j$. As the action of the dual D$p-1$-brane should be the same

$$\frac{T_{p-1}}{T_p} = 2\pi\sqrt{\alpha'}. \tag{4.31}$$

4.4 D-Branes as R-R charges

Further on we will see that D-branes are BPS-states. In general this means that there must be conserved charges. In the present there is a natural set of charges, namely those of the R-R fields. The physical state conditions of the R-R fields G considered as background fields are

$$dG = 0, \quad d * G = 0, \tag{4.32}$$

where we have written the anti-symmetric p-tensor G as an $p+2$-form. The Hodge star $*$ is an isomorphism between the space of p-forms and the space of $D-p$ forms, such that

$$\theta \wedge *G = (\theta, G)\mu \tag{4.33}$$

for any p-form θ. (θ, G) is the inner-product, i.e. contraction over all indices by the space-time metric. μ is the D-volume form with respect to the metric, that is locally equal to $dx_0 \wedge dx_1 \wedge ..dx_D$ for any local orthonormal coordinate system x_i. With G equal to the 2-form $F_{\mu\nu}$, the field strength of Maxwell theory, (4.32) are just Maxwell's equations. This is generalized for the R-R fields. For any p, the p-form G is called field strength. Since $dG = 0$, locally we can write $G = dC$ with C a $p-1$-form the vector potential (equal to A_μ in Maxwell's case). The fields can be coupled to an electric charge J_E and a magnetic charge J_M, which are a $p-1$ and a $d-p-1$-form, by imposing

$$dG = *J_M, \quad d * G = *J_E. \tag{4.34}$$

In Maxwell theory in 4 dimensions both J_E and J_M are 1-forms, representing the electric and magnetic current. The electric source is generally a $p-2$-dimensional object, that is a $p-1$-dimensional submanifold E of space-time. Then one can find a 'electric charge density' function ρ on E, such that for any $p-1$ form ω

$$\int_E \omega \cdot \rho = \int \omega \wedge *J_E \tag{4.35}$$

where the integral on the right is over all of space-time. The field equations for the case $J_M = 0$ can be derived from an effective action

$$S = \frac{1}{2}\int G^{(p)} \wedge *G^{(p)} + \mathrm{i}\int_E C^{(p-1)}\mu \tag{4.36}$$

where the left integral is over all of space-time and the right over the magnetic source only.
The $p = 0, 2, 4, 6, 8$ Dp-branes in type IIA and the $p = -1, 1, 3, 5, 7$ Dp-branes

Figure 3: The one loop amplitude of open strings stretching between two D-branes can also be seen as the exchange of a closed string between the branes.

in type IIB, are electric sources for the $p+1$-form vector potentials $C^{(p+1)}$ with $p+2$ form field strengths $G^{(p+2)}$. This means that they are magnetic charges for the $D-p-1$ forms $*G$. The D8-brane couples to $C^{(9)}$. The physical state conditions imply that there is no propagating string state for this form, but we may consider a background with constant $G^{(1)}0$.
A first guess for the action would be to take $\rho = \rho_p$ constant. Consider a 1-brane in the X_1, X_2-plane parameterized by $(t, x) \mapsto (X^0 = t, X^1 = x, X^2(x))$. It is coupled to a 2-form $C^{(2)}$. The action would be

$$\mathrm{i}\rho_1 \int (C_{01} + C_{02}\partial_x X_2(x))dxdt. \tag{4.37}$$

After a T-duality in the 2-direction we get a 3-form $C'^{(3)}$ with $C'_{012} \sim C_{01}$ and a 0-form $C'_0 \sim C_{02}$[1]. The fluctuations of the brane in the X_2-direction correspond to a gauge field $\partial_x X_2(x) = 2\pi\alpha' F_{12}$. So we can rewrite this action

$$\mathrm{i}\rho_1 \int (C'_{012} + C'_0 2\pi\alpha' F'_{12})dxdt. \tag{4.38}$$

These arguments can be generalized, using again the gauge invariance of $B + 2\pi F$, to find a general action

$$\mathrm{i}\rho_p \int \exp(2\pi\alpha' F + B) \wedge C. \tag{4.39}$$

The exponent is an expansion in 2-forms F and B, and C is the summand over all $C^{(q)}$ for different q. The integral should be read as picking up exactly all terms that are $p+1$-forms, so $C^{(p+1)} + (2\pi\alpha' F + B) \wedge C^{(p-1)} + ...$

4.5 D-brane tension and charge

Consider two parallel Dp-branes. The one loop amplitude of open strings stretching between them can be calculated[18]

$$A = \frac{1}{2} V_{p+1} 2 \int \frac{d^{p+1}k}{(2\pi)^{p+1}} \sum_i \int_0^\infty \frac{dt}{t} \exp(-\frac{1}{2}t(k^2 + M_i^2)) \tag{4.40}$$

The sum runs over all open string states which stretch between the branes. The factor 2 counts both possible orientations. The 1/2 is for real fields. V_{p+1} is the volume of the brane, defined by putting the system in a large box. dt/t is the invariant measure over all one loop diagram, cylinders of circumference t. The trace of $\exp(-tM^2/2)$ was already carried out in §3.4 using $q = \exp(-t/2\alpha')$.

[1] there is a normalization here that will be fixed later.

The mass however is now $M^2 = (N_b + N_f + a)/\alpha' + Y^2(2\pi\alpha')^{-2}$ with Y the D-brane separation (4.20). Integrating the momenta out, this gives

$$A = V_{p+1} 2 \int \frac{dt}{t} (2\pi t)^{-(p+1)/2} \exp\left(-\frac{tY^2}{8\pi^2\alpha'^2}\right) \tag{4.41}$$
$$\left[-8 \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n}\right)^8 + \frac{1}{2\sqrt{q}} \prod_{n=1}^{\infty} \left(\frac{1+q^{n-1/2}}{1-q^n}\right)^8 - \frac{1}{2\sqrt{q}} \prod_{n=1}^{\infty} \left(\frac{1-q^{n-1/2}}{1-q^n}\right)^8\right]$$

The minus sign for the R-sector term is required, as it is a one loop fermion diagram. As a result the whole expression vanishes by the identity of the traces in the NS and R sector. It is interesting to realize that if we interchange the σ-coordinate along the cylinder and τ around the cylinder, we are actually looking at a closed string exchange between the branes. The first two terms without $(-)^F$ correspond to the NS-NS states and the last with $(-)^F$ to R-R states. To focus on the massless closed string states, take the limit $t \to 0$ in all terms involving q. Using standard identities for theta functions, this becomes

$$\begin{aligned} A &= \frac{1}{2}(1-1) V_{p+1} \int \frac{dt}{t} (2\pi t)^{(-p+1)/2} \left(\frac{t}{2\pi\alpha'}\right)^4 \exp\left(-\frac{tY^2}{8\pi^2\alpha'^2}\right) \\ &= (1-1) V_{p+1} 2\pi (4\pi^2\alpha')^{3-p} G_{9-p}(Y^2). \end{aligned} \tag{4.42}$$

where G_{9-p} is the scalar Green function in 9-p dimensions. In a field theory calculation (see [1]) using the effective actions (4.22),(4.36) the contribution from the NS-NS fields and of the R-R fields can be exactly calculated

$$A_{NS-NS} = V_{p+1} T_p^2 G_{9-p}(Y^2), \quad A_{R-R} = -V_{p+1} \rho_p^2 G_{9-p}(Y^2) \tag{4.43}$$

This fixes the tension and charge to

$$T_p^2 = \rho_p^2 = 2\pi(4\pi^2\alpha')^{3-p}, \tag{4.44}$$

which is in line with (4.31).

4.6 Dirac charge quantization

The existence of both magnetic and electric charges imposes a condition on the possible charges for which there is a consistent way to interpret the fields as a gauge theory. In the $D = 4, p = 2$ Maxwell case the argument is as follows. Consider a magnetic charge μ in the origin. It has magnetic current $J_M = \mu\delta(x_1)\delta(x_2)\delta(x_3)dt$. This charge is measured by the flux of F through a sphere S^2 around the origin, using Stokes theorem

$$\int_{S_2} F = \int_B dF = \int_B *J_M = \int_B \mu\delta(x_1)\delta(x_2)\delta(x_3)dxdydz = \mu \tag{4.45}$$

Anywhere outside the origin $dF = 0$, so at least locally it is possible to write $F = dA$. In general it is not possible to find such an A for the entire region without the origin, since it is not contractible. Suppose a loop C divides S^2 in two, and for the upper hemisphere we can write $F = dA$ and for the lower $F = dA'$. Again by Stokes the flux is equal to the difference in the loop integral of A and A' over C

$$\int_{S_2} F = \oint_C A - \oint_C A' \tag{4.46}$$

This loop integral however calculates the phase an electric charge ρ picks up after going around the loop.

$$\exp(\mathrm{i}\rho \oint_C A) \tag{4.47}$$

Since this phase should be the same using A or A', we get the condition

$$\rho\mu = 2\pi n, \quad n \in \mathbb{Z}. \tag{4.48}$$

This can be generalized for the R-R fields. The magnetic charge of a field $G^{(p+2)}$ is a D$6-p$-brane. Taking a $p+2$-sphere S^{p+2} in the transverse space, with its origin a fixed point on the brane, the charge in this point is measured by

$$\int_{S^{p+2}} G^{(p+2)} = \rho_{6-p}. \tag{4.49}$$

In order for the exponent of the action in the path-integral for a p-brane wrapped on a S^{p+1} sphere

$$\exp(\mathrm{i}\rho_p \int C^{(p+1)}) \tag{4.50}$$

to be well defined, one gets the condition

$$\rho_{(6-p)}\rho_p = 2\pi n, \quad n \in \mathbb{Z}. \tag{4.51}$$

Remarkably this is satisfied by (4.44) with $n = 1$. So the D-brane charge is the minimum charge satisfying the Dirac charge quantization condition.

4.7 BPS states

The introduction of D-branes and thus of open strings in the type II theories, breaks some of the super-symmetry. This is not surprising as only the vacuum of the theory is invariant under all super-symmetries. However not all of the symmetries are broken. The boundary conditions for open strings relate the left and right moving spin-fields and thus the space-time super-symmetry operators Q_α and $\tilde{Q}_\alpha$, breaking halve of the 32 super-symmetries. The remaining 16 are generated by

$$Q + \Pi_p \tilde{Q} \tag{4.52}$$

where Π_p is the operator that gives the boundary condition on the spin field

$$\tilde{S}_\alpha(\bar{z})|_{\mathrm{Im}\, z=0} = \Pi_p S_\alpha(z) \tag{4.53}$$

For a D8-brane $\Pi_8 = \Gamma^{11}\Gamma^{\mu}(4.21)$, with μ the transverse direction. It has the property that it anti-commutes with the Γ^{μ} of the transverse directions and commutes with those of the directions parallel to the brane. More generally for a Dp-brane with volume form

$$\omega^{(p+1)} = \frac{1}{(p+1)!}\omega^{(p+1)}_{\mu_0..\mu_p} dx^{\mu_0} \wedge dx^{\mu_1} \wedge ...dx^{\mu_p}, \tag{4.54}$$

one can show by applying several T-dualities that it has the form[1]

$$\Pi_p = \frac{1}{(p+1)!}\omega^{(p+1)}_{\mu_0..\mu_p} \Gamma^{\mu_0}\Gamma^{\mu_1}...\Gamma^{\mu_p}(\Gamma)^p \tag{4.55}$$

where $(\Gamma)^p = 1$ for p even and $(\Gamma)^p = \Gamma$ for p odd. Now consider multiple D-branes, not necessarily parallel nor of the same dimension. For a Dp and a Dp'-brane with associated operators Π_p and $\Pi'_{p'}$ their super-symmetries(4.52) are different. If they have some in common, there must be a combination u^{α} and u'^{α} of (4.52) for which

$$u^{\alpha}(Q + \Pi_p\tilde{Q})_{\alpha} = u'^{\alpha}(Q + \Pi'_{p'}\tilde{Q})_{\alpha}. \tag{4.56}$$

The number of solutions is given by the number of zero eigenvalues of

$$\Pi'_{p'} - \Pi_p \tag{4.57}$$

Consider two branes each parallel to the coordinates axes. It is now possible for open strings stretching between the branes that in some direction one endpoint is constrained on the brane (Neumann-condition), while the other endpoint can move freely in this direction on the other brane (Dirichlet-condition). Because the number of directions in which the strings have both Neumann boundaries and the number of Neumann-Dirichlet boundaries is related by

$$p + p' = \#ND + 2\#NN \tag{4.58}$$

and, since we are either in type IIA or type IIB, p and p' are either both even or both odd, the number $r = \#ND$ of ND directions is always even. $\Pi'_{p'}\Pi_p^{-1}$ is a multiple of $\Gamma^{\nu_1}..\Gamma^{\nu_r}$ with ν_i the directions in which there are both Neumann and Dirichlet conditions. If $r = 2, 6, 10$, $M^2 = -1$, so there are no $+1$ eigenvalues and thus no remaining super-symmetries. For $r = 0$ and $\omega_p = \omega'_{p'}$ $\Pi'_{p'}\Pi_p = \mathbf{1}$ and thus all 16 super-symmetries remain, giving a $N = 1$ space-time super-symmetry. For $r = 4, 8$, $M^2 = 1$ and halve of the eigenvalues are $+1$ reducing the number of super-symmetries to 8.
The N=2 super-symmetry algebra is of the form

$$\begin{aligned}
\{Q_{\alpha}, \tilde{\bar{Q}}_{\beta}\} &= 2(\Gamma^0\Gamma^{\mu})_{\alpha\beta}(P_{\mu} + Q^{NS}_{\mu}/2\pi\alpha'),\\
\{\tilde{Q}_{\alpha}, \tilde{\bar{Q}}_{\beta}\} &= 2(\Gamma^0\Gamma^{\mu})_{\alpha\beta}(P_{\mu} - Q^{NS}_{\mu}/2\pi\alpha'),\\
\{Q_{\alpha}, \tilde{\bar{Q}}_{\beta}\} &= 2Q^R.
\end{aligned} \tag{4.59}$$

Here Q^{NS} is the charge of the NS-NS 2-form B, we will see in chapter 5 that the fundamental string is charged under the B-field. Q^R is the R-R charge, which is the sum over the operators Π_p of the present D-branes. In the rest-frame of some D-brane configuration we can write, ignoring the B-field:

$$\frac{1}{2}\left\{\begin{pmatrix} Q_\alpha \\ \tilde{Q}_\alpha \end{pmatrix}, \begin{pmatrix} \bar{Q}_\beta & \tilde{\bar{Q}}_\beta \end{pmatrix}\right\} = m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \rho_p \begin{pmatrix} 0 & Q^R \\ Q^{R,T} & 0 \end{pmatrix} \tag{4.60}$$

Here μ is the mass-density. The left-hand side of this equation is a positive matrix. For a single D-brane $Q^R = \Pi_p$ and $\mu = T_p\sqrt{|\omega^{(p+1)}|^2}$ with $|\omega^{(p+1)}|^2 = \omega^{(p+1)} * \omega^{(p+1)}$ the local volume density of the D-brane. Locally we can choose coordinates such that $|\omega^{(p+1)}|^2 = 1$. Further $(Q^R)^2 = |\omega^{(p+1)}|^2$. Thus the left-most matrix has eigenvalues $\pm\rho_p$. But as the total left-hand side is always positive, we derive a mass bound

$$m \geq \rho_p \tag{4.61}$$

This is called a BPS-bound. For states that exactly satisfy this bound, called BPS-states, there are zero eigenvectors of the right-hand side of (4.60). They give linear combinations of Q_α's that generate the remaining super-symmetries of this state. In the case of 1 D-brane half of the Q_α-combinations give zero, corresponding to the negative eigenvalues of Q^R. Thus as we saw before there are 16 super-symmetries left. For the combination of 2 D-branes one derives in a similar way

$$\begin{aligned} &\text{for } r = 0,4,8: && \Longrightarrow m \geq \rho_p + \rho_{p'}, \\ &\text{for } r = 2,6,10: && \Longrightarrow m \geq \sqrt{\rho_p^2 + \rho_{p'}^2}. \end{aligned} \tag{4.62}$$

Again the states that exactly satisfy these bounds have remaining super-symmetries. Thus with $m = T_p + T_{p'}$ there are BPS-states for r a multiple of 4.

4.8 Brane-anti brane configurations

Consider again two parallel branes of the same dimension. Flipping the orientation of one of the two branes, changes sign of ω and thus $\Pi'_p = -\Pi_p$. Since the matrix in (4.57) is now $2\mathbf{1}$, which has no zero eigenvalues, this is no longer a BPS-configuration. Although it was first thought that the super-symmetry property of D-branes was crucial in assuring stability and providing self-consistency checks, it was found that there are also stable non-BPS non-perturbative configurations. In fact their existence is required by strong/weak coupling dualities because of the existence of stable weakly-coupled non super-symmetric states. Reference [21] gives an overview of the application of non-BPS states in string theory.

Characteristic for non-BPS D-brane configurations is the reappearance of tachyonic modes. Remember the one-loop open-string amplitude where the $(-1)^F$-term of the NS open-string sector corresponds to the R-R sector of the closed string. Changing the orientation of one of the branes, changes the sign of the

R-R charge and thus the sign of the R-R term in the amplitude. In the open string picture this corresponds with replacing the operator $\frac{1}{2}(1+(-1)^F)$ by $\frac{1}{2}(1-(-1)^F)$ in the NS-sector. This means that in stead of projecting out the tachyon, we project out the massless mode and keep the tachyon as ground state. With a Dp-brane and a D$\bar{p}$-brane, the oppositely oriented anti-brane, on top of each of other, the four different open strings $(p-p, p-\bar{p}, \bar{p}-p, \bar{p}-\bar{p})$ can be thought of having different Chan Paton labels, represented by a matrix. The lowest mass modes are then of the form

$$\begin{pmatrix} A^\mu & T \\ \bar{T} & \bar{A}^\mu \end{pmatrix} \tag{4.63}$$

The lowest modes of the $p-p$ and $\bar{p}-\bar{p}$ open strings give rise to gauge fields A^μ and $\bar{A}^\mu$ on the brane and the anti-brane respectively for the directions μ parallel to the brane, and fluctuations of the D-brane positions for the transverse directions. Mathematically the gauge fields are the connections of line bundles E on Dp and F on D$\bar{p}$. The lowest mass modes of the $p-\bar{p}$ and $\bar{p}-p$ strings give tachyon scalar fields T and $\bar{T}$. The endpoints of these strings are charged under A and $\bar{A}$ but with opposite signs for T and $\bar{T}$, because of their opposite orientation. This shows that T can be regarded as a section of the line bundle $E \otimes F^*$, since its connection is $A-\bar{A}$, and $\bar{T}$ as a section of $E^* \otimes F$. The pair $(T, \bar{T})$ can be thought of being a complex field with modulus $|T|^2 = T\bar{T}$.
After integrating out the massive modes one can argue[22] that the tachyonic potential $V(T)$, which only depends on $|T|$, has a minimum such that

$$2\tau_p + V(T_0) = 0. \tag{4.64}$$

This shows that the minimum describes a state which carries no charge nor any energy. Consequently it is indistinguishable from the super-symmetric vacuum. The process of rolling down the potential to a stable minimum is called *tachyon condensation.* Instead of condensating to the tachyon ground state $T = T_0 e^{i\theta}$ with arbitrary constant θ, one might consider condensation to a tachyonic vortex-solution. This is a configuration of the $(T, \bar{T})$-field which has modulus T_0 at infinity, but with a topological non-trivial twist in its phase. For example on a Dp-D$\bar{p}$ system, it might have a twist when going around a $p-2$ dimensional object on the brane pair. This tachyon condensate is then equal to the standard BPS D$(p-2)$-brane configuration. Other twists in the tachyon field give rise to all possible kinds of lower dimensional branes. In fact it is thought that all known BPS D-branes arise in this way. The construction of D-branes arising from topological-nontrivial twists in the line bundles on a brane-anti brane pair, is remarkably resembled by a standard technique in topological K-theory. In the next chapter we will see that using this technique one can describe mathematically, how all lower dimensional branes of the type IIB superstring arise from a stack of $D9$ and $D\bar{9}$-branes. Something similar can be done for type IIA-branes.

4.9 Descent relations among D-branes

First we will give another perspective[21], from which one can see that brane/anti-brane pairs indeed contains the spectrum of lower dimensional BPS-branes. Consider the action of $(-1)^{F_L}$ on a Dp,D$\bar{p}$-system in either type IIA or type IIB superstring theory, where F_L is the *space-time fermion number* of the space-time fields arising from the left-moving sector of the worldsheet only. It changes sign of the left-moving Ramond sector, but does not change anything else. This definition implies that all R-R fields change sign, while the NS-NS fields remain unchanged. In this way the brane and the anti-brane, being opposite R-R charges, get interchanged. It is interesting to investigate the orbifold induced by the operator $(-1)^{F_L}$.

In general an orbifold is contrived from an existing theory by modding out a symmetry of the target space. The resulting theory consists of two sectors. The *untwisted sector* contains all states of the original theory that are invariant under the symmetry. For the closed strings they are, focusing on the lowest modes, the NS-NS and the NS-R fields. The open string states in the untwisted sector are those invariant under interchanging the brane and the anti-brane. They are the linear combination of a $p-p$ and a similar $\bar{p}-\bar{p}$ string, corresponding to a Chan Paton matrix that is a multiple of the identity, and the linear combination of similar $p-\bar{p}$ and $\bar{p}-p$ strings, corresponding to a symmetric off-diagonal Chan Paton matrix. The *twisted sector* contains states corresponding to fields on the closed string worldsheet, that are invariant but are only periodic up to the symmetry, i.e. under a translation

$$\sigma \to \sigma + 2\pi : \quad |\Psi\rangle \to (-1)^{F_L}|\Psi\rangle. \tag{4.65}$$

As the target space of the orbifold is invariant under the symmetry, consistency requires these strings to be included. In [5] it is explained that the states of the left-moving sector with these twisted boundary conditions, are equal to those of the R-sector, and that if we start with type IIA theory this R-sector is of the same chirality as the right-moving R-sector. Starting with type IIB the twisted left-moving sector is now of opposite chirality. Combining this new left-moving sector with the unchanged right-moving sector gives the R-NS and R-R sector of the closed superstring. Adding the untwisted sector with NS-NS and NS-R sectors, to the twisted sector with R-NS and R-R sectors, the orbifold of type IIA superstring under $(-1)^{F_L}$ gives the type IIB superstring, and vice versa the orbifold of type IIB is type IIA.

The open string states that survive the $(-1)^{F_L}$ modding, are those that are invariant under interchanging the brane and anti-brane. They have Chan Paton matrices which are a multiple of

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.66}$$

Thus only one gauge field $A^\mu + \bar{A}^\mu$ and only one real scalar tachyon $T + \bar{T}$ survives the projection. The result is a single D-brane. This can be seen by the fact that in the transverse directions, of the lowest mass modes corresponding to the fluctuation of the positions of the D-branes, only the sum remains while the difference is projected out. Thus the distance between the brane and anti-brane is fixed and only the system as a whole can be moved in the transverse direction. Note that if we start with a Dp-D$\bar{p}$-brane system with p even for type IIA and odd for type IIB, we end up with a single Dp-brane with p even for type IIB and odd for type IIA! This is different then the possibilities in table 1. It is a non-BPS configuration with a real tachyon field.

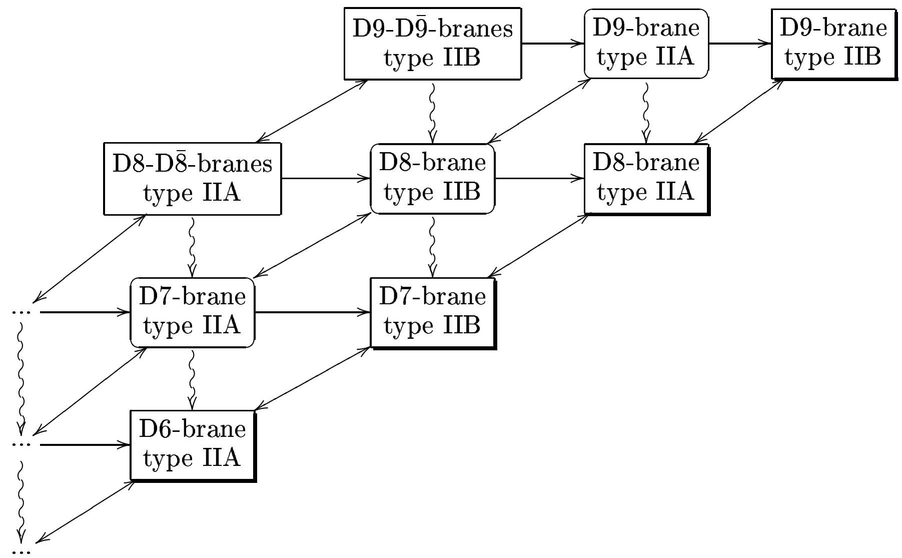

Figure 4: Descent relations among BPS and non-BPS D-brane configurations in type II superstring theories. The horizontal arrows denote modding out $(-1)^F$. The vertical arrows denote the tachyon condensation. The diagonal arrows denote T-dualities.

This trick can be repeated. Note that $(-1)^{F_L}$ is different in type IIA then in type IIB as in their left-moving R-sectors states with different chirality are projected out. Orbifolding again using $(-1)^{F_L}$ switches again between type IIA and type IIB for the closed strings. The open string looses the tachyon mode. So we end up with a single Dp brane with type IIA closed strings for p even and type IIB for p odd, and the usual gauge field on the brane and brane position fluctuations from the open strings. This is again the standard BPS Dp-brane configuration.

Figure 4 gives an overview of all descent relations we have described. The tachyon condensation is here performed in two steps. First one gives the tachyon an instanton-like solution in one direction of the Dp-D$\bar{p}$-branes. That is

- $\mathrm{Im}(T) = 0$

- $\mathrm{Re}(T)$ is independent of time and $p-1$ of the spatial coordinates of the branes
- For the remaining coordinate x:
  $\mathrm{Re}\, T \to T_0$ for $x \to \infty$
  $\mathrm{Re}\, T \to -T_0$ for $x \to -\infty$

thus giving T a jump around $x = 0$. This gives a D$p-1$-brane on the original Dp-brane extending in all directions transverse to the x-direction. It still has one tachyonic mode of freedom. As instead of the limit $T \to T_0$ we could also take $T \to T_0 \exp(i\theta)$ with an arbitrary phase. This is the non-BPS brane with only one real tachyon field T, we found after a single orbifolding of a D$p-1$-D$\overline{p-1}$ system. The second step gives the real tachyon T a kink in another direction, say y, such that

- T is independent of time and $p-2$ of the spatial coordinates of the original brane-anti-brane system.
- $\mathrm{Im}(T) \to T_0$ for $y \to \infty$ $\mathrm{Im}(T) \to -T_0$ for $y \to -\infty$

The result is the standard BPS D$p-2$-brane.

5 K-theory

K-theory was first introduced in the 1950's by A.Grothendieck and further developed in the 1960's by Atiyah and Hirzebruch, who where the first to introduce the K-group of a topological space. Since then it has become a very important tool in many areas of topology, differential geometry and algebra. A very complete introduction in the mathematics of topological K-theory is [11]. A short introduction focused on topics related to spin geometry, such as the ABS-construction, can be found in [16].
The fact that R-R fields are differential forms, satisfying Dirac charge quantization, suggests that they should be described in integer cohomology. In [15] however, Minasian and Moore made the conjecture that the charges of the R-R fields of superstring theory take their values in K-theory. More evidence for this idea was formulated by Witten in [23], where it was shown how Sen's construction of tachyon condensation[21] in brane-anti-brane systems leads naturally to the identification of D-brane charges as elements of $K(X)$, the K-theory of the space-time manifold X. Furthermore it was shown in this paper that a topological condition in K-theory exactly matches the cancellation of a previously noted worldsheet global anomaly. A good introduction treating both the mathematics of K-theory and its application in string theory is [17].
The first paragraphs of this chapter are devoted to explaining all necessary topics in K-theory. The last paragraph shows how all elements of the theory are related to the physics of D-brane systems.

5.1 The Grothendieck group

The most general definition of the theory is rather algebraic, but this is needed to make some generalizations later on. A *monoid* is a set which satisfies all axioms of a group except for the fact that not all elements have an inverse. Think for example of the additive semigroup $\mathbb{Z}_{\geq 0}$ or the multiplicative semigroup $\mathbb{Z}^*\backslash\{0\}$. To an abelian monoid $\mathcal{A}$ one can associate an abelian group $S(\mathcal{A})$. On the direct product $\mathcal{A} \times \mathcal{A}$ one can define an equivalence relation with $(x, y) \sim (x', y')$ if there exists a $z \in \mathcal{A}$ such that

$$x + y' + z = y + x' + z. \tag{5.1}$$

The set $S(\mathcal{A})$, *the symmetrization* of $\mathcal{A}$, is the set of equivalence classes $[(x, y)]$

$$S(\mathcal{A}) = \mathcal{A} \times \mathcal{A} / \sim . \tag{5.2}$$

It is an abelian group because it inherits the abelian monoid structure of the direct product, with addition defined by

$$[(x, y)] + [(x', y')] = [(x + x', y + y')] \tag{5.3}$$

and, since by

$$[(x, y)] + [(y, x)] = [(x + y, x + y)] = [(0, 0)], \tag{5.4}$$

every element $[(x, y)]$ has inverse $[(y, x)]$.
The symmetrization of $\mathbb{Z}_{\geq 0}$ consists of pairs of nonnegative integers with $(x, y) \sim (x', y')$ if and only if $x + y' + z = y + x' + z$ for some z, but this just means $x + y' = y + x'$. In $\mathbb{Z}$ this means $x - y = x' - y'$. So all pairs with equal difference are equivalent. This difference is just a number in $\mathbb{Z}$. Thus $S(\mathbb{Z}_{\geq 0}) = \mathbb{Z}$. Similarly $S(\mathbb{Z}^*\backslash\{0\}) = \mathbb{Q}$.
Less trivial examples arise from additive categories. An additive category is a category with an additive structure and the notion of linear homo- and isomorphisms. Let $\Phi(\mathcal{C})$ be the set of isomorphism classes of the additive category $\mathcal{C}$. It is an abelian monoid with addition defined by $[x] + [y] = [x + y]$. The symmetrization $S(\Phi(\mathcal{C}))$ is called the *Grothendieck group* of $\mathcal{C}$ and is denoted by $K(\mathcal{C})$. Instead of using the consequent notation $[([x], [y])]$ for the equivalence class in $K(\mathcal{C})$ represented by a pair of equivalence classes $[x]$ and $[y]$ in $\Phi(\mathcal{C})$ with $x, y \in \mathcal{C}$, one uses the notation $[x, y]$, or with $[x] = [x, 0]$ and $[y] = [0, y]$, writes it as the difference $[x] - [y]$.
As a simple example, consider the additive category $\mathcal{C}$ of finite-dimensional vector spaces over the field $\mathbb{R}$ or $\mathbb{C}$. The equivalence classes are completely determined by giving the dimension of the vector space. So $\Phi(\mathcal{C}) = \mathbb{Z}_{\geq 0}$ and thus $K(\mathcal{C}) = S(\mathbb{Z}_{\geq 0}) = \mathbb{Z}$.
The most important example is the Grothendieck group of $\mathrm{Vect}(X)$, the additive category of complex vector bundles over a compact manifold X. This is an additive category using the Whitney sum $E \oplus F$, which is the bundle of direct sums of the vector spaces of E and F at each point, and the notion of equivalent

vector bundles (see appendix A.3 for some of the basics of vector bundles). One uses the notation

$$K(X) = K(\mathrm{Vect}(X)) = S(\Phi(\mathrm{Vect}(X))). \tag{5.5}$$

The study of this abelian group for, in general, a topological space X, is the subject of *topological K-theory.* Very important in this context is Swan's theorem which says that if G is a bundle over a compact manifold X there exists a bundle G' such that $G \oplus G'$ is a trivial bundle, that is a bundle which is of the form $X \times \mathbb{C}^k$. We will use the notation I^k for a trivial bundle of rank k. It follows that the following two equivalence relations define the same equivalence classes $[E, F]$ of the Grothendieck group

$$\begin{aligned} & E \oplus F' \oplus G \cong E' \oplus F \oplus G \text{ for some } G \in \mathrm{Vect}(X) \\ \Longleftrightarrow\ & E \oplus F' \oplus I^k \cong E' \oplus F \oplus I^k \text{ for some } k \in \mathbb{Z}_{\geq 0}. \end{aligned} \tag{5.6}$$

A very important property of the K-group is its homotopy invariance. The pull-back of a vector bundle $E \to Y$ under a map $f : X \to Y$ is the vector bundle f^*E over X, whose fiber E_x at $x \in X$ is the vector space $F_{f(x)}$, the fiber of F at $f(x)$. If $f, g : X \to Y$ are homotopic maps, the bundles f^*E and f^*F are equivalent. Similarly an element $[E, F] \in K(Y)$ can be pulled back to $f^*[E, F] = [f^*E, f^*G] \in K(X)$ and for f and g homotopic

$$f^*[E, F] = g^*[E, F]. \tag{5.7}$$

Thus $f^* : K(Y) \to K(X)$ and $g^* : K(Y) \to K(X)$ are identical if f and g are homotopic.
Suppose X is a compact manifold which is contractible to a point. This means that there are maps $f : X \to \mathrm{pt}$ and $g : \mathrm{pt} \to X$ such that $g \circ f$ is homotopic to the identity. Therefore the pull-back under $g \circ f$ on vector bundles over X is an isomorphism. The pull-back can be decomposed in $(g \circ f)^* = f^* \circ g^*$. The pull-back $(g \circ f)^*E$ of any vector bundle $E \in \mathrm{Vect}(X)$ is therefore the pull-back of the vector bundle g^*E over pt, which is necessarily trivial. Since every bundle $E \in \mathrm{Vect}(X)$ is isomorphic to $(g \circ f)^*E$, every bundle $E \in \mathrm{Vect}(X)$ is trivial. As a result the K groups of X and pt are equal, and because $\Phi(\mathrm{Vect}(\mathrm{pt})) = \mathbb{Z}_{\geq 0}$, it follows that

$$K(X) = K(\mathrm{pt}) = \mathbb{Z}. \tag{5.8}$$

5.2 Reduced K-theory

Every K-group contains $\mathbb{Z}$ as a subgroup, generated by the element $[I^1, 0]$. This subgroup thus consists of elements of the form $[I^m, I^n]$. These 'trivial' elements can always be written as the pull-back of an element of $K(\mathrm{pt})$. Let X again be a compact space and consider maps

$$p : X \to \mathrm{pt}, \quad i : \mathrm{pt} \to X \tag{5.9}$$

Then p^* maps to the trivial elements of $K(X)$

$$p^* : K(\mathrm{pt}) = \mathbb{Z} \to K(X). \tag{5.10}$$

Sometimes it is useful to mod out these trivial elements. Therefore we define the *reduced K-group*

$$\tilde{K}(X) = K(X)/\,\mathrm{img}(p^* : \mathbb{Z} \to K(X)). \tag{5.11}$$

As $i^* \circ p^*$ is the identity on $K(X)$, an equivalent definition is given by

$$\tilde{K}(X) = \ker(i^* : K(X) \to \mathbb{Z}). \tag{5.12}$$

Given a vector bundle $E \to X$, the rank of E is a function $\mathrm{rk}\,E$ on X that gives the dimension of the vector space E_x for every $x \in X$. In every connected component this function is constant, so $\mathrm{rk}\,E \in H^0(X, \mathbb{Z}_{\geq 0})$ the space of $\mathbb{Z}_{\geq 0}$-valued locally constant functions on X. The map $\mathrm{rk} : \mathrm{Vect}(X) \to H^0(X, \mathbb{Z}_{\geq 0})$ is defined on $K(X)$ by

$$\mathrm{rk}([E,F]) : K(X) \to H^0(X, \mathbb{Z}); [E,F] \mapsto \mathrm{rk}(E) - \mathrm{rk}(F). \tag{5.13}$$

Note that this *virtual dimension* can also be negative. For instance $\mathrm{rk}[0, I^n] = -n$. We define

$$K'(X) = \ker(\mathrm{rk} : K(X) \to H^0(X, \mathbb{Z})) \tag{5.14}$$

If X is connected $H^0(X, \mathbb{Z}) = \mathbb{Z}$. Furthermore as i^* maps every element of $K(X)$ to the trivial element of the same rank in $K(\mathrm{pt})$, if X is connected $i^* = \mathrm{rk}$ and thus $K'(X) = \tilde{K}(X)$.
So far we have required the manifolds to be compact. The K-group of a noncompact manifold X can be defined as

$$K(X) = \tilde{K}(X^+), \tag{5.15}$$

where X^+ is the one-point compactification of X: $X^+ = X \cup \{\mathrm{pt}\}$. This can be thought of as adding infinity as one point to the manifold. For instance $(\mathbb{R}^k)^+ = S^k$. One usually calculates $\tilde{K}(X^+) = \ker i^*$ with i a map from a point to the added infinity-point in X^+. This means that the pair of vector bundles in $\tilde{K}(X^+)$ have equal rank at infinity, and locally around infinity there is an isomorphism between the two. Later on we will interpret this physically, as the condition that the configuration looks like the vacuum at infinity. For X is compact X^+ is defined as the union of X and a disjoint point. This disjoint point gives another $\mathbb{Z}$ of trivial bundles of different rank at that point. The reduction removes this extra $\mathbb{Z}$. Therefore for compact X $\tilde{K}(X^+) = K(X)$ using the old definition. Thus (5.15) is merely a generalization for noncompact X. This definition of $K(X)$ is is called compactly supported K-theory. There is also a non compactly supported K-theory, defined as the Grothendieck group of $Vect(X)$, that differs only if X is noncompact. For applications in D-brane mechanics, we actually need a combination of both definitions. For the time being $K(X)$ will denote the compactly supported K-theory.

5.3 Higher K-groups

The homotopy invariance is the first parallel between cohomology and K-theory. For a further development of this relation one introduces the higher K-groups. Using the compact support definition the higher K-groups can be defined as

$$K^{-n}(X) = K(X \times \mathbb{R}^n) \text{ with } n \geq 0. \tag{5.16}$$

By definition $K^0(X) = K(X)$ and for $n = 0, 1, 2...$ we get the different rank K-groups. However by the *Bott periodicity theorem*

$$K^{-n-2}(X) = K^{-n}(X) \tag{5.17}$$

So in fact we only have two distinct K-groups, $K(X)$ and $K^{-1}(X)$.
Like in cohomology the K-group $K(X)$ is in fact a ring. The product follows from $[E] \otimes [F] = [E \otimes F]$ for isomorphism classes of $\mathrm{Vect}(X)$. By writing this out using distributivity, for products of $[E, F] = [E] - [F]$ and $[E', F'] = [E'] - [F']$, we get

$$\begin{aligned} [E, F] \otimes [E', F'] &= ([E] - [F]) \otimes ([E'] \otimes [F']) \\ &= [E \otimes E'] - [E \otimes F'] - [F \otimes E'] + [F \otimes F'] \\ &= [E \otimes E' \oplus F \otimes F', E \otimes F' \oplus F \otimes E']. \end{aligned} \tag{5.18}$$

The Chern characters are maps (again see A.3)

$$\mathrm{Ch}_k : \mathrm{Vect}(X) \to H^{2k}(X, \mathbb{Q}). \tag{5.19}$$

The total Chern character is defined

$$\mathrm{Ch} : \mathrm{Vect}(X) \to H^{even}(X, \mathbb{Q}); E \mapsto \sum_k Ch_k(E). \tag{5.20}$$

It is a ring homomorphism between the semi-ring of vector bundles and the cohomology ring, by the properties

$$\begin{aligned} \mathrm{Ch}(E \oplus F) &= \mathrm{Ch}(E) + \mathrm{Ch}(F), \\ \mathrm{Ch}(E \otimes F) &= \mathrm{Ch}(E) \cdot \mathrm{Ch}(F). \end{aligned} \tag{5.21}$$

The Chern character can be made into a ring homomorphism between $K(X)$ and the cohomology ring by

$$\mathrm{Ch} : K(X) \to H^{even}(X, \mathbb{Q}); [E, F] \mapsto \mathrm{Ch}(E) - \mathrm{Ch}(F) \tag{5.22}$$

In fact if X is compact, it induces a ring isomorphism

$$K(X) \otimes \mathbb{Q} \xrightarrow{\cong} H^{even}(X, Q). \tag{5.23}$$

It looks like the K-group and rational cohomology are almost equal. However in multiplying $K(X)$ with $\mathbb{Q}$ one looses all finite subgroups such as $\mathbb{Z}_n$ with

$n \in \mathbb{Z}_{>0}$. It is exactly by those so called torsion elements that K-theory differs from ordinary cohomology.
Let us calculate

$$K^{-1}(\mathrm{pt}) = K(\mathbb{R}^1) = \tilde{K}(S^1) \tag{5.24}$$

On S^1 there are only trivial complex vector bundles. Thus $K(S^1) = \mathbb{Z}$ and $K^{-1}(\mathrm{pt}) = 0$. The reduced higher K-groups are defined by

$$\tilde{K}^{-n}(X) = \ker(i^* : K^{-n}(X) \to K^{-n}(\mathrm{pt})) \tag{5.25}$$

By Bott periodicity it follows that

$$\tilde{K}^{-n-2}(X) = \tilde{K}^{-n}(X). \tag{5.26}$$

Further we have

$$\begin{aligned} K^0(X) &= \tilde{K}^0(X) + K^0(\mathrm{pt}) = \tilde{K}(X) + \mathbb{Z}, \\ K^{-1}(X) &= \tilde{K}^{-1}(X) + K^{-1}(\mathrm{pt}) = \tilde{K}^{-1}(X). \end{aligned} \tag{5.27}$$

5.4 Relative K-Theory

We define the relative K-group of a manifold X and a closed submanifold Y of X by

$$K(X, Y) = \tilde{K}(X/Y) \tag{5.28}$$

with X/Y the manifold X with Y shrunk to a point. One defines $X/\emptyset = X^+$. And thus for all compact and non-compact manifolds X:

$$K(X) = K(X, \emptyset). \tag{5.29}$$

The relative K-group can be described in another equivalent way that will be useful later on. Let $T(X,Y)$ be the set of triples (E, F, α) with $E, F \in \mathrm{Vect}(X)$ and $\alpha : E|_Y \to F|_Y$ an isomorphism of the vector bundles restricted to Y. Two such triples are called *isomorphic* $(E, F, \alpha) \cong (E', F', \alpha')$, if there are isomorphisms $f : E \to E'$ and $g : F \to F'$ such that the following diagram commutes.

$$\begin{tikzcd} E|_Y \arrow[r, "\alpha"] \arrow[d, "f"] & F|_Y \arrow[d, "g"] \\ E'|_Y \arrow[r, "\alpha"] & F'|_Y \end{tikzcd} \tag{5.30}$$

A triple is called *elementary* if E is isomorphic to F and α is homotopic to $\mathrm{Id}_{E|_Y}$ within automorphisms of $E|_Y$. The sum is defined on $T(X,Y)$ by

$$(E, F, \alpha) \oplus (E', F', \alpha) := (E \oplus E', F \oplus F', \alpha \oplus \alpha'). \tag{5.31}$$

Further there is an *equivalence* relation, $(E, F, \alpha) \sim (E', F', \alpha')$, if there are two elementary triples (G, H, β) and (G', H', β') such that

$$(E, F, \alpha) \oplus (G, H, \beta) \cong (E', F', \alpha') \oplus (G', H', \beta'). \tag{5.32}$$

It can be shown that with this equivalence relation, the relative K-group can also be defined as

$$K(X,Y) = T(X,Y)/\sim . \tag{5.33}$$

We use the notation $[E, F, \alpha]$ for the equivalence classes. The inverse of $[E, F, \alpha]$ is given by $[F, E, \alpha^{-1}]$. Now for non-compact X, $K(X) = K(X, \emptyset)$ can be described in terms of triples $[E, F, \alpha]$ where α is a bundle isomorphism in a neighborhood of the infinity of the one-point compactification $X^+ = X/\emptyset$. This verifies our earlier statement that the compactly supported K-group $K(X)$ with X noncompact, is formed by pairs of bundles which are isomorphic at infinity.

5.5 The ABS-construction

The purpose of this paragraph is to show the ABS construction in its full glory, providing a ring isomorphism between a certain ring of Clifford modules and $\tilde{K}(S^n)$. The practical purpose of the ABS construction lies in the fact that it gives an explicit generator of $\tilde{K}(S^n)$, which will have a very natural and important interpretation in the brane/anti-brane system. The reader not interested in the rather algebraic construction of the isomorphism, can skip to the next paragraph where the result and its physical interpretation are explained. The necessary facts about Clifford modules can be found in A.6. For more details one might consult [16].

Let W be a $\mathbb{Z}_2$-graded module over the Clifford algebra $Cl_{r,s}$. This is a module which can be decomposed $W = W^0 \oplus W^1$ such that

$$Cl^i_{r,s} \cdot W^j \subseteq W^{i+j \mod 2}, \quad i,j \in \{0,1\} \tag{5.34}$$

where $Cl^0_{r,s}$ and $Cl^1_{r,s}$ are respectively the subalgebra and linear subspace of $Cl_{r,s}$ generated by even and odd products of the Γ's that generate $Cl_{r,s}$. Restriction to $Cl^0_{r,s}$ makes it a direct sum of two disjoint modules W^0 and W^1. By $Cl^0_{r,s} \cong Cl_{r-1,s}$, these are modules of $Cl_{r-1,s}$. Conversely it can be shown that given only W^0 as a module of $Cl_{r-1,s} \cong Cl^0_{r,s}$, one can reconstruct the $\mathbb{Z}_2$-graded module $W = W^0 \oplus W^1$. This establishes an equivalence between the category of $\mathbb{Z}_2$-graded modules over $Cl_{r,s}$ and the category of ungraded modules over $Cl_{r-1,s}$.

We consider the category of complex $\mathbb{Z}_2$-graded modules over $\mathbb{C}l_{r,s} \cong \mathbb{C}l_n$ with $n = r+s$. For n is even there are two inequivalent irreducible ungraded modules over $\mathbb{C}l_{n-1}$, and that there is only one for n odd. This gives two inequivalent $\mathbb{Z}_2$-graded modules over $\mathbb{C}l_n$ for n is even, namely $W = W^+ \oplus W^-$ and $\tilde{W} = W^- \oplus W^+$ with W^+ and W^- the inequivalent modules over $\mathbb{C}l^0_n \cong \mathbb{C}l_{n-1}$. All elements of the category of $\mathbb{Z}_2$-graded modules over $\mathbb{C}l_n$ with n even, can be decomposed in a number of W's and $\tilde{W}$'s. It is an additive category. The Grothendieck group of this category will be denoted by $\hat{\mathcal{M}}^{\mathbb{C}}_n$. For n is even we thus have $\hat{\mathcal{M}}^{\mathbb{C}}_n = \mathbb{Z} \oplus \mathbb{Z}$. For n is odd there is only one irreducible $\mathbb{Z}_2$-graded module over $\mathbb{C}l_n$, namely $W = W^+ \oplus W^-$ with $W^+ \cong W^-$ the only irreducible module over $\mathbb{C}l^0_n \cong \mathbb{C}l_{n-1}$. Thus for n is odd $\hat{\mathcal{M}}^{\mathbb{C}}_n = \mathbb{Z}$.

One can define the $\mathbb{Z}_2$-graded tensor product $V \hat{\otimes} W$ of $\mathbb{Z}_2$-graded modules $V =$

$V^0 \oplus V^1$ over $\mathbb{C}l_m$ and $W = W^0 \oplus W^1$ over $\mathbb{C}l_n$. The tensor product $V \hat{\otimes} W = (V \hat{\otimes} W)^0 \oplus (V \hat{\otimes} W)^1$ with

$$
\begin{aligned}
(V \hat{\otimes} W)^0 &= (V^0 \otimes W^0) \oplus (V^1 \otimes W^1), \\
(V \hat{\otimes} W)^1 &= (V^0 \otimes W^1) \oplus (V^1 \otimes W^0),
\end{aligned}
\tag{5.35}
$$

is a module over Cl_{m+n} given by

$$
(e \otimes f) \cdot (v \otimes w) = (e \cdot v) \otimes (f \cdot w) \tag{5.36}
$$

where $e \in \mathbb{R}^m \subset \mathbb{C}l_m$ and $f \in \mathbb{R}^n \subset \mathbb{C}l_n$ and thus $e \otimes f \in \mathbb{R}^{m+n} \subset \mathbb{C}l_{m+n}$, and $v \in V^0$ (with $v \in V^1$ one gets a minus switching f and v). The tensor product transcends to a map $\hat{\mathcal{M}}^{\mathbb{C}}_m \otimes \hat{\mathcal{M}}^{\mathbb{C}}_n \to \hat{\mathcal{M}}^{\mathbb{C}}_{m+n}$, which makes

$$
\hat{\mathcal{M}}^{\mathbb{C}}_{\bullet} := \bigoplus_{n \geq 0} \hat{\mathcal{M}}^{\mathbb{C}}_n \tag{5.37}
$$

a graded ring.
The inclusion $i : \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ induces an algebra homomorphism $i : \mathbb{C}l_n \to \mathbb{C}l_{n+1}$. The restriction of the action of $\mathbb{C}l_{n+1}$ to $\mathbb{C}l_n$ then gives a group homomorphism $i^* : \hat{\mathcal{M}}^{\mathbb{C}}_{n+1} \to \hat{\mathcal{M}}^{\mathbb{C}}_n$. Consider now the groups

$$
\hat{\mathcal{M}}^{\mathbb{C}}_n / i^* \hat{\mathcal{M}}^{\mathbb{C}}_{n+1} = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{5.38}
$$

For n is even $\hat{\mathcal{M}}^{\mathbb{C}}_n = \mathbb{Z} \oplus \mathbb{Z}$ is generated by $[W, 0]$ and $[\tilde{W}, 0]$ with $W = W^+ \oplus W^-$ and $\tilde{W} = W^- \oplus W^+$ the two inequivalent irreducible $\mathbb{Z}_2$-graded modules. $i^* \hat{\mathcal{M}}^{\mathbb{C}}_{n+1} = \mathbb{Z}$ is generated by $[W, 0] \oplus [\tilde{W}, 0]$. Modding this out of $\hat{\mathcal{M}}^{\mathbb{C}}_n$, is imposing an equivalence relation $[W, 0] \oplus [\tilde{W}, 0] \sim 0$ or $[W, 0] \sim [0, \tilde{W}]$ (as $[0, \tilde{W}]$ is the inverse of $[\tilde{W}, 0]$). Thus we see that the combination $[W, 0] \oplus [0, \tilde{W}]$ survives the modding out, while $[W, 0] \oplus [\tilde{W}, 0]$ vanishes.
Consider now the relative K-group $K(B^n, S^{n-1})$. Using its description in terms of triples $[E, F, \alpha]$, to every $\mathbb{Z}_2$-graded module $W = W^+ \oplus W^-$ over $\mathbb{C}l_n$ we can associate

$$
\phi(W) = [S^+, S^-, \mu] \in K(B^n, S^{n-1}) \tag{5.39}
$$

where $S^{\pm} = B^n \times W^{\pm}$ are trivial vector bundles, and

$$
\mu : S^+|_{S^{n-1}} \to S^-|_{S^{n-1}}; (x, w) \mapsto (x, x \cdot w) \tag{5.40}
$$

where $x \cdot w$ denotes the action of $x \in S^{n-1} \subset \mathbb{C}l_n$. Note that as vector bundles S^+ and S^- are isomorphic. In fact since the ball B^n is contractible, all vector bundles over it are trivial. The only thing which makes $\phi(W)$ non-trivial is the topological non-trivial winding of μ over S^{n-1}. Topologically B^n/S^{n-1} is the same as the one-point compactification of $\mathbb{R}^n$, i.e. S^n. Thus

$$
K(B^n, S^{n-1}) = \tilde{K}(B^n/S^{n-1}) = \tilde{K}(S^n) = K^{-n}(\mathrm{pt}). \tag{5.41}
$$

For the Atiyah-Bott-Shapiro construction, one shows that $\phi(W)$ only depends on the isomorphism class $[W]$ of graded modules over $\mathbb{C}l_n$, and that ϕ gives a homomorphism

$$\phi : \hat{\mathcal{M}}_n^{\mathbb{C}} \to K(B^n, S^{n-1}); [V, W] \mapsto \phi(V) - \phi(W). \tag{5.42}$$

Furthermore it can be shown that it vanishes for elements of $i^*\hat{\mathcal{M}}_{n+1}^{\mathbb{C}} \subset \hat{\mathcal{M}}_n^{\mathbb{C}}$. The final result is that

$$\phi : \hat{\mathcal{M}}_\bullet^{\mathbb{C}}/i^*\hat{\mathcal{M}}_{\bullet+1}^{\mathbb{C}} \to K^{-\bullet}(\mathrm{pt}). \tag{5.43}$$

is a ring isomorphism between the ring

$$\hat{\mathcal{M}}_\bullet^{\mathbb{C}}/i^*\hat{\mathcal{M}}_{\bullet+1}^{\mathbb{C}} = \bigoplus_{n\geq 0} \hat{\mathcal{M}}_n^{\mathbb{C}}/i^*\hat{\mathcal{M}}_{n+1}^{\mathbb{C}} \tag{5.44}$$

and the ring $K^{-\bullet}(\mathrm{pt})$. One of the things we can read of is that

$$\tilde{K}(S^n) = K^{-n}(\mathrm{pt}) = \begin{cases} \mathbb{Z} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{5.45}$$

The construction may seem like a rather long way to compute these K-groups. The upshot of this method is that it provides explicit generators of the group $\tilde{K}(S^n)$ for n is even. In the quotient space $\hat{\mathcal{M}}_n^{\mathbb{C}}/i^*\hat{\mathcal{M}}_{n+1}^{\mathbb{C}}$ we have $[W, 0] = -[\tilde{W}, 0] = [0, \tilde{W}]$, as the generator $[W, 0] + [\tilde{W}, 0]$ of $i^*\hat{\mathcal{M}}_{n+1}^{\mathbb{C}}$ has been divided out. The generator of $\hat{\mathcal{M}}_n^{\mathbb{C}}/i^*\hat{\mathcal{M}}_{n+1}^{\mathbb{C}}$ is therefore $[W, 0] = [0, \tilde{W}]$. The generator of $\tilde{K}(S^n)$ is its image under ϕ

$$\phi([W, 0]) = \phi(W) = [S^+, S^-, \mu] \tag{5.46}$$

This explicit generator will prove to have a direct physical interpretation. As a check we calculate

$$\phi([0, \tilde{W}]) = -\phi(\tilde{W}) = -[S^-, S^+, -\mu] \tag{5.47}$$

where we used that x acts with opposite sign in W^- compared to its action in W^+, since x anti-commutes with Γ and $\Gamma = \pm 1$ in $W^\pm$. Moreover

$$\mu \circ \mu(x, w) = (x, x \cdot (x \cdot w)) = (x, (x \cdot x) \cdot w) = -(x, w) \tag{5.48}$$

because $(x \cdot x) = -|x|^2$ by definition of $\mathbb{C}l_n$ and $|x|^2 = 1$ because μ is restricted to S^{n-1}. Therefore

$$\phi([0, \tilde{W}]) = -[S^-, S^+, -\mu] = [S^+, S^-, -\mu^{-1}] = [S^+, S^-, \mu] = \phi([W, 0]),$$

thus checking that ϕ is indeed well-defined on $\hat{\mathcal{M}}_n^{\mathbb{C}}/i^*\hat{\mathcal{M}}_{n+1}^{\mathbb{C}}$.

5.6 K-theory and D-branes

In the last paragraph of this chapter we will finally see how K-theory is related to D-brane mechanics. In fact it will become clear that almost all basic elements of K-theory, have a physical interpretation in branes/anti-branes systems.
Remember that the brane/anti-brane system consists of an ordinary D-brane and a second one one top of it, equal to the first one only with opposite orientation. The lowest mass modes of open strings attached to the first brane is a vector particle, which in the parallel direction gives a gauge field A, denoted in the 1-form notation. Open strings on the second, anti-brane give a gauge field $\bar{A}$. The open strings stretching between the branes give two tachyonic real field. Taking a number of brane/anti-brane pairs, the gauge fields are $U(n)$-valued, giving a vector bundle on the branes and on the anti-branes. If the brane is completely identical to the anti-brane, apart from its orientation, the tachyon T condensates to its vacuum expectation value $|T| = T_0$. The resulting configuration is indistinguishable from the vacuum everywhere. Starting with n branes and n anti-branes but with a different $U(n)$-gauge field on the brane then on the anti-brane, the tachyon taking values in the product of the vector bundles on the branes and that on the anti-branes with connection $A \otimes 1 - 1 \otimes \bar{A}$, might have a non trivial twisting around a lower dimensional object on the brane. As we saw these twists can condensate to a configuration with lower dimensional branes on the brane/anti-brane manifold.
We start by considering n Dp-branes stretched out on a manifold X and n D$\bar{p}$-branes on the same manifold. Let E be the vector bundle with connection A, and F be the vector bundle with connection $\bar{A}$. Adding m branes with vector bundle H and m anti-branes with the same vector bundle, will not change the situation, as the tachyon of the m branes and anti-branes will condensate to the vacuum. Thus the configuration with vector bundles E and F is physically equivalent to the situation with bundles $E \oplus H$ and $F \oplus H$. This equivalence is exactly the same as the equivalence relation $(E, F) \sim (E \oplus H, F \oplus H)$ that defines classes $[E, F]$ in the K-group $K(X)$. The claim of Witten et al. is that in type IIB superstring theory, the branes/anti-branes configurations are indeed adequately described by $K(X)$.
If X is noncompact we have to make a more precise prescription. Remember that the definition of the compactly supported K-group for noncompact spaces, defines it to be the reduced K-group of the one-point compactification of X. This can be interpreted as imposing the condition that around infinity there should be an isomorphism between the two vector bundles. This means that locally around infinity the pair (E, F) looks like the vacuum configuration. However if we want to investigate lower dimensional branes arising in a brane/anti-brane pair, we do not want this condition for the direction in which the lower dimensional brane is supposed to arise. We want to have a brane in that direction, not the vacuum!
Let us start with the example of Dp-branes arising in a D9-D$\bar{9}$-system in a flat $\mathbb{R}^{10}$ space-time. We only want a compact-support condition in the transverse directions. The possible vector bundles on $\mathbb{R}^{p+1}$ on which the Dp-brane wraps, are

the same as the possible vector bundles on pt. Therefore the parallel directions do not contribute, and the possible Dp-brane charges are measured by

$$K(\mathbb{R}^{9-p}) = \tilde{K}(S^{9-p}). \tag{5.49}$$

Notice that the previous result

$$\tilde{K}(S^{9-p}) = \begin{cases} \mathbb{Z} & \text{when } p \text{ is odd,} \\ 0 & \text{when } p \text{ is even,} \end{cases} \tag{5.50}$$

exactly matches the condition that there are only type IIB Dp- branes for p is odd.

Let B^{9-p} be the unit ball in transverse space, and S^{8-p} its boundary. As $B^{9-p}/S^{8-p} = S^{9-p}$ we have

$$K(B^{9-p}, S^{8-p}) = \tilde{K}(S^{9-p}). \tag{5.51}$$

The elements of the relative K-group are triples $[E, F, \alpha]$ with E and F bundles on B^{9-p} and α an isomorphism between E and F restricted to S^{8-p}. All bundles on the contractible B^{9-p}, and thus E and F, are trivial. All information in the element $[E, F, \alpha]$ lies in the winding of α around the origin in transverse space. The previous paragraph showed how the ABS-construction gives an explicit generator of $K(B^{9-p}, S^{8-p}) = \mathbb{Z}$. Let $W^{\pm}$ be the two inequivalent irreducible complex representations of $Spin(9-p)$. The generator of $K(B^{9-p}, S^{8-p})$ is given by $[E, F, \alpha]$ with E and F the trivial vector bundles $B^{9-p} \times W^{+}$ and $B^{9-p} \times W^{-}$ and α given by

$$\alpha : E \to F; (x, w) \mapsto \sum_{i=1}^{9-p} x_i \Gamma^i \cdot w. \tag{5.52}$$

This map gives an isomorphism between E and F everywhere except at the origin. It is not homotopic to the isomorphism $E \to F$ defined on all of B^{9-p}. Instead it has a winding around the origin and thus generates $K(B^{9-p}, S^{8-p}) = \tilde{K}(S^{9-p})$.

This description of $\tilde{K}(S^{9-p})$ is remarkably similar to the winding of the tachyon around a Dp-brane on the D9-D$\bar{9}$-system. The vector bundle in which the complex tachyon field T takes its values is $E \otimes F^*$, as it is positively charged under A and negative under $\bar{A}$, and thus the connection of the bundle in which it takes values should be $A \otimes 1 - 1 \otimes \bar{A}$. A section of this bundle can be seen as a bundle map

$$T : F \to E \tag{5.53}$$

that maps each fiber of F linearly on the corresponding fiber of E. Away from the Dp-brane where $E \cong F$ this is an isomorphism, which makes it look like the vacuum. However around the Dp-brane it makes a topological nontrivial twist. The possible windings of the isomorphism over the sphere S^{8-p} in transverse space, are generated by $\alpha^{-1} : F \to E$. In fact it can be extended to the whole transverse space $\mathbb{R}^{9-p}/\{0\}$. The tachyon vortex solution is homotopic with this α^{-1}. It can be written as

$$T(x) = f(\|\vec{x}\|) \sum_{i=1}^{9-p} x_i \Gamma^i, \tag{5.54}$$

where $f(\|\vec{x}\|)$ is a function that is equal to 1 around the origin, and $f(\|\vec{x}\|) = T_0/x$ for $x \to \infty$, such that T condensates to its vacuum expectation value at infinity. In fact we now have taken $T = \alpha$, but since $\alpha^2 = -\|x\|^2 = -1$ this only means we have taken the inverse generator.
The element $[E, F, \alpha] \in K(B^{9-p}, S^{8-p})$ describes a D9-D$\bar{9}$-system which condensates to a single Dp-brane. Taking a multiple of $[E, F, \alpha]$

$$[E \oplus E \oplus ..., F \oplus F \oplus ..., \alpha \oplus \alpha \oplus ...] \tag{5.55}$$

we get the other elements of $K(B^{9-p}, S^{8-p})$. This is equivalent by starting with a multiple of the D9-D$\bar{9}$-system in which a single Dp-brane arises, thus creating a multiple of Dp-branes. We can also make anti(D$\bar{p}$)-branes by reversing the role of E and F. In this way the K-group $\tilde{K}(S^{9-p})$, describing first the possible configurations of D9-D$\bar{9}$-branes on $\mathbb{R}^{10}$ with compact support in $9-p$ directions, now describes a Dp-D$\bar{p}$ system on $\mathbb{R}^{p+1}$.

5.7 D-branes and the Thom isomorphism

Now let us consider a Dp-brane wrapped on a submanifold Y of the space-time manifold X. As before, we do not want the vacuum-at-infinity restriction in the noncompact directions of Y. For a Dp-brane wrapped on $Y \times \mathbb{R}^k$ with Y a $p+1-k$-dimensional compact manifold, we restrict our attention to Y, as again $Y \times \mathbb{R}^k$ is homotopic to $\mathbb{R}^k$ and thus has the same vector bundles on it. So in the following let Y be compact. The normal bundle $N_{Y,X}$ of Y in X is the $9-p$-dimensional vector bundle, such that $TY \oplus N_{Y,X}$ is equal to the restriction of TX as a vector bundle to Y. The local coordinates in $U_i \times \mathbb{R}^{9-p}$ of $N_{Y,X}$, with U_i open in Y, also give local coordinates of a neighborhood of the submanifold Y in X. Let $B(N_{Y,X})$ be the ball bundle of $N_{Y,X}$, the fiber bundle on Y formed by restricting each fiber $\mathbb{R}^{9-p}$ of $N_{Y,X}$ to the ball B^{9-p}. This $B(N_{Y,X})$ gives, by the above identification of $N_{Y,X}$ as a local coordinate neighborhood of Y in X, a so called tubular neighborhood of Y in X. The sphere bundle $S(N_{Y,X})$ is $N_{Y,X}$ with each fiber restricted to the sphere S^{8-p}. It is the boundary of the tubular neighborhood. If the transverse space of the Dp-brane is of the form $\mathbb{R}^{9-p}$, by a generalization of the situation of a Dp-brane wrapped on $\mathbb{R}^{p+1} \subset \mathbb{R}^{10}$, the possible configurations of the D9-D$\bar{9}$ system with a compact support condition in the transverse directions of Y are elements of

$$K(B(N_{Y,X}), S(N_{Y,X})) \tag{5.56}$$

The Dp-brane arising in the brane/anti-brane system on X is thus described by a pair of vector bundles in a tubular neighborhood of Y and an isomorphism at the boundary of this neighborhood, which might have a topological twist. This again has the interpretation of a tachyon field winding around the Dp-brane. As $B(N_{Y,X})/S(N_{Y,X})$ is the one point compactification of $N_{Y,X}$ we have

$$K(B(N_{Y,X}), S(N_{Y,X})) = K(N_{Y,X}) \tag{5.57}$$

similar to $K(B^{9-p}, S^{8-p}) = K(\mathbb{R}^{9-p}) = \tilde{K}(S^{9-p})$ in the $\mathbb{R}^{p+1} \subset \mathbb{R}^{10}$-case. By the *Thom isomorphism* for every vector bundle V over a compact space Y $K(V) \cong K(Y)$. Thus

$$K(B(N_{Y,X}), S(N_{Y,X})) = K(N_{Y,X}) = K(Y). \tag{5.58}$$

It is instructive to look in more detail how this isomorphism is formed. It is similar to a construction in cohomology where there is an isomorphism between the compact supported cohomology $H^*_c(Y)$ and the cohomology $H^*_{cv}(V)$ with compact support along the fiber of V. There the isomorphism is given by

$$\lambda \mapsto \pi^*(\lambda) \wedge \Phi(V) \tag{5.59}$$

for $\lambda \in H^*_c(Y)$, with $\pi : V \to Y$ the projection to the base, and $\Phi(V)$ the Thom class in $H^*_{cv}(V)$. Similarly for the bundle $N_{Y,X}$ one can define a Thom class in $K(B(N_{Y,X}), S(N_{Y,X}))$. The construction of it follows naturally from a generalization of the ABS-construction. There we have in fact $Y = \text{pt}$ such that all bundles over Y are trivial. The element $[E, F, \alpha] \in K(B^{9-p}, S^{8-p})$ given above, explicitly gives an isomorphism

$$K(\text{pt}) \to K(B^{9-p}, S^{8-p}); \quad n \mapsto n \cdot [E, F, \alpha] \tag{5.60}$$

with $n \in \mathbb{Z} = K(\text{pt})$. By $I^n \otimes E = E \oplus E \oplus ..E$ (n times), with I^n the rank n trivial bundle over B^{9-p}, we get

$$n \cdot [E, F, \alpha] = [I^n] \otimes [E, F, \alpha] = [I^n \otimes E, I^n \otimes F, 1 \otimes \alpha] \tag{5.61}$$

for $n \geq 0$ and similarly $[I^n, I^m] \otimes [E, F, \alpha] = (n - m) \cdot [E, F, \alpha]$. Thus the isomorphism is given by

$$K(\text{pt}) \to K(B^{9-p}, S^{8-p}); \quad [G] \mapsto \pi^*[G] \otimes [E, F, \alpha] \tag{5.62}$$

where $\pi*$ is the pull back under $\pi : B^{9-p} \to \text{pt}$, mapping the trivial bundle on B^{9-p} to that on pt. $[E, F, \alpha]$ is the Thom class of $\mathbb{R}^{9-p}$ considered as a vector bundle on pt.

The generalization for $Y \neq \text{pt}$ goes as follows. First of all $N_{Y,X}$ is oriented because both Y and X are. Therefore the local coordinate transformations h_{ij} of the fibers between the coordinates of U_i and U_j, can be chosen to be $SO(9-p)$-functions. We can choose $Spin(9-p)$-functions $\tilde{h}_{ij}$ such that $\tilde{h}_{ij}$ maps to h_{ij} by the standard 2 to 1 covering $Spin(9-p) \to SO(9-p)$. This lifting is not unique, because $\mathbf{1} \in SO(9-p)$ can be lifted to both $\mathbf{1}$ and $-\mathbf{1}$ in $Spin(9-p)$. By definition of a vector bundle the h_{ij} satisfy the cocycle condition

$$h_{ij} h_{jk} h_{ki} = 1. \tag{5.63}$$

If the h_{ij} of a vector bundle can be lifted to $Spin$-functions $\tilde{h}_{ij}$ such that the $\tilde{h}_{ij}$ also satisfy the cocycle condition, we call it a spin bundle. Assume that $N_{Y,X}$ is a spin bundle with $Spin(9-p)$-functions $\tilde{h}_{ij}$. Let W^+ and W^- be the two inequivalent irreducible complex representations of $Spin(9-p)$ for p even. Let $S^{\pm}$ be the spinor bundles on Y of which the fibers are $W^{\pm}$ and for which the local coordinate transformations of the spinors are given by the action of $\tilde{h}_{ij}$ in $W^{\pm}$. The Thom class $\Phi(N_{Y,X})$ of the vector bundle $\pi : N_{Y,X} \to Y$ is given by

$$\Phi(N_{Y,X}) = [\pi^* S^+, \pi^* S^-, \alpha] \in K(B(N_{Y,X}), S(N_{Y,X})) \tag{5.64}$$

The Thom isomorphism $K(Y) \to K(B(N_{Y,X}), S(N_{Y,X}))$ is then given by

$$[G] \mapsto \pi^*[G] \otimes \Phi(N_{Y,X}) = [\pi^*(G \otimes S^+), \pi^*(G \otimes S^-), \alpha]. \tag{5.65}$$

Like in the $Y =$ pt-case, the Thom class generates the possible windings of the tachyon around the Dp brane wrapped on Y. Again we can make multiple Dp-branes and also D$\bar{p}$-branes. However now the Dp-branes wrapped on Y might have other than trivial vector bundles. The Dp-D$\bar{p}$ branes system on Y is classified by $K(Y)$. By the Thom isomorphism the possible configurations of a D9-D$\bar{9}$ system with compact support conditions in $9-p$ directions transverse to Y measured in $K(B(N_{Y,X}), S(N_{Y,X}))$, is equal to the possible Dp-D$\bar{p}$ configurations wrapped on Y, that will arise on it.
There is one more generalization we have to make. Until now we have assumed that the transverse space of Y is of the form $\mathbb{R}^{9-p}$, so that in fact $B(N_{Y,X})$ extends in all transverse directions. Suppose it is not. For example it might be compact. We still have the isomorphism

$$K(Y) \cong K(B(N_{Y,X}), S(N_{Y,X})) \tag{5.66}$$

between the brane/anti-brane configurations on Y and the configurations of the D9-D$\bar{9}$ system in a tubular neighborhood of Y in X. It might be that vector bundles on the tubular neighborhood do not extend to the whole of X. However if $[E, F, \alpha] \in B(B(N_{Y,X}), S(N_{Y,X}))$, we can choose $H \in Vect(B(N_{Y,X}))$ such that $F \oplus H$ is trivial on $B(N_{Y,X})$. By the isomorphism α on $S(N_{Y,X})$ also $E \oplus H$ is trivial at $S(N_{Y,X})$. Now we can extend $[E, F, \alpha] = [E \oplus H, F \oplus H, \alpha \oplus 1]$ to the whole of X, by trivially extending $E \oplus H$ and $F \oplus H$ and α to X. Basically this is done using a map $r : X \to B(N_{Y,X})/S(N_{Y,X}) = N^+_{Y,X}$ that is the identity at $B(N_{Y,X})$ and maps all points in X outside this region to $\infty \in N^+_{Y,X}$. This map is continuous and if X is noncompact it maps $\infty \in X^+$ to $\infty \in N^+_{Y,X}$. Therefore we can pull back elements of $K(X) = \tilde{K}(X^+)$ to $\tilde{K}(N^+_{Y,X}) = K(B(N_{Y,X}), S(N_{Y,X}))$. The composition of the Thom isomorphism $K(Y) \to K(B(N_{Y,X}), S(N_{Y,X}))$ and the pull back $r^* K(B(N_{Y,X}), S(N_{Y,X}))$ is called a Gysin homomorphism. If f is the embedding of Y in X, we denote it by f_*. It has the following nice properties. If f is an embedding of Y in X and g an embedding of Z in Y. The Gysin homomorphism $(f \circ g)_*$ is equal to $f_* \circ g_*$. Furthermore for $[E, F] \in K(Y)$ and $[E', F'] \in K(X)$ we have

$$f_*([E, F] \otimes f^*([E', F'])) = f_*([E, F]) \otimes [E', F'] \tag{5.67}$$

By these properties it becomes clear that nothing is lost by describing a Dp-D$\bar{p}$ branes system, in the K-theory of the D9-D$\bar{9}$ system. The description of the tachyon twist in the previous chapter, only showed the existence of D$p-2$-branes on a Dp-D$\bar{p}$ system. By the taking multiples of this system, one gets multiple D$(p-2)$-branes, and one can also make D$\overline{(p-2)}$-branes by twisting the tachyon the other way around. In this D$(p-2)$-D$\overline{(p-2)}$-branes system one can get D$(p-4)$-branes and so on. Thus any D$(p-2k)$-brane should already be present in the description of the Dp-D$\bar{p}$ system and thus any Dp-brane in the D9-D$\bar{9}$ system. The properties of the Gysin homomorphism show that this "all at once" description is equivalent to the step-by-step description, starting with D9 and D$\bar{9}$ branes, then D7 and D$\bar{7}$-branes until we reach the Dp-brane. Thus taking into account the right boundary conditions, every Dp-brane is already present in the K-group $K(X)$ of the space-time manifold X describing the space filling D9-D$\bar{9}$ systems.
The only restriction left is that we required that the manifold Y on which the D-branes wrap should be a spin manifold. If Y is not a spin manifold the Thom class of it is not well defined, as the lifted $\tilde{h}_{ij}$ do not make a proper vector bundle of S^+ and S^-. Sometimes it is possible to make a vector bundle G on Y which has exactly the same problem of a not closing cocycle, such that the product $G \otimes S^{\pm}$ is a well defined vector bundle. By (5.65) this thus gives a well defined element of the K-group of the tubular neighborhood of Y. This last construction is called a *spin*$^{\mathbb{C}}$-structure. In the next chapter we will see that by an anomaly of the path-integral it is only possible to wrap D-branes on manifolds with a spin or a spin$^{\mathbb{C}}$-structure. This condition is thus exactly matched in K-theory.

5.8 K-theory for type I and type IIA branes

Thus far we have only considered branes/anti-branes systems for type IIB theory. The possible charges in this system were measured by $K(X)$ with suitable boundary conditions in the noncompact directions. The fact that there are only Dp brane charges for p odd is exactly matched by the Bott-periodicity, and the fact that $\tilde{K}(S^{9-p}) = K^{p-9}(\mathrm{pt})$ is trivial for p even and $\mathbb{Z}$ for p odd.
One can now guess how to describe type IIA Dp-D$\bar{p}$ systems in K-theory. The possible charges are now given by $K^{-1}(X)$. The fact that there are now only even Dp-branes is reflected by the fact that

$$K^{-1}(S^{9-p}) = \tilde{K}(S^{10-p}) = K^{p-10}(\mathrm{pt}) \tag{5.68}$$

is $\mathbb{Z}$ for p even and trivial for p odd. More arguments, using the unstable non-BPS D9-branes of type IIA, that these configuration should indeed be measured in $K^{-1}(X)$ can be found in [10]. All other constructions shown above apply in a similar way.
There is an interesting equivalent definition of $K^{-1}(X)$. With X still the 10-dimensional space-time, let i be the embedding of X as $X \times \mathrm{pt}$ in the direct product $X \times S^1$. Then $K^{-1}(X)$ can be defined as the subspace of $\tilde{K}(X)$ formed

by the pull-back of $\tilde{K}(X \times S^1)$ under i. This suggests that the K-theory description of type IIA-branes, comes from considering an 11-dimensional space $X \times S^1$. As type IIA theory is the compactification on a circle of M-theory, this hints at a possible description of M-theory in K-theory. This description is not that straightforward, as for example the branes in M-theory do not carry Chan Paton bundles. Some progress has been made in [6], where it is shown how some aspects of a K-theoretic description of R-R charges can be derived from M-theory by a detailed study of its partition function.
Finally type I branes can be described in $KO(X)$, the Grothendieck group of real vector bundles. This follows naturally from the fact that by the orientifold projection, the gauge bundle in which the tachyon takes its values becomes real. The analysis of type I-branes in $KO(X)$ is quite similar to that of type II-branes in $K^\bullet(X)$. The Bott-periodicity of the higher KO-groups is less simple. Bott-periodicity can be derived by describing the higher K-groups using the same periodicity in the K-group of spheres. For instance

$$\begin{aligned} K^{-n}(\mathrm{pt}) &= K(\mathbb{R}^n) = \tilde{K}(S^n), \\ KO^{-n}(\mathrm{pt}) &= KO(\mathbb{R}^n) = \tilde{KO}(S^n). \end{aligned} \tag{5.69}$$

These groups can be calculated using the ABS-construction. For the KO-groups however, one has to use the real Clifford modules. These have a 8-periodicity, instead of the 2-periodicity of complex Clifford modules. The result is

D-brane	$D9$	$D8$	$D7$	$D6$	$D5$	$D4$	$D3$	$D2$	$D1$	$D0$	$D(-1)$
Transverse space	S^0	S^1	S^2	S^3	S^4	S^5	S^6	S^7	S^8	S^9	S^{10}
$\tilde{KO}(S^n)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

We find back the BPS D9, D5 and D1-brane. The $\mathbb{Z}_2$ charges are stable, but non-BPS branes. The D0-particle was previously known, but the D8-brane, the D7-brane and the D(-1)-instanton are new discoveries of K-theory.

6 Nontrivial B-field and twisted K-theory

Although not explicitly mentioned, the analysis in the previous chapter is only valid for trivial NS-NS B-field, i.e. $H = dB = 0$. The gauge field A on the D-branes can no longer be interpreted as the connection of a standard vector bundle if the B-field is nontrivial. A solution to this problem was already mentioned in [23] for flat B-fields, i.e. B-fields for which the de Rham cohomology class $[H] \in H^3(M, \mathbb{R})$ of $H = dB$ vanishes but for which $[H]$ does not necessarily vanish in the integer cohomology $H^3(M, \mathbb{Z})$.

This chapter gives a review of this article, trying to give some more details in the calculations and arguments involved. The calculations in this chapter are done using various techniques in algebraic topology. The necessary details of this theory are given in the appendix. More information can be found in the references at beginning of it.

6.1 The factors

Because of the use of algebraic topology, it is useful to describe our physical system in a more strict mathematical way. We consider Type II superstring theory on a space-time M, which is an oriented spin manifold, and a (multiple of) D-brane(s) stretched on an oriented submanifold $Q \subset M$. Q is oriented because it is charged under the R-R fields. The embedding of the string in space-time is a map ϕ from the worldsheet Σ to M that maps $\partial\Sigma$ on Q, as open string boundaries are only allowed at the D-brane.
The relevant factors in the path integral for the anomaly calculation are

$$\text{pfaff}(D) \cdot \exp\left(\mathrm{i}\int_\Sigma \phi_* B\right) \cdot \operatorname{Tr}\operatorname{Hol}_{\partial\Sigma}(A). \tag{6.1}$$

Here pfaff(D) is the pfaffian, the square root of the determinant, of the world-sheet Dirac operator D. This factor arises after integrating out the ψ fields in the path integral

$$\text{pfaff}(D) \sim \int \mathcal{D}\psi\mathcal{D}\tilde{\psi}\exp\left(-\frac{1}{4\pi}\int d^2z G_{\mu\nu}\left[\psi^\mu\bar{\partial}\psi^\nu + \tilde{\psi}^\mu\partial\tilde{\psi}^\nu\right]\right). \tag{6.2}$$

B is the 2-form field from the NS-NS sector. We have used the mathematical notation $\phi^* B$ for the pull-back of the space-time 2-form B to the worldsheet

$$(\phi^* B)_{ab} = \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}. \tag{6.3}$$

A is the $U(n)$ gauge field on the n D-branes. Given the background fields $G_{\mu\nu}$, A and B, the three factors are functions on the space

$$X := \text{Map}(\Sigma, M) \times \text{Met}(\Sigma) \tag{6.4}$$

of maps $\phi : \Sigma \to M$ and metrics g on the worldsheet. Actually we will see that these factors are not always well defined, but must be considered as sections of a line bundle over X. In the next sections we will study each factor separately.

6.2 The A-field, line bundles revisited.

The arguments in this chapter often make use of line bundle constructions, such as that on a single D-brane with $U(1)$-gauge field A. Furthermore the physics of the B-field can be best understood by a generalization of the physics of the A-field. Therefore we will give an overview of the arguments involved there.
For simplicity we first restrict to $U(1)$-gauge fields. The physical state conditions for the A-field, give that a free photon A^μ with momentum k^μ, is physically equivalent to a photon $A^\mu + ck^\mu$ with the same momentum. By the correspondence between strings and background-fields, the photon state becomes a Fourier mode of the background field A. The equivalence relation then translates into the equivalence relation for 1-form fields A to $A \sim A + d\lambda$, for any function λ. As we want to have a local field theory on Q , this means that if we cover Q

with a collection of open sets $\{U_i\}$ we can choose different 1-forms A_i on every U_i as long as on the intersections $U_{ij} = U_i \cap U_j$ we can write

$$A_i - A_j = dg_{ij} \tag{6.5}$$

with g_{ij} real functions on U_{ij}. The field strength $F = dA$ is globally defined, i.e. on all of Q, because $dA_i - dA_j = d^2 g_{ij} = 0$. The equation $dF = 0$ follows directly. A good covering of Q is a covering by open sets U_i such that all U_i, U_{ij}, $U_{ijk} = U_i \cap U_j \cap U_k, \ldots$ are contractible. Given a good cover one can show that for any F with $dF = 0$, there are A_i such that $F = dA_i$ on U_i and $A_j - A_i = dg_{ij}$ on U_{ij}. There is however a restriction we have to put on the possible field strengths, in order to make the holonomy well defined. The holonomy of A over a path γ is the phase a point-like object charged under A picks up going around the path. Locally, that is for a path $\gamma \subset U_i$, it is given by

$$\mathrm{Hol}_\gamma(A_i) = \exp(\mathrm{i} \int_\gamma A_i). \tag{6.6}$$

In our situation the endpoints of the open strings are charged under A. This is expressed in the fact that the same factor $\mathrm{Hol}_\gamma(A_i)$ is present in the string path integral. This expression gives a contribution to the generator of translations of the endpoints (like in (4.17)), which gives the state of an open string whose endpoint is transported along γ the same phase $\mathrm{Hol}_(\gamma)(A_i)$. The expression is only gauge invariant, i.e. under $A_i \to A_i + d\lambda_i$, if γ is a closed loop.
For a path going through more than one U_i, the only way to make the expression gauge invariant for closed loops, is cutting up the path in r pieces $\gamma_k \subset U_{i_k}, 1 \leq k \leq r$. Then the expression is

$$\mathrm{Hol}_\gamma(A) = \exp(\mathrm{i}\sum_{k=1}^{r} \int_{\gamma_k} A_{i_k} + \mathrm{i}\sum_{k=2}^{r} g_{i_{k-1}i_k}(v_k) + \mathrm{i}g_{i_r i_1}(v_1)) \tag{6.7}$$

where v_k are the points on γ connecting γ_{k-1} and γ_k. v_1 connects γ_r with γ_1. We have to check that this definition does not depend on the way we cut up the path in pieces, i.e. on the choice of A_i on every path piece. Suppose for instance we have a path that lies on a sphere, and divides it in a upper and lower hemisphere. Suppose further that the upper hemisphere lies in U_i with gauge field A_i, and the lower hemisphere in U_j. For the calculation of the holonomy we can choose to use A_i on the entire path or A_j. By the same argument as in §4.6 (4.46)-(4.48), this means that the integral of F over the entire sphere has to be an integer multiple of 2π. This can be imposed by requiring that $F/2\pi$ lies in a cohomology class $[F/2\pi] \in H^2(Q, \mathbb{Z})$. It can be computed in the following way. Define constants c_{ijk} on every U_{ijk} by

$$g_{ij} + g_{jk} + g_{ki} = 2\pi c_{ijk}. \tag{6.8}$$

The fact that the c_{ijk} are constants, can be seen by taking the derivative of the left-hand side combined with $dg_{ij} = A_i - A_j$. The c_{ijk} define a class in Čech

cohomology, called the first Chern class. The condition for F to have an integral that is an integer multiple of 2π, is exactly that the c-constants are integer, that is $[c] \in H^2(Q, \mathbb{Z})$.
In mathematical terms, this defines a line bundle with connection. A line bundle is a structure which can locally be seen as $U_i \times \mathbb{C}$. The real functions g_{ij} can be used to form $U(1)$-functions $h_{ij} = \exp(\mathrm{i} g_{ij})$ that translate between the local trivializations, in the following manner

$$U_i \times \mathbb{C} \to U_j \times \mathbb{C}; (x, z) \to (x, h_{ij} z) \tag{6.9}$$

This structure is only a line bundle if the c_{ijk} are integer. Then the h_{ij} satisfy the cocycle relation

$$h_{ij} h_{jk} h_{ki} = 1, \quad \text{on every } U_{ijk} \tag{6.10}$$

A section s of a line bundle is an object, which can locally be described as functions $s_i : U_i \to \mathbb{C}$, with transformation property $s_j = h_{ij} s_i$. The connection induced by the A_i is a derivative on sections. $\nabla_A s$ is the product of a 1-form and a section, which is locally described by

$$(\nabla s)_i = \partial s_i - \mathrm{i} A_i s_i \tag{6.11}$$

The phase a section s which satisfies $\nabla_A s = 0$, picks up under going around a path is given by $\mathrm{Hol}_\gamma(A)$.
If we have a multiple of D-branes on top of each other, there are gauge fields A_{kl} for every open string going from the k-th brane to the l-th brane. If there are n branes, the A_{kl} form a n by n symmetric matrix that gives a $U(n)$-connection in a complex vector bundle of rank n. It is actually the product of a symmetric-matrix valued function and a 1-form. The g_{ij} defined in the usual way, are then symmetric-n-by-n-matrix valued functions, and the $h_{ij} = \exp(i g_{ij})$ are therefore $U(n)$-matrices. They give the transformation between the local trivializations $U_i \times \mathbb{C}^n$ and $U_j \times \mathbb{C}^n$ of the vector bundle. The condition for this to be actually a line bundle is again the cocycle condition for h_{ij}. The holonomy in this vector bundle induced by A is a little more complicated. The parallel transport over a path in U_i, $\gamma : [t_b, t_e] \to U_i$, is the solution $M : [t_b, t_e] \to U(n)$ of the differential equation

$$\frac{d}{dt} M(t) = \mathrm{i} A_i(\gamma(t)) M(t), \quad M(t_b) = I \tag{6.12}$$

For a closed path $\gamma \subset U_i$ this defines the holonomy over γ. A solution is given by

$$\begin{aligned} \mathrm{Hol}_\gamma(A_i) = P \exp(\mathrm{i} \int_\gamma A_i) = 1 + \mathrm{i} \int_{t=t_b}^{t_e} A_i(t) \cdot \gamma'(t) dt - \int_{t_b}^{t_e} \int_{t_1}^{t_e} A_i(t_2) \cdot \gamma'(t_2) A_i(t_1) \cdot \gamma'(t_1) dt_2 dt_1 \\ -\mathrm{i} \int_{t_b}^{t_e} \int_{t_1}^{t_e} \int_{t_1}^{t_e} A_i(t_3) \cdot \gamma'(t_3) A_i(t_2) \cdot \gamma'(t_2) A_i(t_1) \gamma'(t_1) dt_3 dt_2 dt_2 \end{aligned} \tag{6.13}$$

The P before the exp stands for path-ordered, which is necessary to make this a solution. An expression for the holonomy over a path going through more than one local coordinate region U_i, can be constructed in the above way by patching together this expression for different path pieces and multiplying with h_{ij} on every transition.

6.3 The pfaffian

The pfaffian of the Dirac world-sheet operator is a real number whose absolute value is well defined, but its sign is not if Q does not admit a spin structure. First we will give a mathematical condition which tells exactly whether a manifold admits a spin structure or not.

Consider the tangent bundle TQ. It is an oriented real vector bundle over Q. Therefore the h_{ij} cocycle of transformations between the trivializations of the tangent bundle, can be taken to be $SO(n)$-valued functions on every U_{ij} ($n = \dim Q$). We say that Q admits a spin structure if the h_{ij} can be lifted to a cocycle $\tilde{h}_{ij}$ of $Spin(n)$-functions, such that $\tilde{h}_{ij} \mapsto h_{ij}$ by the standard 2 to 1 covering map. This is not always possible, since $\mathbf{1} \in SO(n)$ can be lifted to both $\mathbf{1}$ and $\mathbf{-1}$ in $Spin(n)$. Therefore the cocycle condition of $\tilde{h}_{ij}$ changes by

$$h_{ij}h_{jk}h_{ki} = 1, \quad \Longrightarrow \quad \tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki} = (-1)^{w_{ijk}}, \tag{6.14}$$

with w_{ijk} a cocycle of constants in $\mathbb{Z}_2 = \{0,1\}$. It defines a class $w_2(Q) = [w] \in H^2(Q, \mathbb{Z}_2)$ called the second Stiefel-Whitney class of Q. In exact sequences we have

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{(-1)^{\cdot}} Spin(n) \longrightarrow SO(n) \longrightarrow 1 \tag{6.15}$$

which induces the exact sequence

$$\cdots \longrightarrow H^1(Q, Spin(n)) \longrightarrow H^1(Q, SO(n)) \longrightarrow H^2(Q, \mathbb{Z}_2) \longrightarrow \cdots \tag{6.16}$$

So we see that $[h_{ij}] \in H^1(Q, SO(n))$ can be lifted to a class $[\tilde{h}_{ij}] \in H^1(Q, Spin(n))$ if and only if $w_2(Q) = 0$. Thus Q admits a spin structure if and only if $w_2(Q) = 0$.

In [7] it was shown that if we deform the embeddings of Σ in a continuous way by a one-parameter family of embeddings that comes back to itself, so we have a circle S^1 and a map $\Gamma : S^1 \times \Sigma \mapsto M$ with $\Gamma(S^1 \times \partial\Sigma) \subset Q$, that under going around the circle pfaff(D) may pick up a change of sign by

$$\mathrm{pfaff}(D) \to (-1)^{\alpha}\, \mathrm{pfaff}(D) \tag{6.17}$$

with

$$\alpha = \int_{S^1 \times \partial\Sigma} \Gamma^* w_2(Q). \tag{6.18}$$

This is an integral of a $\mathbb{Z}_2$-valued cohomology class. Therefore the expression has to be understood as an integral over a homology class. For some details of this theory, see the appendices. With the standard notation $(.,.)$ for the integral of a cohomology class over a homology class, we get

$$\alpha = ([S^1 \times \partial\Sigma], \Gamma^* w_2(Q)) \tag{6.19}$$

and α takes values in $\mathbb{Z}_2$. Actually Γ should be its restriction to the worldsheet boundary, as $w_2(Q)$ is a class in $H^2(Q,\mathbb{Z}_2)$. So

$$\Gamma : S^1 \times \partial\Sigma \to Q. \tag{6.20}$$

This map can be split in two. First f maps $(t,\sigma) \in S^1 \times \partial\Sigma$ to (γ_t, σ) with γ_t a path $\partial\Sigma \to Q$ defined by

$$\gamma_t(\sigma') = \Gamma(\sigma', t). \tag{6.21}$$

Let us assume that $\partial\Sigma$ is a single closed loop. Then γ_t is an element of LQ the space of all loops in Q. Now we can write

$$\Gamma(t,\sigma) = e \circ f(t,\sigma) = e(\gamma_t, \sigma) = \gamma_t(\sigma) \tag{6.22}$$

with $e : LQ \times S^1 \to Q$ (identifying $\partial\Sigma$ with S^1), the evaluation map

$$e(\gamma, \sigma) = \gamma(\sigma). \tag{6.23}$$

The transgression of Q to LQ is a map $\mathcal{T}$ from $H^q(Q,G)$ to $H^{q-1}(LQ,G)$ such that for all $s \in H_2(LQ,\mathbb{Z})$

$$(s, \mathcal{T}\lambda) = (s \times C, e^*\lambda) \tag{6.24}$$

with C the generator of $H_1(S^1, G)$, i.e. the homology class of S^1 itself. For a de Rham cohomology class λ in LQ it is given by integration of λ over "its own loop":

$$\mathcal{T}\lambda = \int_{S^1} e^*\lambda. \tag{6.25}$$

Now

$$\alpha = ([S^1 \times \partial\Sigma], (e \circ f)^* w_2) = (f_*[S^1 \times \partial\Sigma], e^* w_2). \tag{6.26}$$

The push forward f_* of $[S^1 \times \partial\Sigma]$ is the direct product of the loop $\tilde{\gamma} : t \mapsto \gamma_t$ through the loop space LQ, and $[\partial\Sigma] = C$. Thus

$$\alpha = ([\tilde{\gamma}] \times C, e^* w_2) = ([\tilde{\gamma}], \mathcal{T} w_2). \tag{6.27}$$

The use of this algebraic rewriting, is that we can see pfaff(D) as a section over a line bundle on LQ. The phase pfaff(D) picks up under going around a loop $\tilde{\gamma}$ through LQ, is the holonomy of the line bundle. The value of pfaff(D) is only locally defined, depending on the loop formed by the worldsheet boundary. Let the set $\{U_i\}$ be a covering of LQ and let $\tilde{w}_{ij}$ be the $\mathbb{Z}_2$-cocycle representing $\mathcal{T} w_2 = [\tilde{w}] \in H^2(LQ, \mathbb{Z}_2)$. The first Chern class $[c] \in H^2(LQ,\mathbb{Z})$ of the line bundle is given by $\mathbb{Z}$-constants c_{ijk} such that

$$\tilde{v}_{ij} + \tilde{v}_{jk} + \tilde{v}_{ki} = 2c_{ijk} \tag{6.28}$$

with $\tilde{v}_{ij}$ $\mathbb{Z}$-constants such that $\tilde{v}_{ij} = \tilde{w}_{ij} \mod 2$. The fact that this is the right expression for the first Chern class can be seen as follows. As in (6.8) take

$g_{ij} = \pi\tilde{v}_{ij}$, then the local coordinate transformations of the line bundle with first Chern class $[c]$ are given by $h_{ij} = \exp(\mathrm{i}g_{ij})$. Because the g_{ij} are constant, and $dg_{ij} = A_i - A_j$ we can take a vanishing connection A. The holonomy of this line bundle over a loop $\tilde{\gamma}$ is then given by

$$\begin{aligned} \mathrm{Hol}_{\tilde{\gamma}} &= ([\tilde{\gamma}], h) = \exp(-\mathrm{i}([\tilde{\gamma}], g_{ij})) = \\ &= \exp(\mathrm{i}\pi([\tilde{\gamma}, \tilde{v}])) = (-1)^{([\tilde{\gamma}], \mathcal{T}w_2)}. \end{aligned} \tag{6.29}$$

Here we used that the integral of the $U(1)$-cocycle h_{ij} over the homology cycle $[\tilde{\gamma}]$ corresponding to $\tilde{\gamma}$, is defined as the product the values of h_{ij} in the vertices in U_{ij}, i.e. the points connecting path pieces lying in U_i with path pieces of U_j. Filling in $h_{ij} = \exp(\mathrm{i}g_{ij})$ gives the second term which is defined in the same way, now with summation over the real values of g_{ij} in the vertices. This second term is then equal to the expression in the previous paragraph.
The right-hand side is the same as (6.17) and (6.27), thus checking the claim that the Chern class is given by $[c]$. This construction can be formalized using the following exact sequences.

$$\begin{gathered} 0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0 \quad \Longrightarrow \\ \cdots \longrightarrow H^1(LQ,\mathbb{Z}) \longrightarrow H^1(LQ,\mathbb{Z}_2) \xrightarrow{\tilde{\beta}} H^2(LQ,\mathbb{Z}) \longrightarrow \cdots \end{gathered} \tag{6.30}$$

Then we can write $[c] = \tilde{\beta}([\tilde{w}]) = \tilde{\beta}(\mathcal{T}w_2)$. There is a similar map β in the cohomology of Q.

$$\cdots \longrightarrow H^2(Q,\mathbb{Z}) \longrightarrow H^2(Q,\mathbb{Z}_2) \xrightarrow{\beta} H^3(LQ,\mathbb{Z}) \longrightarrow \cdots . \tag{6.31}$$

This map is usually called Bockstein homomorphism. It can be shown that the transgression homomorphism $\mathcal{T}$ intertwines β and $\tilde{\beta}$

$$\begin{tikzcd} \cdots \arrow[r] & H^2(Q,\mathbb{Z}) \arrow[r] & H^2(Q,\mathbb{Z}_2) \arrow[r, "\beta"] \arrow[d, "\mathcal{T}"] & H^3(Q,\mathbb{Z}) \arrow[r] \arrow[d, "\mathcal{T}"] & \cdots \\ \cdots \arrow[r] & H^1(LQ,\mathbb{Z}) \arrow[r] & H^1(LQ,\mathbb{Z}_2) \arrow[r, "\tilde{\beta}"] & H^2(LQ,\mathbb{Z}) \arrow[r] & \cdots \end{tikzcd} \tag{6.32}$$

There is a standard notation $W_3(Q) = \beta(w_2(Q))$. Thus

$$[c] = \tilde{\beta}(\mathcal{T}w_2) = \mathcal{T}\beta(w_2) = \mathcal{T}W_3(Q). \tag{6.33}$$

The factor pfaff(D) does not only depend on the worldsheet boundary, but more generally on $(\phi, g) \in X$ defined in (6.4). Let $\Psi : X \to LQ$ map the pair (ϕ, g) to loop formed by the restriction of ϕ to the worldsheet boundary. What we have shown is that pfaff(D) is a section of a line bundle over X whose first Chern class $[c] \in H^3(X,\mathbb{Z})$ is given by

$$[c] = \Psi^*\mathcal{T}W_3(Q). \tag{6.34}$$

The generalization to a worldsheet boundary consisting of more than one loop is straightforward. The result of taking each worldsheet boundary over a different cycle is still given by (6.17) with α now the sum $\alpha_1 + \alpha_2 + ..\alpha_r$, where α_i is the contribution of the i'th worldsheet boundary $\partial\Sigma_i$.

$$\alpha_i = ([S^1 \otimes \partial\Sigma_i], \Gamma^* w_2). \tag{6.35}$$

Here Γ is every time its restriction to $\partial\Sigma_i$. Mathematically pfaff(D) is the section of the line bundle over X, which is the pull-back under Ψ of a line bundle over $LQ^r = LQ \times LQ \times ..LQ$. Ψ is the map $X \to LQ^r$ giving the loops formed by each worldsheet boundary. The line bundle over LQ^r has first Chern class $\mathcal{T}W_3(Q) \oplus .. \mathcal{T}W_3(Q)$ (r equal terms). Although $W_3(Q)$ is $\mathbb{Z}_2$-valued, because β is a homomorphism $W_3(Q) + W_3(Q) = \beta(w_2(Q) + w_2(Q)) = 0$, if we have two loops forming the worldsheet boundary, the contributions of in the sign ambiguity do not necessarily cancel, as each $\mathcal{T}W_3(Q)$ is an element of a different copy of $H^2(LQ, \mathbb{Z})$ in the cohomology of LQ^r.

6.4 Spin$^{\mathbb{C}}$-structures

It seems that in order for the pfaff(D)-factor to be well defined we have to require that D-branes only wrap on spin manifolds. After all, the factors in the path integral have to be real functions, not just sections of a line bundle, to give unambiguous results in calculations. There is however an alternative option.
For this it is useful to take a closer look at spinor bundles of Q. As we explained the problem of lifting the h_{ij} $SO(n)$-functions of the tangent bundle TQ to $Spin(n)$-functions is that the cocycle changes to

$$\tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki} = \exp(\mathrm{i}\pi w_{ijk}). \tag{6.36}$$

As we saw before the cocycle closes only if $w_2(Q) = [w] \in H^2(Q, \mathbb{Z}_2)$ vanishes. If it does not, take v_{ijk} to be $\mathbb{Z}$-constants such that $w_{ijk} = v_{ijk} \mod 2$. If v_{ijk} is a cocycle, i.e. $v_{ijk} + v_{jkl} + v_{kli} + v_{lij} = 0$, we can find real functions g_{ij} such that

$$g_{ij} + g_{jk} + g_{ki} = \pi v_{ijk}. \tag{6.37}$$

Define the $U(1)$-functions $h_{ij} = \exp(\mathrm{i}g_{ij})$. The use of this is that it exactly separates the problem in the spinor bundle. Namely if we divide the $\tilde{h}_{ij}$ by h_{ij}. The cocycle condition becomes

$$\frac{\tilde{h}_{ij}}{h_{ij}} + \frac{\tilde{h}_{ij}}{h_{ij}} + \frac{\tilde{h}_{ij}}{h_{ij}} = \exp(\mathrm{i}\pi w_{ijk})\exp(-\mathrm{i}v_{ijk}) = 1. \tag{6.38}$$

The h_{ij} do not define a line bundle on Q because they also do not satisfy the cocycle condition. The first Chern class of it would be given by constants $c_{ijk} = v_{ijk}/2$, as can be seen by comparing (6.37) with (6.8). It is the square root of a bundle, in the sense that we can write

$$\mathcal{L} = \mathcal{L}^{\frac{1}{2}} \otimes \mathcal{L}^{\frac{1}{2}} \tag{6.39}$$

where $\mathcal{L}^{\frac{1}{2}}$ is the pseudo line bundle with local coordinate transformations h_{ij}, and $\mathcal{L}$ is a true line bundle with local coordinate transformations h_{ij}^2 and first Chern class $[v] \in H^2(Q, \mathbb{Z})$.
The problem with the pseudo line bundle $\mathcal{L}^{\frac{1}{2}}$ is that its holonomy is not well defined. If we take a loop γ, move it through Q such that it comes back to itself it sweeps out a surface S in Q. Then γ can be seen as the boundary of the beginning and the ending of S. However we may choose a different division in pieces for the beginning boundary and the end boundary. By methods explained in the appendices the holonomy calculated using both divisions differs by a factor $\exp(\mathrm{i}2\pi(S, [c]))$, the exponent of the integral of $[c] \in H^2(Q, \mathbb{R})$ over S. This calculation will be done later on more generally for even less standard line bundles. For $[c] \in H^2(Q, \mathbb{Z})$ the expression vanishes, thus verifying that the holonomy is well-defined for a first Chern class $[c]$ given by integers c_{ijk}. Now however, going around S the holonomy might pick up a change of sign. This problem sounds familiar. In fact the sign it picks up is given by

$$\exp(\mathrm{i}\pi(S, v)) = \exp(\mathrm{i}\pi(S, w)) = (-1)^{(S,w)} \tag{6.40}$$

Suppose γ forms the boundary of the string worldsheet. The moving around of the embedding of the worldsheet Σ through M, is again given by a map $\Gamma : S^1 \times \Sigma \to M$ such that Γ restricted to the worldsheet boundary is a map $S^1 \times \partial\Sigma \to Q$. $\Gamma(S^1 \times \Sigma)$ is the closed surface the boundary sweeps out through Q when moving and coming back to itself. With $S = \Gamma_*[S^1 \times \partial\Sigma]$ the homology cycle of $S^1 \times \partial\Sigma$ pushed forward to space-time, we get the same factor as in (6.17) combined with (6.27). The ambiguity in $\mathrm{pfaff}(D)$ is exactly the same as the ambiguity in the holonomy of the line bundle $\mathcal{L}^{\frac{1}{2}}$ around the worldsheet boundary!
Notice that the product of those two expressions cancel precisely. If on a single D-brane we take the line bundle formed by the gauge field A, to be equal to $\mathcal{L}^{\frac{1}{2}}$, the product of $\mathrm{pfaff}(D)$ and $\mathrm{Hol}_{\partial\Sigma}(A)$ is well-defined. In fact we can take any ordinary line bundle $\mathcal{M}$ on Q, or a vector bundle E in the case of multiple D-branes. The holonomy of ordinary line bundles and vector bundles is well-defined. As we remarked the holonomy of the product factors. Therefore the factor $\mathrm{Tr}\,\mathrm{Hol}_{\partial\Sigma}(A)$ gets the right sign ambiguity that cancels that of $\mathrm{pfaff}(D)$, if we take A to be the connection in the product $E \otimes \mathcal{L}^{\frac{1}{2}}$, with E any ordinary vector bundle.
Remember that this construction is only possible if the $v_{ijk} = w_{ijk} \mod 2$ form a cocycle. Only then there are g_{ij} such that (6.37) holds. The question is thus whether we can lift $w_2(Q) \in H^2(M, \mathbb{Z}_2)$ to a class $[v] \in H^2(M, \mathbb{Z})$. By the exact sequence (the same as (6.31))

$$\cdots \longrightarrow H^2(Q, \mathbb{Z}) \longrightarrow H^2(Q, \mathbb{Z}_2) \xrightarrow{\beta} H^3(LQ, \mathbb{Z}) \longrightarrow \cdots \tag{6.41}$$

this is only possible if $\beta(w_2(Q)) = W_3(Q) = 0$.
As we saw in (6.38) the problem in the "would-be" spinor bundle $S(Q)$ formed by local coordinate transformations $\tilde{h}_{ij}$, is solved by dividing $\tilde{h}_{ij}$ by the h_{ij} of

$\mathcal{L}^{\frac{1}{2}}$. This is the same as taking the direct product $S(Q)\otimes\mathcal{L}^{-\frac{1}{2}}$, where $\mathcal{L}^{-\frac{1}{2}}$ is the pseudo line bundle with local coordinate transformations h_{ij}^{-1}. This product is called a spin$^{\mathbb{C}}$-structure. It is the same structure as we encounter in K-theory. There the Thom class of the normal bundle N of Q, on which the D-brane wraps, in the space-time manifold M, can not be defined if the normal bundle did not have a spin structure. In the mean time we have have learned that this is if $w_2(N) \neq 0$. Furthermore the Stiefel-Whitney classes $w_i(X) \in H^i(X, \mathbb{Z}_2)$ can be shown to have the property

$$(1 + w_1(M) + w_2(M) + ..) = (1 + w_1(Q) + w_2(Q) + ..)(1 + w_1(N) + w_2(N) + ..)$$

for $TM = TQ \oplus N$, with the usual notation $w_i(X) = w_i(TX)$. Thus

$$\begin{aligned} w_1(M) &= w_1(Q) + w_1(N), \\ w_2(M) &= w_1(Q)\cdot w_1(N) + w_2(Q) + w_2(N). \end{aligned} \tag{6.42}$$

The first Stiefel-Whitney class $w_1(X)$ of a manifold X vanishes if X is oriented. Thus $w_1(M) = w_1(Q) = w_1(N) = 0$. Since the total space-time manifold M has a spin structure, this is a basic assumption of type II superstrings, $w_2(M) = 0$. Therefore $w_2(Q) = w_2(N)$ (remember that all Stiefel-Whitney classes are $\mathbb{Z}_2$-valued). Consequently the normal bundle N admits a spin structure if and only if Q does.

If $w_2(N) \neq 0$ it is still possible in K-theory to define an element in $K(M)$ that can be interpreted as the vector bundle of Dp-branes wrapped on Q. If we can find a pseudo vector bundle G on Q, such that the product $G\otimes S^{\pm}$, with $S^{\pm}$ the pseudo spinor bundles of N, is well defined, then this product can be extended in a tubular neighborhood of Q. The K-theory class of it can then be mapped by the Gysin homomorphism to give an element of $K(M)$. The pseudo vector bundle G can be constructed as the product of any ordinary vector bundle E on Q and the $\mathcal{L}^{-\frac{1}{2}}$ we constructed above. In this way the usual correspondence between $K(Q)$ and $K(B(N), S(N))$ by the Thom isomorphism(5.65) still holds. The Thom class however is now defined as

$$[\pi^*(S^+ \otimes \mathcal{L}^{-\frac{1}{2}}), \pi^*(S^- \otimes \mathcal{L}^{-\frac{1}{2}}), \alpha] \in K(B(N), S(N)). \tag{6.43}$$

This construction is of course only possible for $W_3(Q) = 0$. Thus the possible D-brane charges predicted by K-theory, are only those of D-branes wrapped on a manifold on which a spin$^{\mathbb{C}}$-structure is allowed. (The usual mathematical definition also declares any ordinary spin manifold to have a spin$^{\mathbb{C}}$-structure.) This exactly matches the condition for cancellation of the anomaly in the product of pfaff(D) and $\mathrm{Tr}\,\mathrm{Hol}_{\partial\Sigma}(A)$. It is a very nontrivial check of the K-theory/D-brane correspondence.

6.5 The B-field

For the B-field the physical state conditions for closed strings imply that there is a gauge freedom $B \sim B + d\mu$ for any 1-form μ. Just as for the A-field, this means that the B field is defined only locally. With $\{U_i\}$ a covering of M, we have different 2-forms B_i on every U_i, with on every intersection U_{ij}

$$B_i - B_j = d\Lambda_{ij} \tag{6.44}$$

The field strength H is globally defined, so for H is a 3-form on M with for every U_i: $H = dB_i$. If $\{U_i\}$ is a good covering we can also find real functions a_{ijk} on U_{ijk} and constants m_{ijkl} on U_{ijkl} such that we get the following diagram

$$\begin{array}{c|ccccc}
\Omega^3 & H & & & & \\
\Omega^2 & & B_i & & & \\
\Omega^1 & & & \Lambda_{ij} & & \\
\Omega^0 & & & & a_{ijk} & \\
\mathbb{R} & & & & & 2\pi m_{ijkl} \\
\hline
 & M & U_i & U_{ij} & U_{ijk} & U_{ijkl}
\end{array}
\qquad
\begin{array}{l}
H = dB_i \\
B_i - B_j = d\Lambda_{ij} \\
\Lambda_{ij} + \Lambda_{jk} + \Lambda_{ki} = da_{ijk} \\
a_{ijk} - a_{jkl} + a_{ikl} - a_{ijl} = 2\pi m_{ijkl}
\end{array} \tag{6.45}$$

This is the standard way to show the equivalence between classes in de Rham and Čech cohomology. Under this equivalence $[H]$ in the de Rham cohomology $H^3(M, \mathbb{R})$ corresponds to $[2\pi m]$ in Čech cohomology. We require $[m] \in H^3(M, \mathbb{Z})$, because if we move the worldsheet through space-time it sweeps out a 3 dimensional volume V with the worldsheet at the beginning and at the end as boundary. If those two worldsheets are embedded by maps ϕ and ϕ' the difference in the factor can be calculated by

$$\int_\Sigma \phi^* B - \int_\Sigma \phi'^* B = \int_V H \tag{6.46}$$

But if we move the worldsheet such that it comes back to itself, we don't want the factor to change. Therefore the integral of H over all closed volumes must be an integer multiple of 2π, i.e. $[m_{ijkl}] \in H^3(M, \mathbb{Z})$. This is again the argument of Dirac's quantization.
We have not been very precise yet how the integral of B is defined. The foregoing argument is only valid within one coordinate region U_i. Given an embedding $\phi : \Sigma \to M$ and a covering $\mathcal{U} = \{U_i\}$ we get a covering by $V_i = \{\phi^{-1}(U_i)\}$ of the worldsheet. It is always possible to make a good covering $\{U_i\}$ of space-time, such that $\{V_i\}$ is a good covering of Σ. We then make a triangulation of the worldsheet relative to this covering. This is a formal sum of triangles with each triangle entirely in at least one coordinate region V_i, such that the homology class of this sum is equal to that of the worldsheet. As in (A.25) this triangulation can be separated in triangles, edges of the triangles and vertices, the endpoints of the vertices.
$t^{(0)}_i$ are the triangles in each V_i,
$t^{(1)}_{ij}$ are the edges in each V_{ij},
$t^{(2)}_{ijk}$ are the vertices in V_{ijk}.

We now define the path integral factor to be

$$\exp(\mathrm{i}\phi^* B) := \exp(\mathrm{i}(t^{(0)}, \Phi^* B) + \mathrm{i}(t^{(1)}, \Phi^* \Lambda) - \mathrm{i}(t^{(2)}, \Phi^* a)). \tag{6.47}$$

Here every time a summation over the missing indices is assumed. This is a generalization of the expression of the holonomy of a line bundle. It is well defined on equivalence classes of quadruples $(B_i, \Lambda_{ij}, a_{ijk}, m_{ijkl})$ with equivalence relation

$$\begin{pmatrix} B_i \\ \Lambda_{ij} \\ a_{ijk} \\ m_{ijkl} \end{pmatrix} \sim \begin{pmatrix} B_i + d\mu_i \\ \Lambda_{ij} + \mu_i - \mu_j + d\rho_{ij} \\ a_{ijk} + \delta\rho_{ijk} - 2\pi\tilde{a}_{ijk} \\ m_{ijkl} + \delta\tilde{a}_{ijkl} \end{pmatrix} \tag{6.48}$$

for any $\mu \in \check{C}^0(M, \Omega^1), \rho \in \check{C}^1(M, \Omega^0)$ and $\tilde{a} \in \check{C}^2(M, \mathbb{Z})$. Notice that the equations in (6.45) are also invariant under this equivalence relation. This can be summarized in

$$\begin{array}{c|ccccc} \Omega^3 & H & & & & \\ \Omega^2 & & B_i & & & \\ \Omega^1 & & \mu_i & \Lambda_{ij} & & \\ \Omega^0 & & & \rho_{ij} & a_{ijk} & \\ \mathbb{Z} & & & & \tilde{a}_{ijk} & m_{ijkl} \\ \hline & M & U_i & U_{ij} & U_{ijk} & U_{ijkl} \end{array} \tag{6.49}$$

This is a generalization of the classification of line bundles with connection. These are classified by triples (A_i, g_{ij}, c_{ijk}) which satisfy the usual relations for line bundles, i.e.

$$[c] \in H^2(M, \mathbb{Z}), (\delta g)_{ijk} = c_{ijk}, \text{ and } A_i - A_j = dg_{ij}, \tag{6.50}$$

with equivalence relation

$$(A_i, g_{ij}, c_{ijk}) \sim (A_i + d\lambda_i, g_{ij} + \lambda_i - \lambda_j + dr_{ij}, c_{ijk} + (\delta r)_{ijk}) \tag{6.51}$$

for any real functions λ_i and $\mathbb{Z}$ constants r_{ij}. These classes are elements of $H^1(M, \mathbf{U}_1 \to \Omega^1)$[4]. This notation reflects the fact that we can also classify by pairs (A_i, h_{ij}) with A_i as before and $h_{ij} = \exp(\mathrm{i}g_{ij})$ a cocycle of $U(1)$-functions. Consider now equivalence classes $[(B, \Lambda, a, m)]$ of quadruples, satisfying the equations in (6.45) and with $[m] \in H^3(M, \mathbb{Z})$. The space of these equivalence classes is denoted by $H^2(M, \mathbf{U}_1 \to \Omega^1 \to \Omega^2)$. Suppose we have an equivalence class which is represented by (B, Λ, a, m) and suppose $[m] = 0$ in $H^3(M, \mathbb{Z})$. We can write $\mu = \delta\tilde{a}$ for some $\tilde{a} \in \check{C}^2(M, \mathbb{Z})$. Then $\delta(a + 2\pi\tilde{a}) = 0$ so there is a $\rho \in \check{C}^1(M, \Omega^0)$ with $\delta\rho = a + 2\pi\tilde{a}$. Because $d\delta\rho = \delta\Lambda$, we have $\delta(d\rho - \Lambda) = 0$ and thus a $\mu \in \check{C}^0(M, \Omega^1)$ such that $\delta\mu = d\rho - \Lambda$. Finally because $\delta(d\mu - B) = 0$, there is a $\tilde{B} \in \Omega^1(M)$ such that $\tilde{B} = B_i - d\mu_i$. To summarize:

$$\begin{aligned} m_{ijkl} &= \delta\tilde{a}_{ijkl}, \\ a_{ijk} &= -2\pi\tilde{a}_{ijk} + \delta\rho_{ijk}, \\ \Lambda_{ij} &= -\delta\mu_{ij} + d\rho_{ij}, \\ B_i &= B + d\mu_i. \end{aligned} \tag{6.52}$$

This shows that this quadruple is equivalent to $(B,0,0,0)$. In fact there is an exact sequence like the one for line bundles with connection in (A.47)

$$0 \longrightarrow H^2(M,\mathbb{R}) \longrightarrow H^2(M,\mathbf{U}_1 \to \Omega^1 \to \Omega^2) \longrightarrow H^3(M,\mathbb{Z}) \longrightarrow 0. \tag{6.53}$$

which means

$$H^2(M,\mathbf{U}_1 \to \Omega^1 \to \Omega^2) = H^3(M,\mathbb{Z}) \oplus H^2(M,\mathbb{R}) \tag{6.54}$$

In the case $(B,0,0,0)$ the definition (6.47) simplifies to

$$\exp(\mathrm{i}\phi^* B) = \exp(\mathrm{i}\int \phi^* B). \tag{6.55}$$

Very often one considers only backgrounds in which $[m] = [H]/2\pi$ vanishes in $H^3(M,\mathbb{Z})$ and then this is the right factor in the path integral. Because in general the definition of $\exp(\mathrm{i}\phi^* B)$ at least locally (i.e. in a contractible region where again we can write $H = dB$) gives the right factor, and it has the right gauge invariance $B \sim B + d\mu$, this should be the right factor in general.
A special case concerns the restriction to configurations in which $[m]$ and thus $[H]$ vanish in $H^3(M,\mathbb{R})$. This does not automatically imply that $[m]$ vanishes in $H^3(M,\mathbb{Z})$. Still we can write $m_{ijkl} = \delta\tilde{a}_{ijkl}$, but the $\tilde{a}$ need no longer be integer. Within an arbitrary quadruple equivalence class, the a_{ijk} can in general no longer be gauged to zero. But at least we can take them to be equal to these constants $\tilde{a}_{ijk}$. Still Λ can be gauged to zero and we can write $H = dB$ globally, so $B_i = B$ with one $B \in \Omega^1(M)$.
As remarked before, one can check easily that $\exp(\mathrm{i}\phi^* B)$ is well defined on the equivalence classes in $H^2(M,\mathbf{U}_1 \to \Omega^1 \to \Omega^2)$, as long as S is a closed surface. Furthermore one can check that it is independent of the triangulation of S and of the chosen $t^{(0)}, t^{(1)}$ and $t^{(2)}$. Both calculation will actually follow from the more general calculation with boundary. If we want to introduce D-branes, we have to admit open strings and thus worldsheets with a boundary. In the following subparagraph we will explicitly check whether the definition of $\exp(\mathrm{i}\phi^* B)$ is still invariant under the equivalence relation of the quadruples. And whether it is still independent of the chosen triangulation. The calculations are rather algebraic and use a special notation explained in the appendices. Readers not interested in verifying the derivation of the results, are strongly advised to jump ahead to the summary of the results.

6.6 The anomaly calculation

First of all the introduction of a worldsheet boundary, means that the construction of the triangulation is no longer straightforward. First let us give a triangulation of the worldsheet boundary. The $\{U_i\}$ can be chosen such that the U_i give a good covering of M and $U_i \cap Q$ a good covering of Q and $V_i = \phi^* U_i$ a good covering of Σ and $V_i \cap \partial\Sigma$ a good covering of $\partial\Sigma$. We start with a triangulation subject to the covering $V_{ij} \cap \partial\Sigma$, which is a formal sum γ of edges.

This can be divided in

$c_i^{(0)}$ are all path pieces in V_i, $\gamma_i^{(0)}$ the endpoints of these pieces.

$c_{ij}^{(1)}$ are the endpoints lying in V_{ij}, $\gamma_{ij}^{(1)}$ is the sum of these endpoints in V_{ij}.

In the following we use the Čech boundary operator

$$(\check{\partial} s)_{i_0 i_1 .. i_q} = \sum_i s_{i i_0 i_1 .. i_q}. \tag{6.56}$$

It maps between Čech cochains with $q+2$ indices to cochains with $q+1$ indices. For instance $\check{c}^{(1)} = \gamma^{(1)}$ and $\check{c}^{(0)} = \gamma^{(0)}$. A frequently used argument goes as follows. $\check{\partial}\gamma = 0$ (by definition), therefore we can find a $c^{(0)}$ such that $\check{\partial} c^{(0)} = \partial\Sigma$, giving the path pieces in V_i. Furthermore it is important that the Čech boundary operator $\check{\partial}$ commutes with the ordinary boundary operator ∂. Define $\gamma^{(0)} = \partial c^{(0)}$. Then $\check{\partial}\gamma^{(0)} = \partial\check{\partial} c^{(0)} = 0$, therefore we can find a $c^{(1)}$ such that $\check{\partial} c^{(1)} = \gamma^{(0)}$, giving the vertices in V_{ij}. The above division of the triangulation of the worldsheet boundary is constructed in this way. The triangulation of a closed worldsheet is also divided in this way.

We start with a triangulation s of Σ subject to the V_i. Then we define $t^{(0)}$ such that $\check{\partial} t^{(0)} = s$. They give the triangles in every V_i. For a closed worldsheet we now define $s^{(0)} = \partial t^{(0)}$ and $\check{\partial} t^{(0)} = \partial\Sigma = 0$ gives a $t^{(1)}$ such that $\check{\partial} t^{(1)} = s^{(0)}$, giving the edges in V_{ij}. With boundary however $\partial\Sigma \neq 0$. But if we define $s^{(0)} = \partial t^{(0)} - c^{(0)}$, i.e. we subtract the edges lying on the boundary, we have $\check{\partial} s^{(0)} = 0$. Thus we have a $t^{(1)}$ such that $\check{\partial} t^{(1)} = s^{(0)}$. If we calculate

$$\check{\partial}\partial t^{(1)} = -\partial c^{(0)} = -\gamma^{(1)} \tag{6.57}$$

These are the vertices on the worldsheet boundary, they obstruct the lifting of $\partial t^{(1)}$ to $t^{(2)}$. To get rid of them we have to add $\gamma^{(1)}$, that is define $s^{(1)} = \partial t^{(1)} + \gamma^{(1)}$. Then $\check{\partial} s^{(1)} = 0$. Finally we define $t^{(2)}$ such that $\check{\partial} t^{(2)} = s^{(1)}$. To summarize:

$$\begin{array}{ll} t_i^{(0)} & \text{the triangles in } V_i \\ s_i^{(0)} = \partial t_i^{(0)} - c_i^{(0)} & \text{the edges in } V_i \\ t_{ij}^{(1)}, \text{ with } \check{\partial} t^{(1)} = s^{(0)} & \text{the edges in } V_{ij} \\ s_{ij}^{(1)} = \partial t_{ij}^{(1)} + c_{ij}^{(1)} & \text{the vertices in each } V_{ij} \\ t_{ijk}^{(2)}, \text{ with } \check{\partial} t^{(2)} = s^{(1)} & \text{the vertices in } V_{ijk} \end{array} \tag{6.58}$$

with every time the objects at the worldsheet boundary subtracted.

Now we ought to check whether our definition of $\exp(\mathrm{i}\phi^* B)$ is still invariant under the equivalence relation for the quadruples. In the following we omit the pull-backs Φ^* in the pairing $(.,.)$, which denotes the integral of cochains of 2-forms, 1-forms and real functions over cochains of triangles, edges and vertices, with implicit summation over the indices. Important are the relations

$$(\check{\partial} a, b) = (a, \delta b), \quad (\partial a, b) = (a, db) \tag{6.59}$$

where δ is the Čech coboundary operator, and the second equation is just Stokes theorem.
The contributions from a gauging (6.48) are

$$\begin{aligned} (t^{(0)}, d\mu) &= (\partial t^{(0)}, \mu) = (c^{(0)}, \mu) + (s^{(0)}, \mu), \\ (t^{(1)}, -\delta\mu + d\rho) &= -(\check{\partial} t^{(1)}, \mu) + (\partial t^{(1)}, \rho) = -(s^{(0)}, \mu) + (s^{(1)}, \rho) - (c^{(1)}, \rho), \\ -(t^{(2)}, \delta\rho - 2\pi\tilde{a}) &= -(\check{\partial} t^{(2)}, \rho) + (t^{(2)}, 2\pi\tilde{a}) = -(s^{(1)}, \rho) + (t^{(2)}, 2\pi\tilde{a}). \end{aligned} \tag{6.60}$$

The sum of these contributions is

$$(c^{(0)}, \mu) - (c^{(1)}, \rho) + (t^{(2)}, 2\pi\tilde{a}). \tag{6.61}$$

The last term of this vanishes in the exponent, so we have restricted the change to an integration over the worldsheet boundary. In fact with μ_i playing the role of connection and $\exp(-\mathrm{i}\rho_{ij})$ the local coordinate transformations, it can be seen as the holonomy of a line bundle over the worldsheet boundary given by the triple $(\mu_i, -\rho_{ij}, \delta\rho_{ijk})$:

$$\mathrm{Hol}_{\partial\Sigma}(\mu, -\rho) = \exp(\mathrm{i}(c^{(0)}, \mu) + \mathrm{i}(c^{(1)}, \rho)). \tag{6.62}$$

There now seems to be a way to solve the problem of the broken symmetry. Accompany the gauge transformation of the B-field by a gauge transformation in the A-field $A_i \to A_i - \mu_i$ and $g_{ij} + \rho_{ij}$. The product of the holonomy of a $U(1)$-gauge field A and $\exp(\mathrm{i}\phi^* B)$ is then invariant under this combined gauging. However since there are no restrictions on the ρ_{ij} and μ_i, the A-field is no longer a connection in a line bundle. If we start with a triple (A_i, g_{ij}, c_{ijk}) satisfying the usual conditions for a line bundle, after a gauging $A'_i = A_i - \mu_i, g'_{ij} = g_{ij} + \rho_{ij}$ we have

$$\begin{aligned} A'_i - A'_j = dg_{ij} - \mu_i + \mu_j &= dg'_{ij} - \mu_i + \mu_j - d\rho_{ij}, \\ g'_{ij} + g'_{jk} + g'_{ki} &= c_{ijk} + \rho_{ij} + \rho_{jk} + \rho_{ki} \end{aligned} \tag{6.63}$$

Notice that the contribution on the right-hand sides are the change by gauging of Λ_{ij} and a_{ijk}. So consider triples (A_i, g_{ij}, c_{ijk}) of 1-forms, real functions and $\mathbb{Z}$-constants, satisfying

$$\begin{aligned} A_i - A_j &= dg_{ij} - \Lambda_{ij}, \\ \delta g_{ijk} &= 2\pi c_{ijk} + a_{ijk}, \\ \delta c_{ijk} &= m_{ijkl}. \end{aligned} \tag{6.64}$$

After the combined gauging together with $c_{ijk} \to c'_{ijk} = c_{ijk} + \tilde{a}_{ijk}$, the triple still satisfies these conditions. However the last line should ring a bell. It says that m_{ijkl} is the coboundary of c, thus $[m]$ vanishes in cohomology. The c are a cochain on Q. It trivializes m only on Q. Thus $[m]$ vanishes in $H^3(Q, \mathbb{Z})$ but not necessarily in $H^3(M, \mathbb{Z})$.

For a possible solution of this, we first rewrite the triples and quadruples of the A and the B-field. Define

$$h_{ij} = \exp(ig_{ij}), \quad \zeta_{ijk} = \exp(\mathrm{i}a_{ijk}) \tag{6.65}$$

Then the equivalence classes of quadruples (B, Λ, a, m) can also be described as equivalence classes of (B, Λ, ζ) satisfying

$$\begin{aligned} B_i - B_j &= \Lambda_{ij}, \\ \Lambda_{ij} + \Lambda_{jk} + \Lambda_{ki} &= \mathrm{i}\zeta_{ijk} d\zeta_{ijk}^{-1}, \\ \delta\zeta_{ijkl} &= 1 \end{aligned} \tag{6.66}$$

with equivalence relation

$$\begin{pmatrix} B_i \\ \Lambda_{ij} \\ \zeta_{ijk} \end{pmatrix} \sim \begin{pmatrix} B_i + d\mu_i \\ \Lambda_{ij} + \mu_i - \mu_j + \mathrm{i}q_{ij} dq_{ij}^{-1} \\ \zeta_{ijk} q_{ij} q_{jk} q_{ki} \end{pmatrix}. \tag{6.67}$$

for any local 1-forms μ_i and local $U(n)$-functions q_{ij}.
For the A-field we generalize to n D-branes, and thus A becomes a $U(n)$-connection in a vector bundle. The triples (A, g, c) consists of functions A_i on U_i whose values are the product of a 1-form and a symmetric $n \times n$-matrix, and functions g_{ij} and constants c_{ijk} with values symmetric $n \times n$-matrices. Then $h_{ij} = \exp(\mathrm{i}g_{ij})$ are $U(n)$-valued functions. $U(n)$-bundles are classified by pairs (A, h) satisfying

$$A_i - h_{ij} A_j h_{ij}^{-1} = \mathrm{i}h_{ij} dh_{ij}^{-1}, \quad h_{ij} h_{jk} h_{ki} = 1, \tag{6.68}$$

with equivalence relation

$$(A_i, h_{ij}) \sim (A_i - \mathrm{i}l_i dl_i^{-1}, l_j h_{ij} l_i^{-1}). \tag{6.69}$$

for any local $U(n)$ function l_i.
The definition of the twisted line bundle is then

$$\begin{aligned} A_i - h_{ij} A_j h_{ij}^{-1} &= \mathrm{i}h_{ij} dh_{ij}^{-1} - \Lambda_{ij}, \\ h_{ij} h_{jk} h_{ki} &= \zeta_{ijk}. \end{aligned} \tag{6.70}$$

The Λ_{ij} have to be read as Λ_{ij} times the $n \times n$ identity matrix $I_{n\times n}$, which is the generator of the subgroup $U(1) \subset U(n)$. Similar ζ_{ijk} is $\zeta_{ijk} \cdot I_{n\times n}$. This is consistent under the combined gauging (6.67) and

$$\begin{aligned} A_i &\to A_i - \mu_i, \\ h_{ij} &\to q_{ij} h_{ij}. \end{aligned} \tag{6.71}$$

If we look at the last in (6.70) and take the determinant on the right and left-hand side

$$\det h_{ij} \det h_{jk} \det h_{ki} = \zeta_{ijk}^n. \tag{6.72}$$

The ζ^n_{ijk} arises because this is the determinant of $\zeta_{ijk} \cdot I_{n\times n}$. Now the left-hand side is a coboundary of $U(1)$-functions. It trivializes ζ^n_{ijk}. Thus $[\zeta^n_{ijk}] = 0 \in H^2(Q, \mathbf{U}_1)$. By the usual isomorphism $H^2(Q, \mathbf{U}_1) \cong H^3(Q, \mathbb{Z})$ this means $[\zeta^n_{ijk}] = n[H] = 0$. These situations are possible when $H^3(Q, \mathbb{Z})$ contains torsion subgroups. For example if $H^3(Q, \mathbb{Z}) = \mathbb{Z}_n$ and $[H]$ is the generator of $\mathbb{Z}_n$, we get $n[H] = 0$. If we look at the image of $[H]$ under the standard homomorphism $H^3(Q, \mathbb{Z}) \to H^3(Q, \mathbb{R})$, we get $n[H]_\mathbb{R} = 0$. With real coefficients this means $[H]_\mathbb{R} = 0/n = 0$. So $[H]$ vanishes in $H^3(Q, \mathbb{R})$. We have already looked at this flat-but-torsion case, and derived that in this case we can write $H = dB$ everywhere on Q, can gauge $\Lambda = 0$ and choose a_{ijk} to be real constants. In our new description this gives that ζ_{ijk} are $U(1)$-constants.
Now let us see whether in this combined system of the A field and torsion B-field, the product

$$\exp(\mathrm{i}\int_\Sigma \phi^* B)\,\mathrm{Tr}\,\mathrm{Hol}_{\partial\Sigma}(A, h) \tag{6.73}$$

is indeed gauge invariant. We have to be more careful now about the holonomy of the A-field. Since we are now in non-abelian gauge theory it is defined by the alternating product of path ordered exponential of $\mathrm{i}\int_{\gamma_i} A_i$ along the path piece γ_i in U_i and h_{ij} in the vertex in U_{ij} in the order the path passes the path pieces and the vertices. The direct product of a $U(n)$ (vector) bundle (A, h) with a $U(1)$ (line) bundle $(\tilde{A}, \tilde{h})$, is a $U(n)$ bundle with connection $A_i + \tilde{A}_i I_{n\times n}$ and transition functions $h_{ij}\tilde{h}_{ij}$. The holonomy of this product factors in a product of the (path ordered) holonomy of (A, h) and the holonomy of $(\tilde{A}, \tilde{h})$, since the $U(1)$-functions $\tilde{h}_{ij}$ and the $U(1)$-generators $A_i \cdot I_{n\times n}$ commute with all the factors in the holonomy of the product bundle. Therefore if we apply the gauge transformation (6.71), the holonomy of the twisted $U(n)$ bundle gets multiplied by a factor that is equal to the holonomy of the pseudo line bundle formed by $(-\mu, q)$. With

$$(c^{(1)}, q) = \exp(\mathrm{i}(c^{(1)}, \rho)) \tag{6.74}$$

this exactly cancels the factor $\exp(\mathrm{i}\int_\Sigma \phi^* B)$ picks up after the gauge (see (6.61)). So for the $n[H] = 0$ case we have restored the gauge invariance of the B-field triples (B, Λ, ζ).
Another problem is that we do not know how our definition depends on the triangulation of the surface. Consider therefore a second triangulation s' of Σ. The triangulations of Σ induce two different triangulations of the worldsheet boundary $\gamma = \partial s$ and $\gamma' = \partial s'$. The triangulation s' can be divided in $s'^{(l)}$ and $t'^{(l)}$ with $l = 0, 1, 2$. For all objects

$$\Delta s^{(l)} = s'^{(l)} - s^{(l)}, \quad \Delta t^{(l)} = t'^{(l)} - t^{(l)} \tag{6.75}$$

denotes the difference in the triangulation. First we relate the two triangulations of $\partial\Sigma$ with each other. As γ and γ' are triangulations of the same closed object $\partial\Sigma$

$$[\gamma] = [\gamma'] = [\partial\Sigma] \in H_1(\partial\Sigma) \tag{6.76}$$

Thus $\gamma' - \gamma$ is the boundary of a triangulation $\tilde{s}$ of a surface. Of course there are no surfaces in $\partial\Sigma$, but $\tilde{s}$ should be considered as the formal sum of maps of triangles into $\partial\Sigma$. For a better picture, one can also push forward γ and γ' to Q. Then $\phi_*\gamma$ and $\phi_*\gamma'$ are the boundary of some surface triangulation in Q, which can be thought of as the surface over which one moves $\partial\Sigma$ back to itself, but with a changed triangulation. We restrict however to $\partial\Sigma$ and its covering $V_i \cap \partial\Sigma$.

In the same way as the division of the triangulation s of Σ with boundary γ, we can now divide the triangulation $\tilde{s}$ with boundary $\Delta\gamma$ by

$$\begin{aligned} \tilde{t}^{(0)} \text{ such that } \check{\partial}\tilde{t}^{(0)} &= \tilde{s}, \\ \tilde{s}^{(0)} &:= \partial\tilde{t}^{(0)} - \Delta c^{(0)}, \\ \tilde{t}^{(1)} \text{ such that } \check{\partial}\tilde{t}^{(1)} &= \tilde{s}^{(0)}, \\ \tilde{s}^{(1)} &:= \partial\tilde{t}^{(1)} + \Delta c^{(1)}, \\ \tilde{t}^{(2)} \text{ such that } \check{\partial}\tilde{t}^{(2)} &= \tilde{s}^{(1)} \end{aligned} \tag{6.77}$$

All $\tilde{t}^{(\cdot)}$ and $\tilde{t}^{(\cdot)}$ lie on $\partial\Sigma$.

Now turning back to triangulations in Σ. The push forward of the triangulation $\tilde{s}$ to Σ, remains located on the boundary $\partial\Sigma$. We say that s and s' are triangulations of the same surface if $\Delta s - \tilde{s}$ is the boundary of a volume triangulation v, i.e. $\partial v = \Delta s - \tilde{s}$. In space-time M this means that $\phi_*(s' - \tilde{s})$ and $\phi_* s$ are both triangulations of $\phi(\Sigma)$. In a similar way as before we calculate

$$\begin{aligned} \check{\partial} v = 0, &\implies \exists w^{(0)} : \check{\partial} w^{(0)} = v, \\ v^{(0)} &= \partial w^{(0)} - \Delta t^{(0)} + \tilde{t}^{(0)}, \\ \check{\partial} v^{(0)} = 0, &\implies \exists w^{(1)} : \check{\partial} w^{(1)} = v^{(0)}, \\ v^{(1)} &= \partial w^{(1)} + \Delta t^{(1)} - \tilde{t}^{(1)}, \\ \check{\partial} v^{(1)} = 0, &\implies \exists w^{(2)} : \check{\partial} w^{(2)} = v^{(1)}, \\ v^{(2)} &= \partial w^{(2)} - \Delta t^{(2)} + \tilde{t}^{(2)}, \\ \check{\partial} v^{(2)} = 0, &\implies \exists w^{(3)} : \check{\partial} w^{(3)} = v^{(2)}, \end{aligned} \tag{6.78}$$

Using this we calculate the change in the exponent of $\exp(\mathrm{i}\phi_* B)$

$$\begin{aligned} (\Delta t^{(0)}, B) &= (\partial w^{(0)} - \check{\partial} w^{(1)} + \tilde{t}^{(0)}, B) = (w^{(0)}, H) - (w^{(1)}, -d\Lambda) + (\tilde{t}^{(0)}, B) \\ (\Delta t^{(1)}, \Lambda) &= (-\partial w^{(1)} + \check{\partial} w^{(2)} + \tilde{t}^{(1)}, \Lambda) = -(w^{(1)}, d\Lambda) + (w^{(2)}, da) + (\tilde{t}^{(1)}, \Lambda) \\ -(\Delta t^{(2)}, a) &= -(\partial w^{(2)} - \check{\partial} w^{(3)} + \tilde{t}^{(2)}, a) = -(w^{(2)}, da) - (w^{(3)}, 2\pi m) - (\tilde{t}^{(2)}, a) \end{aligned}$$

The term $(w^{(0)}, H)$ vanishes because it is the integral over the image of a 3-simplex (tetrahedron) in Σ, thus over a two dimensional surface. In space-time M this means that the push forward $\phi_* w^{(0)}$ has no 3-dimensional volume. For the same reason $(\tilde{t}^{(0)}, B)$ vanishes, because $\tilde{t}^{(0)} \subset \partial\Sigma$ is 1-dimensional.

$2\pi(w^{(3)}, m)$ vanishes in the exponent, because it is an integer multiple of 2π. So the total change comes down to

$$(\tilde{t}^{(1)}, \Lambda) - (\tilde{t}^{(2)}, a) \tag{6.79}$$

Now we want to show that this change is cancelled by the change of the trace of holonomy of the twisted $U(n)$ bundle. For this we first have to describe the triangulation of the worldsheet boundary in a different manner. First we restrict the possible triangulations, to those for which all edges only go in the direction of the orientation of $\partial\Sigma$, thus excluding triangulations which go back and forth on $\partial\Sigma$. The going a back and forth can be shown to have no contribution to total holonomy. If we parallel transport a vector over a path and go back in the same way, the vector comes back to itself. Now separate γ in all r edges $E_\alpha, \alpha = 1, ..r$. Thus

$$\gamma = \sum_{\alpha=1}^{r} E_\alpha. \tag{6.80}$$

For a general triangulation every $c_i^{(0)}$ might consist of multiple E_α. We define a map f from the index set $\{1, 2, ..r\}$ to the index set of the covering $V_i \cap \partial\Sigma$, such that $E_\alpha \subset V_{f(\alpha)} \cap \partial\Sigma$. The σ_α denote the vertex connecting $E_{\alpha-1}$ and E_α, so in fact E_α is the interval $[\sigma_\alpha, \sigma_{\alpha+1}]$. The holonomy around the world sheet boundary is then

$$\begin{aligned} h_{f(1)f(r)}(\sigma_r)\cdot\mathrm{Hol}_{E_r}(A_f(r)) \cdot h_{f(r-1),f(r)}(\sigma_{r-1})\cdot \\ \mathrm{Hol}_{E_{r-1}}(A_{f(r-1)}) \cdot ....h_{f(2)f(1)}(\sigma_2) \cdot \mathrm{Hol}_{E_1}(A_f(1)). \end{aligned} \tag{6.81}$$

If we divide one E_α in two edges assigning them both to the coordinate region $f(\alpha)$, the holonomy does not change.
If we have a second triangulation (E', f') of $\partial\Sigma$, its vertices σ'_α that are not vertices of (E, F), always lie on some edge of (E, f). If we break up the edges of (E, f) at every point, where there is a vertex of (E', f'), the holonomy of (E, f) does not change. The other way around if we break up the edges of (E', f') at vertices of (E, f) the holonomy of (E', f') does not change. In both cases after the breaking up we have the same set of edges, only combined with different maps f and f'. So all we have to do is to check what happens if we change the map f into f' by changing the coordinate region assigned to every edge. The factors $\mathrm{Hol}_{E_\alpha}(A_{f(\alpha)})$ are given by the solutions of the differential equation

$$\frac{\partial}{\partial\sigma} M_i(\sigma, \sigma_0) = \mathrm{i} A_i(\sigma) M_i(\sigma, \sigma_0), \quad M_i(\sigma_0, \sigma_0) = I \tag{6.82}$$

for $U(n)$-functions M_i. Here $A_i(\sigma)$ is, as in all previous calculations, the pull back $\phi^* A_i$ to $\partial\Sigma$, that is

$$\phi^* A_i(\sigma) = A_\mu(\phi(\sigma)) \frac{\partial\phi^\mu(\sigma)}{\partial\sigma} \tag{6.83}$$

with derivation in the direction along the boundary. The holonomy over the edge $E_\alpha = [\sigma_\alpha, \sigma_\alpha + 1]$ is given by

$$\mathrm{Hol}_{E_\alpha}(A_{f(\alpha)}) = M_{f(\alpha)}(\sigma_{\alpha+1}, \sigma_\alpha). \tag{6.84}$$

If $[\sigma_0, \sigma]$ lies both in V_i and in V_j, define

$$\tilde{M}(\sigma, \sigma_0) = h_{ij}(\sigma) \exp(-\mathrm{i} \int_{\sigma_0}^{\sigma} \Lambda_{ij}) M_j(\sigma, \sigma_0) h_{ij}^{-1}(\sigma_0). \tag{6.85}$$

It satisfies

$$\begin{aligned} \frac{\partial}{\partial \sigma} \tilde{M}(\sigma, \sigma_0) &= \left[(d h_{ij}(\sigma)) h_{ij}^{-1}(\sigma) - i \Lambda_{ij}(\sigma) + \mathrm{i} h_{ij} A_j h_{ij}^{-1} \right] \tilde{M}(\sigma, \sigma_0) \\ &= \mathrm{i} \left[\mathrm{i} h_{ij}(\sigma) h_{ij}^{-1}(\sigma) - \Lambda_{ij}(\sigma) + h_{ij} A_j h_{ij}^{-1} \right] \tilde{M}(\sigma, \sigma_0) \\ &= \mathrm{i} A_i \tilde{M}(\sigma, \sigma_0). \end{aligned} \tag{6.86}$$

and $\tilde{M}(\sigma_0, \sigma_0) = I$, thus $\tilde{M}(\sigma, \sigma_0) = M_i(\sigma, \sigma_0)$. Therefore, if the edge E_α belongs to different coordinate regions $f(\alpha)$ and $f'(\alpha)$, we can write

$$\begin{aligned} \mathrm{Hol}_{E_\alpha}(A_{f(\alpha)}) = h_{f(\alpha) f'(\alpha)}(\sigma_{\alpha+1}) \exp(-\mathrm{i} \int_{\sigma_\alpha}^{\sigma_{\alpha+1}} \Lambda_{f(\alpha) f'(\alpha)}) \\ \mathrm{Hol}_{E_\alpha}(A_{f'(\alpha)}) h_{f(\alpha) f'(\alpha)}^{-1}(\sigma_\alpha). \end{aligned} \tag{6.87}$$

In the holonomy of E using coordinate regions f we have the factor

$$\begin{aligned} h_{f(\alpha+1) f(\alpha)}(\sigma_{\alpha+1}) \mathrm{Hol}_{E_\alpha}(A_{f(\alpha)}) h_{f(\alpha) f(\alpha-1)} = \\ h_{f(\alpha+1) f(\alpha)}(\sigma_{\alpha+1}) h_{f(\alpha) f'(\alpha)}(\sigma_{\alpha+1}) \exp(-\mathrm{i} \int_{\sigma_\alpha}^{\sigma_{\alpha+1}} \Lambda_{f(\alpha) f'(\alpha)}) \mathrm{Hol}_{E_\alpha}(A_{f'(\alpha)}) \\ h_{f'(\alpha) f(\alpha)} h_{f(\alpha) f(\alpha-1)} = \\ h_{f(\alpha+1) f'(\alpha)} \zeta_{f(\alpha+1) f(\alpha) f'(\alpha)}(\sigma_{\alpha+1}) \exp(-\mathrm{i} \int_{\sigma_\alpha}^{\sigma_{\alpha+1}} \Lambda_{f(\alpha) f'(\alpha)}) \mathrm{Hol}_{E_\alpha}(A_{f'(\alpha)}) \\ \zeta_{f'(\alpha) f(\alpha) f(\alpha-1)}(\sigma_\alpha) h_{f'(\alpha) f(\alpha-1)} \end{aligned}$$

So if we want to replace the assigned coordinate region $f(\alpha)$ of E_α by $f'(\alpha)$ we have to multiply with the $U(1)$-factors

$$\zeta_{f(\alpha+1) f(\alpha) f'(\alpha)}(\sigma_{\alpha+1}) \zeta_{f'(\alpha) f(\alpha) f(\alpha-1)}(\sigma_\alpha) \exp(-\mathrm{i} \int_{\sigma_\alpha}^{\sigma_{\alpha+1}} \Lambda_{f(\alpha) f'(\alpha)})$$

to keep the holonomy the same. Thus the holonomy calculated with edges E_α and coordinate region assignment f' multiplied with

$$\prod_{\alpha=1}^{r} \zeta_{f(\alpha+1) f(\alpha) f'(\alpha)}(\sigma_{\alpha+1}) \zeta_{f'(\alpha) f(\alpha) f(\alpha-1)}(\sigma_\alpha) \exp(-\mathrm{i} \int_{\sigma_\alpha}^{\sigma_{\alpha+1}} \Lambda_{f(\alpha) f'(\alpha)})$$

we get the holonomy using the assignment f. Now all we have to do is show that this factor is equal to the change in $\exp(\mathrm{i} \int_\Sigma \phi^* B)$. Similar to the holonomy it is easy to show that if we divide triangles in the triangulation without changing the coordinate region assignment, the factor does not change. The sum of integrals of B_i over the broken up triangles is of course the same as B_i over the original triangle. Further $t^{(1)}$ and $t^{(2)}$ do not change, as there are no new edges and vertices between different coordinate regions. We showed that the factor only depends on the triangulation of the boundary. The division of triangles cause division of edges on the boundary. So also there it is possible to break up the edges without changing the B-factor. Therefore we can focus on a fixed triangulation γ with different coordinate region assignments. From picture **??** it then follows that the change (6.79) in the B-field factor, being

$$\exp(\mathrm{i}(c^{(0)}, \Lambda) - \mathrm{i}(c^{(1)}, a)) = \exp(\mathrm{i}(c^{(0)}, \Lambda))(c^{(1)}, \zeta), \tag{6.88}$$

is the same as we just calculated for the holonomy of the twisted line bundle.

6.7 Summary of the results

We have shown that the product of factors

$$\exp(\mathrm{i} \int_\Sigma \phi^* B) \operatorname{Tr} \operatorname{Hol}_{\partial\Sigma}(A, h) \tag{6.89}$$

in the path integral, is well defined if the B-field is formed by triples $(B_i, \Lambda_{ij}, \zeta_{ijk})$ of 2-forms, 1-forms and $U(1)$-functions satisfying

$$\begin{aligned} B_i - B_j &= \Lambda_{ij}, \\ \Lambda_{ij} + \Lambda_{jk} + \Lambda_{ki} &= \mathrm{i}\zeta_{ijk} d\zeta_{ijk}^{-1}, \\ \delta\zeta_{ijkl} &= 1 \end{aligned} \tag{6.90}$$

and the A-field is a connection in the twisted vector bundle formed by the pair (A_i, h_{ij}) of 1-forms and $U(n)$-functions satisfying

$$\begin{aligned} A_i - h_{ij} A_j h_{ij}^{-1} &= \mathrm{i} h_{ij} d h_{ij}^{-1} - \Lambda_{ij}, \\ h_{ij} h_{jk} h_{ki} &= \zeta_{ijk}. \end{aligned} \tag{6.91}$$

Here Λ_{ij} and ζ_{ijk} are to be read as the $U(1)$ factor times the $n \times n$ identity matrix. The product of path integral factors is independent of the chosen triangulation and is invariant under the combined gauging

$$\begin{pmatrix} B_i \\ \Lambda_{ij} \\ \zeta_{ijk} \end{pmatrix} \sim \begin{pmatrix} B_i + d\mu_i \\ \Lambda_{ij} + \mu_i - \mu_j + \mathrm{i} q_{ij} d q_{ij}^{-1} \\ \zeta_{ijk} q_{ij} q_{jk} q_{ki} \end{pmatrix}, \tag{6.92}$$

and

$$\begin{aligned} A_i &\to A_i - \mu_i, \\ h_{ij} &\to q_{ij} h_{ij}. \end{aligned} \tag{6.93}$$

with μ_i any 1-forms and q_i any $U(1)$-functions. However from (6.72) it follows that the definition of the twisted vector bundle makes ζ^n trivial as cohomology

class in $H^2(Q, \mathbf{U}_1)$. Thus this only brings a solution to the anomaly problem if $[\zeta^n] = 0$. By the isomorphism $H^2(Q, \mathbf{U}_1) \cong H^3(Q, \mathbb{Z})$ this is only if $n[H] = 0$. In this case we can write $H = dB$ with the same B everywhere on Q, we can choose all $\Lambda_{ij} = 0$ and the ζ_{ijk} to be $U(1)$-constants.
The twisted line bundles have a nice interpretation in the context of adjoint bundles. These are vector bundles, whose fibers consist of $n \times n$-matrices and of which the transformation on coordinate region overlaps is given by the action of $U(n)$-valued functions in the following manner

$$M \to h_{ij} M h_{ij}^{-1} \tag{6.94}$$

This makes that the bundle is invariant under the multiplication of the h_{ij} by any $U(1)$-function. So in fact the h_{ij} have values in $U(n)/U(1) = SU(n)/\mathbb{Z}_n$. If one maps the $U(n)$ functions into $U(n)/U(1)$ and lifts them back into $U(n)$ they may have picked up a factor by a $U(1)$-function. However this does not the change the cohomology class of ζ in $H^2(M, \mathbf{U}_1)$.
The natural generalization for B-fields with $n[H] \neq 0$, lies in taking $n \to \infty$. To be more precise we use a infinite-dimensional separable Hilbert space $\mathcal{H}$, and take the h_{ij} to have values in $U(\mathcal{H})$, the group of unitary operators on $\mathcal{H}$. By a theorem of Kuiper $U(\mathcal{H})$ is contractible. Therefore as the ζ_{ijk} are a cocycle, one can always find $U(\mathcal{H})$-functions h_{ij} with coboundary ζ_{ijk}. The generalization of the adjoint bundles can be made using bundles on Q, with as fiber the C^*-algebra $\mathcal{K}$ of compact operators on $\mathcal{H}$. With

$$Ad(h) : \mathcal{K} \to \mathcal{K}; C \mapsto hCh^{-1} \tag{6.95}$$

the structure group $Aut(\mathcal{K})$ of the bundle is isomorphic to $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$ by the map $Ad : PU(\mathcal{H}) \to Aut(\mathcal{K})$. By a theorem of Dixmier and Douady the isomorphism classes of such bundles are given by elements of $H^3(Q, \mathbb{Z})$, exactly by mapping $U(\mathcal{H})$ functions h_{ij} with coboundary equal to ζ_{ijk}, where $[\zeta] \cong [H]$ by the isomorphism $H^2(Q, \mathbf{U}_1) \cong H^3(Q, \mathbb{Z})$. This lead to the proposal of Bouwknegt and Mathai[3] that for this situation we should look at these bundles, and that the K-theory of the algebra of sections of this line bundles should describe the D-brane configurations. Furthermore in [4] it is shown that the equivalence classes of triples (B, Λ, ζ) of the B-field classified in the space $H^2(Q, \mathbf{U}_1 \to \Omega^1 \to \Omega^2)$, also classify isomorphism classes of Dixmier-Douady sheaves with connective structure. It seems natural that the local coordinate description of this connective structure is given by the A-fields with twisted coboundary conditions in the above manner. However this must be worked out. Moreover the global worldsheet anomaly cancellation has to be explicitly checked for this case. For the $n[H] = 0$ case this has been done above and in [12]. In the calculation we never made use of the fact that the ζ are constant or the Λ zero, as in the $n[H] = 0$ case. It seems there is no obstruction to generalizing this calculation to the case with $h_{ij} \in U(\mathcal{H})$. The mathematical details have to be worked out.
As noticed in [12] and [7] the anomaly problem can be rephrased by saying that the factors in the path integral are not functions on the space of world sheet

embeddings and metrics, rather sections of a line bundle over them. The line bundle can be constructed from a line bundle over loop space. The local coordinate regions in this loops space are given by different triangulations of the loop. Already in '85 by Gawedzki [14] in the context of Wess-Zumino-Witten models it was shown, that the integral of B over a surface with boundary, defined in the above way, depends on the triangulation of the boundary in such a way, that the transformation between different triangulations of the boundary, is given by the gluing maps in the line bundle over loop space switching between the same triangulations. In this way if $\pi : X \to LQ$ is the map from the configuration space X of embeddings and metrics to LQ, mapping the embedding of $\partial\Sigma$ to LQ, and $\mathcal{L}$ is the line bundle over loop space, the pull back $\pi^*\mathcal{L}$ is a line bundle over X and the B-field factor is a section in this line bundle. The first Chern class of the line bundle $\mathcal{L}$ is given by a homomorhpism $H^2(Q, \mathbf{U}_1 \to \Omega^1 \to \Omega^2) \to H^1(LQ, \mathbf{U}_1 \to \Omega^1)$ which is described in more detail in [4]. It is not very hard to show that the dependence calculated above is indeed the same as in [4]. If the anomaly calculation for the A-field were completed for the general $n \to \infty$, Dixmier-Douady bundle, case this would give an alternative description of this line bundle. The trace of the path ordered exponent of $\mathrm{i} \int A$ around a loop would be a section of it.
The pfaff(D)-factor also takes values in a section over loop space with first Chern class $\mathcal{T}W_3(Q)$. For the product of the three factor to be a well defined function on X this gives the following condition

$$W_3(Q) + \beta(\alpha) + [H]_Q = 0 \tag{6.96}$$

where α give the twisted coboundary condition on the h_{ij} of the $U(n)$-bundle ($U(\mathcal{H})$-bundle?)

$$h_{ij} h_{jk} h_{ki} = \alpha_{ijk} \tag{6.97}$$

and β is the Bockstein $\beta : H^2(Q, \mathbf{U}_1) \to H^3(Q, \mathbb{Z})$. When this condition is satisfied the product of the three factors are a section in a trivial line bundle.

A Algebraic topology

The following appendices contain a brief exposition of the algebraic topology that is used in the main text. Some basic knowledge about manifolds, differential forms and homotopy groups is assumed. The first appendix defines the de Rham and Čech cohomology and their relation. The second appendix explains singular homology. A broader introduction into these subjects can be found in [2] and [13]. The next appendix reviews the main facts about vector bundles. Appendix A.4 gives a more detailed overview of the theory of complex line bundles, explaining holonomy and the fact that complex line bundles are given by classes of $H^1(M, \mathbf{U}_1)$, while the category of complex line bundles with connection is classified by $H^1(M, \mathbf{U}_1 \to \Omega^1)$. This last notation and some more details can be found in [4]. The last appendix treats higher rank vector bundles, or $U(n)$-bundles.

A.1 de Rham and Čech cohomology

Let M be a smooth manifold. There are two ways to define the cohomology space $H^q(M,\mathbb{R})$. In *de Rham cohomology* we use the differential complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \tag{A.1}$$

with $\Omega^q(M)$ the space of q-forms. The q-th de Rham cohomology ($q \geq 0$) is then defined as

$$H^q_{dR}(M) := \ker(d : \Omega^q \to \Omega^{q+1}) / \operatorname{img}(d : \Omega^{q-1} \to \Omega^q). \tag{A.2}$$

q-forms λ with $d\lambda = 0$ are called *closed* and q forms with $\lambda = d\omega$ for some $q-1$ form *exact*. So de Rham cohomology consists of closed forms modulo the exact forms. $H^0_{dR}(M)$ are the locally constant functions.
For the description in *Čech cohomology* we make use of a good cover of M. This is an open cover $\{U_i\}$ such that all finite intersections

$$U_{i_1 i_2 i_3 \ldots i_s} := U_{i_1} \cap U_{i_2} \cap U_{i_3} \cap \ldots U_{i_s} \tag{A.3}$$

with $s \geq 1$, are contractible. For manifolds such coverings can always be found. Then we can make the following complex

$$\check{C}^0(M,G) \xrightarrow{\delta_0} \check{C}^1(M,G) \xrightarrow{\delta_1} \check{C}^2(M,G) \xrightarrow{\delta_2} \cdots \tag{A.4}$$

with $\check{C}^q(M,G)$ consisting of maps α that assign an element $\alpha_{i_0 i_1 \ldots i_q}$ of the abelian group G to every intersection of $q+1$ different U_i's. We use the convention that interchanging any of the indices leads to a change of sign

$$\alpha_{i_0 i_1 .. i_k i_{k+1} .. i_q} = -\alpha_{i_0 i_1 .. i_{k+1} i_k .. i_q} \tag{A.5}$$

The coboundary operator $\delta_q : \check{C}^q(M,G) \to \check{C}^{q+1}(M,G)$ is defined as

$$(\delta_q \alpha)_{i_0 i_1 \ldots i_{q+1}} := \sum_{j=0}^{q+1} (-1)^j \alpha_{i_0 i_1 \ldots i_{j-1} i_{j+1} \ldots i_{q+1}}, \tag{A.6}$$

omitting the j-th term. One can check that $\delta_{q+1} \sim \delta_q = 0$. The q-th Čech cohomology ($q \geq 0$) can then be defined as

$$H^q(M,G) := \ker(\delta_q) / \operatorname{img}(\delta_{q-1}). \tag{A.7}$$

Elements of $\alpha \in \check{C}^q(M,G)$ with $\delta_q \alpha = 0$, are called *cocycles*, and elements that can be written as $\delta_{q-1}\beta$ *coboundaries*. So the Čech cohomology spaces consist of the cocycles modulo the coboundaries. $H^0(M,G)$ consists of cocycles that assign an element of G to every connected component of M.
The group G need not be a fixed group but may depend on the $U_{i_0 i_1 .. i_q}$. For instance the elements of $\alpha \in \check{C}^q(M,\Omega^p)$ assign to every intersection $U_{i_0 i_1 .. i_q}$ an

element $\alpha_{i_0 i_1 .. i_q} \in \Omega^p(U_{i_0 i_1 .. i_q}$ of p-forms on that intersection.
One can show that for manifolds the cohomology of de Rham and Čech, using $\mathbb{R}$ as the abelian group G, are the same. So $H^q_{dR}(M) = H^q(M, \mathbb{R})$. For this we need the Poincaré lemma, which says that $H^q_{dR}(U) = 0$ for $q \geq 1$ if U is contractible. This means that every closed q-form on U is exact. Further we need the fact that $H^q(M, \Omega^p) = 0$ for all $p, q \geq 0$. This means that every cocycle of p forms is a coboundary.
Let us take $H^2(M)$ as an example. If F is a closed 2-form, the restriction of F to every U_i can be written as dA_i with A_i a 1-form on U_i, because the U_i are contractible (Poincaré lemma). The set A_i can be seen as an element of $\check{C}^1(M, \Omega^1)$, and $(\delta_1 A)_{ij} = A_j - A_i$ as an element of $\check{C}^2(M, \Omega^1)$. It is a closed 1-form on every U_{ij}, therefore we can write $A_i - A_j = dg_{ij}$ with g_{ij} a 0-form (smooth function) on U_{ij}. g_{ij} is an element of $\check{C}^2(M, \Omega^0)$. $c_{ijk} = (\delta_2 g)_{ijk} = g_{jk} - g_{ik} + g_{ij}$ is closed so is a constant function on U_{ijk}, therefore $c_{ijk} \in \check{C}^2(M, \mathbb{R})$. One can show that $\delta_3 c = 0$. In this way the class $[F]$ in $H^2_{dR}(M)$ corresponds to a class $[c_{ijk}]$ in $H^2(M, \mathbb{R})$.
The other way around, suppose c_{ijk} is a cocycle of $\mathbb{R}$ constants. Seen as a cocycle of smooth functions (0-forms) it is then a coboundary, so $c_{ijk} = g_{ij} + g_{jk} + g_{ki}$ for some $g \in \check{C}^1(M, \Omega^0)$ of smooth functions g_{ij}. dg is a cocycle since $dg_{ij} + dg_{jk} + dg_{ki} = dc_{ijk} = 0$. Therefore $dg_{ij} = A_i - A_j$ for some $A \in \check{C}(M, \Omega^1)$. Applying the same trick again we get a global 2-form F with $F = dA_i$.
In general the correspondence between a de Rham and a Čech cohomology class can be computed in the following diagram:

$$\begin{tikzcd}
\cdots & \cdots & \cdots & \cdots & \\
\Omega^2(M) \arrow[r] \arrow[u, "d"] & \check{C}^0(M,\Omega^2) \arrow[r, "\delta"] \arrow[u, "d"] & \check{C}^1(M,\Omega^2) \arrow[r, "\delta"] \arrow[u, "d"] & \check{C}^2(M,\Omega^2) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
\Omega^1(M) \arrow[r] \arrow[u, "d"] & \check{C}^0(M,\Omega^1) \arrow[r, "\delta"] \arrow[u, "d"] & \check{C}^1(M,\Omega^1) \arrow[r, "\delta"] \arrow[u, "d"] & \check{C}^2(M,\Omega^1) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
\Omega^0(M) \arrow[r] \arrow[u, "d"] & \check{C}^0(M,\Omega^0) \arrow[r, "\delta"] \arrow[u, "d"] & \check{C}^1(M,\Omega^0) \arrow[r, "\delta"] \arrow[u, "d"] & \check{C}^2(M,\Omega^0) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
 & \check{C}^0(M,\mathbb{R}) \arrow[r, "\delta"] \arrow[u, "d"] & \check{C}^1(M,\mathbb{R}) \arrow[r, "\delta"] \arrow[u] & \check{C}^2(M,\mathbb{R}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots
\end{tikzcd}$$

where the arrows $\Omega^q(M) \to \check{C}^0(M, \Omega^q)$ maps a q-form to the cocycle that assigns this same q-form to every U_i. Filling in the representative of the a cohomology class in the leftmost column, working your down the diagonal gives the corresponding Čech cocycle in the bottom row. One can even define another equivalent cohomology consisting of diagonals of this diagram and a map

$$\begin{aligned} D : \Omega^q(M) \oplus \check{C}^0(M, \Omega^{q-1}) \oplus \check{C}^1(M, \Omega^{q-2}) \oplus ..\check{C}^q(M, \mathbb{R}) \to \\ \Omega^{q+1}(M) \oplus \check{C}^0(M, \Omega^q) \oplus \check{C}^1(M, \Omega^{q-1}) \oplus ..\check{C}^{q+1}(M, \mathbb{R}) \end{aligned} \quad (A.8)$$

between subsequent diagonals given by

$$D : (\lambda, \lambda^{(0)}, \lambda^{(1)}, .., \lambda^{(q)}) \mapsto (d\lambda, \lambda - d\lambda^{(0)}, \delta\lambda^{(0)} + d\lambda^{(1)}, ..\delta\lambda^{(q)}) \quad (A.9)$$

with an alternating sign before the d. Then again $D^2 = 0$ and the cohomology consists of equivalence classes of diagonals with $D(...) = 0$ modulo the diagonals that are in the image of D. We have $[\lambda] = [(\lambda, \lambda^{(0)}, \lambda^{(1)}, ..\lambda^{(q)})] = [\lambda^{(q)}]$. For instance the case of the degree 2 cohomology is given by

$$\begin{array}{c|cccc} \Omega^2 & F & & & \\ \Omega^1 & .. & A_i & & \\ \Omega^0 & .. & .. & g_{ij} & \\ \mathbb{R} & & .. & .. & c_{ijk} \\ \hline & \mathrm{M} & U_i & U_{ij} & U_{ijk} \end{array} \quad (A.10)$$

with $[F] \cong [(F, -A_i, g_{ij}, -c_{ijk})] \cong [c_{ijk}]$. We need the extra minus signs in every even column if we want to have $F = dA_i$ etc.
A very useful technique in cohomology uses a short exact sequence (i.e. the image of one map is the kernel of the following)

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \quad (A.11)$$

of abelian groups A, B and C. So here f is injective, g surjective and $\ker g = f(A)$. This then gives the following long exact sequence

$$\begin{aligned} &H^0(M,A) \xrightarrow{f^*} H^0(M,B) \xrightarrow{g^*} H^0(M,C) \to \\ &\to H^1(M,A) \xrightarrow{f^*} H^1(M,B) \xrightarrow{g^*} H^1(M,C) \to \\ &\to H^2(M,A) \xrightarrow{f^*} \cdots \end{aligned} \quad (A.12)$$

which proceeds between all cohomology groups. Let $[\gamma] \in H^q(M, C)$, by the short exact sequence there is a $\beta \in \check{C}^q(M, C)$ such that $\gamma = g(\beta)$. Because $g(\delta\beta) = \delta g(\beta) = 0$, there is a $\alpha \in \check{C}^{q+1}(M, A)$ such that $f(\alpha) = \delta\beta$. Furthermore because $f(\delta\alpha) = \delta f(\alpha) = 0$, $\delta\alpha = 0$ so $[\alpha] \in H^{q+1}(M, A)$. This defines the map $H^q(M, C) \to H^{q+1}(M, A)$. The exactness of the sequence can be checked by similar arguments.

A.2 Integrals of cohomology classes and homology

The definition of a de Rham cohomology class of degree q is such, that its integral over a closed (i.e. compact and no boundaries) oriented (sub)manifold S of dimension q is well defined, because it is invariant under addition of $d\lambda$, with λ any $q-1$-form. For a Čech cohomology class it is not directly clear how to define its integral over a manifold. Let us take again the example of degree 2. Let S be a closed oriented surface. Divide the surface in patches such that every patch S_j lies inside one of the U_i. We can refine the covering such that in every open set U_i lies at most one patch, and renumber the patches and open sets such that $S_i \subset U_i$ for every patch S_i.

$$\int_S [F] = \sum_i \int_{S_i} dA_i = \sum_i \oint_{\partial S_i} A_i. \tag{A.13}$$

We can choose the patches to be triangular. All edges border on two triangles, say S_i and S_j, the integral of A_i and A_j are in opposite directions.

$$\int_{\text{edge}} A_i - A_j = \int_{\text{edge}} dg_{ij}. \tag{A.14}$$

This integral is equal to the difference of g_{ij} in both endpoints of the edge, the vertices. One can make the triangulation such that at every vertex three edges start. If these three edges lie between the patches S_i, S_j and S_k in clockwise order (with respect to the orientation of the surface) its contribution to the integral is $g_{ij} + g_{jk} + g_{ki} = c_{ijk}$. So we can write

$$\int_S [F] = \sum_{\text{vertices}} c_{ijk}. \tag{A.15}$$

In this way we have expressed the integral of the cohomology class in terms of its Čech cycle. In particular if the Cech cohomology class can be written as an integer valued cocycle, so it can be a lifted to a class in $H^2(M, \mathbb{Z})$ we know that the integral is integer valued. To be able to define the analogue for Čech cohomology classes with coefficients not in $\mathbb{R}$ or $\mathbb{Z}$, we need to apply the former procedure in a more formal way.
The subset

$$\Delta_q := \{\sum_{i=0}^{q} t_j P_j | \sum_{i=0}^{q} t_j = 1, t_j \geq 0\} \tag{A.16}$$

of $\mathbb{R}^q$ with basis $\{P_j\}$, is called the *standard q-simplex.* So for $q = 0, 1, 2, 3, ...$ we have a point, a line, a triangle, a tetrahedron, etc. A *singular q simplex* is a continuous map $\Delta_q \to M$. We define the space $S_q(M, \mathbb{Z})$ of *singular q-chains* to be the space of formal sums of singular q simplices, that is arbitrary linear combinations of simplices with $\mathbb{Z}$-coefficients. We define the *boundary operator* $\partial : S_q(M, \mathbb{Z}) \to S_{q-1}(M, \mathbb{Z})$ by

$$(\partial s)(t_0, t_1, ..t_{q-1}) = \sum_{i=0}^{q} (-1)^i s(t_0, t_1, .., t_{i-1}, 0, t_{i+1}, ...t_{q-1}). \tag{A.17}$$

Again we have $\partial^2 = 0$. Then the *singular homology* with $\mathbb{Z}$-coefficients of degree q is given by

$$H_q(M, \mathbb{Z}) = \ker(\partial : S_q \to S_{q-1}) / \operatorname{img}(\partial : S_{q+1} \to S_q) \tag{A.18}$$

A singular q-chain s with $\partial s = 0$ is called a *cycle* and the image δs of a q chain a *boundary*. The boundary of a q-chain σ consists of $q-1$-simplices, called *faces*, which either have the same orientation as σ or the opposite but with a minus sign. An oriented triangulation of a closed oriented submanifold Q of M is a combination of simplices $\sigma_j : \Delta_q \to Q$ which must be orientation preserving maps into Q and cover all of Q. The overlapping faces of two bordering simplices in this triangulation, either have the same orientation but opposite signs, so they vanish in the boundary of the total triangulation, or opposite orientation but the same sign. Then one can show that one can add some 'degenerate' simplices to the triangulation, which cancel these non vanishing faces. So roughly speaking, in homology opposite oriented faces cancel out, and therefore the total boundary of the triangulation vanishes. Thus a closed oriented submanifold Q of dimension q gives a homology class $[Q] \in H_q(M, \mathbb{Z})$. For non-oriented closed submanifolds, the boundary of the triangulation will not vanish, since not all faces are opposite oriented. But it does vanish if we take $\mathbb{Z}_2$ coefficients in stead of $\mathbb{Z}$.
Let $s = \sum_j g_i \sigma_i$ be a singular q-chain with $g_i \in \mathbb{Z}$ and σ_i singular q-simplices. The integral of a q-form λ over s is defined as

$$(s, \lambda) = \sum_i g_i \int_{\sigma_i} \lambda. \tag{A.19}$$

Let ω be a $q-1$-form, then

$$\begin{aligned}(s, d\omega) &= \sum_i g_i \int_{\sigma_i} d\omega \\ &= \sum_i g_i \int_{\partial \sigma_i} \omega \\ &= (\partial s, \omega).\end{aligned} \tag{A.20}$$

Therefore we can define the integral of a de Rham cohomology class $[\lambda] \in H^q_{dR}(M)$ over a homology cycle class $[s] \in H_q(M, \mathbb{Z})$ by the pairing $([s], [\lambda]) = (s, \lambda)$. This coincides with the usual definition. So

$$(s, \lambda) = \int_S \lambda. \tag{A.21}$$

As we saw in the 2-dimensional example, the integral of a Čech cohomology class over a singular q-cycle, becomes a sum of the vertices lying in the intersections $U_{i_0 i_1 .. i_q}$. For the general case (the q-cycle need not be a triangulation of a manifold, and we want to be able to work with coefficients other than $\mathbb{Z}$), we need a formal way of deriving what combination of vertices we have to use. Given a good cover $\mathcal{U} = \{U_i\}$ of the manifold M, for every class in $c \in H_q(M, G)$ it is possible to find a q-cycle s in the class c, so $c = [s]$, such that every simplex of s lies entirely in one of the U_i. This can be done by breaking up the simplices that are too big, without changing the homology class. We say $s \in S_q^{\mathcal{U}}(M, \mathbb{Z})$. We can write s as a cochain $t_j^{(0)}$ of singular q-chains such that $t_i^{(0)} \subset U_i$. This cochain is thus a member of $\check{C}^0(M, S_q)$. We have the following exact sequence

$$0 \longleftarrow S_q^{\mathcal{U}}(M, \mathbb{Z}) \xleftarrow{\check{\partial}} \check{C}^0(M, S_q) \xleftarrow{\check{\partial}} \check{C}^1(M, S_q) \xleftarrow{\check{\partial}} \cdots \tag{A.22}$$

where $\check{\partial}$ is the *Čech boundary operator*

$$(\check{\partial} s)_{i_0 i_1 .. i_q} = \sum_i s_{i i_0 i_1 .. i_q}. \tag{A.23}$$

It is important here that the boundary operator ∂ and the Čech boundary operator $\check{\partial}$ commute. Then with $s_i^{(0)} := \partial t_i^{(0)}$

$$\check{\partial} s^{(0)} = \partial \check{\partial} t^{(0)} = \partial s = 0. \tag{A.24}$$

So $s_i^{(0)} \in \check{C}^0(M, S_{q-1})$ can be lifted to an element $t_{ij}^{(1)} \in \check{C}^1(M, S_{q-1})$ with $\check{\partial} t^{(1)} = s^{(0)}$. We define $s^{(1)} := \partial t^{(0)}$. Repeating this trick a few times we derive an element $s^{(q-1)} \in \check{C}^{q-1}(M, S_0)$. By construction this element has $\check{\partial} s^{(q-1)} = 0$, so it can again be lifted to an element $t^{(q)} \in \check{C}^q(M, S_0)$ which is a combination $\sum_j g_{i_0 i_1 .. i_q, j} v_{i_0 i_1 .. i_q, j}$ of vertices $v_{..}$ in every $U_{i_0 i_1 .. i_q}$ with coefficients $g_{..}$ in $\mathbb{Z}$. We define $s_{i_0 i_1 .. i_q}^{(q)} = \sum_j g_{i_0 i_1 .. i_q, j}$. Let us take again the example in which s is the triangulation of some closed oriented surface. For this case

$$\begin{array}{c|cccc}
S_2 & s & t_i^{(0)} & & \\
S_1 & .. & s_i^{(0)} & t_{ij}^{(1)} & \\
S_0 & .. & .. & s_{ij}^{(1)} & t_{ijk}^{(2)} \\
\mathbb{Z} & .. & .. & .. & s_{ijk}^{(2)} \\
\hline
 & M & U_i & U_{ij} & U_{ijk}
\end{array} \tag{A.25}$$

where $s_i^{(0)}, s_{ij}^{(1)}$ are combinations of edges and vertices respectively, and $s_{ijk}^{(2)}$ gives the sum of the coefficients of the vertices in every U_{ijk}. Combing this with diagram (A.10), we see that we can calculate the integral as the integral of F over s, the A_i over the $s_i^{(0)}$, the g_{ij} over the $s_{ij}^{(1)}$, or summing the $c_{ijk} \cdot s_{ijk}^{(2)}$. So in general if starting with a q-cycle s and a q-form λ, $s^{(p)}$ is in $\check{C}^p(M, S_{q-p})$ and $\lambda^{(p)}$ in $\check{C}^p(M, \Omega^{q-p})$. Define

$$(s^{(p)}, \lambda^{(p)}) := \sum_{i_0 < i_1 < .. i_p} (s_{i_0 i_1 .. i_p}^{(p)}, \lambda_{i_0 i_1 .. i_p}^{(p)}). \tag{A.26}$$

Then

$$\int_s \lambda = (s, \lambda) = (s^{(p)}, \lambda^{(p)}) \tag{A.27}$$

for all $0 \leq p \leq q$. And in particular we now know how to integrate a Čech cohomology class over s. Even if a Čech cohomology class $[\alpha]$ of degree q has coefficients other than $\mathbb{R}$, so there is no corresponding class of q-forms in de Rham cohomology, the integral is given by

$$([s], [\alpha]) = (s^{(q)}, \alpha). \tag{A.28}$$

This is well defined on the classes by the following variant of Stokes' theorem(A.20)

$$(s^{(q)}, \delta\beta) = (\check{\partial} s^{(q)}, \beta) \tag{A.29}$$

with β a $q-1$ Čech cochain.

A.3 Vector bundles

First let us recall the definition of a *fiber bundle*. A fiber bundle is a quadruple (E, M, F, π) of smooth manifolds F, the *bundle*, M, the *base*, and F the *fiber* and a map $\pi : E \to M$, the *projection*. The bundle is required to be locally trivializable, that is for every $x \in M$ there is a neighborhood U such that there is a diffeomorphism τ, the trivialization, between $\pi^{-1}(U) \subset E$ and $U \times F$ such that $\pi \circ \tau = \pi$. The pre-image $\pi^{-1}(\{x\})$ of a point $x \in M$, is called the fiber at x and denoted E_x. Two fiber bundles E and E' on the same base space M and with equal fibers F are *equivalent* if there is a diffeomorphism $E \to E'$ that commutes with the respective projections π and π'.
The base space can be covered by open sets U_i and trivializations τ_i. If U_i and U_j overlap in $U_{ij} = U_i \cap U_j$, one can define a map

$$\phi_{ij} : U_{ij} \times F \to U_{ij} \times F; (u, f) \mapsto \tau_j \circ \tau_i^{-1}(u, f), \tag{A.30}$$

which 'translates' the local trivialization on U_i to that of U_j. By definition on a triple overlap $U_{ijk} = U_{ij} \cap U_{jk} \cap U_{ik}$

$$\phi_{ki} \circ \phi_{jk} \circ \phi_{ij} = \mathrm{Id}\,. \tag{A.31}$$

This is the same cocycle condition as for Čech cocycles. If for the same covering a different set trivializations τ_i' are given, with $\lambda_i = \tau_i' \circ \tau_i^{-1}$ we have

$$\phi_{ij}' = \tau_j' \circ \tau_i'^{-1} = \lambda_j \circ \phi_{ij} \lambda_i^{-1}. \implies \phi_{ij} \sim \lambda_j \circ \phi_{ij} \circ \lambda_i^{-1}. \tag{A.32}$$

This is the same equivalence relation as in Čech cohomology, now for the non-commutative group of diffeomorphisms. A collection of open sets U_i covering M and diffeomorphisms ϕ_{ij} on $U_{ij} \times F$, with $\pi \circ \phi = \phi$ and satisfying the cocycle condition, is called a collection of local coordinates. Given such a collection one can reconstruct the original bundle F. A second collection (U_i, ϕ_{ij}') using the same covering, produces an equivalent fiber bundle if and only if the ϕ_{ij}' can be related to the ϕ_i by (A.32) for some set λ_i of diffeomorphisms on $U_i \times F$ with $\pi \circ \lambda_i = \pi$.
A fiber bundle is called a *real vector bundle* if its fiber is a real vector space

and the maps ϕ_{ij} are linear isomorphisms in every point of U_{ij}. As every finite-dimensional real vector space is isomorphic to $\mathbb{R}^k$ for some k, we simply take $F = \mathbb{R}^k$. Analogously we define the complex vector bundle with $F = \mathbb{C}^k$. Here k is called the rank of the bundle E, denoted $\mathrm{rk}\, E$. If the base M is not connected it is possible to take different rank vector spaces at the different connected components of M. Then rk is a local constant function on M.
Note that the ϕ_{ij} in some point u, denoted $\phi_{ij}(u)$, is an element of $GL(k, \mathbb{R})$ or $GL(k, \mathbb{C})$ the spaces of linear isomorphism on real and complex vector spaces. So in fact ϕ_{ij} is a map U_{ij} to $GL(k, \mathbb{R})$ or $GL(k, \mathbb{C})$. Vector bundles on M are called equivalent if they are equivalent as fiber bundles and the diffeomorphism is a linear isomorphism on every fiber. The local coordinates of two equivalent vector bundles, (U_i, ϕ_{ij}) and (U_i, ϕ'_{ij}) are again related by (A.32) with the λ_i now maps on U_i to $GL(k, \mathbb{R})$ or $GL(k, \mathbb{C})$. A vector bundle is called *orientable*, if one can choose local coordinates such that $\det \phi_{ij} > 0$ on all U_{ij}. The local coordinates can always be chosen such that the ϕ_{ij} are in $O(k)$ for real bundles and in $U(k)$ for complex bundles. For orientable bundles they can be chosen to be in $SO(k)$ and $SU(k)$. This shows that if the vector bundle is endowed with a metric, that is a smoothly varying positive definite symmetric bilinear form on each fiber, one can choose local coordinates such that the standard base in each fiber is orthonormal with respect to this metric.
The various constructions that can be made using vector spaces, can be transferred to the level of vector bundles. For instance the direct sum $E \oplus F$ of two vector bundles E and F on M is the bundle obtained by taking the direct sum of the fibers at each point of M. In the same way one defines the direct product $E \otimes F$ of two vector bundles. The fibers of the bundle E^* are the dual vector spaces of the fibers of E.
Finally we define the space of sections $\Gamma(E)$, that are maps $s : M \to E$ with $\pi \circ s = \mathrm{Id}$. In local coordinates they can be given as maps $s_i : U_i \to \mathbb{R}^k$ or $s_i : U_i \to \mathbb{C}^k$, which are related by $s_j = h_{ij} s_i$ on each intersection U_{ij}.
The space of all vector bundles over a smooth manifold X, is denoted by $\mathrm{Vect}(X)$. If $f : X \to Y$ is a smooth map, the pull-back $f^* : \mathrm{Vect}(Y) \to Vect(X)$ is defined as

A.4 Line bundles

Hermitian complex line bundles, or equivalently principal $U(1)$ bundles, are up to isomorphism (hermitian means that isomorphisms leave the metric invariant) given by a cocycle class $[h_{ij}] \in H^1(M, \mathbf{U}_1)$. The h_{ij} are smooth $U(1)$-valued functions on every intersection U_{ij}, giving the transformation between the local trivializations $U_i \times \mathbb{C}$ and $U_j \times \mathbb{C}$ of the bundle. The space of smooth $U(1)$-valued functions will be denoted by $\mathbf{U}_1(M)$ to distinguish it from $U(1)$ itself. The short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi\cdot} \Omega^0(M) \xrightarrow{\exp(\mathrm{i}.)} \mathbf{U}_1(M) \longrightarrow 1 \,, \tag{A.33}$$

gives

$$.. \to H^1(M,\Omega^0) \longrightarrow H^1(M,\mathbf{U}_1) \longrightarrow H^2(M,\mathbb{Z}) \longrightarrow H^2(M,\Omega^0) \to ..$$

and because both $H^1(M,\Omega^0)$ and $H^2(M,\Omega^1)$ are trivial, this gives an isomorphism between $H^1(M,\mathbf{U}_1)$ and $H^2(M,\mathbb{Z})$. If g_{ij} is a cochain of real functions such that $h_{ij} = \exp(\mathrm{i}g_{ij})$, then

$$g_{ij} + g_{jk} + g_{ki} = 2\pi c_{ijk} \tag{A.34}$$

with c_{ijk} a $\mathbb{Z}$-valued cocycle, whose class, called the *first Chern class*, corresponds to the class $[h_{ij}]$ under this isomorphism. $2\pi c_{ijk}$ gives a class in $H^2(M,\mathbb{R})$ and by

$$\begin{array}{c|cccc} \Omega^2 & F & & & \\ \Omega^1 & .. & A_i & & \\ \Omega^0 & .. & .. & g_{ij} & \\ \mathbb{R} & & .. & .. & 2\pi c_{ijk} \\ \hline & \mathrm{M} & U_i & U_{ij} & U_{ijk} \end{array} \tag{A.35}$$

this gives a class $[F]$ in de Rham cohomology called the curvature. The 1-forms A_i define a connection on the space $\Gamma(L)$ of sections of the line bundle L. The sections can be described on every U_i by a function $s_i : U_i \to \mathbb{C}$, transforming $s_j = h_{ij}s_i$ on U_{ij}. The connection is then

$$(\nabla s)_i = ds_i - \mathrm{i}A_i s_i \text{ on every } U_i \tag{A.36}$$

This defines an element of $\Gamma(L) \otimes \Omega^1(M)$. The *hermitian line bundles with connection* are up to isomorphism given by an equivalence class of pairs (A_i, h_{ij}) satisfying

$$A_j - A_i = \mathrm{i}d\log(h_{ij}), \quad h_{ij}h_{jk}h_{ki} = 1, \tag{A.37}$$

the equivalence given by

$$(A_i, h_{ij}) \sim (A_i + \mathrm{i}\lambda_i^{-1}d\lambda_i, \lambda_j h_{ij}\lambda_i^{-1}) \text{ for any } \lambda \in \check{C}^0(M,\mathbf{U}_1). \tag{A.38}$$

The space of these equivalence classes is denoted by $H^1(M,\mathbf{U}_1 \to \Omega^1)$. Consider also the equivalence classes of triples (A_i, g_{ij}, c_{ijk}) with equivalence relation

$$(A_i, g_{ij}, c_{ijk}) \sim (A_i + dl_i, g_{ij} - l_j + l_i + dr_{ij}, c_{ijk} + \delta r_{ijk}) \tag{A.39}$$

for any $l \in \check{C}^0(M,\Omega^0)$ and any $r \in C^1(M,\mathbb{Z})$. With identifications $h_{ij} = \exp(\mathrm{i}g_{ij})$ and $\lambda_i = \exp(-\mathrm{i}l_i)$, it is easily checked that the space of equivalence classes of triples satisfying $A_j - A_i = -dg_{ij}$ and (A.34), is the same as $H^1(M,\mathbf{U}_1 \to \Omega^1)$.

Let $\pi : L \to M$ be a hermitian line bundle with connection $\nabla = d + \mathrm{i}A_i$, and $\gamma : [0,1] \to M$ be a path through M. The parallel transport of a vector v of the fiber $L_{\gamma(0)}$ over γ is a path P_v through L with $\pi \circ P_v = \gamma$ and $P_v(0) = v$ satisfying the differential equation

$$\left[\frac{d}{dt} - \mathrm{i}A_i(\gamma(t)) \cdot \gamma'(t)\right] P_{v,i}(t) = 0 \tag{A.40}$$

where $P_{v,i}$ is P_v in local coordinates. So if $\gamma(t)$ lies in the coordinate region U_i, $\gamma(t) \times P_{v,i}(t) \in U_i \times \mathbb{C}$ give the local coordinates of $P_v(t)$. If $\gamma([t_0, t_1]) \subset U_i$, the local solution of (A.40) is given by

$$P_{v,i}(t) = \exp(\mathrm{i} \int_{t_0}^{t} A_i(\gamma(t))dt) \cdot P_{v,i}(t_0) \tag{A.41}$$

for $t \in [t_0, t_1]$. Let $\gamma(0) \times z$ be the local coordinates of v in $U_i \times \mathbb{C}$. If γ is a closed loop, Hol_γ that maps z to $P_{v,i}(1)$, is called the *holonomy* around γ. Given a triangulation of γ

$$\begin{array}{c|ccc} S_1 & \gamma & c_i^{(0)} & \\ S_0 & .. & \gamma_i^{(0)} & c_{ij}^{(1)} \\ G & .. & .. & \gamma_{ij}^{(1)} \\ \hline & \mathrm{M} & U_i & U_{ij} \end{array} \tag{A.42}$$

(so $c_i^{(0)}$ gives the segments of γ that lie in U_i and $c_{ij}^{(1)} = \sum_l g_{ij,l} v_{ij,l}$ the vertices $v_{ij,l} \in U_{ij}$), and using (A.41) and $P_{v,j} = h_{ij} P_{v,i}$ one sees that

$$\mathrm{Hol}_\gamma = \exp(\mathrm{i}(c^{(0)}, A))(c^{(1)}, h) \tag{A.43}$$

where $(c^{(1)}, h)$ is calculated with $U(1)$-coefficients so with products in stead of sums:

$$(c^{(1)}, h) = \prod_{i<j} \prod_l h_{ij}(v_{ij,l})^{g_{ij,l}}. \tag{A.44}$$

It is easy to see that this expression is indeed invariant under the equivalence relation (A.38). Furthermore one can check that it is independent of the triangulation of γ. Now suppose we can write $\gamma = \partial s$, that is γ consists of one or more path components forming the border of a surface s. By (A.22) we have $s = \check{\partial} t^{(0)}$. Because $\check{\partial}\partial t^{(0)} = \gamma$, we can take $c^{(0)} = \partial t^{(0)}$, and because now $\partial c^{(0)} = 0$, $c^{(1)} = 0$. This is a very algebraic way of saying that the border of an oriented surface can be written as a sum of closed paths each inside the local U_i patches. The holonomy becomes

$$\mathrm{Hol}_\gamma = \exp(\mathrm{i}(\partial t^{(0)}, A)) = \exp(\mathrm{i}(t^{(0)}, dA)) = \exp(\mathrm{i} \int_s F) \tag{A.45}$$

Because $[F] = 2\pi[c]$ and $[c] \in H^2(M, \mathbb{Z})$ this is equal to one. This shows that Hol_γ only depends on the homology class $[\gamma] \in H_1(M, \mathbb{Z})$.
Suppose $[c] \in H^2(M, \mathbb{Z})$ vanishes. This means $[h] \in H^1(M, \mathbf{U}_1)$ vanishes, so we can write $h_{ij} = \lambda_j \lambda_i^{-1}$ with some $\lambda_i \in \check{C}^0(M, \mathbf{U}_1)$. Consequently in a vector bundle with connection we can gauge h_{ij} to 1, but then $A_i - A_j = 0$ so if M is connected we have a global A and the holonomy is given by

$$\mathrm{Hol}_\gamma = \exp(\mathrm{i} \int_\gamma A). \tag{A.46}$$

For this reason the holonomy in general is often given by the formula on the righthand side. The argument can be summarized in the following exact sequence

$$0 \longrightarrow H^1(M,\mathbb{R}) \longrightarrow H^1(M,\mathbf{U}_1 \to \Omega^1) \longrightarrow H^1(M,\mathbf{U}_1) \longrightarrow 1, \tag{A.47}$$

with a global $A \in H^1(M,\mathbb{R})$ giving a bundle with connection $[(A,1)] \in H^1(M, \mathbf{U}_1 \to \Omega^1)$ with vanishing first Chern class in $H^1(M,\mathbf{U}_1)$.

A.5 U(n) bundles

For (higher rank) vector bundles things are more complicated. The h_{ij} are now a cocycle of $U(n)$-valued functions. We can choose hermitian matrices g_{ij} such that $h_{ij} = \exp(\mathrm{i} g_{ij})$. In the same way as for a line bundle we can construct 'hermitian matrix'-valued 1-forms A_i on the U_i, a global 2-form F out of the direct product of $\Omega^2(M)$ and the space of maps from M to the hermitian matrices, and constant hermitian matrices c_{ijk} on every U_{ijk}, which have to satisfy $\exp(\mathrm{i}c_{ijk}) = 1$ for the h_{ij} to be a cocycle. They satisfy similar relations as for the line bundle (A.37)

$$F = dA_i, \quad A_j = h_{ij} A_i h_{ij}^{-1} + \mathrm{i} h_{ij}^{-1} dh_{ij}, \quad h_{ij} h_{jk} h_{ki} = 1 \tag{A.48}$$

The connection is the same as in (A.36)

$$(\nabla s)_i^\alpha = ds_i^\alpha + \mathrm{i} A_{\beta,i}^\alpha s_i^\beta \tag{A.49}$$

only the sections s in local coordinates s_i^α have an index $1 \le \alpha \le n$, and the A_i are matrices with indices α and β with summation over β assumed. The local solutions $P_{v,i}$ of the parallel transport of a vector v, is no longer a simple expression as in (A.41).

$$\begin{aligned} P_{v,i}(t) &= \mathrm{P}\exp(-\mathrm{i}\int_{t_0}^t A_i(\gamma(t'))dt') \cdot P_{v,i}(t_0) = (1 - \mathrm{i}\int_{t_0}^t A_i(\gamma(t'))dt' \\ &- \frac{1}{2}\int_{t'=t_0}^t \int_{t''=t_0}^{t'} A_i(\gamma(t')) \wedge A_i(\gamma(t''))dt'dt'' + ..) \cdot P_{v,i}(t_0) \end{aligned} \tag{A.50}$$

where P stands for path ordered. The local solutions $P_{v,i}$ can be patched together by $P_{v,j} = h_{ij} P_{v,i}$. The mapping $v_i \mapsto P_{v,i}(1)$ with $\gamma(0) \times v_i$ the local coordinates of $v \in L_{\gamma(0)}$, is a linear map $\mathbb{R}^n \to \mathbb{R}^n$ which can be seen to be independent of the chosen local coordinate i.

A.6 Clifford Algebras

This appendix tries to give some basic notions about Clifford algebras and modules. More information can be found in [16]. The Clifford algebra $Cl_{r,s}$ is the algebra over $\mathbb{R}$ generated by the vector space $\mathbb{R}^{r+s}$ and the identity 1, subject to the relations

$$v \cdot w + w \cdot v = -2q(v,w)1 \tag{A.51}$$

for any $v, w \in \mathbb{R}^{r+s}$. $v \cdot w$ is the Euclidean inner product and q is a symmetric bilinear form on $\mathbb{R}^{r+s}$:

$$q(v,w) = v_1 w_1 + v_2 w_2 + ..v_r w_r - v_{r+1} w_{r+1} - v_{r+2} w_{r+2} - ..v_{r+s} w_{r+s}. \quad \text{(A.52)}$$

The standard basis of $\mathbb{R}^{r+s}$ is denoted by Γ_i. Thus

$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = \begin{cases} 2\delta_{ij} & i \leq r \\ -2\delta_{ij} & i < r \end{cases} \quad \text{(A.53)}$$

This shows that the relations given in (3.16) generate the Clifford algebra $Cl_{1,D-1}$. Further we define

$$\Gamma = \Gamma^1 \Gamma^2 .. \Gamma^n, \quad \text{(A.54)}$$

with $n = r + s$, which has the properties

$$\Gamma^2 = (-1)^{\frac{n(n+1)}{2}+s}, \quad x\Gamma = (-1)^{n-1} \Gamma x. \quad \text{(A.55)}$$

for any $x \in \mathbb{R}^n$. The spin group is the subset of $Cl_{r,s}$ defined by

$$Spin(r,s) = \{v_1 v_2 ... v_k | q(v_i, v_i) = \pm 1 \text{ for all } i \text{ and } k \text{ is even}\} \quad \text{(A.56)}$$

It is a double cover of the group $SO(r,s)$, of linear transformations with det $= 1$ which leave q invariant. This can be expressed in the following exact sequence

$$0 \to \mathbb{Z}_2 \to Spin(r,s) \to SO(r,s) \to 1 \quad \text{(A.57)}$$

The complex Clifford algebra is the complex algebra

$$\mathbb{C}l_{r,s} = Cl_{r,s} \otimes_{\mathbb{R}} \mathbb{C}. \quad \text{(A.58)}$$

For all $r + s = n$ we have

$$\mathbb{C}l_{r,s} \cong \mathbb{C}l_{n,0} \quad \text{(A.59)}$$

So we will usually just talk about $\mathbb{C}l_n := \mathbb{C}l_{n,0}$. In the complex Clifford algebra we can define

$$\begin{aligned} \Gamma_{\mathbb{C}} &= \mathrm{i}^{\frac{n}{2}} \Gamma && \text{for } n \text{ even,} \\ \Gamma_{\mathbb{C}} &= \mathrm{i}^{\frac{n+1}{2}} \Gamma && \text{for } n \text{ odd.} \end{aligned} \quad \text{(A.60)}$$

For all n it has the following properties

$$\Gamma_{\mathbb{C}}^2 = 1, \quad x\Gamma = (-1)^{n+1} \Gamma_{\mathbb{C}} x \quad \text{(A.61)}$$

for all $x \in \mathbb{C}^n$.

A representation of the Clifford algebra $Cl_{r,s}$ is a homomorphism of real algebras between $Cl_{r,s}$ and the algebra of linear transformations of a finite dimensional vector space V.

$$\rho : Cl_{r,s} \to \mathrm{Hom}(V,V) \quad \text{(A.62)}$$

This space V is called a Clifford module and can be either a real or a complex vector space, giving a real or a complex representation of the Clifford algebra. The Clifford module is called reducible if it can be written as a direct sum

$$V = V_1 \oplus V_2 \tag{A.63}$$

of subspaces V_1, V_2 which are invariant under the action of $Cl_{r,s}$, i.e. $\rho(\phi)V_i \subseteq V_i$ for all $\phi \in Cl_{r,s}$. The module is irreducible if it is not reducible. Every reducible representation can be written as a direct sum of irreducible representations

$$V = V_1 \oplus V_2 \oplus ..V_m. \tag{A.64}$$

Two real (complex) Clifford modules V and W, are called equivalent if there is a real (complex) linear isomorphism $T : V \to W$ that intertwines the action of the Clifford module, i.e. for all $\phi \in Cl_{r,s}$ the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\rho(\phi)"] & W \arrow[d, "\rho(\phi)"] \\ V \arrow[r, "T"] & W \end{tikzcd} \tag{A.65}$$

Similarly a representation of the complex Clifford algebra $\mathbb{C}l_{r,s}$ is a homomorphism of complex algebras between $\mathbb{C}l_{r,s}$ and $\mathrm{Hom}(V,V)$. Here V has to be a complex vector space in order for $\mathrm{Hom}(V,V)$ to be a complex algebra. In fact the representations of $\mathbb{C}l_{r,s}$ are in a one-one correspondence with the complex representations of $Cl_{r,s}$. The same notion of equivalence between representations of a complex Clifford algebra exists.
The number of inequivalent irreducible representations is not very big. Let $\nu_{r,s}$ denote the number of inequivalent irreducible real representations of $Cl_{r,s}$. Then

$$\nu_{r,s} = \begin{cases} 2 & \text{if } r+1 = s \mod 4, \\ 1 & \text{otherwise} \end{cases} \tag{A.66}$$

$\nu_{r,s} = 2$ corresponds to the cases (cf. (A.55)) in which $\Gamma^2 = 1$ and Γ commutes with all elements of $Cl_{r,s}$. The two inequivalent representations then either have $\rho(\Gamma) = 1$ or $\rho(\Gamma) = -1$. These two options correspond precisely with the two inequivalent representations. We can make use of the following algebra isomorphism (which holds for all r, s)

$$Cl^0_{r+1,s} \cong Clr, s. \tag{A.67}$$

Here $Cl^0_{p,q}$ denotes the subalgebra of $Cl_{p,q}$ of all linear combinations of even products of the Γ_i's (including the identity). $Cl^1_{p,q}$ denotes the set of all other elements of $Cl_{p,q}$. Let ρ^0 and ρ^1 be the two inequivalent representations of $Cl_{r,s}$ with $\nu_{r,s} = 2$ in respectively the spaces W^0 and W^1. Under the isomorphism we get a representation in the space $W^0 \oplus W^1$ of $Cl^0_{r+1,s}$. This representation can be extended to $Cl_{r+1,s}$. In fact if (ρ, W) is the irreducible representation of $Cl_{r+1,s}$

it has two eigenspaces of Γ with eigenvalues $\pm$i or ± 1. As Γ anti-commutes with all elements of $Cl^1_{r+1,s}$, these elements switch between the eigenspaces. But restricting to $Cl^0_{r+1,s}$ separates W in the two eigenspaces, which are now inequivalent representations of $Cl^0_{r+1,s}$. Namely by the condition $\nu_{r,s} = 2$, n is odd and thus Γ of $Cl_{r+1,s}$ is an even product, so $\Gamma \in Cl^0_{r+1,s}$. Therefore the two eigenspaces must be the two inequivalent representations W^0, W^1 of $Cl^0_{r+1,s} \cong Cl_{r,s}$. The decomposition $W = W^0 \oplus W^1$ has the following property

$$Cl^i_{r,s} \cdot W^j \subseteq W^{i+j \mod 2} \tag{A.68}$$

for all $i, j \in \{0,1\}$. This is the definition of a $\mathbb{Z}_2$-graded Clifford module of $Cl_{r+1,s}$. For $\nu_{r,s} = 1$ a $\mathbb{Z}_2$-graded Clifford module of $Cl_{r+1,s}$ consists of two equivalent irreducible representations of $Cl^0_{r+1,s} \cong Cl_{r,s}$.
The dimension d_n of an irreducible real representation of $Cl_{r,s}$ $(n = r+s)$ is given by

$$d_1 = 2,\ d_2 = d_3 = 4,\ d_4 = d_5 = d_6 = d_7 = 8,\ d_8 = 16, \quad d_{m+8k} = 2^{4k} d_m$$

For the number $\nu^{\mathbb{C}}_n$ of inequivalent irreducible representations of $\mathbb{C}l_n$ and their complex dimension $d^{\mathbb{C}}_n$, we have

$$\begin{array}{llll} \nu^{\mathbb{C}}_n & = 2, \quad d^{\mathbb{C}}_n & = 2^{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \\ \nu^{\mathbb{C}}_n & = 1, \quad d^{\mathbb{C}}_n & = 2^{\frac{n}{2}} & \text{if } n \text{ is even.} \end{array} \tag{A.69}$$

For $\nu^{\mathbb{C}}_n = 2$ again (cf. (A.61)) $\Gamma^2_{\mathbb{C}} = 1$ and $\Gamma_{\mathbb{C}}$ commutes with all elements of $\mathbb{C}l_n$, so either $\rho(\Gamma) = 1$ or $\rho(\Gamma) = -1$.
Very important for physical applications are the so called spinor representations which are the restrictions of Clifford representations to $Spin(r,s)$. It useful to notice that $Spin(r,s) \subset Cl^0(r,s)$. By the discussion above the restriction splits the representation in two inequivalent spinor representations if $\nu_{r-1,s} = 2$.
In physics the bilinear form q is equal to minus the metric $\eta^{\mu\nu} = \text{diag}\{-1,1,1,...1\}$ to arrive at

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}. \tag{A.70}$$

So in D space-time dimensions we look at $Cl_{r,s}$ with $r = D-1$ and $s = 1$. We usually start with the complex Clifford algebra $\mathbb{C}l_{1,D-1}$. The elements of an irreducible representation of this algebra are called *Dirac spinors.* For D is even the restriction to $Spin(1, D-1)$ splits the representation in two inequivalent representations. The elements of these two spaces are called *Weyl spinors* of positive $(\rho(\Gamma_{\mathbb{C}}) = 1)$ and negative $(\rho(\Gamma_{\mathbb{C}}) = -1)$ chirality.

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