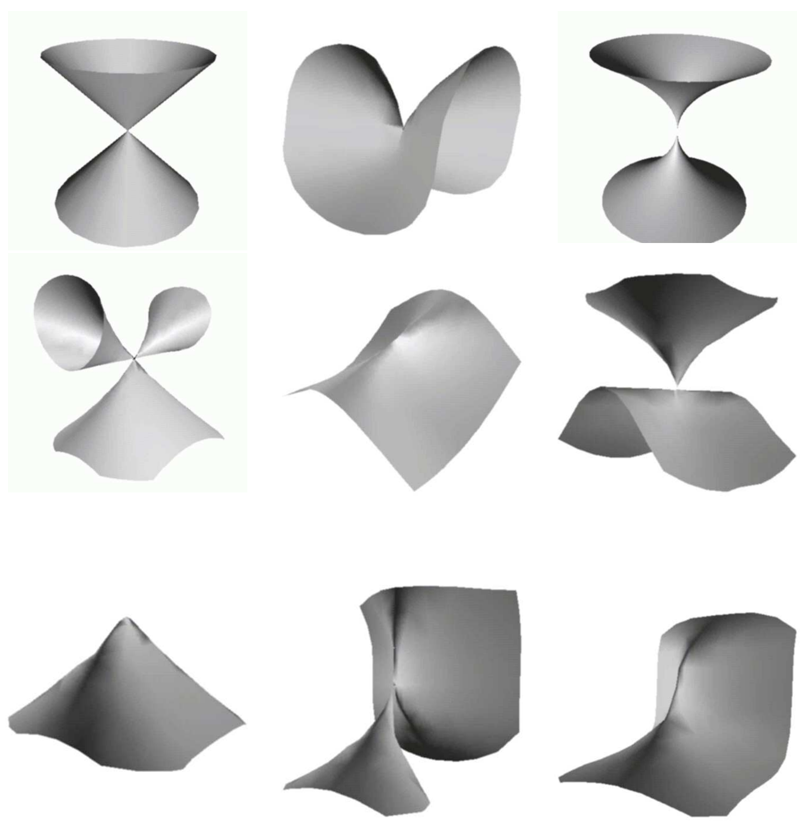
M-Theory Compactifications on Manifolds of G_2-holonomy

W.A.J. Vijvers



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The figures on the title-page are real slices of four-dimensional complex surfaces that contain the following types of ADE singularities:

First row: A_1, A_2 and A_3 singularity
Second row: D_4-, D_4+ and D_5 singularity
Third row: E_6, E_7 and E_8 singularity

Images taken from `http://www.amsta.leeds.ac.uk/~rjm/parade/`.

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Wouter Anton Johan Vijvers

Scriptiebegeleider:
Prof. Dr. Jan de Boer

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Instituut voor Theoretische Fysica Amsterdam
Faculteit der Natuurwetenschappen, Wiskunde en Informatica

UNIVERSITEIT VAN AMSTERDAM

Aan mijn ouders

Abstract

This thesis gives a description of one possible way to do "M-theory Phenomenology". M-theory is the conjectured eleven-dimensional theory that is supposed to provide a complete and unified description of quantum mechanics and gravity. Doing phenomenology in this context means searching for ways to reduce the theory to an effective four-dimensional field theory that resembles the Standard Model as close as possible. In our case, we do this by making the extra seven dimensions compact and small enough to be unobservable in current experiments. Demanding $\mathcal{N}=1$ supersymmetry for the four-dimensional effective field theory forces the seven extra dimensions to constitute a manifold of G_2-holonomy.

After considering the compactification of M-theory on a smooth seven-dimensional G_2-manifold it will become clear that singularities in the compact dimensions are needed to realize in the four-dimensional effective theory the two basic features of the Standard Model: non-Abelian gauge groups and chiral fermions. So-called ADE singularities are needed to generate non-Abelian gauge symmetry and isolated singularities support chiral fermions. We will discuss the merits and problems of this type of model and compare it to other possible ways of obtaining realistic four-dimensional physics from M-theory and its ten-dimensional counterpart String Theory.

Contents

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I love deadlines.
I especially like the whooshing sound they make as they go flying by.

Douglas Adams

Chapter 1

Introduction

1.1 Motivation

Ever since the beginning of science, people have been looking for more unified ways to describe their knowledge of the universe. Describing the physical laws in terms of less underlying principles adds to our feeling of having understood Nature better, because it reduces a sense of arbitrariness. Although satisfying, the ultimate justification of these efforts must come in the form of mathematical consistency and experimental verification. Although it is impossible to determine up front whether a completely unified description of all observable phenomena - a Theory of Everything - exists, the latest research in Theoretical Physics seems to indicate that at the very least something very special is going on at the moment: we are discovering the pieces of an eleven-dimensional theory that might provide us with such a description! This theory is called *M-theory*.

The purpose of this thesis is in essence to explain how this theory, whose development is in a large part being pushed forward by the need for mathematical consistency, might ultimately be put to experimental test. The body of the work is devoted to explaining one way to connect this theory to our current knowledge of the fundamental interactions, namely by *compactifying M-theory on a manifold of G_2-holonomy*. Other possibilities for reaching the same objective are shortly discussed and compared to this method. It is our hope that this thesis will be a good starting point for someone who is interested in doing research on this subject.

1.2 Historical Perspective on Unification

We pick up the story of unification at the first half of the last century. In the beginning of the 20th century people saw an enormous change in the way we think about Nature with the introduction of General Relativity and Quantum Mechanics. It took some time for people to come to grips with these two elusive theories, whose development in the beginning was mainly pushed forward by the need for mathematical consistency. But ever since experimental data have confirmed the predictions made by these theories, no one doubts their validity as descriptions of the Universe at large and small scales, respectively. However, after a number of years of gathering data, something seemed off: new elementary particles were discovered almost by the day! This immediately raised the question why the supposedly fundamental building blocks of Nature came in such huge numbers and why their properties seemed to be unrelated. It was not until the early 1970s, when the Standard Model of Elementary Particles was constructed, that people found an answer to these questions. In the Standard Model, the only fundamental matter particles are three families of only two quarks and two leptons (not counting so-called color charge). Most of the particles found before were shown to be composite particles, whose properties could be understood in terms of the properties of the quarks and leptons. Besides these fermions (which is the collective name for the matter particles), the Standard Model contains a second group of particles. These are called gauge bosons and are responsible for the interactions between the matter particles.

Today, more than 30 years later, it is clear that no matter what theory we eventually find to describe Nature, we know for sure that it *must* in some limit and at low energies contain the Standard Model. The reason for this strong statement is that since its incarnation in the early 1970s, huge amounts of experimental data have confirmed this model to very high accuracy. Furthermore, using the Standard Model many predictions have been made and to this date no discrepancies between these predictions and the outcome of experiments have been found. One might then ask, if this is such a well-functioning model, then why would we look for something else? Although it has been recognized to be a successful description of the fundamental particles, a number of unsatisfactory features of the Standard Model have made many people believe that it cannot be the final description. Some of the most important problems with the Standard Model are that it contains 26 free parameters (leading to a sense of arbitrariness), it provides no explanation for the hierarchy in the relative strengths of the fundamental interactions and there is no explanation for charge quantization. But the most important problem of all is that it is impossible to include quantum mechanical description of gravity in the Standard Model, treating the gravitational interaction on the same footing as the other three interactions.

To counter some of these problems, a number of different ideas have been put forward since then. One of these ideas is that of Grand Unification, in which all the particle interactions (i.e. the bosons) are unified into a single description[1]. Not only are the interactions given a unified description, but

[1]For the people that know the Standard Model: Grand Unified theories are basically non-Abelian gauge theories with a single gauge group that contains the Standard Model

also the quarks and leptons (the fermions or matter particles) are unified into a single field, meaning that they can be seen as basically different realizations of a single object. Grand Unified Theories are known to solve some of the problems of the Standard Model: they contain less free parameters and provide an explanation for the hierarchy problem and charge quantization.

Supersymmetry is another possible extension of the symmetries present in the Standard Model and provides a further unification in the description of the fundamental particles. Supersymmetry was first suggested about 20 years ago and, crudely speaking, is an (approximate) symmetry of nature which transforms bosons into fermions and vice versa. So, in a way it is another unification of the way we describe Nature. But this time the bosons and fermions are put on an equal footing. Note the word approximate in parentheses. To this date no sfermions or bosinos (these are the names of the so-called super-partners of the Standard Model particles) have been discovered yet, so supersymmetry cannot be an exact symmetry of nature. At a certain energy-scale, currently out of reach for experiments, supersymmetry should be broken. The new Large Hadron Collider (LHC) accelerator, currently under construction at CERN in Geneva, can possibly change this situation, because it should be able to reach the energies at which most people believe supersymmetry gets broken. [55] is an excellent starting point for someone who knows the Standard Model, to learn more about its supersymmetric extension.

The supersymmetry transformations we just described are so-called *global* transformations, i.e. transformations that change all bosons into fermions (and vice versa) at the same time. If we modify the concept of supersymmetry in such a way that it describes *local* supersymmetry transformations (i.e. if we modify it to look like a sort of gauge symmetry version of supersymmetry), it turns out that the resulting theory will also contain a description of gravity! Such a theory is known as a *Supergravity Theory*. Supergravity theories have been shown to exist in four to eleven spacetime dimensions. This is the first time that we see the mathematical possibility that there are more dimensions than we observe in our everyday life. For the reader that is familiar with quantum field theories, but not with supergravity theories, we recommend [72] as a starting point.

In the 1960s, experimental data seemed to indicate that the quarks inside a nucleus move around as if they are attached to each other with tiny strings. This started theoretical research in String Theory as a possible description of this phenomenon called the *asymptotic freedom* of Quantum Chromo Dynamics (QCD), the theory of the strong interaction. A number of mathematical problems and the advent of the Standard Model resulted in

group as a subgroup. Examples of used groups are $SU(5)$, $SO(10)$ and E_6. The quarks and leptons are combined in a single multiplet that transforms in a representation of the grand unified group.

String Theory	Open / Closed	Chiral	SUSY
Type IIA	- / $\surd$	-	$\mathcal{N}=2$
Type IIB	- / $\surd$	$\surd$	$\mathcal{N}=2$
$E_8 \times E_8$ Heterotic	- / $\surd$	$\surd$	$\mathcal{N}=1$
$SO(32)$ Heterotic	- / $\surd$	$\surd$	$\mathcal{N}=1$
Type I	$\surd$ / $\surd$	$\surd$	$\mathcal{N}=1$

Table 1.1: Overview some of the basic features of the five String Theories

this theory only being studied by a limited number of theoretical physicists for some 20 years. But in the early 1980s, the mathematical problems were resolved and String Theory (as a theory of both open and closed strings) was recognized to be a possible theory of quantum gravity instead of just a description of the strong interactions. But it turns out that these theories can only be made mathematically consistent in ten spacetime dimensions! The fact that we do not observe these extra dimensions can be explained by taking these dimensions to be so small that they are unobservable. Not too long after this "First String Theory Revolution" of the early 1980s, theorists learned how to make the extra six dimensions compact to get a four-dimensional theory that looks quite like a $\mathcal{N}=1$ supersymmetric version of the Standard Model. These are the famous *Calabi-Yau compactifications* of the $E_8 \times E_8$ Heterotic String Theory.

Some of the basic properties of the five known String Theories are given in table 1.1. Just like the Standard Model, String theories are *perturbative* theories, with the expansion given in orders of the so-called string coupling $g_s^{L-1} = e^{2(L-1)\phi}$ $(L = 0, 1, 2, ..)$ and L corresponding to the number of 'loops' in the expansion. Soon after their birth it was also discovered that if we take a low energy limit of the five string theories, we find supergravity theories in ten dimensions.

But the quest for unification was not over, because it was still unsatisfactory that there were apparently five different candidates for a Theory of Everything. In 1995, the famous string theorist Edward Witten discovered that these five string theories are actually related to each other and to *eleven-dimensional supergravity* through so-called dualities. He found strong circumstantial evidence that the five string theories can be brought together under an eleven-dimensional hood and gave this unifying theory the tentative name *M-theory*. This name is deliberately ambiguous. According to Witten, the M can be taken to mean either Mother, Magical, Mystery, Matrix or Membrane, according to taste. The first three options have a clear justification. The fourth name was added later, when in [10] the conjecture was made that M-theory can be described as a Matrix Theory. For an introduction to this theory, we recommend [70]. And as we shall see in chapter 3 the last name was introduced for a very specific reason as well.

This thesis is concerned with describing in what way we can compactify M-theory on a seven-dimensional space in such a way that the four-dimensional theory coming out looks like the Standard Model (or a supersymmetric version of it), analogously to the much studied Calabi-Yau compactifications. These seven-dimensional spaces turn out to be complicated beasts called *G_2-manifolds.* For a nice non-technical overview of current research into string theory and M-theory and many references to the literature, see [23].

1.3 Outline

We start in chapter 2 by introducing the reader to the mathematics used in the rest of this thesis. This treatment centers around two topics: singularities and G_2-manifolds. It is a long chapter, but will give the reader a solid background with which he can get through the rest of the thesis. Then in chapter 3, the reader is introduced to M-theory. The bulk of this chapter actually consists of a description of the low energy limit of M-theory: eleven-dimensional supergravity. It also contains a description of the Kaluza-Klein mechanism for doing compactifications. At the end of this chapter we outline the methods one can use to learn more about M-theory itself.

In chapter 4, the real work begins. There we compactify eleven-dimensional supergravity to four dimensions. We will see that if we demand the four-dimensional theory to be $\mathcal{N}=1$ supersymmetric, the compactification manifold has to have G_2-holonomy. But the theory coming out of this compactification will look nothing like the Standard Model. It only has Abelian gauge symmetry and no charged chiral fermions. In chapter 5 we solve this problem by considering *singular* M-theory compactifications. Then, finally, in chapter 6 we discuss the advantages and disadvantages of these models, some phenomenology and give some suggestions for further research.

Chapter 2

Geometrical Preliminaries

The aim of this chapter is to introduce the mathematical concepts which are needed in order to understand the rest of this thesis. Since it would take several books to cover all the subjects comprehensively, we try to keep to the bare essentials. Luckily, should the information in this chapter not be enough, books and review articles that introduce the necessary topics thoroughly, do exist. [61], [30] and [17] are good examples. Chapters 12 and 15 of [36] and [37] are good general references as well, although they are mainly aimed at introducing Calabi-Yau manifolds. For more on $K3$, see [8] and for more about G_2-manifolds and singularities, see [39] or the authoritative book [49]. For the reader that feels somewhat uncertain about his knowledge of Lie groups, it might be a good idea to study chapter 9 on the Cartan classification of semi-simple Lie groups in [45] first. Knowledge of ordinary (real and complex) differential geometry is assumed in this chapter.

2.1 Tools from Differential Geometry

Although we assume preexisting knowledge of differential geometry, some concepts are so essential to understanding the later chapters (and may not be included in a standard issue geometrical toolbox) that we present them here.

2.1.1 Killing vector fields and the isometry group

Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection on it. A Killing vector field K is defined as a vector field which generates an isometry of the manifold M. This means that if we move over the manifold in the direction of K, the the metric does not change. In this sense, it represents a symmetry of the manifold. To find the condition for a vector field to be a Killing vector field, we impose that the metric is invariant under an infinitesimal transformation $x^m \to x^m + \epsilon K^m$. A small calculation [61] shows that this condition locally is

$$(\mathcal{L}_K g)_{mn} = 0, \tag{2.1}$$

with $\mathcal{L}$ the Lie-derivative. We will often use another equivalent form of the *Killing equation*, which is valid for all Levi-Civita connections,[1]

$$\nabla_m K_n^{(i)} + \nabla_n K_m^{(i)} = 0. \tag{2.2}$$

Here we have given the Killing vector fields an extra index (i), because this equation generally has a number of solutions. Because the covariant derivative is a linear operator, a linear combination of two Killing vector fields, $aK^{(i)} + bK^{(j)}$ with $a, b \in \mathbb{R}$, is again a Killing vector field. The same goes for the commutator of two Killing vector fields, because of the following property of the Lie-derivative:

$$\mathcal{L}_{[X,Y]} g = [\mathcal{L}_X, \mathcal{L}_Y]\, g, \tag{2.3}$$

for any two vector fields X and Y. If we now take X and Y to be Killing vector fields, this equation becomes zero. This implies that the commutator is also a Killing vector field. These properties show that the Killing vector fields form a Lie algebra,

$$[K^{(i)}, K^{(j)}] = f^{ij}{}_k K^{(k)}. \tag{2.4}$$

The Killing vector fields are thus also the generators of a group, the *isometry group*, of all the symmetry transformations on the manifold. The dimension of the isometry group can be bigger than the dimension of the manifold itself, but the maximum is related to its dimension by $\dim(G) = m(m+1)/2$. A manifold with this number of Killing vector fields is called a *maximally symmetric space*.

A minimal way to construct a manifold M with a certain isometry group G is by demanding that G acts freely and transitively on it, as in that case M is just isomorphic to G. If G acted transitively but not necessarily freely, the space would still consist of a single group orbit, but we could have non-trivial isotropy groups H_p at points $p \in M$. Identifying under actions within the isotropy group reduces the dimension to the original value for free group action. Now if G is a Lie group, the isotropy group H_p is a Lie subgroup of G and the coset G/H_p will admit a differentiable structure and hence be a manifold. Such a manifold is called a *homogeneous space*. Note that it doesn't matter which choice $p \in M$ we make, as all isotropy groups are conjugate to each other and conjugate subgroups are always isomorphic to each other. Some examples of symmetric spaces are: $\mathbb{R}^n$ with the Euclidean metric, $S^n \cong SO(n+1)/SO(n)$ with the round metric and $\mathbb{C}P^n \cong U(n+1)/U(n)$ (see section 2.2) with the so-called Fubini-Study metric.

[1] The previous form was also valid for more general connections.

2.1.2 Holonomy

As we will find out in later chapters, the concept of holonomy is of central importance. Very roughly speaking, the holonomy group is a geometric characteristic of a manifold that holds some information about the amount of symmetry it has. The smaller the holonomy group, the more symmetry the manifold possesses. Requiring that a manifold has a certain holonomy group usually leads to a restriction of its curvature and topology. Let us start by giving the definition of the holonomy group of a manifold.

Definition 2.1 (Holonomy group) *Let (M, g) be a Riemannian m-dimensional manifold, ∇ be the metric connection*[2] *on M and let*

$$C_p := \{c(t)|0 \le t \le 1, c(0) = c(1) = p\} \tag{2.5}$$

be the set of all closed loops with base-point p. The connection gives us a notion of parallel transport of vector fields over our manifold. Around a closed loop $c(t)$, this induces a transformation (acting on X from the right)

$$P_c : X \in T_p(M) \to X_c = X \cdot P_c \in T_p(M). \tag{2.6}$$

The holonomy group at p, $\mathrm{Hol}_p(g)$ *is defined as the set of all possible P_c, i.e.*

$$\mathrm{Hol}_p(g) := \{P_c|c(t) \in C_p\}. \tag{2.7}$$

The group product of two group elements $P_c \cdot P_c'$ is given by the transformation generated by first transporting the vector along the loop c' and then along the loop c. It is easily shown that this product satisfies the axioms of a group, with the inverse being given by transporting in the opposite direction and the unit element by the constant loop $c(t) = p$. One important property of the holonomy group is that if M is simply-connected, its holonomy group is connected. This is because on a simply-connected manifold any loop is contractible to the constant loop. The corresponding family of parallel transports is a continuous path in $\mathrm{Hol}_p(g)$ joining any P_c to the unit element. If M is not simply-connected, we can also define the *local holonomy group* by considering only loops that are contractible to the constant loop, but most of the time we are just interested in the total holonomy group.

From this definition it seems that the concept of a holonomy group depends on the point p, but the following proposition shows that it is a well-defined geometric property of a manifold (under the assumption that it is connected).

Proposition 2.2 *Let $p, q \in M$. If M is arcwise connected, there is a curve $a(t)$ connecting p and q, which defines a map $\tau_a : T_p(M) \to T_q(M)$ by*

[2] A metric connection is a Levi-Civita connection which satisfies $\nabla g = 0$.

parallel transport along a. Then $\mathrm{Hol}_q(g) = \tau_a^{-1}\mathrm{Hol}_p(g)\tau_a$, *so they are all conjugate to each other and thus*

$$\mathrm{Hol}_p(g) \cong \mathrm{Hol}_q(g). \tag{2.8}$$

Therefore we can simply write $\mathrm{Hol}(g)$.

We note that if M is flat ($R_{\mu\nu\rho\sigma} = 0$) everywhere on a *smooth* and simply-connected manifold, the holonomy group is trivial, $\mathrm{Hol}(g) = \{1\}$, and vice versa. Later on we will see that if the space is not smooth (i.e. if it contains singularities) the holonomy group can be discrete. And we already know that this statement also holds for non-simply-connected manifolds. The orbifold with the A_1 singularity we describe in section 2.2.4 is an example of a space with $\mathbb{Z}_2$ holonomy.

Furthermore, if we would not demand ∇ to be a metric connection, the holonomy group would be $\mathrm{Hol}(g) \subset GL(m, \mathbb{R})$.[3] But in any practicable application we do take ∇ to be metric-compatible. This implies that the connection preserves the length of vectors, with which we find that

- $Hol(g) \subset O(m)$ if (M, g) real and of dimension m
- $Hol(g) \subset SO(m)$ if (M, g) real, orientable and Riemannian
- $Hol(g) \subset SO(m-1, 1)$ if (M, g) real, orientable and Lorentzian.

$SO(m)$ (assuming Riemannian geometry) is the biggest holonomy group a Riemannian manifold can have. As said, smaller holonomy groups are more restrictive and indicate the presence of more symmetry. So if we want to analyze more interesting spaces using holonomy, we want to consider subgroups of $SO(m)$ as holonomy groups. In 1955 Berger proved the following theorem, thereby constructing a classification of all the subgroups of $SO(m)$ which are possible holonomy groups of manifolds.

Theorem 2.3 (Berger) *Let* (M, g) *be a simply-connected m-dimensional Riemannian manifold that is not isometric to a Riemannian product* $(M_1 \times M_2, g_1 \times g_2)$ *with* $\dim(M_i) > 0$ *(i.e. is irreducible) and whose Riemann curvature satisfies* $\nabla R \neq 0$, *then exactly one of the cases as listed in table 2.1 holds.*

Strictly speaking, the last three columns are not part of this theorem, but these properties are listed here for future reference. This theorem will more or less be our guide for the rest of the chapter, in which we treat the cases which are important for our specific applications.

[3] A generalization of Bergers classification (see the theorem below) is available in case we do not consider metric connections. Part of this list was already discovered by Berger in his original work, but recent research indicates that his list was incomplete. For details, see [58].

Case	Dimension	$Hol(g)$	Name	Ricci flat	Kähler
1.	$m = n$	$SO(n)$	Riemannian	$\surd$	-
2.	$m = 2n$ $(n \geq 2)$	$U(n)$	Kähler	-	$\surd$
3.	$m = 2n$ $(n \geq 2)$	$SU(n)$	Calabi-Yau	$\surd$	$\surd$
4.	$m = 4n$ $(n \geq 2)$	$Sp(n)$	Hyper-Kähler	$\surd$	$\surd$
5.	$m = 4n$ $(n \geq 2)$	$Sp(n).Sp(1)^*$	Quat. Kähler	-	-
6.	$m = 7$	G_2	Exceptional	$\surd$	-
7.	$m = 8$	$Spin(7)$	Exceptional	$\surd$	-

Table 2.1: Bergers classification of possible holonomy groups. *) Notation for $Sp(n) \times Sp(1)/\mathbb{Z}_2$.

But before moving on, some remarks have to be made about this classification. First of all, the statement that Kähler manifolds are not Ricci flat does not mean that they *cannot* be Ricci flat. They can be, but if they are, their holonomy is further restricted to lie in $SU(n)$. This is because for Ricci-flat metrics the $U(1)$ part of the connection vanishes. If this is the case it thus becomes a Calabi-Yau manifold, which is already listed under case 3. Because we do not want to wander too far of the main path leading to the introduction of G_2-manifolds, we have reserved Appendix B for a more detailed treatment of Kähler and Calabi-Yau manifolds. Note that these manifolds are very important in String Theory and that G_2-manifolds can be constructed in a way very similar to how we construct these. So it might be worth it to read this appendix.

Second of all, one of the assumptions in theorem 2.3 is that $\nabla R \neq 0$. A manifold with this property is called *non-symmetric*. If a manifold (M, g) on the other hand *is* symmetric (i.e. if its Riemann curvature tensor satisfies $\nabla R = 0$), it can be shown that it is actually isomorphic to a homogeneous space: $M \cong G/H$. A symmetric manifold written as such has $\mathrm{Hol}(g) = H$. Note that it might be necessary to take G to be some properly chosen subgroup of G to make this identification. Building on work done for the classification of compact semi-simple lie groups, Cartan already in 1927 constructed the classification of all possible holonomy groups of symmetric manifolds. Details about this classification, a specification of the way we have to choose the proper subgroup of G and proofs of the statements we just made can be found in section 3.3 of [49].

Third of all, by demanding the metric to be irreducible we are basically excluding product manifolds. If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, then the product metric $g_1 \times g_2$ simply has holonomy $\mathrm{Hol}(g_1 \times g_2) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2)$. So reducible metrics have holonomy groups which are products of the groups listed in table 2.1. Finally (although we will not meet them often), we note that quaternionic Kähler manifolds always admit

Einstein metrics.

2.1.3 Calibrated Geometry

Calibrated submanifolds are a certain kind of volume-minimizing submanifolds of Riemannian manifolds. In 1982 calibrated geometry was given a solid basis by Harvey and Larson [40] and since then they have started to play an important role in supersymmetric compactifications. Their existence is also closely related to conditions for special holonomy and Bergers classification, as special holonomy manifolds usually come equipped with one or more natural calibrations. We will encounter several examples of such calibrations.

We begin by defining the notion of a minimal submanifold. First of all, the volume of a Riemannian m-dimensional manifold (M, g) is just the integral of its invariant volume form over M,

$$\mathrm{Vol}(M) := \int_M \mathrm{vol}_M \tag{2.9}$$

Now consider an immersion $\imath : N \to M$, which defines a n-dimensional submanifold N of M. We call N a *minimal submanifold* if its volume is stationary under small variations of its immersion $\imath$. The condition for N to be minimal is a second order equation on $\imath$.

Note that a minimal submanifold does not necessarily have a minimal area. For example, the equator of the two-sphere S^2 is a 1-dimensional minimal submanifold, but does not have minimal length amongst the lines of latitude. Also note that a 1-dimensional minimal submanifold is called a *geodesic*. We now define a calibration to see how it relates to minimal submanifolds.

Definition 2.4 (Calibration) *Let (M, g) be an m-dimensional Riemannian manifold and φ be an n-form on M. φ is called a* calibration *if it satisfies*

$$d\varphi = 0 \quad \text{and} \quad \int_N \varphi \leq \mathrm{Vol}(N) \tag{2.10}$$

for all n-dimensional submanifolds N of M.

With the use of this definition we can define a

Definition 2.5 (Calibrated submanifold) *A submanifold N of M is called a* calibrated submanifold *or* φ-submanifold *if*

$$\int_N \varphi = \mathrm{Vol}(N), \tag{2.11}$$

with φ as defined above.

From these definitions, we can see that a φ-calibrated submanifold has the minimal volume in its homology class. To understand this, note that any other manifold in the homology class of N can be written as $N' = N + \partial C$, with C some $(n+1)$-dimensional submanifold of M. If we now integrate the calibration over N', we find

$$\int_{N'} \varphi = \int_N \varphi + \int_{\partial C} \varphi = \int_N \varphi + \int_C d\varphi = \int_N \varphi = \mathrm{Vol}(N). \tag{2.12}$$

But because φ is a calibration, we have $\int_{N'} \varphi \leq \mathrm{Vol}(N')$ and thus

$$\mathrm{Vol}(N) \leq \mathrm{Vol}(N'). \tag{2.13}$$

Note that this proof only holds for the case that N is a *compact* submanifold. For a non-compact calibrated submanifold a similar statement holds: they are *locally* volume-minimizing in their homology class. This statement is harder to prove, though.

One of the advantages of considering calibrated submanifolds instead of minimal submanifolds is that with a given φ, the condition for an immersed submanifold N to be calibrated is a set of *first-order* equations on the immersion $\imath$. These are often easier to solve than the second-order equations that determine whether a submanifold is minimal. Because of this, many examples of minimal submanifolds can be found with techniques from calibrated geometry.

For reasons that will become clear later on, calibrated submanifolds are often called *supersymmetric cycles*. Cycles - and in particular the way they intersect with each other - are also the subject of the next subsection.

2.1.4 Intersection Numbers

In later chapters we will encounter the concept of intersection numbers multiple times. Therefore, we will now give their definition. For this, suppose that M is an oriented n-dimensional manifold and A and B are two piecewise smooth cycles on M of dimensions k and $n-k$ respectively. If we now take a point $p \in A \cap B$ and any two bases for $T_p(A) \subset T_p(M)$ and $T_p(B) \subset T_p(M)$, then A and B are said to *intersect transversally at* p if together these bases form a basis of $T_p(M)$. If this holds for all $p \in A \cap B$ the two cycles are simply said to intersect transversally. If $A \cap B = 0$, then the cycles automatically intersect transversally.

Now take A and B to be *oriented* cycles and suppose that $p \in A \cap B$ is a point of transversal intersection (it will be clear in a minute what happens if there are points of intersection at which the cycles do not intersect transversally). Now, let $v_1, \ldots, v_k \in T_p(A)$ be an oriented basis for $T_p(A) \subset T_p(M)$ and $w_1, \ldots, w_{n-k} \in T_p(B)$ be an oriented basis for $T_p(B) \subset T_p(M)$.

Definition 2.6 (Intersection number) *Define the* intersection index $\imath_p(A \cdot B)$ *of A with B at p to be $+1$ if $v_1, \ldots, v_k, w_1, \ldots, w_{n-k}$ is an oriented basis of $T_p(M) = T_p(A) \oplus T_p(B)$ and -1 if it is not. Now, if A and B intersect transversally everywhere, we define the* intersection number *to be*

$$\#(A \cdot B) = \sum_{p \in A \cap B} \imath_p(A \cdot B). \tag{2.14}$$

Note that in this definition we took as a prerequisite that the cycles A and B intersect transversally. Furthermore, the sum in (2.14) suggests that A and B intersect in isolated points, while at this point it is not clear that this is necessarily the case. These two points do not pose a problem because of the following. It can be shown that the intersection number is a topological invariant and is in particular invariant under deformations of the two cycles within their respective homology classes. In other words, if $A, A' \in H_k(M, \mathbb{Z})$ are elements of the same homology class, then

$$\#(A' \cdot B) = \#(A + \partial C \cdot B) = \#(A \cdot B) + \#(\partial C \cdot B) = \#(A \cdot B) \tag{2.15}$$

for some $(k+1)$-dimensional submanifold $C \subset M$. Note that this equation contains the non-trivial statements that the intersection number is linear and that if a cycle is homologous to zero, its intersection number is zero. What this also implies is that if we have cycles that do not intersect transversally in isolated points, we can deform them within their homology classes until they do and thereupon calculate their intersection number using formula (2.14). For more background material, proofs of these statements and applications we refer the reader to [38].

We have now gathered almost all the tools that we need in this thesis when dealing with smooth spaces. But as we indicated in chapter 1 this will not be enough: the interesting physics is located at singularities in the internal space. A good way to describe spaces with singularities is using algebraic geometry, where the spaces are given by algebraic equations.

2.2 Basic Algebraic Geometry

In algebraic geometry the basic object is a variety. The simplest way to describe a variety is as the solution set of a number of polynomial equations in a complex space. If we want to describe non-compact spaces, we can use polynomials in $\mathbb{C}^n$, but in order to describe compact varieties we need polynomials in $\mathbb{C}P^n$, the complex projective space. Because $\mathbb{C}P^n$ plays such a central role in algebraic geometry, the first thing we need to do is define it.

Definition 2.7 (Complex Projective Space) *Let $z \in \mathbb{C}^{n+1}$ be $z \neq 0$. Now define an equivalence relation $z' \sim z$ where z and z' are equivalent if* *there exists an* $\lambda \neq 0 \in \mathbb{C}$ *such that* $z' = \lambda z$. *The* complex projective space $\mathbb{C}P^n$ *is defined to be the set*

$$\mathbb{C}P^n := \{z \in \mathbb{C}^{n+1} - \{0\}\}/\sim, \tag{2.16}$$

with $\sim$ *the given equivalence relation.*

It can easily be shown (see for example [36]) that $\mathbb{C}P^n$ is a compact complex manifold and that all of its closed submanifolds are compact as well.

One of the reasons we introduce algebraic geometry is that we can use it to describe singular spaces. But as a manifold is by definition a smooth space, we need to introduce the analogue of a manifold in this setting. This analogue is the algebraic variety we just described.

Definition 2.8 (Algebraic variety) *The set of common zeros of a finite number of homogeneous polynomials in* $\mathbb{C}P^n$ *is called an* algebraic set. *An algebraic set which is not the union of two algebraic sets is called an* irreducible *algebraic set. An open subset of an irreducible algebraic set is called an* algebraic variety.

Explicitly, what this definition says is that any variety can be described by a set of equations $p_0(z_0, \ldots, z_n) = \ldots = p_r(z_0, \ldots, z_n) = 0$, with the $\{p_i\}_{i=1,\ldots,r}$ a set of homogeneous polynomials of some degree k_i and $r \leq n$. Homogeneity of the polynomials means that they have to obey the condition $p_i(\lambda z_0, \ldots, \lambda z_n) = \lambda^{k_i} p_i(z_0, \ldots, z_n)$. They have to have this property for the equations $p_i = 0$ to be well-defined on $\mathbb{C}P^n$.

That it is possible to describe a smooth manifold as an algebraic variety, was proven by Chow:

Theorem 2.9 (Chow) *Any complex submanifold of* $\mathbb{C}P^n$ *can be realized as the zero locus of a finite number of homogeneous polynomial equations in* $\mathbb{C}P^n$.

A submanifold defined as such is known as a *complete intersection of hypersurfaces.*

We would like to have a condition that expresses whether a variety is actually a manifold. Intuitively we can say that a singular variety is a manifold on which we cannot define a unique tangent space at every point. Since tangent spaces are generated by derivatives, we can expect to be able to give such a criterium in terms of derivatives of the polynomials defining the variety. The condition for the polynomials to define a submanifold M is as follows. If we define the $n \times r$ matrix

$$H_{ij}(p) := \left(\frac{\partial p_i}{\partial z_j}\bigg|_p \right), \quad i = 1, \ldots, r; \; j = 1, \ldots, n \tag{2.17}$$

the condition for M to be smooth is that

$$\operatorname{rank}(H_{ij}(p)) \equiv \operatorname{rank}(H_{ij})(p) = \operatorname{rank}(H_{ij}) = r. \tag{2.18}$$

What we mean by this equation is that its rank must be maximal *everywhere*. If a space has less then maximal rank somewhere, it is *singular*. This condition can alternatively be stated in terms of the r-form

$$P := dp_1 \wedge \ldots \wedge dp_r. \tag{2.19}$$

If this form vanishes nowhere on the hypersurface, then M is a submanifold of $\mathbb{C}P^n$. The idea is that if this is the case, the p_i can be chosen as r of the coordinates on $\mathbb{C}P^n$.

We hope to have made it clear that with methods from algebraic geometry it is possible to say a lot of sensible things about singular varieties. But please note that for example the study of Calabi-Yau manifolds (appendix B) depends in large parts on methods from algebraic geometry as well. As our introduction to this subject ha been quite short and sketchy, we refer the reader interested in learning more to [38] and relevant parts of [36] and [64]. We continue by treating varieties that have singularities in isolated points.

2.2.1 Isolated Singularities

We already hinted a couple of times at the fact that later on (in chapter 5) we are going to need spaces with singularities to obtain realistic physics from M-theory compactifications. Specifically, in order to create chiral fermions we need *isolated* singularities. In this section we the codimension of a singularity and then define some of the most common isolated singularities.

The 'size' or 'gravity' of a singularity is customarily indicated by its codimension. Basically, the codimension of a singularity is nothing more than the number of non-singular dimensions in its direct vicinity. One way to define this mathematically is

Definition 2.10 (Codimension) *Define the map $H : \mathbb{C}^n \to \mathbb{C}^r$ by the matrix from equation (2.17). Then the* codimension *of a singularity s is defined as*

$$\operatorname{codim}(s) := \dim(\ker(H(s))) \tag{2.20}$$

So for example a variety of dimension d which contains a codimension $d-1$ singularity is singular along a line. If one were to fold a page of this thesis, such a codimension one singularity in a two-dimensional variety would appear.

Like we said, an especially important class of singularities are isolated singularities.

Definition 2.11 (Isolated singularity) *A singularity of maximal codimension, i.e. of codimension d if it is embedded in a variety of dimension d, is called an* isolated singularity.

We define two types of isolated singularities, because these are prevalent in later chapters. The first type define is probably the simplest as well.

Definition 2.12 (Conical singularity) *Let (M, g) be a real Riemannian m-dimensional manifold. A point $q \in M$ is said to be a* conical singularity *in M if there is a neighborhood U_q of q such that on $U_q \backslash \{q\}$, the line element (metric) takes the form*

$$ds^2 = dr^2 + r^2 d\Omega_N^2, \tag{2.21}$$

where Ω_N^2 is the metric on a $(m-1)$-dimensional manifold N.

If g can globally be written in this form, M is said to be a *cone* over N. The manifold N is called the *base* of the cone. In the special case that N is a round sphere S^{m-1}, the singularity is just a coordinate singularity, which can be avoided by choosing a different coordinate system[4].

Another frequently encountered isolated singularity is the orbifold:

Definition 2.13 (Orbifold) *A (real) orbifold is a space which admits an open covering $\{U_i\}$, such that each patch is diffeomorphic with $\mathbb{R}^m/\Gamma_i$, with Γ_i a discrete group acting on the manifold M.*

All points $p_0 \in U_i$ which are stable under the action of Γ_i, i.e. all the points for which $gp_0 = p_0, g \in \Gamma_i$ are singular points of the orbifold, called *orbifold* or *quotient* singularities. In most physics texts an orbifold is defined as a space of the form M/Γ, with M a manifold an Γ a discrete group. That global definition is a special case of the definition we give here.

Also note that in a similar way we can define a complex orbifold by requiring that each patch is biholomorphic to $\mathbb{C}^m/\Gamma_i$ and that the induced transition functions are holomorphic. We can also define a metric on $\mathbb{C}^m/\Gamma_i$ by the natural inherited metric on $\mathbb{C}^m$. Furthermore, when a complex orbifold can be embedded in $\mathbb{C}P^n$, it can also be viewed as an algebraic variety.

We conclude by saying that the definition of the holonomy group which was given in section 2.1 can be extended to spaces with orbifold (or any other kind of isolated) singularities. If there is an isolated singularity somewhere in the space, we simply take all paths which do not pass through the singularity and define the holonomy group with respect to that set of paths.

2.2.2 ADE singularities

Complex two-dimensional orbifolds of the form $\mathbb{C}^2/\Gamma$ will become very important when trying to construct non-Abelian gauge groups in M-theoretical model building. With coordinates (u, v) of $\mathbb{C}^2$ written as a column vector, $SU(2)$ acts in the natural way on the coordinates:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} a & b^* \\ -b & a^* \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad |a|^2 + |b|^2 = 1; \quad a, b \in \mathbb{C}. \tag{2.22}$$

[4]The standard spherical coordinate system in $\mathbb{R}^3$ is the obvious example.

It turns out that just as holonomy groups could be classified, it is possible to make a classification of all the possible discrete subgroups of $SU(2)$ and thus all the possible orbifold singularities. This classification, which was completed by Du Val in [27], gives a curious relation between the discrete subgroups of $SU(2)$ and the Dynkin diagrams of the simply-laced, semi-simple Lie groups A_n, D_k, E_6, E_7 and E_8. Later, in chapter 5, we hope make this relation somewhat clearer for the case of an A_n singularity. As we see, there are two infinite series corresponding to $SU(n) = A_{n-1}$ $(n \geq 2)$ and $SO(2k) = D_k$ and three corresponding to the exceptional Lie groups of type E. This is what gives the orbifold singularities $\mathbb{C}^2/\Gamma_{ADE}$ their name *ADE singularity*[5].

Below we give an explicit formulation of the classification by giving the generators of the discrete subgroups. As globally defined orbifolds are non-compact spaces, they cannot be written as hypersurfaces in $\mathbb{C}P^3$. But in each case it turns out to be possible to describe the orbifold $\mathbb{C}^2/\Gamma_{ADE}$ as a hypersurface in $\mathbb{C}^3$ instead. This means that they are not varieties as defined in section 2.2, but if we are keen on using this term anyway, we could call these spaces *non-compact varieties* or *affine varieties*. The (non-homogeneous) equation for each hypersurface is also given. To simplify the notation we define

$$\eta := e^{\frac{2\pi i}{5}}, \quad \epsilon := e^{\frac{2\pi i}{8}}, \quad \zeta_n := e^{\frac{\pi i}{n}} \tag{2.23}$$

for the fifth, eighth and $2n^{\text{th}}$ root of one respectively.

A_{n-1} singularity: The discrete group $\Gamma_{A_{n-1}}$ is generated by

$$\begin{pmatrix} \zeta_n^2 & 0 \\ 0 & \zeta_n^{-2} \end{pmatrix}. \tag{2.24}$$

$\Gamma_{A_{n-1}}$ is isomorphic to $\mathbb{Z}_n$, the *cyclic group* of order n. The equation for $\mathbb{C}^2/\Gamma_{A_{n-1}}$ as a hypersurface in $\mathbb{C}^3$ is

$$x^2 + y^2 + z^n = 0. \tag{2.25}$$

D_{n+2} singularity: The discrete group $\Gamma_{D_{n+2}}$ has two generators:

$$\begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.26}$$

$\Gamma_{D_{n+2}}$ is isomorphic to $\mathbb{D}_n$, the *binary dihedral group* of order n. The equation for $\mathbb{C}^2/\Gamma_{D_{n+2}}$ as a hypersurface in $\mathbb{C}^3$ is

$$x^2 + y^2 z + z^{n+1} = 0. \tag{2.27}$$

[5] Note that sometimes, the singularities are called Du Val singularities after the the discoverer of the classification.

E_6 **singularity:** The discrete group Γ_{E_6} has three generators: those of D_4, combined with

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon^7 & \epsilon^7 \\ \epsilon^5 & \epsilon \end{pmatrix}. \tag{2.28}$$

Γ_{E_6} is isomorphic to $\mathbb{T}$, the *binary tetrahedral group* of order 24. The equation for $\mathbb{C}^2/\Gamma_{E_6}$ as a hypersurface in $\mathbb{C}^3$ is

$$x^2 + y^3 + z^4 = 0. \tag{2.29}$$

E_7 **singularity:** The discrete group Γ_{E_7} has all the generators of E_6 combined with

$$\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^7 \end{pmatrix}. \tag{2.30}$$

Γ_{E_7} is isomorphic to $\mathbb{O}$, the *binary octahedral group* of order 48. The equation for $\mathbb{C}^2/\Gamma_{E_7}$ as a hypersurface in $\mathbb{C}^3$ is

$$x^2 + y^3 + yz^3 = 0. \tag{2.31}$$

E_8 **singularity:** Finally, the discrete group Γ_{E_8} has only two generators. They are given by

$$-\begin{pmatrix} \eta^3 & 0 \\ 0 & \eta^2 \end{pmatrix}, \frac{1}{\eta^2 - \eta^3}\begin{pmatrix} \eta + \eta^4 & 0 \\ 0 & -\eta - \eta^4 \end{pmatrix}. \tag{2.32}$$

Γ_{E_8} is isomorphic to $\mathbb{I}$, the *binary icosahedral group* of order 120. The equation for $\mathbb{C}^2/\Gamma_{E_8}$ as a hypersurface in $\mathbb{C}^3$ is

$$x^2 + y^3 + z^5 = 0. \tag{2.33}$$

This singularity is sometimes also called a Kleinian singularity, because this case was actually already worked out by Felix Klein in 1884.

We would like to remind the reader of the fact that nine representations of ADE singularities are shown on the cover of this thesis. These are created by taking the coordinates x, y, z in the defining equations to be real and can thus be seen real slices of the codimension four singular affine varieties we just defined.

After all this work done to describe and classify singularities, we turn to the question of how to excise them from a space. What we will try to do is remove the singularity and replace it with a suitable smooth space that becomes singular again in a certain part of its moduli space. This means that we can use methods from differential geometry to describe the physics on the space and take the singular limit to investigate the physics associated with the singularity.

2.2.3 Singularity resolution

The resolution of singularities is the art of creating a smooth manifold from a singular space. One of the reasons that this is important is that it provides a way to construct manifolds of special holonomy (e.g. Calabi-Yau or G_2-manifolds) when a singular description is at hand. Such a singular description is often easier to find (for example in the form of an orbifold singularity) than constructing a manifold with certain properties directly. There are two ways of resolving a singularity. It can be done by *blowing up* or by *deforming* the singular variety in which it is embedded into a smooth space. Although these two methods sometimes look alike, there is a difference. Making this distinction is important when we are describing phenomena like Mirror Symmetry and topology changing processes (see chapter 6 for a brief description of these phenomena).

Intuitively, we can view the procedure of blowing up as removing a neighborhood of the singularity from the space and gluing in a smooth manifold with the right properties instead. The mathematical definition may not be completely transparent (and we will not even try to give it in the most precise way), but the example in the next subsection will hopefully clear up the procedure a lot.

In the previous subsection we described the ADE singularities as affine varieties in $\mathbb{C}^3$. Before we can describe the blow-up procedure for singular varieties, we need to define the blow-up of a point in $\mathbb{C}^n$. For simplicity we take this point to be the origin. Blow-ups of points in manifolds, submanifolds and varieties can be understood using generalizations of this procedure. If we take $(z_1, \ldots, z_n)$ to be Euclidean coordinates on $\mathbb{C}^n$ and $[l_1, \ldots, l_n]$ to be homogeneous coordinates on $\mathbb{C}P^{n-1}$, we can define the set

$$\widetilde{\mathbb{C}}^n := \left\{ \big((z_1, \ldots, z_n), [l_1, \ldots, l_n]\big) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} \;\middle|\; z_i l_j = z_j l_i, \ \forall i, j \ \right\}. \tag{2.34}$$

There are two ways to look at this space. Firstly, for any given fixed point l in $\mathbb{C}P^{n-1}$, the space describes a line in $\mathbb{C}^n$, i.e. it is of the form $\mathbb{C}^1 \times \{l\} \cong \mathbb{C}^1$. It is quite simple to see this for the cases with small n by writing down all the relations for the coordinates and picking a specific point, but it is true for all dimensions. So we can regard $\widetilde{\mathbb{C}}^n$ as a complex line bundle over $\mathbb{C}P^{n-1}$, $\widetilde{\mathbb{C}}^n \xrightarrow{\pi} \mathbb{C}P^{n-1}$ with fibre $\mathbb{C}^1$. Secondly, if we fix a point $z \in \mathbb{C}^n$ that is not the origin, the equations can similarly be seen to describe a point in $\mathbb{C}P^{n-1}$. If we on the other hand take $z = 0$, we find the whole $\mathbb{C}P^{n-1}$ instead of just a point.

The previous statements imply that the line bundle $\widetilde{\mathbb{C}}^n \xrightarrow{\pi} \mathbb{C}P^{n-1}$ can be pointwise identified with $\mathbb{C}^n$, except that the origin has been replaced by the manifold $\mathbb{C}P^{n-1}$. Because of this $\widetilde{\mathbb{C}}^n$ is called the *blow-up* of $\mathbb{C}^n$ at the origin. The space $\mathbb{C}P^{n-1}$ that replaces the origin is called an *exceptional divisor*. Together with the blow-up we always give a projection Π back onto the original space, representing the reversed procedure of *blowing-down*. In our case this map is given by

$$\Pi : \widetilde{\mathbb{C}^n} \to \mathbb{C}^n, \tag{2.35}$$

with the action given simply by $\Pi(z, l) = z$.

In the previous subsection we described a number of singular varieties as hypersurfaces embedded in $\mathbb{C}^3$. A natural thing to do after describing the blow-up procedure for $\mathbb{C}^n$ is to extend this to the *blow-up of a singular variety.* Consider a hypersurface $A \subset \mathbb{C}^n$ that is singular in the origin. Then the blow-up $\widetilde{A}$ is defined as the subset $\widetilde{A} \subset \widetilde{\mathbb{C}^n}$ given by the closure of the inverse image of the projection,

$$\widetilde{A} := \overline{\Pi^{-1}(A - \{0\})}. \tag{2.36}$$

The closure of a set is defined as the unique smallest closed set that contains the given set.

We do not go into any further details about blowing-up singular varieties, but we hope that this procedure will be made clearer by explicitly working out the blow-up of the A_1-singularity in the next subsection. The following theorem[6], given in a slightly paraphrased form, shows the applicability of the blow-up procedure to singular varieties:

Theorem 2.14 (Hironaka) *Let X be a singular algebraic variety. Then there exists a nonsingular variety $\widetilde{X}$, which is the result of a finite sequence of blow-ups of X.*

Another way to get rid of a singularity is to find a "nearby" manifold (i.e. one that looks very much like the original) which is smooth, but deforms into the original singular variety by changing some parameter(s). Usually this is done in an algebraic variety by adding some (small) terms in the defining polynomials. The example in the next subsection will hopefully make this procedure clear without giving an exact definition.

2.2.4 An example: the A_1 singularity

With conclude this section on algebraic geometry with an example which will hopefully make some of the things introduced clearer. The beauty of this example is that despite the fact that it is the simplest example, it displays almost all of the concepts we introduced.

Just as in section 2.2.2 we begin by considering $\mathbb{C}^2$ with complex coordinates (u, v). We then mod out by Γ_{A_1}, which has only one generator that acts on the coordinates as

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} -u \\ -v \end{pmatrix}. \tag{2.37}$$

[6]For proving this theorem (which he did in 1964) Heisuke Hironaka received the 1970 Fields Medal, one of the highest distinctions in mathematics.

We see that Γ_{A_1} is just $\mathbb{Z}_2$, so the A_1 singularity is just the orbifold $\mathbb{C}^2/\mathbb{Z}_2$. The only fixed point of the transformation (2.37) is the origin. In the origin the tangent space is ill-defined, so if we want to know what the holonomy of this orbifold is, we have to consider only path that do not pass through the origin. Because the orbifold is flat everywhere but in the origin, a vector will come back onto itself if it is transported along any path that does not end on the image (under $\mathbb{Z}_2$) of its starting point. If we transport it along a path that does end on the image, it direction gets inverted. In other words, this orbifold has $\mathbb{Z}_2$ holonomy. As non-trivial holonomy is an indication for non-trivial curvature, we could say that in the origin there is a curvature singularity.

In section 2.2.2 we also gave algebraic descriptions of the singularities. Let us see how we find one for the current example. Note that u and v are double-valued on $\mathbb{C}^2/\mathbb{Z}_2$, so they are not suited to use as coordinates. The combinations

$$z_1 = u^2, \quad z_2 = v^2, \quad z_3 = uv \tag{2.38}$$

are single-valued. We can use any two of these as local coordinates, but we have to keep in mind that they are not independent and obey the equation

$$z_3^2 - z_1 z_2 = 0. \tag{2.39}$$

We now again make a linear change of variables $x = z_1 - z_2$, $y = i(z_1 + z_2)$, $z = 2z_3$, so that we can now write this equation as

$$x^2 + y^2 + z^2 = 0. \tag{2.40}$$

This is exactly the equation for $\mathbb{C}^2/\mathbb{Z}_2$ as a hypersurface in $\mathbb{C}^3$ that was given in (2.25) with $n = 2$. Now call this hypersurface S.

Blow-up of the A_1 singularity

S is a hypersurface in $\mathbb{C}^3$, so first we define $\widetilde{\mathbb{C}}^3$ as a subset of $\mathbb{C}^3 \times \mathbb{C}P^2$ as in (2.34) and then the blow-up of S as $\widetilde{S} = \overline{\Pi^{-1}(S - \{0\})}$. Now consider following a path towards the origin in S. From the discussion below (2.34) it should be clear[7] that the point we land on in the exceptional divisor $\mathbb{C}P^2$ of $\mathbb{C}^3$ depends on the direction in which we approach the origin. Now take this path to be defined by the line (tz_0, tz_1, tz_2) with $t \in \mathbb{C}$ and the z_i of course still obeying $z_0 z_1 - z_2^2 = 0$. If we follow this particular path to the origin, the point we land on will be $[l_0, l_1, l_2] \in \mathbb{C}P^2$ with the l_i satisfying $l_0 l_1 - l_2^2 = 0$ as well. It can be shown [8] that this hypersurface is $\mathbb{C}P^1 \cong S^2$. So we see that the exceptional divisor of $\widetilde{S}$ in the origin is a two-sphere. This is depicted in figure 2.1.

[7] Note that a path through S is of course also a path through $\mathbb{C}^3$, which means that this discussion applies to this case as well.

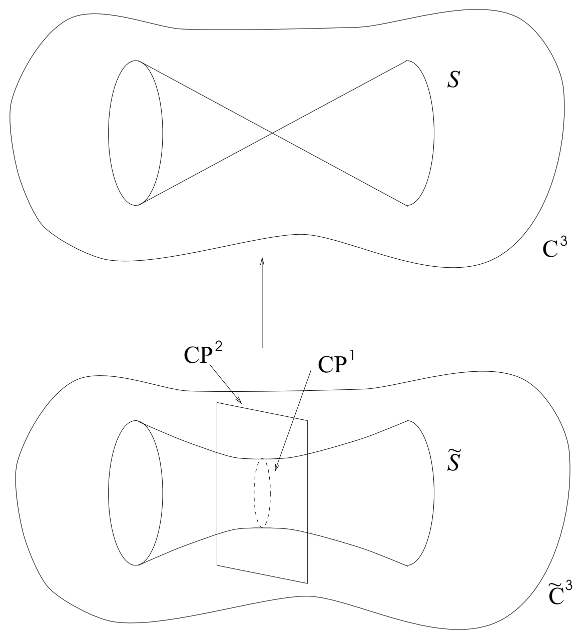

Figure 2.1: Blow-up of the A_1 singularity. Figure taken from [8].

Deformation of the A_1 singularity

The next thing we can do is slightly deform the defining equation (2.40) by adding a small term on the right hand side,

$$x^2 + y^2 + z^2 = \varepsilon^2. \tag{2.41}$$

By doing this the singularity gets replaced by a 2-sphere of radius ε. This can be seen by taking x, y, z and ε to be real. Note that this is exactly the exceptional divisor of the blow-up!

So in the case of an A_1-singularity the two resolution procedures lead to the same smooth space. But like we said, generally this will not be the case. One of the reasons we use this example is that enables us to introduce the Eguchi-Hanson metric, which will be important later on.

Eguchi-Hanson Metric

We now have the equation for the resolved A_1 singularity, but we do not have a metric on it yet. By resolving the singularity the holonomy group got enlarged from $\mathbb{Z}_2$ to $SU(2)$, so the metric should exhibit this as well. Note that since $SU(2) \cong Sp(1)$, this space is both Calabi-Yau and hyperkähler [8]. In 1978, Eguchi and Hanson found a Ricci-flat metric on this space, which is given by

$$ds^2_{EH} = \frac{1}{1-(\varepsilon/r)^4}dr^2 + r^2\left(\sigma_x^2 + \sigma_y^2 + (1-(\varepsilon/r)^4)\sigma_z^2\right). \tag{2.42}$$

In this metric $\varepsilon \in \mathbb{R}^+$ and the σ_i are 1-forms that are left-invariant under $SU(2)$ that are given by

$$\begin{aligned} \sigma_x &= \cos\psi d\theta + \sin\psi \sin\theta d\phi, && (2.43)\\ \sigma_y &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi, && (2.44)\\ \sigma_z &= d\psi + \cos\theta d\phi. && (2.45)\end{aligned}$$

The ranges of all the coordinates in the metric are $\varepsilon \leq r < \infty$, $0 \leq \psi < 2\pi$, $0 \leq \theta < \pi$ and $0 \leq \phi < 2\pi$.

At first sight there appears to be a singularity in the metric at $r = \varepsilon$, while we claimed that it is the metric on a resolved space. This apparent singularity is an artifact of the chosen coordinate system and is removable by going to a different coordinate system. To see this, make the transformation $u^2 = r^2[1-(\varepsilon/r)^4]$. After this and in the limit $u \to 0$ (corresponding to $r \to \varepsilon$), the metric takes the form $ds^2 \approx \frac{1}{4}(du^2 + u^2 d\psi^2)$ for fixed θ and ϕ. So with the periodicity $0 \leq \psi < 2\pi$ we see that this is just the flat metric on $\mathbb{R}^2$ in polar coordinates.

We now note that we can use the σ_i to write the metrics on the 2-sphere and the 3-sphere in a particularly compact way,

$$\begin{aligned} ds^2_{S^2} &= \sigma_x^2 + \sigma_y^2, && (2.46)\\ ds^2_{S^3} &= \sigma_x^2 + \sigma_y^2 + \sigma_z^2 && (2.47)\\ &= d\theta^2 + \sin^2\theta d\phi^2 + (d\psi^2 + \cos\theta d\phi)^2, && (2.48)\end{aligned}$$

but in the S^3-metric we have to let ψ run from 0 to 4π to cover S^3 completely.

If we take the limit $\varepsilon \to 0$ (or alternatively $r \to \infty$) of the metric (2.42) the metric asymptotes to the form $ds^2 \approx dr^2 + r^2(\sigma_x^2 + \sigma_y^2 + \sigma_z^2)$. This seems to be a cone over the 3-sphere, but because ψ runs from 0 to 2π this is actually a cone over $S^3/\mathbb{Z}_2$, with the $\mathbb{Z}_2$-action given by $\psi \to \psi + 2\pi$. Logically, a metric with the property that in a certain asymptote it is a cone is called *asymptotically conical.* In figure 2.1 you can see a graphical representation of the blow-up of the A_1 singularity.

2.3 G_2-manifolds

All the introduction in this chapter more or less builded up to the introduction of the most important space that we are going to need: the G_2-manifold. As stated in the introduction, this is the space we need obtain a realistic particle model from compactifications of M-theory. The study of G_2-manifolds can be done at a highly mathematical level, but we *try* to keep our treatment as physical as possible. The leading work on exceptional holonomy (which as we saw means having either G_2 or $Spin(7)$ holonomy) is [49], which can be consulted for a more rigorous treatment.

We start this section by defining the group G_2, move on to defining G_2-manifolds, then list a number of their most important properties and finally describe shortly a couple of methods for constructing them.

2.3.1 The group G_2

G_2 is the smallest of the so-called *exceptional Lie groups*. These exceptional groups are those semi-simple groups in the Cartan classification that do not belong to one of the infinite series of the $SU(r+1)$ (or A_r), $SO(2r+1)$ (or B_r), $Sp(r)$ (or C_r) or $SO(2r)$ (or D_r) type. The other exceptional groups are E_6, E_7, E_8 and F_4. G_2 is a subgroup of $SO(7)$, so it acts in a natural way on $\mathbb{R}^7$. It has some extra structure imposed on it, though, as can be seen from its definition.

Definition 2.15 (The group G_2) *Let $(y_1, \ldots, y_7)$ be coordinates on $\mathbb{R}^7$. Write $\mathrm{d}y_{ij\ldots l}$ for the exterior form $\mathrm{d}y_i \wedge \mathrm{d}y_j \wedge \cdots \wedge \mathrm{d}y_l$ on $\mathbb{R}^7$. If we define a three-form by*

$$\begin{aligned} \varphi_0 &= \mathrm{d}y_{123} + \mathrm{d}y_{145} + \mathrm{d}y_{167} + \mathrm{d}y_{246} - \mathrm{d}y_{257} - \mathrm{d}y_{347} - \mathrm{d}y_{356} && (2.49) \\ &\equiv \frac{1}{3!}\varphi_{ijk}\mathrm{d}y_{ijk}, && (2.50) \end{aligned}$$

the exceptional Lie group G_2 *is that subgroup of $SO(7)$ that preserves φ_0.*

Although the appearance of the three-form in this definition might look a bit arbitrary, there is more to it than meets the eye. The components φ_{ijk} of φ_0 are actually the structure constants of the imaginary octonions $\mathbb{O}$. What this means is that the basis vectors σ_i in $\mathrm{Im}(\mathbb{O})$ satisfy

$$\sigma_i\sigma_j = -\delta_{ij} + \varphi_{ijk}\sigma_k, \quad i, j, k = 1, \ldots, 7. \tag{2.51}$$

The specific form (2.49) just corresponds to a particular choice of basis. In other words, G_2 is the automorphism group of the octonion algebra, which means that acting with G_2 on elements of the algebra again produces the whole algebra. Note that the group G_2 also preserves the Hodge dual of φ_0,

$$*\varphi_0 = \mathrm{d}y_{4567} + \mathrm{d}y_{2367} + \mathrm{d}y_{2345} + \mathrm{d}y_{1357} - \mathrm{d}y_{1346} - \mathrm{d}y_{1256} - \mathrm{d}y_{1247}, \tag{2.52}$$

the Euclidean metric

$$g_0 = \mathrm{d}y_1^2 + \cdots + \mathrm{d}y_7^2, \tag{2.53}$$

and the orientation on $\mathbb{R}^7$. Further properties of G_2 are that it is compact, connected, semi-simple, 14-dimensional and has rank 2 (as the subscript indicates[8]).

Later on, the so-called *branching rules* of for the decomposition of representations of $SO(7)$ under G_2 will become important. In general, if we have a Lie subgroup $A \subset B$ and a certain representation D_B of B, we can create a representation D_A^B of A by restriction of D_B to A. This representation of A will however not be irreducible in general. If it turns out to be reducible, we can use for example the standard Clebsch-Gordan decomposition (or possibly the theory of characters) to write this as a direct sum of irreducible representations. The way representations of the original group decompose in such a way under a subgroup, are called branching rules. For the group G_2 as a subgroup of $SO(7)$, the branching rules of the most important representations are given by:

$$
\begin{aligned}
SO(7) &\supset G_2 \\
\mathbf{21} &= \mathbf{14} \oplus \mathbf{7} && (2.54)\\
\mathbf{7} &= \mathbf{7} && (2.55)\\
\mathbf{8} &= \mathbf{7} \oplus \mathbf{1} && (2.56)
\end{aligned}
$$

Here the **21**, **7** and **8** are the adjoint, vector and spinor representations of $SO(7)$ respectively. Under G_2 these decompose in the way indicated into the fourteen-dimensional adjoint representation **14**, the (fundamental) vector representation **7** and the singlet **1**. Especially the branching rule of the spinor representation will be important later on. Because we never work with explicit representations in this thesis, we do not provide the details of the deduction of these branching rules. The interested reader can consult [49] for this.

Other than the way given above, G_2 can alternatively be defined as that subgroup of $SO(7)$ that allows for a covariantly constant spinor representation. If we take this viewpoint, i.e. if we start with a covariantly constant spinor $\nabla\theta = 0$, then we can construct the G_2-invariant three-form (2.49) by taking its coefficients to be

$$\varphi_{ijk} := \theta^\dagger \Gamma_{ijk} \theta. \tag{2.57}$$

As such φ_0 is then called the induced three-form. For details, see [5].

2.3.2 G_2-manifolds defined

This simplest way to define a G_2-manifold at this point is as a seven-dimensional Riemannian manifold that has G_2 as its holonomy group. A

[8]Note that for all the groups in the Cartan classification the rank is given by the subscript.

mathematical definition of a G_2-manifold as given by Joyce in [48] is stated in terms of G-structures, instead of holonomy. This is a somewhat broader definition than one just given and can be shown to reduce to reduce to meaning "having holonomy G_2". It goes as follows.

First of all, a *principle bundle* is basically a bundle whose fibre F is identical to its structure group G (and is therefore often called the G-bundle)[9] Furthermore, the *frame bundle* F of an oriented manifold X (in our case of dimension seven) is a particular example of a principle bundle over X whose fibre at $p \in X$ is the set of isomorphisms between $T_p(X)$ and $\mathbb{R}^7$, which is $GL(7, \mathbb{R})$. A frame is a set of seven linearly independent sections, so this fibre is the set of all possible frames at p. Now we are ready to define a G_2-structure.

Definition 2.16 (G_2-structure) *A* G_2-structure *on X is a principal subbundle of the frame bundle of X, with structure group G_2.*

Each G_2-structure gives rise to a three-form φ and a metric g on X, such that every tangent space of X admits an isomorphism with $\mathbb{R}^7$, identifying φ and g with φ_0 and g_0 of (2.49) respectively. We take over the abuse of notation (as Joyce calls it himself) of calling (φ, g) the G_2-structure.

Definition 2.17 (Torsion) *Let (φ, g) be a G_2-structure and ∇ the Levi-Civita connection of g. Then $\nabla\varphi$ is called the* torsion *of the G_2-structure. If $\nabla\varphi = 0$, the structure is said to be* torsion-free.

With these definitions out of the way, we are finally ready to define a G_2-manifold.

Definition 2.18 (G_2-manifold) *A triple (X, φ, g) is called a* G_2-manifold *if X is a seven-dimensional manifold and (φ, g) is a torsion-free G_2-structure on X.*

The following nameless but important theorem gives us the correspondence between the picture we developed for manifolds whose holonomy group is G_2 and the more mathematical definition given above.

Theorem 2.19 *Let X be a 7-manifold and (φ, g) a G_2-structure on X. Then the following are equivalent:*

1. $\mathrm{Hol}(g) \subseteq G_2$, *and φ is the induced 3-form,*
2. (φ, g) *is torsion-free,* $\nabla\varphi = 0$,
3. $\mathrm{d}\varphi = \mathrm{d} * \varphi = 0$ *on X.*

[9] For an in-depth treatment of principle bundles, see [61].

This theorem shows that the holonomy group of a G_2-manifold is not necessarily G_2, but can also be a subgroup of it. To avoid confusing language and because we almost exclusively deal with cases in which $\mathrm{Hol}(g) = G_2$, we take the word G_2-manifold (unless stated otherwise) to mean "having exactly holonomy G_2".

Note that the equivalence of case 3 in theorem 2.19 is not at all obvious. Nevertheless, this set of first-order equations is considerably simpler to solve than the equations in case 2. Also note that both φ and $*\varphi$ are calibrations on a G_2-manifold. The corresponding three-dimensional φ-submanifolds are called *associative submanifolds* and the four-dimensional $*\varphi$-submanifolds are *co-associative submanifolds.*

After this definition we give an overview of general properties of G_2-manifolds in order to get a better understanding of them.

2.3.3 Properties of G_2-manifolds

In this subsection we list the properties of G_2-manifolds that we are going to need in quite a matter of factly way. We omit the proofs that are too involved. These proofs can be found in [49] and references therein. To begin with, an important proposition is

Proposition 2.20 *If (X, g) is a Riemannian manifold with* $\mathrm{Hol}(g) = G_2$*, that then X is a* spin manifold *and its space of parallel spinors has dimension one.*

What we already saw in (2.54) is that there is a singlet in the spinor representation. This implies that we can define exactly one parallel spinor satisfying $\nabla^S \eta = 0$ on a manifold of G_2 holonomy. We will not show that X is a spin manifold.

We will give a short presentation of the proof originally given in [41] of the proposition

Proposition 2.21 *A Riemannian spin manifold admitting a non-zero parallel spinor is Ricci flat.*

$\diamond$ **Proof:** To prove this proposition, we note that if there is a parallel spinor, $\nabla^S \eta = 0$, then we also have

$$0 = [\nabla^S_m, \nabla^S_n]\eta = \frac{1}{4} R_{mnpq}\Gamma^{pq}\eta. \tag{2.58}$$

If we contract this again with a gamma-matrix Γ^n and then use the Bianchi identity for R_{mnpq} (which asserts that components totally antisymmetric in $[npq]$ are zero), and the relation $\Gamma^n\Gamma^{pq} = \Gamma^{npq} - \Gamma^p\delta^{qn} + \Gamma^q\delta^{pn}$, we find

$$\begin{aligned} R_{mnpq}\Gamma^n\Gamma^{pq}\eta &= 0 \\ \Rightarrow \mathcal{R}_{mn}\Gamma^n\eta &= 0 \\ \Rightarrow \mathcal{R}_{mn} &= 0, \end{aligned} \tag{2.59}$$

because we assumed the parallel spinor η to be non-zero. □

In section 3.6 of [49] a proof is given of the converse statement. In other words, it is also true that a Ricci-flat Riemannian manifold is always spin and admits at least one covariantly constant spinor. If we combine the two statements from propositions 2.20 and 2.21, we find the important property

Corollary 2.22 *If (X, g) is a Riemannian seven-dimensional manifold that has $\mathrm{Hol}(g) = G_2$, then its metric g is Ricci flat.*

The proofs of the following two propositions are beyond the scope of this thesis and are thus omitted.

Proposition 2.23 *If (X, g) is a compact manifold with g a metric of holonomy G_2. Then the fundamental group $\pi_1(X)$ is finite.*

In order to understand the second proposition we need to know what *isotopic* means. An isotopy is defined as a homotopy of one embedding of a manifold (in another manifold) to another, such that at every time it is an embedding. Because a homotopy can be understood to mean a continuous deformation, we can think of isotopy as being basically a smooth deformation of the manifold.

Proposition 2.24 *The moduli space of metrics with holonomy G_2 on a compact 7-manifold X, up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $b^3(X)$.*

So basically, generic metrics of G_2 holonomy have $b^3(X)$ parameters determining the size and shape of the manifold.

The following lemma will be important later on and also has a fairly short proof, so we include it for completeness.

Lemma 2.25 *A Killing vector on a Ricci-flat compact manifold M is covariantly constant.*

⋄ **Proof:** Let K_n be a Killing vector on M. By contracting the defining equation (2.2) with g^{mn} we find $\nabla^m K_m = 0$. Now, if we let a covariant derivative act, we get

$$\nabla^m \nabla_m K_n + \nabla^m \nabla_n K_m = 0. \tag{2.60}$$

Using (2.58), $\nabla^m K_m = 0$ and the assumed Ricci-flatness, we can now write this as

$$\begin{aligned}
0 &= \nabla^m \nabla_m K_n + \nabla^m \nabla_n K_m && (2.61)\\
&= \nabla^m \nabla_m K_n + [\nabla^m, \nabla_n] K_m + \nabla_n \nabla^m K_m && (2.62)\\
&= \nabla^m \nabla_m K_n + R^m{}_{nmp} K^p && (2.63)\\
&= \nabla^m \nabla_m K_n + \mathcal{R}_{np} K^p && (2.64)\\
&= \nabla^m \nabla_m K_n && (2.65)
\end{aligned}$$

If we now multiply this last equation by K_m and integrate it over our compact manifold M, the result becomes

$$\int_M K_m \nabla_n \nabla^n K^m = -\int_M (\nabla_n K_m)(\nabla^n K^m) = 0. \tag{2.66}$$

So, because this holds for any compact[10] M, in the end this means that $\nabla_n K_m = 0$. $\square$

Now recall that a manifold of G_2-holonomy is Ricci flat. This means that a Killing vector on a compact G_2-manifold is covariantly constant. But G_2-holonomy is incompatible with the existence of covariantly constant vector fields, as we have seen that the **7** of SO(7) decomposes under G_2 as $\mathbf{7} \to \mathbf{7}$. In particular, there is no singlet and thus no chance of finding a covariantly constant vector. So this automatically proofs the following proposition:

Proposition 2.26 *A manifold of G_2-holonomy has no Killing vectors and hence no continuous symmetries, i.e. the isometry group of a G_2-manifold is trivial.*

Again without a proof, we state that a compact G_2-manifold has a trivial first cohomology group, $H^1(X, \mathbb{R}) = \{0\}$. We define the r-th Betti number as the dimension of the r-th cohomology group,

$$b^r(X) := \dim H^r(X) = \dim H_r(X) = b_r(X), \tag{2.67}$$

where we have used the fact that the homology group is dual to the cohomology group. The triviality of the first cohomology group, together with the connectedness of X and the Poincaré duality (which implies that $b^r = b^{7-r}$) enables us to write down all of its Betti numbers

$$b^0 = b^7 = 1; \quad b^1 = b^6 = 0; \quad b^2 = b^5, \quad b^3 = b^4 \text{ arbitrary} \tag{2.68}$$

So the most important topological information about manifolds with G_2 holonomy is encoded in the pair of numbers (b^2, b^3). The above values imply that the Euler characteristic

$$\chi(X) = \sum_{r=0}^{7} (-1)^r b^r = 0. \tag{2.69}$$

This concludes our short review of some of the most essential general properties of G_2-manifolds. For more background, consult [49].

[10] Note that we have taken M to be compact, because otherwise the partial integration would lead to boundary terms and Killing vector fields typically do not vanish at a boundary (e.g. at infinity), so in the non-compact case this lemma generically does not hold.

Topology of X	Base manifold Y	Isometry group
$S^4 \times \mathbb{R}^3$	$\mathbb{C}P^3$	$SO(5)$
$\mathbb{C}P^2 \times \mathbb{R}^3$	$\frac{SU(3)}{U(1)\times U(1)}$	$SU(3)$
$S^3 \times \mathbb{R}^4$	$S^3 \times S^3$	$SU(2)^3$

Table 2.2: Topology and base manifolds of Asymptotically Conical (AC) spaces, as constructed by [16] and [34] and reproduced by [42].

2.3.4 Construction of G_2-manifolds

We have now seen what a G_2-manifold is and have gotten to know a number of their important properties, but we still do not know how to construct them. Although Bergers theorem was formulated in 1955, it took until 1987 for the first local G_2-holonomy metric to be constructed [15]. This local metric said nothing about the possible global structure of the manifold. Then two years later it was Bryant and Salamon [16] and independently Gibbons, Page and Pope [34] who then constructed *complete* metrics of G_2-holonomy for a number of *non-compact* manifolds. The method of their construction essentially came down to analyzing the Ricci flatness equations $\mathcal{R}_{mn} = 0$ for a certain ansatz for the metric. The disadvantages of this method are that the Einstein equations are a relatively complicated set of second-order equations and that the method does not allow generalizations, as it depends on making a certain ansatz. All the examples created in [16] and [34] are asymptotically conical smooth manifolds, which can be regarded as resolutions of conical singularities over various base-manifolds Y. The currently known base manifolds that can be used in this construction are $\mathbb{C}P^3, SU(3)/(U(1)\times U(1))$ and $S^3 \times S^3$ (see table 2.2). We list these metrics explicitly under the description of the Hitchin construction below.

Owing to Dominic Joyce, we now have a systematic way of constructing *compact* G_2-manifolds [46]. This method is inspired by and very similar to the Kummer construction of a $K3$ surface (see Appendix B). Unfortunately, this construction only provides a way to construct local G_2-holonomy metrics and no explicit complete metrics are known on any of the Joyce-manifolds. Also, none of these manifolds appear to have a limit in which they develop isolated singularities. So another method of constructing compact manifolds of G_2 holonomy would be beneficial. One possible method was suggested by Kovalev in [52]. It is an elegant method, but again it provides no way to construct a complete and compact metric and it is still unclear if these manifolds can develop isolated singularities. Yet another method was developed by Hitchin in [42]. This seems to be a promising and interesting approach to the problem, but only explains how to construct non-compact G_2-holonomy metrics. We will say something more about Hitchins construction below, but not before we have explained how the Joyce construction works.

Joyce construction of compact G_2-manifolds

As Joyce clearly explains himself in [46], his construction can be divided into four steps.

Step 1 Take the 7-torus T^7 and a flat G_2-structure (φ_0, g_0) and choose a finite group Γ of isometries of T^7, preserving (φ_0, g_0). Take the quotient T^7/Γ to get a singular, compact orbifold.

Step 2 If the group Γ is well chosen, there is a method of resolving the singularities in a natural way. We now have a compact manifold X with a map $\pi : X \to T^7/\Gamma$.

Step 3 Write down a 1-parameter family of G_2-structures (φ_t, g_t) on X which depends on $t \in (0, \epsilon)$, such that these have small torsion if t is small. As $t \to 0$, the G_2-structure converges to the singular structure $\pi * (\varphi_0, g_0)$.

Step 4 Prove that for sufficiently small t, the G_2-structure can be deformed to a G_2-structure $(\tilde{\varphi}_t, \tilde{g}_t)$ with zero torsion. Finally, show that $\tilde{g}_t$ is a metric with holonomy G_2 on the compact 7-manifold X.

As was explained in [39], the fixed points introduced in step 1 are A_1-singularities and be resolved in step 2 basically by gluing in copies of the Eguchi-Hanson space.

In Part II of [48], a large number of examples is given of manifolds constructed by following this procedure. We reproduce none of these here as the reader can consult the original article or the book [49] for a detailed description of them. We do want to note at this point however that one of the examples is constructed by resolving the singularities of some orbifold of $T^3 \times K3$. In other words, it has the form $\widetilde{T^3 \times K3/\Gamma}$, with Γ some suitably chosen discrete group. The reason we mention this is that near the singularities, this orbifold can locally be described as $\mathbb{R}^3 \times \mathbb{C}^2/\Gamma_{A_1}$, which contains the simplest ADE singularity that we have described in such depth in this chapter. This is our first indication that it is possible to embed general ADE singularities in manifolds of G_2-holonomy.

The Betti numbers (b^2, b^3) of all the the G_2-manifolds constructed in this way (including all the ones constructed after [48]) are listed in table 2.3. There are in total 252 different sets of Betti numbers and thus 252 topologically distinct G_2-manifolds. According to Joyce, this is probably only a small fraction of the possible Betti numbers of all compact, simply-connected seven-dimensional manifolds with holonomy G_2.

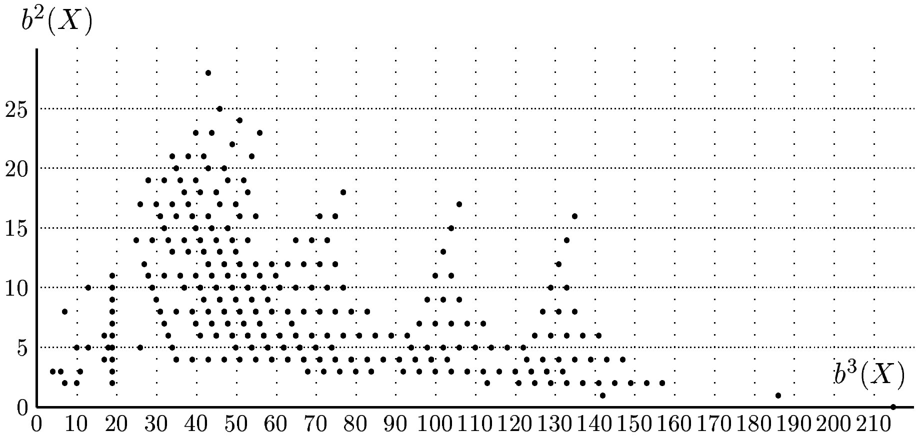

Table 2.3: Betti numbers (b^2, b^3) of compact G_2-manifolds. Taken from [46].

Hitchin construction of non-compact G_2-manifolds

Quite recently, Hitchin published a new method [42] of constructing G_2 metrics on non-compact Asymptotically Conical (AC) spaces. Very schematically, in this construction we take the base of the conical space to be a homogeneous space $Y = G/K$. Then it turns out that on the space $\mathcal{P}$ of G-invariant differential form on Y, we can define a symplectic structure. On this space we can use a Hamiltonian dynamical system to find a G_2-holonomy metric. The beauty of his method is that we are able to construct a metric by solving a set of first-order differential equations (the Hamilton equations) with certain boundary conditions, instead of solving the (second-order) Einstein equations using a specific ansatz. So the Hitchin construction has two big advantages: the mathematics involved is simpler and it can be generalized. For more information about this method, we refer the reader to [6] and references therein.

Like we said in the introduction to this section, [16] and [34] constructed explicit metrics on three different smooth non-compact asymptotically conical G_2-manifolds. Using the Hitchin construction, we can reproduce these metrics exactly. This is a strong validation of the soundness of this construction. For future reference, we list these metrics below in their full glory. Note that all these spaces have the same topology as indicated in figure 2.2. In this figure X is the total space, Y is the base of the cone, T is the "tip of the cone" and F is the collapsing fibre.

First of all, the metric that is asymptotic to a *cone over* $\mathbb{C}P^3$ is given by

$$ds^2_{CP} = \frac{dr^2}{1-(r_0/r)^4} + \frac{r^2}{4}(1-(r_0/r)^4)(du_i + \epsilon_{ijk}A_j u_k)^2 + \frac{r^2}{2}d\Omega_4^2, \quad (2.70)$$

where $d\Omega_4^2$ is the standard round metric on S^4 with $SO(5)$ isometry, normalized to have $\mathcal{R}_{rs} = 3g_{rs}$ $(r, s \in \{1,2,3,4\})$. The u_i are a set of coordinates

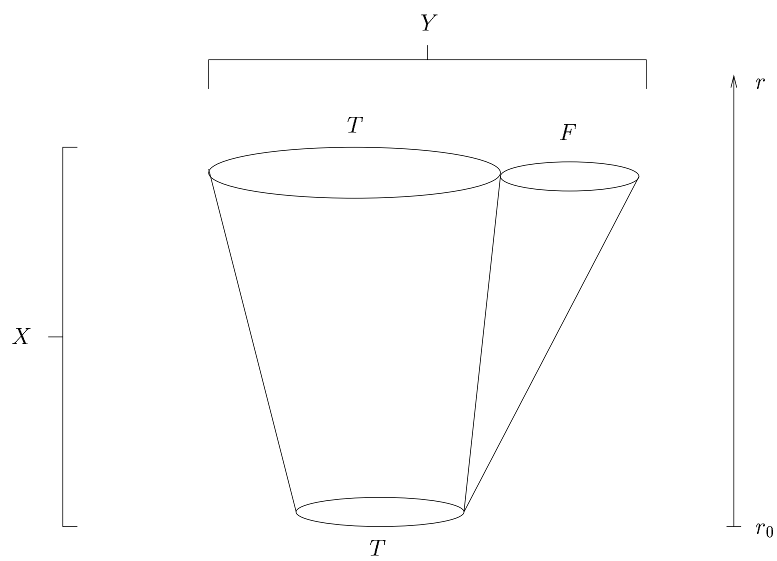

Figure 2.2: Topology of asymptotically conical spaces. X is the total space, Y is the base of the cone, T is the "tip of the cone" and F is the collapsing fibre.

of $\mathbb{R}^3$ that a subject to the condition $\sum_i (u_i)^2 = 1$ and $r_0 \in \mathbb{R}^+$. A_i is an $SU(2)$ gauge field on S^4, carrying unit instanton number. We see that if we take $r = r_0$, the S^4 remains at finite size, while the fibre S^2 shrinks to zero size. Topologically, this space looks like figure 2.2, with $X = S^4 \times \mathbb{R}^3$, $Y = \mathbb{C}P^3$, $T = S^4$ and $F = S^2$.

If we would replace in (2.70) the metric $d\Omega_4^2$ by a metric with $SU(3)$ isometry on $\mathbb{C}P^2$, we have the second case of a metric on a *cone over* $SU(3)/U(1)^2$. The collapsing fibre is the same S^2. For a detailed description of the geometry of both these spaces, see [9] and the original papers [16] and [34].

The metric that approaches a *cone over* $S^3 \times S^3$ is given by

$$ds^2_{SS} = \frac{dr^2}{1-(r_0/r)^3} + \frac{1}{9}r^2[1-(r_0/r)^3](\nu_1^2+\nu_2^2+\nu_3^2) + \frac{r^2}{12}(\sigma_1^2+\sigma_2^2+\sigma_3^2) \quad (2.71)$$

where $\nu_i = \Sigma_i - \frac{1}{2}\sigma_i$ and Σ_i, σ_i are two sets of $SU(2)$ left-invariant one-forms we defined in equation (2.43). The parameter r_0 here determines the size of the S^3 generated by the ν_i. If now $r \to r_0$, we see that one S^3 shrinks to zero size, while the other remains finite. So here both T and F are S^3.

It is interesting to note that for these three manifolds the "tip-of-the-cone" T is a calibrated cycle. In the asymptotically conical metrics with base manifolds $\mathbb{C}P^3$ and $SU(3)/U(1)^2$, $T = S^4$ and $T = \mathbb{C}P^2$ are their respective $*\varphi$-calibrated co-associative four-cycles. For the metric with base $S^3 \times S^3$, the S^3 that remains of finite size is a φ-calibrated associative three-cycle.

Kovalev construction of compact G_2-manifolds

The last method of constructing G_2-manifolds we would like to mention was recently suggested by Alexei Kovalev in [52]. He uses a kind of "generalized connected sum" of two asymptotically cylindrical Calabi-Yau manifolds W_1 and W_2 to create a smooth compact G_2-manifold X. The way this can be pictured is that the asymptotically cylindrical spaces with W_1 and W_2 at their ends are glued "back-to-back" to form X,

$$X \cong (W_1 \times S^1) \cup (W_2 \times S^1). \tag{2.72}$$

Although this method is very elegant, it remains to be seen whether these spaces can be deformed to include also isolated singularities. It appears to have only led to the creation of a few examples of compact G_2-manifolds, but this might change in the future.

2.4 Conclusions

In this chapter we provided a lot of background material for the remainder of this thesis. We started with introducing isometry, holonomy and calibrated geometry and then mainly used tools from algebraic geometry to study singularities. This was all aimed at being able to describe (singular) G_2-manifolds, which were introduced in section 2.3.2. After studying them in detail, let us make a couple of general of remarks about the challenges which have to be faced when dealing with G_2-manifolds. The two most important challenges are

- the lack of an existence theorem for G_2 holonomy metrics,
- dealing with singularities and constructing spaces with the right type of singularities.

The first problem basically boils down to the fact that there is no theorem analogous to Yau's theorem B.7 for manifolds of G_2-holonomy, ensuring that (under certain favorable conditions) a metric with G_2 holonomy can be found. This means that for now the analysis of (physics) problems involving G_2-manifolds has to be done on a case-by-case basis for the specific known examples of such metrics.

The second problem has to do with the fact that we need singular spaces to obtain realistic physics from G_2-compactifications and we do not yet have sufficient methods to construct G_2-manifolds with all the needed singularities. This has in a large part to do with the fact that all the tools from algebraic geometry that can normally be used to study singular spaces (which was done very successfully for singular Calabi-Yau manifolds over the years), cannot be used on G_2-manifolds. The reason is of course that G_2-manifolds do not admit a complex structure and can hence not be described as varieties (which are inherently complex objects). More explicitly, as we will see in chapter 5 we need isolated singularities to obtain the most interesting physics. Furthermore, in chapter 3 we explain that compactness is needed in order to have any hope of finding realistic physics from M-theory compactifications. But no compact manifolds with isolated singularities have been constructed up till now. Examples of non-compact G_2-manifolds with isolated singularities *have* been found as we saw in the previous subsection, but these can only be serve as local models of M-theory compactifications.

The statement that these are true challenges is underlined by the fact that G_2-manifolds have been known to be interesting spaces to do Kaluza-Klein reduction on for more than twenty years (in the early eighties people already knew that you needed a space of G_2-holonomy to compactify eleven-dimensional supergravity to four dimensions while preserving $\mathcal{N}=1$ supersymmetry), but only in the last couple of years we have (partially) learned how to deal with these problems. This has just as much to do with our greater understanding of M-theory as with our improved knowledge of methods to construct G_2-manifolds[11]. But it should be clear that still great mathematical challenges lie ahead and that the progress in finding realistic M-theory compactifications has been slower than many people would have liked, because of these challenges.

[11]Note that the same comments actually also apply to the other type of exceptional holonomy manifolds, those with $Spin(7)$ holonomy. These have their own applications (similar to those of G_2-manifolds), but will not go into them at all.

Chapter 3

M-Theory Essentials

3.1 Introduction

This chapter is intended as a short introduction to M-theory. This discussion will be focused on the part of M-theory we know best: its low-energy effective theory. This low-energy limit is the eleven-dimensional supergravity theory we briefly mentioned in chapter 1. The way we introduce this theory is by first reviewing the general procedure for Kaluza-Klein dimensional reduction. After this review, it will be clear that compactification of theories in higher dimensions provide a way to construct a purely geometrical description of the fundamental interaction and that something special is going on in eleven dimensions. But it is also clear that eleven-dimensional supergravity cannot be the final answer, because this theory is not renormalizable and thus ill-behaved at short distances (i.e. at high energies). Therefore, after describing eleven-dimensional supergravity in detail, we describe some key features of M-theory.

General references for section 3.2 are [36] and [29]. These references are a bit outdated by now, but they still contain nice introductions to the basics of Kaluza-Klein supergravity. The material in section 3.3 can be found in many places, but [64] and [67] are excellent places to learn more. For a more accessible introduction to M-theory, see for example [54].

3.2 Eleven-dimensional Supergravity

We explained in the introduction that supergravity is a locally supersymmetric field theory that contains a description of gravity. Supergravity theories exist in dimensions four to eleven[1] and provide the low-energy limits of all

[1]In three dimensions, it is possible to define a supergravity theory, but then the metric is non-dynamical. This theory a so-called Chern-Simons theory, whose non-trivial field configurations are purely topological. In *two* dimensions even this is not possible anymore, so the theory becomes trivial.

superstring theories and M-theory in 10 and 11 dimensions respectively. An explanation for the existence of these extra dimensions may be given by the Kaluza-Klein mechanism. The authors of [29] make this statement even stronger: "Our research in this area has convinced us that the only way to do supergravity is via Kaluza-Klein and that the only viable Kaluza-Klein theory is supergravity." Motivated by these encouraging words, we begin by describing the general procedure of Kaluza-Klein theory.

3.2.1 Kaluza-Klein Theory

The basic idea of Kaluza-Klein theory is that a relatively simple theory in a high number of dimensions can give rise to a fairly complicated theory in low dimensions if some of the dimensions form a compact space. The internal (gauge) symmetries of the low-dimensional effective field theory in this scheme are generated by geometric symmetries (isometries) of the compact space.

As explained in [29] the general procedure for doing Kaluza-Klein compactifications is as follows. We start with a Einstein-Hilbert action in $D = 4 + k$ dimensions with coordinates z^D $(D = 0, \dots, D-1)$, describing the dynamics of the metric $g_{MN}(z)$, combined with a generic action describing a collection of matter fields collectively indicated by $\vec{\Phi}(z)$. So, we consider an action that looks like[2]

$$S = \int d^D z \sqrt{g}\, [\mathcal{R} + \mathcal{L}(\vec{\Phi})]. \tag{3.1}$$

The corresponding set of field equations will contain the Einstein equations with the energy-momentum tensor given in terms of the fields $\vec{\Phi}$. We will then look for classical solutions of these equations for which the metric describes a product spacetime $M = M_4 \times X$, i.e.

$$\langle g_{MN}(z) \rangle = \begin{pmatrix} \mathring{g}_{\mu\nu}(x) & 0 \\ 0 & \Delta(x)\mathring{g}_{mn}(y) \end{pmatrix}. \tag{3.2}$$

Here $\mathring{g}_{\mu\nu}(x)$ is a metric on M_4 that has Lorentzian signature and depends only on the four-dimensional coordinates x^μ. $\mathring{g}_{mn}(y)$ is the Euclidean metric on the internal k-dimensional space X with coordinates y^m. We have included a so-called *warp factor* $\Delta(x)$ in this metric. It is possible to construct consistent theories for certain choices of the warp-factor, but we will only work with the case in which the warp-factor is trivial

$$\Delta(x) = 1. \tag{3.3}$$

Then we demand M_4 to be maximally symmetric, meaning that it is homogeneous and isotropic about every point[3]. This implies that it has

[2] Note that we are not very careful with factors and constants here.

[3] Since current cosmological (WMAP) data [68] indicates that this is the case up to fluctuations of order 10^{-5}, this is not an unreasonable assumption to make.

Name	Λ	Isometry	$E > 0$ theorem	SUSY
dS	> 0	SO(1,4)	-	-
Minkowski	0	Poincaré	$\surd$	$\surd$
AdS	< 0	SO(2,3)	$\surd$	$\surd$

Table 3.1: Summary of properties of maximally symmetric spaces.

constant curvature and is in fact an Einstein space, $\mathring{R}_{\mu\nu} = \Lambda \mathring{g}_{\mu\nu}$. Depending on the sign of the cosmological constant Λ, an Einstein space with Lorentzian signature can be either de Sitter (dS), Minkowski or anti-de Sitter (AdS) space. Some of the most essential properties of these three spaces are listed in Table 3.1. To ensure stability of the vacuum we need a positive energy theorem and to have better control of the theory, we would like it to be supersymmetric. So this restricts to values of the cosmological constant of $\Lambda \leq 0$.

It is at this point necessary to say a few words about this restriction on the value of the cosmological constant. The reason is that current data [68] seems to quite convincingly indicate[4] that we live in a de Sitter universe (i.e. that the cosmological constant is small, but finitely positive). Many ideas exist about ways to describe theories on de Sitter space, but these theories - be it String Theory, M-theory, (Conformal) Field Theory or Supergravity - all seem to have in common that it is quite hard to make them consistent and that it is even difficult to precisely define basic things like entropy and local particle properties. For an overview of this subject and a discussion of these problems, see [79]. We would like to note, though, that it is possible to find de Sitter solutions in supergravity if we take a different set of prerequisites. It is for example possible to find de Sitter vacua if we consider warped compactifications [44]. As this could be the subject of a thesis by itself and the value of the cosmological constant is quite small, we consider for the most part of this thesis only solutions with $\Lambda = 0$. It is our hope that quantum effects beyond the tree-level approximation we do below and effects from the full M-theory will somehow modify the cosmological constant to its observed value. This is also related to the problem of supersymmetry breaking, as these quantum effects should also generate a superpotential for the various fields that forces the fields to choose a specific non-supersymmetric vacuum.

Now that we put some restrictions on M_4, let us do the same for X. For X we demand that it obeys the Einstein equations and is compact. As in any quantum mechanical theory, the spacing between states with momentum in the extra dimensions is inversely proportional to the volume of these

[4]Note that this was first discovered by observing Type IA supernova explosions. WMAP reconfirmed this result and (amongst other things) more accurately determined the cosmological constant.

dimensions. So, compactness of X is needed to ensure that the spectrum of Kaluza-Klein states is non-continuous. The easiest way to achieve this is by taking X to be an Einstein space, $\mathcal{R}_{mn} = cg_{mn}$, as well. Furthermore, as we will explain shortly, gauge symmetries in the low-energy effective theory come from isometries of the compact space. So we would like X to have continuous symmetries that lead to interesting physics. Now two theorems become important.

Theorem 3.1 *A compact Einstein space with Euclidean signature and $c < 0$ has no continuous symmetries.*

Theorem 3.2 *A complete Einstein space with Euclidean signature and $c > 0$ is always compact.*

The first of these theorems was proven in [80] and the second in [60]. So we conclude that we should look for solutions with $c \geq 0$. If we find a solution with $c > 0$, we have constructed a solution that exhibits so-called *spontaneous compactification*, meaning that the compactness of the extra dimensions is an outcome of the theory instead of an input.

Now let us see how to determine the massless four-dimensional spectrum of particles in this setting. To do this, we consider small fluctuations of the fields around the classical solutions

$$g_{MN}(z) = \langle g_{MN}(z) \rangle + h_{MN}(z), \tag{3.4}$$
$$\vec{\Phi}(z) = \langle \vec{\Phi}(z) \rangle + \vec{\phi}(z). \tag{3.5}$$

We now substitute these into the equations of motion, keep terms linear in the fluctuations and solve these equations for the fluctuations. Now, regardless of the exact form of the Lagrangian (3.1), we can always split it in an interacting part and a part that contains standard kinetic terms (i.e. $(\partial\phi)^2$ terms for bosonic fields and $\partial\!\!\!/\phi$ terms for fermionic fields). These terms can then be split into a four-dimensional and a k-dimensional part. If we now do a Fourier transform to momentum space, we can recognize that the four-dimensional mass-operator M^2 is essentially given by the Fourier transform of the k-dimensional kinetic term. We can then write the solution as an expansion in the eigenfunctions of this mass-operator[5] as

$$\vec{\phi}(z) = \sum_i \varphi^{(i)}(x) Y^{(i)}(y), \tag{3.6}$$

where

$$M^2 Y^{(i)}(y) = m_i^2 Y^{(i)}(y). \tag{3.7}$$

In the following we only keep terms with $m_i = 0$.

[5] Note that it might be necessary to make a modification of these terms before we can do this. See chapter 14 of [36] for details.

We now turn to the question of how gauge symmetries arise in this setting. The D-dimensional theory we start with has D-dimensional general coordinate invariance. If we take spacetime to be $M_4 \times X$, part of this symmetry will get broken. If however X retains part of this symmetry, i.e. if it has a non-trivial isometry group G generated by Killing vector fields $K^{(i)}(y)$, massless representations of this unbroken symmetry group will appear as gauge-bosons in the four-dimensional theory. This can be made explicit by considering the Kaluza-Klein ansatz for the fluctuations of the metric with mixed components,

$$h_{\mu n}(z) = \sum_{i=1}^{\dim(G)} A_\mu^{(i)}(x) K_n^{(i)}(y) + \cdots , \tag{3.8}$$

where the $K_n^{(i)}(y)$ are the Killing vector fields on X.[6] To see that the $A_\mu^{(i)}(x)$ really appear as spin one gauge fields in four dimensions, we calculate how they transform under the coordinate transformations that correspond to gauge transformations in the effective theory. First of all, under a general infinitesimal coordinate transformation $z^M \to z^M + \xi^M$, the metric can be shown [73] to transform as

$$\delta g_{MN} = \mathcal{L}_\xi g_{MN} = g_{NP} \nabla_M \xi^P + \xi^P \nabla_P g_{MN} + g_{MP} \nabla_N \xi^P. \tag{3.9}$$

As was explained in subsection 2.1.1, the isometry group is generated by the Killing vectors, so we now consider the special infinitesimal local coordinate transformation

$$\xi^M(z) = \left(0, \sum_i \varepsilon^{(i)}(x) K^{(i)m}(y)\right). \tag{3.10}$$

By noting that $g_{\mu n} = h_{\mu n}$ (because $\langle g_{\mu n} \rangle = 0$) and inserting (3.8) in (3.9), we see that the off-diagonal part of the metric transforms under (3.10) as[7]

$$\begin{aligned} \delta g_{\mu n} &= g_{np} \nabla_\mu \xi^p + \xi^p \nabla_p h_{\mu n} + h_{\mu p} \nabla_n \xi^p & (3.11) \\ &= K_n^{(i)} \nabla_\mu \varepsilon^{(i)} + \varepsilon^{(i)} A_\mu^{(j)} \left[K^{(i)p} \nabla_p K_n^{(j)} + K_p^{(j)} \nabla_n K^{(i)p} \right] & (3.12) \end{aligned}$$

We now use the fact that the $K^{(i)}$ are Killing vector fields (2.2), the definition of the Lie bracket $[X, Y]_m = X^p \nabla_p Y_m - Y^p \nabla_p X_m$ (in which the ordinary derivatives can be and have been replaced by covariant derivatives) and (2.4) to write this as

$$\begin{aligned} \delta g_{\mu n} &= K_n^{(i)} \partial_\mu \varepsilon^{(i)} + \varepsilon^{(i)} A_\mu^{(j)} \left[K^{(i)p} \nabla_p K_n^{(j)} - K^{(j)p} \nabla_p K_n^{(i)} \right] & (3.13) \\ &= K_n^{(i)} \partial_\mu \varepsilon^{(i)} + \varepsilon^{(i)} A_\mu^{(j)} \left[K^{(i)}, K^{(j)} \right]_n & (3.14) \\ &= K_n^{(i)} \partial_\mu \varepsilon^{(i)} + \varepsilon^{(i)} A_\mu^{(j)} f^{ij}{}_k K_n^{(k)}. & (3.15) \end{aligned}$$

[6]Note that in (2.65) it is shown that Killing vector fields are zero-eigenvalue eigenfunctions of the k-dimensional Laplacian, so they in fact do correspond to massless fluctuations.

[7]We now use the summation convention for repeated indices (i), (j), etc. as well.

If we now remember that a variation of course has the property that $\delta g_{\mu n} = \delta A_\mu^{(i)} K_n^{(i)} + A_\mu^{(i)} \delta K_n^{(i)}$, we see that $K^{(i)}$ and $A_\mu^{(i)}$ transform as

$$\delta A_\mu^{(i)}(x) = \partial_\mu \varepsilon^{(i)}(x) + f^{ij}{}_k \varepsilon^{(j)}(x) A_\mu^{(k)}(x) \quad (3.16)$$

$$\delta K_m^{(i)}(y) = 0. \quad (3.17)$$

This shows that $A_\mu(x)$ transforms exactly as a non-Abelian gauge field of a gauge group G.

So, to summarize Kaluza-Klein theory provides a way to reconcile the apparent discrepancy between the observed number of spacetime dimensions and the number of dimensions that we are led to by theoretical considerations. Getting the cosmological constant to obtain the right value in this setting is still a difficult problem. And if the compact extra dimensions form a manifold with non-trivial isometry group G, then the massless spectrum of the effective four-dimensional theory will contain gauge bosons that generate a gauge-group G. This is an exciting prospect, because this seems to open up the possibility of realizing "Einsteins dream" of finding a purely geometric description of the fundamental interactions. As we will see in chapter 4, using the Kaluza-Klein procedure only in the way we describe in this section is not enough to fulfill this dream, though.

3.2.2 The isometry group and the dimension of spacetime

As we saw in the previous subsection, the gauge group of the low energy effective field theory can be identified with the isometry group of the compact space. In 1981 Witten showed [75] that demanding this isometry group to be large enough to be able to contain the Standard Model group gives a lower limit on the number of extra dimensions. We repeat this reasoning here.

Like before we take our spacetime to be of the form $M = \mathbb{R}^{3,1} \times X$, with dimension $\dim(M) = \dim(\mathbb{R}^{3,1}) + \dim(X) \equiv 4 + k$. We want the isometry group G of X to contain or be equal to the Standard Model group G_{SM}:

$$G \supseteq G_{SM} = SU(3) \times SU(2) \times U(1). \quad (3.18)$$

In section 2.1.1 we explained that the minimal way to construct a manifold with a given isometry group G is by taking it to be a homogeneous space $X = G/H$. So if we take in this construction $G = G_{SM}$ and H to be a maximal subgroup of it, X will be the space of minimal dimension we are looking for.

As explained in [36], if H would contain one of the factors of G_{SM} (i.e. $SU(3)$, $SU(2)$ or $U(1)$) as an exact subgroup, identifying one of the factors of G under one of these groups would then produce nothing but a trivial action. This would reduce the isometry group to something smaller than the Standard Model group (i.e. that factor would be removed). The maximal suitable subgroup of G is [75] $H = SU(2) \times U(1) \times U(1)$, but for the reason just given we cannot make the identification in the straight-forward way. The $SU(2)$ factor should be regarded as an isospin subgroup of $SU(3)$. The generators of the two $U(1)$ factors should be linear combinations of the hypercharge generator of $SU(3)$, an arbitrary generator of $SU(2)$ and the $U(1)$ of G_{SM}. Now that we have established the right homogeneous space, calculating the dimension of G/H is simple:

$$\dim(G) = \dim(SU(3) \times SU(2) \times U(1)) = 8 + 3 + 1 = 12, \quad (3.19)$$
$$\dim(H) = \dim(SU(2) \times U(1) \times U(1)) = 3 + 1 + 1 = 5. \quad (3.20)$$

So the dimension of the homogeneous space G/H and the minimal dimension of X is

$$\min(k) = \dim(G/H) = \dim(G) - \dim(H) = 7. \quad (3.21)$$

This important result tells us that the minimal dimension in which we can hope to construct the Standard Model group as a Kaluza-Klein reduction of the gravitational field is $\dim(M) = 4 + 7 = 11$. It is a surprising fact that this number of dimensions coincides with the maximal dimension one can define a supergravity theory in. Had the interactions of the fundamental particles obeyed a different gauge symmetry, we would not have been able to pursue this type of construction.

Note that this result holds if we consider the Kaluza-Klein mechanism to be the only mechanism responsible for generating gauge symmetry. There are many other well-known mechanisms to do this. For example, the five String Theories are perfectly consistent theories in *ten* spacetime dimensions and lead to symmetry groups that easily contain the Standard Model group. The phenomenologically most attractive of these theories, the Heterotic String Theory, has $E_8 \times E_8$ or $SO(32)$ gauge symmetry. In compactifications of the Heterotic String it is often a more interesting question of how to break this symmetry to the Standard Model than it is to generate more gauge symmetry. But still, the considerations made in this section add to the feeling that something special is going on in eleven dimensions.

3.2.3 Action of eleven-dimensional supergravity

As we briefly described in the introduction, in eleven dimensions there is a unique supergravity theory. This theory is $\mathcal{N} = 1$ supersymmetric, which means the it has only one supersymmetry generator Q_α. This generator is a Majorana spinor and in eleven dimensions such a spinor has 32 components (see Appendix B of [64]). This generator satisfies the algebra

$$\{Q_\alpha, \bar{Q}_\beta\} = -2P_M \Gamma^M_{\alpha\beta}. \quad (3.22)$$

The massless bosonic fields in the supersymmetry multiplet of a D-dimensional theory form representations of $SO(D-2)$. Note that massive fields form representations of $SO(D-1)$ and the way we go from the massive to the massless case is by using the massless field equations, i.e. by going on-shell. This removes one polarization of the field, resulting in the $SO(D-2)$ representation. In eleven dimensions, there are 32 supersymmetries and when we have a theory with this number of supersymmetries, the multiplet is unique and only known on-shell [72]. In eleven dimensions the multiplet contains two bosonic representations of $SO(9)$: a traceless symmetric two-index (spin 2) tensor g_{MN} and an anti-symmetric three-index (spin 1) tensor C_{MNP}. The first is of course the *graviton* and has $\frac{1}{2} \cdot 9 \cdot 10 - 1 = 44$ components. The *anti-symmetric tensor* is most easily written as a three-form and has $\binom{9}{3} = \frac{1}{3!} \cdot 9 \cdot 8 \cdot 7 = 84$ components. So the bosonic part of the multiplet in total contains $44 + 84 = 128$ states. The fermionic part contains only a single (spin 3/2) Majorana vector-spinor field (or Rarita-Schwinger spinor) ψ_M, called the *gravitino*. The spinor index is usually suppressed and takes 16 values. The vector index of course takes 9 values. The total number of independent components is reduced by a condition on the trace [64], leading to in total $16 \cdot 9 - 16 = 128$ states in the fermionic sector. So the massless irreducible representation contains 256 states, half of which are bosonic and half of which are fermionic. The fact that there are as much bosonic as fermionic degrees of freedom is a natural feature of any supersymmetric theory, as the generators (3.22) transform bosons into fermions and vice versa.

Not only the field content of eleven-dimensional supergravity is unique. The action we can make with these fields is unique[8] as well and has been constructed already quite some time ago in [21]. The complete action, indicated by S_{11}, contains a purely bosonic part S_B and a fermionic part S_F, or in other words $S_{11} = S_B + S_F$. The bosonic part of this action, written conveniently in differential geometric notation, is

$$S_B = \frac{1}{2\kappa_{11}^2} \int_{M_{11}} \left[\mathcal{R} * 1 - \frac{1}{2} G \wedge *G - \frac{1}{6} C \wedge G \wedge G\right]. \tag{3.23}$$

Here κ_{11} is the eleven-dimensional gravitational coupling constant (related to the eleven-dimensional Newtons constant by $2\kappa_{11}^2 = 16\pi G_N^{11}$) and G is the field strength of the three-form gauge field,

$$G_{MNPQ} = 4\partial_{[M} C_{NPQ]}, \quad \text{or} \quad G = dC. \tag{3.24}$$

The first term is the Einstein-Hilbert action in eleven dimensions and the second term can be seen to be the eleven-dimensional analogue of the standard Maxwell action. This might become clearer if we write these terms of

[8] Without going into too much detail, we note that this is basically a consequence of the fact that theories get more restricted if we add more supersymmetries. In eleven dimensions the large number of supersymmetries (32) is enough to reduce the number of possible consistent theories to one.

the action in the 'standard' form

$$S_{EH} + S_M = \frac{1}{2\kappa_{11}^2} \int d^{11}z\sqrt{g}\big[\mathcal{R} - \frac{1}{48}G_{MNPQ}G^{MNPQ}\big]. \tag{3.25}$$

The third term in (3.23) is a so-called Chern-Simons interaction term and will prove to be very important later on[9]. It can similarly be written in components (which are contracted with the totally anti-symmetric tensor), but we will not do so as we will not gain much with it.

The fermionic part of the eleven-dimensional supergravity action is somewhat more complicated. It is given by

$$\begin{aligned} S_F \;&=\; \frac{1}{2\kappa_{11}^2} \int d^{11}z\sqrt{g}\Big[\bar{\psi}_M \Gamma^{MNP} \nabla_N \big(\frac{\omega + \hat{\omega}}{2}\big)\psi_P \qquad (3.26) \\ &-\; \frac{1}{192}\left(\bar{\psi}_M \Gamma^{MNPQRS}\psi_N + 12\bar{\psi}^P \Gamma^{RS}\psi^Q\right)\left(G_{PQRS} + \hat{G}_{PQRS}\right)\Big]. \end{aligned}$$

In this action some new notation is introduced, which we shall now explain. First of all, the the eleven-dimensional gamma matrices Γ^M appear with multiple indices. This stands for a totally antisymmetric product

$$\Gamma^{N_1 \ldots N_p} = \Gamma^{[N_1} \cdots \Gamma^{N_p]}. \tag{3.27}$$

Furthermore, $\bar{\psi}_m := i\psi_M^\dagger \Gamma^0$ and ω_{MAB} is the spin connection, a definition of which can for example be found in [36]. $\nabla_M(\omega)$ is the covariant derivative which acts on ψ_M as

$$\nabla_M(\omega)\psi_N = \partial_M \psi_N + \frac{1}{4}\omega_{MAB}\Gamma^{AB}\psi_N. \tag{3.28}$$

Finally, a hat over a field indicates the supercovariant version of the field. A supercovariant field is defined in such a way that its supersymmetry variation does not involve derivatives of the infinitesimal Grassmannian parameter of the transformation. The supercovariant versions of the spin connection and the field strength are defined as

$$\hat{\omega}_{MAB} \;:=\; \omega_{MAB} + \frac{1}{8}\bar{\psi}_P \Gamma_{MAB}{}^{PQ}\psi_Q, \tag{3.29}$$

$$\hat{G}_{MNPQ} \;:=\; G_{MNPQ} + 3\bar{\psi}_{[M}\Gamma_{NP}\psi_{Q]}. \tag{3.30}$$

The fact that these fields appear in the action in the combinations $\frac{1}{2}(\omega + \hat{\omega})$ and $\frac{1}{2}(G + \hat{G})$ ensures that only the supercovariant fields $\hat{\omega}$ and $\hat{G}$ appear in the equations of motion.

[9]Note that although the action presented here is unique and of the form (3.23) at tree-level, higher order quantum effects force us to modify the action to maintain consistency. Specifically, at one loop complicated terms need to be added to cancel discrete-symmetry anomalies coming from this Chern-Simons term. See [29] for a short discussion.

To obtain the equations, we vary the action with respect to the fields g_{MN}, C_{MNP} and ψ_M as usual. Because of the complexity of the action, this is quite a lengthy and tedious exercise, which we will not reproduce here. The outcome of this calculation as given in [29] is

$$\mathcal{R}_{MN}(\hat{\omega}) - \frac{1}{2}g_{MN}\mathcal{R}(\hat{\omega}) = \frac{1}{12}\left(\hat{G}_{MPQR}\hat{G}_N{}^{PQR} - \frac{1}{8}g_{MN}\hat{G}_{PQRS}\hat{G}^{PQRS}\right)\tag{3.31}$$

$$\Gamma^{MNP}\Big[\nabla_N(\hat{\omega}) - \frac{1}{288}(\Gamma_N{}^{QRST} - 8\delta_N^Q\Gamma^{RST})\hat{G}_{QRST}\Big]\psi_P = 0,\tag{3.32}$$

$$\nabla_M(\hat{\omega})\hat{G}^{MN_1N_2N_3} + \frac{1}{1152}\epsilon^{N_1\cdots N_{11}}\hat{G}_{N_4\cdots N_7}\hat{G}_{N_8\cdots N_{11}} = 0.\tag{3.33}$$

These equations might look a bit intimidating, so let us say a few words about them. The first of these equations are just the Einstein equations with the energy-momentum tensor given by some expression that is of quadratic order in both G and ψ. The second bears resemblance with the massless Dirac equation we are familiar with, if we forget about the terms involving G. If we squint a bit, the third equation will start to look like a sophisticated version of the Maxwell equations with source[10].

If we take the fermion-field to vanish (which we will often do), these equations simplify a great deal. Equation (3.32) will then vanish identically and in addition we can see from (3.29) and (3.30) that we can drop all the hats. The third equation (3.33) will then take on a particularly clean appearance if we write it in differential geometric notation:

$$d * G + \frac{1}{2}G \wedge G = 0.\tag{3.34}$$

Besides obeying this equation, we note that because G is exact it is also automatically closed, and hence satisfies

$$dG = 0 \quad \text{or} \quad \partial_{[M}G_{NPQR]} = 0.\tag{3.35}$$

These equations represent the Bianchi identities for G.

We already know that this action has $\mathcal{N}=1$ supersymmetry and that it should have eleven-dimensional general covariance, because it is a theory of gravity. In the next subsection we will see that this complicated action has a number of other symmetries as well.

3.2.4 Symmetries

The total supergravity action $S_{11} = S_B + S_F$ and the equations of motion (3.31)-(3.33) possess a number of important symmetries. In this subsection

[10] Readers who own a copy of the *Hitchhikers Guide to the Galaxy* are advised to look up the standard techniques to lift the Somebody Else's Problem Field that surrounds these equations.

we list what these symmetries are. To simplify notation, we now rewrite the metric as an elfbein, using

$$g_{MN}(z) =: e^A{}_M(z) e^B{}_N(z) \eta_{AB}. \tag{3.36}$$

For the same reason we did not derive the equations of motion, do not prove these symmetries explicitly. We simply state that the action and equations of motion are invariant under the following transformations [29]:

1. $D=11$ General Coordinate invariance (with parameter ξ^M)

$$\delta e^A{}_N = e^A{}_M \partial_M \xi^N + \xi^N \partial_N e^A{}_M \tag{3.37}$$
$$\delta C_{MNP} = 3 C_{Q[MN} \partial_{P]} \xi^Q + \xi^Q \partial_Q C_{MNP} \tag{3.38}$$
$$\delta \psi_M = \psi_N \partial_M \xi^N + \xi^N \partial_N \psi_M \tag{3.39}$$

2. Local SO(1,10) Lorentz transformations (with anti-symmetric parameter $\alpha_{AB} = -\alpha_{BA}$)

$$\delta e^A{}_M = -\alpha^A{}_B e^B{}_M \tag{3.40}$$
$$\delta C_{MNP} = 0 \tag{3.41}$$
$$\delta \psi_M = -\frac{1}{4} \alpha_{AB} \Gamma^{AB} \psi_M \tag{3.42}$$

3. Three-form Gauge transformation (with an anti-symmetric parameter $\Lambda_{MN} = -\Lambda_{NM}$)

$$\delta e^A{}_M = 0 \tag{3.43}$$
$$\delta C_{MNP} = 3 \partial_{[M} \Lambda_{NP]} \tag{3.44}$$
$$\delta \psi_M = 0 \tag{3.45}$$

   The second of these transformations can more cleanly be written as $\delta C = \mathrm{d}\Lambda$, if we regard Λ as a two-form $\Lambda = \frac{1}{2} \Lambda_{MN} \mathrm{d}z^M \wedge \mathrm{d}z^N$.

4. Local $\mathcal{N}=1$ Supersymmetry (anti-commuting parameter η)

$$\delta e^A{}_M = -\frac{1}{2} \bar{\eta} \Gamma^A \psi_M \tag{3.46}$$
$$\delta C_{MNP} = -\frac{3}{2} \bar{\eta} \Gamma_{[MN} \psi_{P]} \tag{3.47}$$
$$\delta \psi_M = \Big[\nabla_M(\hat{\omega}) - \frac{1}{288} \big(\Gamma_M{}^{QRST} - 8 \delta^Q_M \Gamma^{RST} \big) \hat{G}_{QRST} \Big] \eta \tag{3.48}$$

5. Odd number of space or time reflections together with

$$C_{MNP} \to -C_{MNP} \tag{3.49}$$

The gauge symmetry of the C-field is actually quite easy to show, so we will do that now. Because $\delta C = d\Lambda$, we immediately see that $\delta G = d^2\Lambda = 0$. Because the G-flux does not transform, all the terms in the action that contain only G (and not C) do not contribute to the variation. So we immediately see that the action transforms as

$$\begin{aligned}
\delta S_{11} &\sim \int_{M_{11}} d\Lambda \wedge G \wedge G && (3.50)\\
&= \int_{M_{11}} d(\Lambda \wedge G \wedge G) && (3.51)\\
&= \int_{\partial M_{11}} \Lambda \wedge G \wedge G && (3.52)\\
&= 0. && (3.53)
\end{aligned}$$

In going to the last line, we assumed the variation to vanish at the boundary, i.e. at infinity. We postpone solving the equations of motion and a further discussion of these symmetries to later chapters. We will instead continue by showing what lies beyond the supergravity approximation and briefly introduce a number of concepts that are important for learning more about M-theory.

3.3 Towards M-theory

We now have mentioned M-theory several times, but we still do not know what this theory is. Although we would have liked to be able to thoroughly explain this in a few pages, the size of the subject, the many different perspectives from which it is being investigated and the uncertainty about its precise formulation makes this impossible. So, this section will only give the reader a nodding acquaintance with the subject and will mainly serve as a guide to the literature. The analysis in the remainder of this thesis will primarily be based on the low-energy supergravity approximation, so this little bit of extra information is enough for our purpose. The outline of this section is that we first quickly review what kind of considerations led to the introduction of M-theory and then describe a number of tools that are being used to study the theory.

3.3.1 M-theory $\leftrightarrow$ Type IIA String Theory duality

We start this section by considering the simplest possible Kaluza-Klein compactification of eleven-dimensional supergravity, namely the one on S^1. We consider only the bosonic part of the eleven-dimensional supergravity action (3.23) and take $z^{10} \equiv y$ to be periodic over $2\pi R$. In such a compactification we expect the eleven-dimensional metric g^{11}_{MN} to split into a ten-dimensional metric $g^{11}_{\mu\nu}$, a gauge field A_μ coming from $g^{11}_{\mu 10}$ and a scalar ϕ coming from

$g^{11}_{10,10}$. This scalar will be called the dilaton. We use the following ansatz for the metric to exhibit this splitting:

$$ds^2 = g^{11}_{MN} dz^M dz^n = e^{4\phi/3} g^{10}_{\mu\nu} dx^\mu dx^\nu + e^{-2\phi/3}(dy + A_\mu dx^\mu)^2. \tag{3.54}$$

This is a generic metric ansatz for a spacetime of the form $\mathbb{R}^{10} \times S^1$ that is invariant under translations in the periodic dimension. The factors with the peculiar powers of the dilaton are included to make the final answer look familiar.

Of course, the bosonic field content of eleven-dimensional supergravity also contains the three-form C_{MNP}. We expect this field to similarly split into a ten-dimensional three-form $C_{\mu\nu\rho}$ and a two-form $B_{\mu\nu}$. From general Kaluza-Klein theory we know that states with momentum in the compact dimension will have masses of size

$$m_n^2 = \frac{n^2}{R^2}. \tag{3.55}$$

We now insert the ansatz for the metric and the C-field into the action and integrate over y to obtain a ten-dimensional effective action. This effective action will then contain terms for both the massless and massive fields. If we now take $R \to 0$, the massive fields become infinitely massive and will decouple from the physics. Therefore we discard the massive field. If we do this and carry out the integration over y, we will arrive at the following effective ten-dimensional action:

$$\begin{aligned} S_{10} &= \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{g} \big[e^{-2\phi} (\mathcal{R} + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} C_{\mu\nu\rho} C^{\mu\nu\rho}) \\ &- \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{48} G_{\mu\nu\rho\sigma} G^{\mu\nu\rho\sigma} \big] \\ &- \frac{1}{4\kappa_{10}^2} \int B_2 \wedge G_4 \wedge G_4, \end{aligned} \tag{3.56}$$

where we have defined $\kappa_{10}^2 = \kappa_{11}^2 / 2\pi R$. This is exactly the Type IIA supergravity action in ten dimensions!

We explained in chapter 1 that the main development leading to the concept of M-theory was the discovery of dualities. One of the ways to study dualities is by looking at the low-energy limits of the dual theories. As we mentioned, these low-energy limits are given by supergravity theories. The circle compactification of eleven-dimensional supergravity we just described looked like an innocent exercise in Kaluza-Klein compactification, but is actually indicative of something much deeper. To understand what is going on here, look at figure 3.1.

One of the fundamental scales in String Theory is set by the Regge slope α', which related to the string-length by $\alpha' = l_s^2$. The limit in which $\alpha' \to 0$, indicated in figure 3.1 by (b.), is basically the long-distance or low-energy

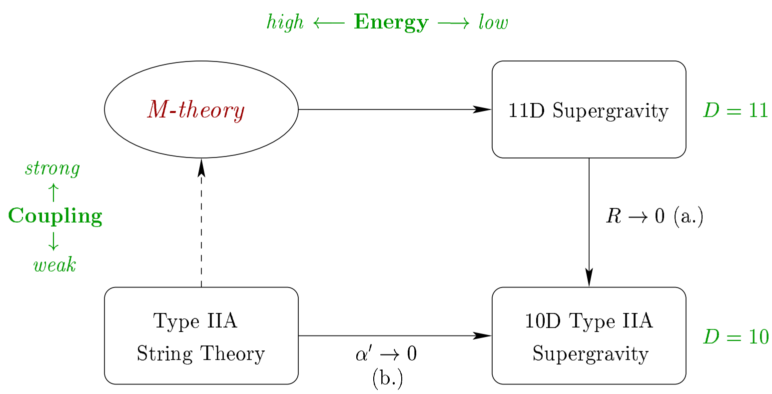

Figure 3.1: Relation between M-theory, Type IIA String Theory and their low-energy limits

limit of String Theory in which the strings start looking and behaving like point particles. If we take this limit in Type IIA String Theory, we end up with the effective Type IIA supergravity action given above. Furthermore, we have just seen that if we take route (a.) from the eleven-dimensional supergravity theory, we end up at this same theory. Now note that [54] the radius of the compact dimension is related to the string coupling constant g[11] and the eleven-dimensional Planck length, roughly by

$$g \sim \left(\frac{R}{l_p^{11}}\right)^{3/2}. \tag{3.57}$$

Basically because of this, it was conjectured in [76] that if we take the *strong coupling limit* of Type IIA String Theory, an extra dimension appears and we will end up with some eleven-dimensional theory, whose low-energy limit is eleven-dimensional supergravity. But even if we know this, we still do not know what this theory is, because we only know string theory well as a perturbative theory and this perturbative expansion stops to be valid if we go to strong coupling.

Then also in [76] it was conjectured that *all* string theories are related in a similar way to each other and to eleven-dimensional supergravity. This was the birth of M-theory and sparked the so-called "second string theory revolution". So, the duality between M-theory and Type IIA String Theory is only one of many dualities that relate the different string theories to each other and to M-theory. A small number of dualities directly involving M-theory directy is listed in table 3.2. For more information about dualities, see for example [64], [54], [8] and [67].

[11] Note that in string theory g is not a free parameter, but is actually given by the expectation value of the dilation, $g = \exp(\langle\phi\rangle)$.

M-theory on	Effective Theory	Dual To
T^2	IIB on S^1	IIA on S^1 (T)
$S^1 \times S^1/\mathbb{Z}_2$	Ho on S^1	He on S^1 (T)
$K3$	M on $K3$	He on T^3 (S)
$T^5/\mathbb{Z}_2$	M on $T^5/\mathbb{Z}_2$	IIB on $K3$ (T)

Table 3.2: Some duality relations directly involving M-theory.

3.3.2 Tools to study non-perturbative effects

The Standard Model and all the five String Theories are in essence *perturbative* theories. This means that interactions of the fundamental constituents, i.e. the particles and strings, are calculated by making an expansion in their corresponding coupling constants. Such an expansion is only valid for a small number of interacting degrees of freedom and for small values of the coupling constant. But we know already from quantum field theory that there are many *non-perturbative effects* that are essential to understanding the observed behavior of the fundamental particles and the way they interact. The most well-known phenomena are quark confinement, the Higgs mechanism and dynamical symmetry breaking. Besides these quantum effects, we can already at the classical level see that it is important to know how to describe non-perturbative physics if we look at topologically non-trivial field configurations like solitons, instantons and monopoles. As the five String Theories and M-theory should in a certain limit contain the Standard Model, the mentioned non-perturbative effects are with no doubt present in these theories as well. In all likelihood, in these theories other non-perturbative effects are important as well. Until not too long ago it was quite hard to say anything useful about such effect in String Theory, but in the last few years new methods based on supersymmetry and D-branes have greatly enhanced our ability to learn more about the non-perturbative physics of String Theory and M-theory.

One of the best tools for testing dualities (beyond their low-energy field theory) and one of the best probes into the non-perturbative physics, are so-called BPS states. As explained in [67], BPS states are states that are invariant under only a part of the supersymmetry algebra and are characterized by two important properties: they belong to a so-called short multiplet of the supersymmetry algebra and their mass is completely determined by their charge. These two properties have analogues in the representation theory of the Lorentz group. For example, a massless representation of the Lorentz group is smaller than that of a massive state (compare the two polarisations of the photon to the three polarisations of a massive vector). The analogue of the second property is that a spin 1 representation of the Lorentz group that contains only two states, must necessarily be massless.

BPS states are extremely important for studying non-perturbative effects, because the BPS property is independant of the values of the moduli in the theory. Because the string coupling $g_s = e^{\langle\phi\rangle}$ is one of such moduli, the BPS property will carry over to the non-perturbative regime. Therefore these states can be used as important probes into the non-perturbative physics and provide us with a way to test dualities beyond the level of their low energy effective theories. We refer to [67] for more details.

Another important non-perturbative effect is the appearance of D-branes. We will not give a treatment of these objects, because this can be found in many places in the literature. [63]/[64] is an excellent starting point. The only thing we mention is that D-branes in String Theory are object defined by the property that open strings can end on them. M-theory also contains brane states, but these are slightly different. A very handwaving way to see that they are present, is by first noting that a (one-form) gauge field in for example quantum electrodynamics couples naturally to the one-dimensional world-line of a particle (electrically) charged under the gauge group. Generalizing this, we infer that there must be a two-dimensional object in M-theory, whose three-dimensional world-volume (electrically) couples naturally to the three-form gauge field C. There exist indeed solitonic solutions to the classical equations of motion of eleven-dimensional supergravity that have this property: they are the so-called *M2-branes* or M-theory membranes. This is the reason why one of the explanations of the name M-theory is Membrane Theory. We further note (without explaining) that M-theory also contains objects that are the magnetic duals of the M2-branes. These are five-dimensional objects, analogously called *M5-branes*. For more about brane-solutions of M-theory and how and when these objects are BPS states, see for example [69].

Chapter 4

Smooth M-Theory Compactifications

4.1 Introduction

In this chapter we explicitly carry out the Kaluza-Klein compactification of eleven-dimensional supergravity on a smooth seven-dimensional manifold X to obtain an effective four-dimensional field theory. Any $\mathcal{N} \geq 2$ supersymmetric theory in four dimensions is CPT-invariant and will therefore not allow for chiral fermions [5]. On the other hand, we would still like a supersymmetric effective theory, because we expect supersymmetry to be broken at a scale that is much lower than the compactification scale. This is augmented by the fact that supersymmetric compactifications are understood much better than non-supersymmetric compactifications. So, these considerations lead us to the requirement of $\mathcal{N}=1$ supersymmetry for the effective four-dimensional theory. If we make this demand in section 4.3 it will finally become clear why G_2-manifolds are so important to us: they are the compactification manifolds that preserve exactly $\mathcal{N}=1$ supersymmetry!

We set the stage by making a number of assumptions on the structure of the vacuum. Then we will how under these assumptions $\mathcal{N}=1$ supersymmetry leads to X having G_2-holonomy. Then we will determine the massless spectrum of the theory with the method of section 3.2.1 to find that this first effort does not do much for us in terms of obtaining realistic particle physics. Most notably, the massless spectrum does not contain chiral fermions that are charged under a non-Abelian gauge group. To be precise, the effective theory resulting from the smooth G_2 compactification will be four-dimensional $\mathcal{N}=1$ supergravity coupled to $b_2(X)$ Abelian vector multiplets and $b_3(X)$ massless neutral chiral multiplets. This will not make us abandon the idea of compactifications on G_2-manifolds, because it will turn out that singular G_2-manifolds can fix the problems we find here. General references for this chapter are [29] and [5].

4.2 Ricci flat Supergravity vacua

As we have seen, at low energies M-theory admits a description in terms of eleven-dimensional supergravity. In the Kaluza-Klein analysis we perform in this chapter, we make the assumption that the full eleven-dimensional spacetime is a Riemannian product of the form $(M_{11}, g) = (M_4 \times M_7, g_4 \times g_7)$, so we do not allow for a warp-factor. This means of course that we assume the background metric to be of the form

$$\langle g_{\mu\nu} \rangle = \mathring{g}_{\mu\nu}(x), \quad \langle g_{mn} \rangle = \mathring{g}_{mn}(y), \quad \langle g_{\mu n} \rangle = 0, \tag{4.1}$$

where we take g_{mn} to be a *irreducible* metric. The supergravity approximation of M-theory is only valid on smooth spacetimes whose smallest length scale is much larger than the eleven dimensional Planck length. This implies that the supergravity description is only valid if the internal space M_7 is smooth and large compared to the eleven dimensional Planck length. On the other hand, M_7 also has to be compact and small enough for the massive Kaluza-Klein modes coming from it to be so heavy as to be unobservable. As we have seen in chapter 3 the eleven-dimensional $\mathcal{N}=1$ supergravity supermultiplet contains three fields: the graviton g_{MN}, a gravitino ψ_M and a three-form gauge field C_{MNP}. The first step in obtaining realistic compactifications is to determine possible vacua of M-theory. A *vacuum* of M-theory is defined as a tuple $(M, \langle g \rangle, \langle C \rangle, \langle \psi \rangle)$, such that $\langle g \rangle$, $\langle C \rangle$ and $\langle \psi \rangle$ satisfy the equations of motion (3.31)-(3.33) of eleven-dimensional supergravity.

As said, these equations too complicated to solve in all generality, so we have to make a number of simplifying assumptions. As we explained in section 3.2.1, we demand the four-dimensional part of the vacuum to be Lorentz-invariant and maximally symmetric, so it should be one of the spaces in table 3.1. An important consequence of the requirement of maximal symmetry is that the vacuum expectation value of any fermion field has to vanish. In particular, the vacuum expectation value of the gravitino field should be zero:

$$\langle \psi_M \rangle = 0 \tag{4.2}$$

This means that the equation of motion for the ψ-field (3.32) is trivially satisfied, so we can completely ignore it. The equations for the metric (3.31) and the C-field (3.33) thus carry all the information about the vacuum. And in these equations we can now drop all the hats, as explained in chapter 3. Because these are relatively complicated set of second-order equations, we make a further simplification by setting $\langle G \rangle = 0$. Solutions with $\langle G \rangle \neq 0$ also exist (for example the Freund-Rubin solutions [31]), but these solutions are considerably more complicated and have some problematic properties. See chapter 6 for a discussion. If we make this assumption, the equation of motion for C,

$$d * G + \frac{1}{2} G \wedge G = 0, \tag{4.3}$$

is trivially satisfied.

The only non-trivial equations that remain are the Einstein equations with vanishing energy-momentum tensor,

$$\mathcal{R}_{MN} - \frac{1}{2} g_{MN} \mathcal{R} = 0. \tag{4.4}$$

But if we contract this equation with g^{MN}, we find that $\mathcal{R} = 0$, so we have to conclude that

$$\mathcal{R}_{MN} = 0. \tag{4.5}$$

In other words, the complete eleven-dimensional spacetime has to be *Ricci-flat*. Now consider the Christoffel symbols

$$\Gamma^{S}_{MN} = \frac{1}{2} g^{PS} \Big\{ \frac{\partial g_{NP}}{\partial x^M} + \frac{\partial g_{MP}}{\partial x^N} - \frac{\partial g_{MN}}{\partial x^P} \Big\}. \tag{4.6}$$

It is not hard to see from this equation by looking at the various possibilities, that because of our assumption of a Riemannian product spacetime (i.e. because of $\langle g_{\mu n} \rangle = 0$), all the mixed components of the Christoffel symbols, such as $\Gamma^{\sigma}_{m\nu}$, vanish. This implies that

$$\mathcal{R}_{\mu\nu} \equiv \mathcal{R}^{P}{}_{\mu P \nu} = \mathcal{R}^{\rho}{}_{\mu\rho\nu} \tag{4.7}$$

and that $\mathcal{R}_{\mu\nu}$ does not depend on the metric g_7. Similarly we find that $\mathcal{R}_{mn}$ is strictly the Ricci tensor of (M_7, g_7) and does not depend on g_4. So, the Ricci curvature tensor splits completely in a four and a seven-dimensional part. This means that the assumptions we have made lead us to vacua that are Ricci-flat in both the macroscopic and the internal part of spacetime,

$$\mathcal{R}_{\mu\nu} = 0, \tag{4.8}$$

$$\mathcal{R}_{mn} = 0. \tag{4.9}$$

We also see from table 3.1 that this fixes M_4 to be Minkowski space $(\mathbb{R}^4, \eta)$.

We would like to note that the fact that the macroscopic part of spacetime is four-dimensional Minkowski space is a *consequence* of our assumption that the G-flux vanishes. If on the other hand we had assumed the vacuum to be of the form $(M_{11}, g) = (\mathbb{R}^4 \times M_7, \eta \times g_7)$, with $\mathbb{R}^4$ Minkowski space and M_7 arbitrary but compact, $\langle G \rangle = 0$ and Ricci-flatness of M_7 would have followed by demanding the vacuum to be supersymmetric. See [19] for a proof of this statement. If we allow for a warp-factor, we find different conditions on the G-flux [51]. Carrying on the program in the way we do in this chapter, would then lead to G_2-manifolds with torsion.

In the next subsection we will learn important things about the structure of the internal space M_7 from supersymmetry considerations, but we want to mention that at this point there are still many possible vacua. We could for example take M_7 to simply be the seven-torus T^7. Why this does not lead to interesting theories will be clear after reading the next section. We will list a number of interesting possibilities for other Ricci-flat vacua with some information about their relevance in subsection 4.5 (cf. table 4.1).

4.3 Supersymmetry and G_2 holonomy

By doing the above analysis, we have established that the four dimensional spacetime should be Minkowski space. In this subsection we will try to learn something more about the internal space from supersymmetry considerations. Since eleven-dimensional supergravity is $\mathcal{N}=1$ supersymmetric, the vacuum should be supersymmetric as well. This means that we demand the supersymmetry transformations (3.46) (with $\hat{\omega}=\omega$ and $\hat{G}=G$) acting on the vacuum to vanish,

$$\delta e_M^A = -\frac{1}{2}\bar{\eta}\Gamma^A \langle\psi_M\rangle = 0, \tag{4.10}$$

$$\delta C_{MNP} = -\frac{3}{2}\bar{\eta}\Gamma_{[MN}\langle\psi_{P]}\rangle = 0, \tag{4.11}$$

$$\delta\psi_M = \nabla_M^S(\omega)\eta - \frac{1}{288}(\Gamma_M^{PQRS} - 8\delta_M^P\Gamma^{QRS})\langle G_{PQRS}\rangle\eta = 0. \tag{4.12}$$

But because our vacuum has $\langle\psi_M\rangle = 0$, equations (4.10) and (4.11) vanish identically. So only equation (4.12) leads to a non-trivial condition on the vacuum. Of course our current vacuum also has $\langle G\rangle = 0$, so equation (4.12) further simplifies to

$$\nabla_M(\omega)\eta = 0. \tag{4.13}$$

On a product manifold like the one we are compactifying on, the Γ-matrices can be rewritten as

$$\Gamma^\alpha = \gamma^\alpha \otimes \mathbb{1} \tag{4.14}$$

$$\Gamma^a = \gamma_5 \otimes \gamma^a, \tag{4.15}$$

where Γ^α and Γ^a satisfy the four and seven dimensional Clifford algebra, respectively. In a way similar to how we saw that the Christoffel symbols with mixed indices vanish, we can see that the mixed components of the spin connection vanish. Because of these two facts, the covariant derivative $\nabla = \nabla_M dz^M$ decomposes into a four and a seven dimensional part,

$$\begin{aligned}
\nabla &= (\partial_M + \frac{1}{4}\omega_{MAB}\Gamma^{AB})dz^M && (4.16)\\
&= (\partial_\mu + \frac{1}{4}\omega_{\mu\alpha\beta}\Gamma^{\alpha\beta})dx^\mu + (\partial_m + \frac{1}{4}\omega_{mab}\Gamma^{ab})dy^m && (4.17)\\
&= (\partial_\mu + \frac{1}{4}\omega_{\mu\alpha\beta}\gamma^{\alpha\beta}\otimes\mathbb{1})dx^\mu + (\partial_m + \frac{1}{4}\omega_{mab}\mathbb{1}\otimes\gamma^{ab})dy^m && (4.18)\\
&= \nabla_4\otimes\mathbb{1} + \mathbb{1}\otimes\nabla_7^S. && (4.19)
\end{aligned}$$

As explained in [64] and [72], in eleven dimensions the smallest possible spinor representation of $SO(1,10)$ has 32 real components. By compactifying on a product manifold, this tangent space group is explicitly broken to

$SO(1,3) \times SO(7)$. This inspires us to decompose the 32 component spinor as $\mathbf{32} = \mathbf{4} \otimes \mathbf{8}$ like

$$\eta(x,y) = \epsilon(x) \otimes \theta(y). \tag{4.20}$$

Because η is an anti-commuting object and because we want ϵ to have the usual statistics associated with a spinor in four dimensions (i.e. be anti-commuting), we have to demand that θ is a commuting object. We do not have to be troubled by the question of what it means for a spinor to commute, because this is an object living in the internal space and our intuition is purely based on observations in the macroscopic part of spacetime. We can use these two decompositions to find a solution to (4.13), which is now written as

$$\nabla_4 \epsilon(x) \otimes \theta(y) + \epsilon(x) \otimes \nabla_7 \theta(y) = 0. \tag{4.21}$$

But in Minkowski space we can always find a set of four constant spinors, so the condition for the vacuum to be supersymmetric finally becomes

$$\nabla_7 \theta(y) = 0. \tag{4.22}$$

Taking all these considerations into account, the total number of solutions of (4.13) will thus be four times the number the number of covariantly constant spinors on M_7. We now assume that the supersymmetry in the four-dimensional effective theory is generate by generation Q^I_α $(I = 1, .., \mathcal{N})$, where α is a spinor index and $\mathcal{N}$ indicates the number of supersymmetries. But as explained in [64] and [72] in $D=4$ a spinor-index α runs from 1 to 4. But this then also implies that $\mathcal{N}$ is equal to the number of solutions of (4.22), i.e. *$\mathcal{N}$ is given by the number of covariantly constant spinors on M_7*. So we are now looking for a Ricci-flat, seven-dimensional manifold that admits exactly one parallel spinor. Luckily, we know from Proposition 2.20 and Corollary 2.22 that seven-manifolds with exactly one covariantly constant spinor exist: they are the G_2-manifolds we worked so hard for to thoroughly introduce in chapter 2. We can make this statement even stronger. We know from the discussion above (2.57) that the group G_2 can alternatively be defined as that subgroup of $SO(7)$ that has exactly one covariantly constant spinor. So, here we are indeed forced to consider manifolds of G_2-holonomy if we consider supersymmetric M-theory compactifications on irreducible manifolds[1].

After the work done in the last two sections, we now have enough information about the vacuum to be able to determine the field content. In the next subsection we will perform the Kaluza-Klein analysis of M-theory on a manifold $(\mathbb{R}^4 \times X, \eta \times g)$, with X a compact G_2-manifold, to do just this.

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[1]See below for a short discussion about compactification on reducible or product manifolds.

4.4 Kaluza-Klein reduction on a G_2-manifold

After having established the properties of the vacuum in the two previous subsections, we will now derive the field content of the effective four-dimensional theory. Following the Kaluza-Klein procedure, we do this by considering small fluctuations of the fields around their classical vacuum expectation value[2],

$$\begin{aligned} g_{MN}(x,y) &= \langle g_{MN}(x,y)\rangle + \delta g_{MN}(x,y), & (4.23)\\ C_{MNP}(x,y) &= \langle C_{MNP}(x,y)\rangle + \delta C_{MNP}(x,y), & (4.24)\\ \psi_M(x,y) &= \langle \psi_M(x,y)\rangle + \delta\psi_M(x,y) = \delta\psi_M(x,y), & (4.25)\end{aligned}$$

where in the last line we have used $\langle\psi_M\rangle = 0$. Reviewing quickly section 3.2.1, we now look for fluctuations of the fields that are also solutions of the equations of motion. To find these, we substitute the fluctuations into the field equations and keep terms that are first order in the fluctuations. After this we use the Kaluza-Klein ansatz (i.e. after expanding the fluctuations in eigenfunctions of the four dimensional mass-operator) and subsequently determine the massless field-content by only keeping the zero-eigenvalue eigenfunctions.

4.4.1 Expansion of the C-field

To begin with, we substitute $C = \langle C\rangle + \delta C$ in the equation of motion (3.34) and expand to first order. Because we have put $\langle G\rangle = \langle dC\rangle = 0$ and the equation of motion involves only G and not C, the only terms that survive are the ones involving the fluctuations. Keeping only terms linear in the fluctuations, we find

$$d * \delta G + \frac{1}{2}\delta G \wedge \delta G \approx d * \delta G = d * d\delta C = 0. \tag{4.26}$$

Taking the Hodge dual of zero will of course also yield zero, so we can write this equation as

$$(*d*)d\delta C = d^\dagger d\delta C = 0. \tag{4.27}$$

Now remember that a solution to the equation of motion is invariant under the variation (3.44). This will thus in particular be true for δC. We will use this gauge symmetry to put δC in a particular gauge. For this note that any three-form $C^{(3)}$ can be written as

$$C^{(3)} = d\alpha^{(2)} + d^\dagger\beta^{(4)} + \gamma^{(3)}, \tag{4.28}$$

[2] Note that we now switch to a slightly different notation. Do not confuse the notation of these fluctuations with the variation of the fields under a certain (symmetry) transformation.

with $\alpha \in \Omega^3(X)$, $\beta \in \Omega^4(X)$ and $\gamma \in H^3(X)$. If we now act with $d^\dagger$ on this equation, we find $d^\dagger C^{(3)} = d^\dagger d\alpha$. If we now take $C^{(3)} = \delta C$ and use its gauge symmetry to make the transformation $\delta C \to \delta C - d\alpha$, we see that we can always put δC into the so-called *Lorentz gauge*,

$$d^\dagger \delta C = 0. \tag{4.29}$$

If we choose this gauge, then equation (4.27) can be written as

$$(d^\dagger d + dd^\dagger)\delta C = \Delta_{11}\delta C = 0, \tag{4.30}$$

which is the eleven-dimensional Laplace equation.

Since we have taken spacetime to be a Riemannian product $\mathbb{R}^{3,1} \times X$, the eleven-dimensional Laplacian separates in a four- and a seven-dimensional part:

$$\Delta_{11}\delta C = \Delta_4 \delta C + \Delta_7 \delta C = 0. \tag{4.31}$$

From this equation we see that the seven-dimensional Laplacian can be identified with the mass-operator for the fields in the four-dimensional effective theory coming from C. The Kaluza-Klein ansatz now tells us to expand the fluctuations in a complete set of mass-eigenstates,

$$\begin{aligned}\delta C(x,y) = &\sum_i p^i(x)\Omega^i(y) + \sum_j A^j(x) \wedge \omega^j(y) \\ &+ \sum_k B^k(x) \wedge a^k(y) + \sum_l H^l(x) g^l(y).\end{aligned} \tag{4.32}$$

In this expression $p^i(x)$, $A^j(x)$, $B^k(x)$ and $H^l(x)$ are zero-, one-, two- and three-forms on $\mathbb{R}^{3,1}$ and

- $\Omega^i(y) \in \Omega^3(X)$, such that $\Delta_7 \Omega^i(y) = \lambda^i_{(3)}\Omega^i(y)$,
- $\omega^j(y) \in \Omega^2(X)$, such that $\Delta_7 \omega^j(y) = \lambda^j_{(2)}\omega^j(y)$,
- $a^k(y) \in \Omega^1(X)$, such that $\Delta_7 a^k(y) = \lambda^k_{(1)} a^k(y)$
- and $H^l(y) \in \Omega^0(X) = \mathcal{F}(X)$, such that $\Delta_7 H^l(y) = \lambda^l_{(0)} H^l(y)$.

Since we are interested in the massless spectrum right now, we only consider solutions with $\lambda^i_{(3)} = \lambda^j_{(2)} = \lambda^k_{(1)} = \lambda^l_{(0)} = 0$. Hodge's theorem tells us [61] that the space of harmonic p-forms on X, $\mathrm{Harm}^p(X)$, is isomorphic to the p-th cohomology group of X, $\mathrm{Harm}^p(X) \cong H^p(X)$. Consequently, the number of massless $(3-p)$-form fields in four dimensions is given by the p-th Betti number of X,

$$\dim(\mathrm{Harm}^p(X)) = \dim(H^p(X)) = b^p(X). \tag{4.33}$$

With this knowledge, we can now rewrite the expansion as

$$\delta C(x,y) = \sum_{i=1}^{b^3(X)} p^i(x)\Omega^i(y) + \sum_{j=1}^{b^2(X)} A^j(x)\wedge\omega^j(y)$$
$$+\sum_{k=1}^{b^1(X)} B^k(x)\wedge a^k(y) + \sum_{l=1}^{b^0(X)} H^l(x)g^l(y) + [\ \ldots \text{massive states} \ldots\ ]. \quad (4.34)$$

In section 2.3.3 we determined the Betti-numbers for manifolds of G_2-holonomy. From equation (2.68) we know that $b^0(X) = 1$ and $b^1(X) = 0$ and that $b^2(X)$ and $b^p(X)$ depend on the specifics of the G_2-manifold. This implies that we can reduce the expansion to

$$\begin{aligned}\delta C(x,y) = &\sum_{i=1}^{b^3(X)} p^i(x)\Omega^i(y) + \sum_{j=1}^{b^2(X)} A^j(x)\wedge\omega^j(y)\\ &+H(x)g(y) + [\ \ldots \text{massive states} \ldots\ ].\end{aligned} \quad (4.35)$$

To make this expression even clearer, we write it in terms of the components of the massless fields (i.e. with all the indices explicitly included),

$$\delta C_{mnp}(x,y) = \sum_{i=1}^{b^3(X)} p^{(i)}(x)\Omega^{(i)}_{mnp}(y), \quad (4.36)$$

$$\delta C_{\mu np}(x,y) = \sum_{j=1}^{b^2(X)} A^{(j)}_\mu(x)\omega^{(j)}_{np}(y), \quad (4.37)$$

$$\delta C_{\mu\nu\rho}(x,y) = H_{\mu\nu\rho}(x). \quad (4.38)$$

From these equations we can easily read off what the massless fields in four dimensions are that come from the three-form gauge field C. From (4.36) we see that there are $b^3(X)$ real four-dimensional scalars. But because of the symmetry (3.49) of the C-field these transform not as ordinary scalars, but as *pseudo-scalars*. Furthermore, equation (4.37) shows that the four-dimensional theory contains $b^2(X)$ vector fields $A^{(i)}_\mu(x)$. If we look at how these vectors transform under the transformation (3.44), we see that these are in fact *Abelian gauge fields*. In equation (4.38) we have dropped the $g(y)$ factor, because any harmonic function on a compact manifold must be constant. This constant has been absorbed into $H(x)$. But more importantly, on-shell a three-form in four dimensions does not have any dynamical degrees of freedom (for a discussion of on-shell counting, see section 3.2.3). So, $H(x)$ does not make any contribution to the four-dimensional physics.

Since C and g are part of the same supermultiplet in eleven dimensions, we expect the four-dimensional fields found above to have superpartners coming from the Kaluza-Klein expansion of g. This believe is strengthened by the fact that a generic G_2-holonomy metric contains $b^3(X)$ moduli (see proposition 2.24), so we expect another $b^3(X)$ scalars to come from the expansion of the metric, which can then combine with their superpartners from the C-field to form supermultiplets. But let us not get too far ahead of ourselves and just start with the Kaluza-Klein expansion of g.

4.4.2 Expansion of the metric

We now repeat the procedure followed above for the equation of motion (3.31). This means that we should substitute (4.23) and (4.24) into this equation and again expand to first order. Because the energy-momentum tensor (the right hand side) is quadratic in the fluctuations of C, we again end up with a condition for Ricci-flatness,

$$\mathcal{R}_{MN}(g_{PQ}) = 0, \tag{4.39}$$

but this time evaluated using the perturbed metric (4.23). So what we now need to calculate is the variation of the Ricci tensor to first order. This calculation is done in appendix C with the result that the first-order variation is given by the so-called Lichnerowicz operator (C.15). It is important to note that during this calculation we have gone to a specific gauge for metric. It is the so-called *harmonic gauge*, in which the metric satisfies

$$\nabla^M \delta g_{MN} - \frac{1}{2}\nabla_N \delta g^M{}_M = 0. \tag{4.40}$$

In the case at hand, we have that $\mathring{\mathcal{R}}_{\mu\nu} = \mathring{\mathcal{R}}_{mn} = \mathring{\mathcal{R}}_{\mu n} = 0$, so the Lichnerowicz operator simplifies to

$$\triangle_L \delta g_{MN} = \nabla^2 \delta g_{MN} - 2\mathring{R}_{MPNQ}\delta g^{PQ} = 0. \tag{4.41}$$

We will now study the solutions of this equation for three different sets of indices.

1. We begin by investigating the equation for $\delta g_{\mu\nu}$. In this case the equation of motion simplifies to

$$\begin{aligned} \triangle_L \delta g_{\mu\nu} &= \nabla^2 \delta g_{\mu\nu} - 2\mathring{R}_{\mu P \nu Q}\delta g^{PQ} \\ &= \nabla^2 \delta g_{\mu\nu} - 2\mathring{R}_{\mu\rho\nu\eta}\delta g^{\rho\eta} \\ &= (\nabla_4^2 + \nabla_7^2)\delta g_{\mu\nu} = 0, \end{aligned} \tag{4.42}$$

   where $\mathring{R}_{\mu\rho\nu\eta} = 0$ because the four-dimensional part of spacetime is just Minkowski space. Again ∇^2 splits into two parts, because our vacuum is a product space. We now make the Kaluza-Klein ansatz

$$\delta g_{\mu\nu}(x, y) = \sum_i h^i_{\mu\nu}(x) t^i(y), \tag{4.43}$$

with $\{t^i\}$ a set of eigenfunctions of ∇_7^2. Substituting this in the equation of motion gives

$$\begin{aligned}\triangle_L \delta g_{\mu\nu} &= \sum_i \left[(\nabla_4^2 h^i_{\mu\nu})t^i + h^i_{\mu\nu}\nabla_7^2 t^i\right] \\ &= \sum_i \left[(\nabla_4^2 h^i_{\mu\nu} + \lambda^i h^i_{\mu\nu})t^i\right] = 0. \end{aligned} \tag{4.44}$$

So now we see that the massless field coming from this ansatz (the unique solution with $\lambda^j = 0$) in four dimensions is a symmetric $(0,2)$ tensor that obeys the Laplace equation (on a curved manifold) $\nabla_4^2 h_{\mu\nu} = 0$. This field can be identified with the *four-dimensional graviton.*

2. We now turn to the equation of motion for $\delta g_{\mu n}$. We explained in section 3.2.1 that these are the fluctuations from which the non-Abelian gauge fields in four dimensions originate, so this is a particularly important case. Now the Lichnerowicz equation again simplifies to the Laplace equation in curved space, because the vacuum Riemann tensor with mixed indices vanishes. Explicitly, we get

$$\nabla^2 \delta g_{\mu n} = (\nabla_4^2 + \nabla_7^2)\delta g_{\mu n} = 0. \tag{4.45}$$

As we explained in section 3.2.1, we now have to make the ansatz

$$\delta g_{\mu n}(x,y) = \sum_i A^i_\mu(x) K^i_n(y) + \cdots , \tag{4.46}$$

where $K^i_n(y)$ $(i = 1, \dots, \dim(G)$ are Killing vector on X and G is its isometry group. But proposition 2.26 showed that a G_2-manifold admits no Killing vectors! Therefore, we have to conclude that *no massless modes arise from* $\delta g_{\mu n}$. In particular, we will not find any Yang-Mills gauge fields in the four-dimensional effective theory. The only gauge symmetry we have is generated by the Abelian gauge fields coming from the C-field.

3. Finally, we focus on possible massless modes coming from δg_{mn}. The Lichnerowicz equation for the fluctuations tangent to X is

$$\nabla^2 \delta g_{mn} - 2\mathring{R}_{mpnq}\delta g^{pq} = 0. \tag{4.47}$$

Here ∇^2 is still the full $D=11$ operator, which of course again can be split as $\nabla_{11}^2 = \nabla_4^2 + \nabla_7^2$. If we do this, we see that the Lichnerowicz operator in seven dimensions, $\triangle_{L,7}$, can be identified with the mass-operator in four dimensions,

$$(\nabla_4^2 + \triangle_{L,7})\delta g_{mn} = 0. \tag{4.48}$$

So now we again make the Kaluza-Klein ansatz, but instead of expanding in eigenstates of the Laplacian, we expand in eigenstates of the Lichnerowicz operator:

$$\delta g_{mn}(x,y) = \sum_i s^i(x) h^i_{mn}(y), \tag{4.49}$$

where

$$\triangle_L h^i_{mn}(y) = \lambda^i h^i_{mn}(y). \tag{4.50}$$

Substituting (4.49) in (4.48) gives us

$$(\nabla_4^2 s^i(x) - \lambda^i s^i(x)) h^i_{mn}(y) = 0, \tag{4.51}$$

so the zero eigenvalue modes of the seven-dimensional Lichnerowicz operator $\triangle_{L,7}$ give us massless scalars in four dimensions. It can be shown [5] that if we define a three-form on X by

$$\omega_{mnp} := \varphi_{n[pq} h_{r]}{}^n, \tag{4.52}$$

with φ the G_2-invariant three-form on X, the following equivalence can be established:

$$\triangle_L h = 0 \Leftrightarrow \triangle \omega = 0. \tag{4.53}$$

This implies that there are exactly $b^3(X)$ massless scalars coming from the expansion of the metric components tangent to X. We can combine these scalars with the pseudo-scalars we found from the expansion of C into complex scalars like

$$\Phi^i(y) = s^i(y) + \imath p^i(y) \quad (i = 1, ..., b^3(X)). \tag{4.54}$$

We would like to note that in [5] it is shown that these complex scalars can be rewritten in a natural way like

$$\Phi^i(y) = \int_{\alpha_i} [\varphi + \delta\varphi + iC]. \tag{4.55}$$

Here the α_i form a basis of the third homology group and φ is the G_2-invariant three-form and $\delta\varphi$ are the fluctuations of the metric rewritten in terms of φ.

In principle we could now repeat the same procedure for the gravitino field ψ_M to determine its massless spectrum. But because our vacuum is $\mathcal{N} = 1$ supersymmetric in four dimensions, we can completely determine the field content already with our current knowledge. All the bosonic fields we have found will get superpartners from the Kaluza-Klein reduction of ψ_M. We will find $b^2(X) + b^3(X)$ Majorana spinors plus a superpartner of the four-dimensional graviton, the gravitino. These spinors combine with the bosonic fields we found into $b^2(X)$ vector multiplets and $b^3(X)$ scalar multiplets (each of which contains a scalar and a pseudo-scalar), as was originally found by [62].

4.5 Other Ricci flat compactifications

On of the assumptions we made was that the metric of the compact manifold X is irreducible. As explained in chapter 2 this basically excludes compactification manifolds that are products themselves. Generically such manifolds have a smaller holonomy group, leaving more supersymmetry unbroken and therefor leading to effective field theories with $\mathcal{N} > 1$ supersymmetry. We would like to mention here that there is at least one other way to obtain $\mathcal{N}=1$ supersymmetry in four dimensions.

As we just mentioned, there are a number of other solutions to the equations of motion that are compatible with the assumption of Ricci-flatness, but not with the irreducibility of the compact metric. For example, we can use the simplest possible seven-manifold, the seven-torus T^7, to compactify supergravity. This manifold is Ricci-flat and has a trivial holonomy group. Because the holonomy group is trivial, every spinor is covariantly constant and thus we get maximal $\mathcal{N}=8$ supersymmetry in four dimensions. This theory was constructed in [20]. Another example of a well-known Ricci-flat vacuum is $\mathbb{R}^4 \times K3 \times T^3$, whose four-dimensional effective theory was constructed in [28]. $K3$ has holonomy group $SU(2)$ (see Appendix B), which implies that half of the supersymmetry is broken upon compactification on $K3$. The T^3 again conserves all supersymmetry, so we find $\mathcal{N}=4$ supersymmetry in four dimensions.

Of course we have already seen that the compactification of eleven-dimensional supergravity yields Type IIA supergravity in ten dimensions. This ten-dimensional theory can thereupon be compactified further down to four dimensions on a Calabi-Yau three-fold to yield a $\mathcal{N}=2$ supersymmetric theory (the precise form of which depends on the chosen CY_3. A very important other compactification of eleven-dimensional supergravity down to ten dimensions was constructed by Hořava and Witten [43] in 1996. They conjectured that if supergravity in $D=11$ is compactified on the orbifold $S_1/\mathbb{Z}_2$ (which is isomorphic to the unit interval, causing spacetime to have two boundaries), we find the strong coupling, low-energy limit of the Heterotic $E_8 \times E_8$ String Theory in ten dimensions. It is this theory that received the most attention of people doing String Theory phenomenology before the "Second String Theory Revolution" in 1995, because if we compactify this theory on a CY_3 manifold, we get a $\mathcal{N}=1$ supersymmetric field theory in four dimensions. What makes this theory even more interesting is that such compactifications lead to chiral fermions in the four-dimensional theory! These results are summarized in table 4.1.

Vacuum	Effective theory
$\mathbb{R}^{10} \times S^1$	Type IIA String Theory
$\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$	$E_8 \times E_8$ Heterotic String Theory
$\mathbb{R}^4 \times T^7$	$\mathcal{N}=8$ SO(8) supergravity
$\mathbb{R}^4 \times K3 \times T^3$	$\mathcal{N}=4$ supergravity coupled to 22 vector multiplets
$\mathbb{R}^4 \times CY_3 \times S^1$	$\mathcal{N}=2$ supergravity
$\mathbb{R}^4 \times CY_3 \times S^1/\mathbb{Z}_2$	$\mathcal{N}=1$ supergravity

Table 4.1: Other (reducible) Ricci-flat M-theory vacua

4.6 Final Considerations

So to sum up our results until now, by compactifying M-theory (or more accurately eleven-dimensional supergravity) on a manifold of G_2-holonomy we have found an effective $\mathcal{N}=1$ supersymmetric four-dimensional supergravity theory, coupled to $b^2(X)$ Abelian vector multiplets and $b^3(X)$ neutral chiral multiplets. This clearly indicates that we are still far from our goal of obtaining a realistic effective field theory from by compactifying M-theory. Two of the basic ingredients of the Standard Model, non-Abelian gauge symmetry and chiral fermions charged under this symmetry, are not present in the effective four-dimensional theory. As we will see in the following chapters, it is possible to get these features from G_2-compactifications, but we need another complication to accomplish this. The compactification space should be a *singular* space with G_2-holonomy. The next chapter is devoted to studying such compactifications. The only thing left is to make a few comments about how the discussion in this chapter might be extended.

Low energy effective action

In principle it will be possible to derive a low-energy effective action for the massless fields in the compactified theory right now. Because we already know that we will not find the physics we are looking for, we will not pursue this road. We refer to [64], chapter 17.4, for the general procedure of constructing such an effective action and [62] for some words about this.

Wittens no-go theorem

First of all, we did all this work to compactify eleven-dimensional supergravity on a G_2-manifold only to find that no interesting physics can ever result from these kinds of compactifications. To make things worse we now bring up that we should not at all be surprised to get this result: Witten already proved in 1983 [74] that no chiral matter can be found if we compactify eleven-dimensional supergravity on any *smooth* seven-dimensional manifold.

This gives us an even stronger indication that we should be studying singularities, branes or some other defects in our G_2-manifolds where the chiral fermions might live. This is again exactly what we are going to do in the upcoming chapter.

But if we knew this up front, then why do all this work? Well, the analysis in this chapter taught us a lot about how to solve the equations of motion of eleven-dimensional supergravity, showed us the explicit connection between supersymmetry and manifolds of G_2 holonomy and pointed out why flux-free solutions get the most attention. We will be able to build on these results in what lies ahead.

Anomalies

An important thing to check is whether the theory we obtained after compactification is free of anomalies. A theory is said to contain an anomaly if a certain symmetry of the classical theory is no longer present after quantization. A thorough examination of the presence of anomalies in all the mentioned M-theory compactifications and a good introduction into the subject of anomalies is given in the impressive thesis [59] and references therein. We simply mention that the theories we study are free of anomalies.

Chapter 5

M-theory on Singular G_2-manifolds

5.1 Introduction

We have seen in the previous chapters that in our search for realistic effective four dimensional physics within the M-theory framework, we were led to compactifications on G_2-manifolds. Although this setting held much promise, carrying out the actual Kaluza-Klein compactification showed us that it could not live up to it. It was already hinted at that we need singularities or D-branes to have any hope of constructing a non-Abelian lower dimensional gauge theory with chiral matter fields. In this chapter we will explore the influence of (metric) singularities on the the lower-dimensional effective theories of M-theory, while in the next chapter we will show the importance of D-branes.

The presence of singularities complicates the analysis greatly. To begin with, the supergravity description of M-theory is only valid on smooth manifolds. So, if we want to work with singular spaces, we have to use new techniques that stem from the full M-theory framework. For example, in this chapter we explain how non-Abelian gauge symmetry is generated by M2-branes wrapping cycles in ADE singular spaces. Secondly, the construction of spaces with the right geometry is not an easy feat. Using Joyce's construction it is possible to create compact manifolds with embedded ADE singularities, but a compact space of G_2-holonomy containing an isolated singularity has not yet been constructed. And isolated singularities are needed to find chiral fermions in the four dimensional effective physics. Therefore, we are forced to use local descriptions of the spaces around a singularity. This is usually no problem as most of the physics is determined by the local properties of the space in the neighborhood of the singularity. And non-compact G_2-metrics with isolated singularities have been constructed (as cones over $\mathbb{C}P^3$, $SU(3)/U(1)^2$ and $S^3 \times S^3$). As a matter of fact, the asymptotically conical spaces described in section 2.3.4 which are smooth at generic points of their moduli space, become exactly these conical spaces if we go to special points in their moduli space.

The structure of this chapter is as follows. In section 5.2, we explain in detail how non-Abelian gauge symmetry arises from ADE singularities. Then in section 5.3 we will see that chiral fermions live at isolated singularities of the compactification manifold. Finally, we briefly explain how these two constructions can possibly be combined into a M-theory model that leads to realistic particle physics.

5.2 Non-Abelian Gauge Symmetry From ADE Singularities

What will be made clear in this section is that the ADE singularities that we introduced in section 2.2.2 can be used to generate non-Abelian gauge symmetry in M-theory compactifications. Since our ansatz throughout this thesis has been that spacetime of the form $\mathbb{R}^{3,1} \times X$, with X a G_2-manifold, we would like to find out how to compactify on an X that contains ADE singularities. Although it is possible to construct via the Joyce construction compact G_2-manifolds with embedded ADE singularities, the most important physical implications of their presence can be understood by just considering a small neighborhood of these singularities. So it will suffice for now to just make a local analysis of M-theory on an non-compact space with an ADE singularity. These non-compact spaces are far easier to construct. In other words, we now take X be of the form $X \simeq \mathbb{R}^3 \times \mathbb{C}^2/\Gamma_{ADE}$, so that we are actually "compactifying" M-theory on $\mathbb{R}^{6,1} \times \mathbb{C}^2/\Gamma_{ADE}$ to seven dimensions. In section 5.3 we instead take $X = Q \times \mathbb{C}^2/\Gamma_{ADE}$ and see what properties the three-dimensional manifold Q has to have in order to define a (non-compact) G_2-manifold with isolated singularities.

As the name implies, the gauge groups that can be constructed using ADE singularities are one of $SU(N)$, $SO(2k)$ and $E_{6,7,8}$. We will mainly focus on the $SU(N)$ case, but note that a similar analysis can be done for the $SO(2k)$ case (by considering the singular limit of a so-called Atiyah-Hitchin manifold). Currently no geometry is known that leads in a similar way to one of the $E_{6,7,8}$ groups. Typically the gauge groups constructed in this way in M-theory model building function as a GUT-group, which should later somehow be broken down to the Standard Model group. Because the exceptional group E_6 has been used to construct fairly realistic Grand Unified Theories, it would be very interesting if such a metric could be constructed.

We start by describing in great detail a smooth four-dimensional metric on a resolved A_{N-1} singularity. The space this metric describes will turn out to contain $(N-1)$ two-spheres. The M-theory membranes that wrap around these two-spheres will become massless if we deform the metric in such a way that is becomes singular. These will consequently be responsible for generating the non-Abelian $SU(N)$ gauge symmetry in the effective seven-dimensional theory. The logic of this section is based on [66], but is expanded to far greater detail.

5.2.1 The geometry of gravitational instantons

In 1978 Gibbons and Hawking found a family of classical solutions to the four-dimensional Einstein equations that they called "gravitational multi-instantons" [33]. Without going into too much detail, they were given this name because these solutions (i.e. metrics) obeyed equations that were very similar to ones that instantons in non-Abelian gauge theories obey[1]. Later it was realized that these metrics were valuable as M-theory backgrounds as well, because they give an explicit formulation of the A_{N-1} singularities that are responsible for the symmetry enhancement mechanism described in this section.

This four-dimensional non-compact metric describing a collection of N instantons at locations $\vec{r}_i$ ($i = 1, \ldots, N$) has $SU(2)$ holonomy and is given by

$$ds^2 = V(\vec{r})d\vec{r}^2 + V(\vec{r})^{-1}(dx^4 + \vec{A} \cdot d\vec{r})^2, \tag{5.1}$$

where $\vec{r} = (x^1, x^2, x^3)$, $d\vec{r}^2$ is the Euclidean metric on $\mathbb{R}^3$ and x^4 is the fourth coordinate. The two functions V and $\vec{A}$ are defined by

$$V(\vec{r}) = \epsilon + \frac{1}{2}\sum_{i=1}^{N} \frac{R}{|\vec{r} - \vec{r}_i|} \tag{5.2}$$

and

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} V, \tag{5.3}$$

where the curl and gradient are the usual operators in $\mathbb{R}^3$. From (5.1) we see that the metric is invariant under a combined reparametrization of x^4 and a gauge transformation on $\vec{A}$ like

$$x'^4 = x^4 + \lambda(x^m), \quad A'_n = A_n - \partial_n \lambda(x^m) \tag{5.4}$$

As we can see, the metric contains a number of parameters: ϵ, R and the $\vec{r}_i$'s. We will study the metric for ϵ fixed to either $\epsilon = 0$ or $\epsilon = 1$, this parameter is not to be treated as a modulus. At first sight this seems to lead to a $(3N+1)$-dimensional moduli space, but not all variations of these parameters lead to inequivalent metrics. Rigid motion of the N instantons (i.e. a combination of translations and translations leaving invariant the orientation

[1]To name one of these similarities: these solutions have a self-dual Riemann tensor (meaning $R_{\mu\nu ab} = \frac{1}{2}\epsilon_{abcd}R_{\mu\nu}{}^{cd}$), just as Yang-Mills instantons obey $F = *F$.

of the instantons with respect to each other) produces an equivalent metric. This removes six degrees of freedom, so the dimension of the moduli space is actually [26]

$$\dim(\mathcal{M}) = 3N - 3 - 3 + 1 = 3N - 5, \quad N > 2. \tag{5.5}$$

For the case $N = 2$, things go a bit differently. We simply state that in this case we have a three-dimensional moduli space[2]. For a calculation of the metric on these moduli spaces, as originally found in [53], see [26] and references therein. We will now analyze the geometry of this space for different values of these parameters to get a better understanding of the geometry.

Asymptotically Locally Euclidean spaces

We start with by putting $\epsilon = 0$ and studying the geometry for different values of N.

$N = 1$ In the case of a single instanton it is easiest to work in a spherical coordinate system which is centered around the instanton (i.e. we put the instanton in the origin). With this choice, the potential (5.2) is simply $V(r) = R/2r$. Rewriting the metric in these coordinates gives us

$$\begin{aligned} ds^2 &= \frac{R}{2r}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \\ &+ \frac{2r}{R}\left(dx^4 + A_r dr + rA_\theta d\theta + r\sin\theta A_\phi d\phi\right)^2 . \end{aligned} \tag{5.6}$$

To find a solution to (5.3) we can make the simplifying assumption that $A_r = A_\theta = 0$, as we can always use the transformation (5.4) to write $\vec{A}$ in a more general gauge. Under this assumption, equation (5.3) in spherical coordinates is reduced to a set of differential equations for A_ϕ given by

$$\frac{\partial}{\partial\theta}(\sin\theta A_\phi) = -\frac{R\sin\theta}{2r}, \quad \frac{\partial}{\partial r}(rA_\phi) = 0. \tag{5.7}$$

These are easily solved to give

$$A_\phi = \frac{R\cos\theta}{2r\sin\theta} + \frac{Rf(\phi)}{2r\sin\theta}. \tag{5.8}$$

We choose $f = 0$ and put this solution back into (5.6). To make its geometry clearer, we again make a coordinate transformation by

[2] As we shall see below, this case actually corresponds to the Eguchi-Hanson space. This space has one modulus that determines the size of the two-sphere at the tip of the cone and two (trivial) moduli that correspond to rotations of this two-sphere.

putting $r = \frac{1}{2R}\rho^2$ and $x^4 = \frac{R}{2}\psi$. This transforms our metric into the form

$$ds^2 = d\rho^2 + \frac{1}{4}\rho^2 \left[d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta d\phi)^2\right]. \tag{5.9}$$

So this metric appears to have a conical singularity in the origin, where $\rho = 0$. But if we take ψ to be periodic with a period of 4π, then the part in the square brackets is exactly the round metric on S^3 given in (2.46). But as we discussed underneath definition 2.12, if the base manifold of a cone is a round sphere, the singularity is only a coordinate singularity and the space is just flat space. So what we just found is that if we periodically identify $x^4 = x^4 + 2\pi R$, the metric is completely smooth and just a fancy way to describe *four-dimensional Euclidean space* $\mathbb{R}^4$.

$N = 2$ It should not come as a surprise that if we consider the case with two instantons, the geometry becomes more complicated and interesting. With two instantons, we can center our coordinate system around either one of the instantons. For any which choice, if we take the limit $r \to 0$ we expect the influence of the second instanton to become negligible and the metric to approach the form (5.9). This implies that just as in the $N = 1$ case, x^4 must have periodicity $2\pi R$ to make the metric free of conical singularities at the locations $\vec{r}_1$ and $\vec{r}_2$.

If we now take the limit $r \to \infty$, the separation between the instantons becomes negligible and the metric should approach that of a single instanton with potential $V = R/r$. The solution for A_ϕ is now two times (5.8) and the metric is asymptotically again of the form (5.9) if we make the coordinate transformation $\psi = x^4/R$. Because of the periodicity of x^4, ψ this time has periodicity 2π. If it would have been 4π, the space would again be a cone over S^3. We see instead that this space is asymptotically a cone over $S^3/\mathbb{Z}_2$, with the $\mathbb{Z}_2$ action given by $\psi \to \psi + 2\pi$.

In summary, we now have a smooth space of $SU(2)$ holonomy which is asymptotically a cone over $S^3/\mathbb{Z}_2$. If the reader feels like the similarity between this space and the one we described in section 2.2.4 is to great to be a coincidence, he is right: this space is just the *Eguchi-Hanson space* EH_2 (2.42) in a different coordinate system. For the coordinate transformation that shows this equivalence, see [65]. For this and a number of other metrics reducing to the Eguchi-Hanson metric (including solutions with non-zero cosmological constant), see [56]. Recall from section 2.2.4 that EH_2 is the metric on a resolved A_1 singularity and that it contains a two-sphere at its tip.

$N \geq 2$ In a way similar to the one given above we can argue that the case with general N is an asymptotically conical space as a boundary the so-called *Lens space* $S^3/\mathbb{Z}_N$. All these spaces are completely smooth if $x^4 = x^4 + 2\pi R$ and all the $\vec{r}_i$ are spatially separated. Because a cone over $S^3/\mathbb{Z}_N$ is diffeomorphic to Euclidean space identified under a discrete group, $\mathbb{R}^4/\mathbb{Z}_N$, we call these spaces *Asymptotically Locally Euclidean (ALE)*. Just as EH_2 (which is the simplest example of an ALE space), the multi-ALE space contains $(N-1)$ two-spheres and gives a metric on a resolved A_{N-1} singularity. How we can understand this statement is explained in the next subsection.

It is interesting to note that the topology of the multi-ALE space has the same general structure as the asymptotically conical G_2-metrics as depicted in figure 2.2. In this case the tip of the cone is $T = S^2$, the collapsing fibre $F = S^1$, the asymptotical base is $Y = S^3/\mathbb{Z}_n$ and the whole space is asymptotic to $X = \mathbb{R}^4/\mathbb{Z}_n$.

Multi-centre Taub-NUT spaces

After having analyzed the metric (5.1) with $\epsilon = 0$ in detail, we now do the same for the multi-centered Taub-NUT geometry, which is nothing more than (5.1) with $\epsilon = 1$. The geometry changes completely if we make this change. This is already clear from the single instanton ($N = 1$) case, where the geometry is not that of 4-dimensional Euclidean space anymore.

$N = 1$ Starting with the potential $V = 1 + R/2r$ obviously changes nothing in the calculation of A_ϕ. After inserting the same solution into the metric and again making the coordinate transformation $r = \frac{1}{2R}\rho^2$, $x^4 = \frac{R}{2}\psi$ we see that *in the limit* $r \to 0$ the metric again approaches that of (5.9). By the same argument made before, x^4 therefore still has to have periodicity $2\pi R$. If we take the limit $r \to \infty$ the metric takes a different form. In the r-coordinate it asymptotically becomes

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + (dx^4)^2 + R\cos\theta dx^4 d\phi. \qquad (5.10)$$

The first two terms constitute a Euclidean $\mathbb{R}^3$ metric and because x^4 is periodic over $2\pi R$, the third term is a flat metric on a circle S^1 of constant radius R. From the form of the metric (5.1) we see that the radius of the circle for general r is given by

$$a(\vec{r}) = \frac{R}{\sqrt{V(\vec{r})}}. \qquad (5.11)$$

The above implies that topologically the space is asymptotic to a cylindrical space $\mathbb{R}^3 \times S^1$. Geometrically it is not simply a trivial bundle, as the last term shows that the S^1 is fibered over $\mathbb{R}^3$ in a non-trivial way. Normally, if the base space of a fibre bundle is contractible to a point, it should be possible to make a coordinate transformation

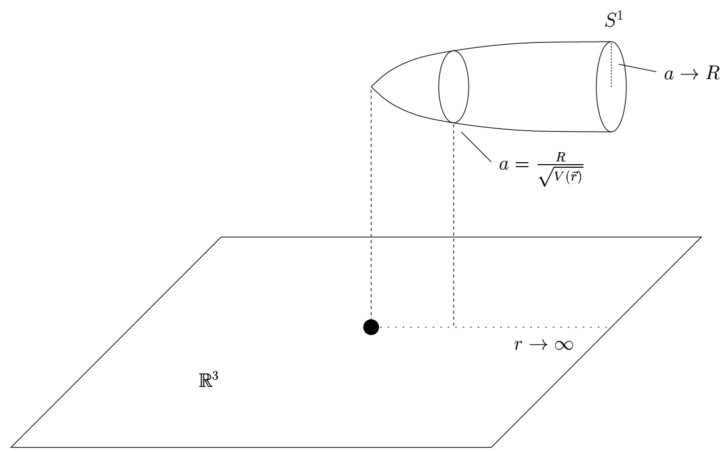

Figure 5.1: Graphical representation of the single-centre Taub-NUT metric

that removes a twist in the bundle. Here this is impossible, because the metric (5.10) is clearly only valid for $r \to \infty$ and in the complete metric the topological defect in the origin (the instanton) prevents us from making such a transformation[3]. In figure 5.1 the geometry of this space is shown schematically.

$N > 1$ We have just seen that in the single-centre Taub-NUT space we have a circular fourth dimension which goes to constant size far away from the defect and shrinks to zero size at the position of the defect. When we have a space with multiple instantons, (5.11) is obviously still true, which means that the S^1 shrinks to zero size at all the locations $\vec{r}_i$. If we consider a generic path between two of the instantons and erect an S^1 of radius a at every point along the path, we see that this corresponds to the topology of a two-sphere. Naively we would expect there to be $\frac{1}{2}N(N-1)$ different two-spheres, corresponding to the total number of "links" between N distinct points. However, in the next subsection we will see that the multi-centered Taub-NUT space topologically also only contains $(N-1)$ homologically non-trivial two-spheres instead.

Note that close to the instantons the space locally again looks like flat Euclidean space. Far away from the configuration of instantons however, the geometry is more complicated, like in the ALE case. It is again a twisted S^1 bundle over S^2, but it is not an ALE space, because the S^1 has finite size. It is not an ALE space, but an example of an *Asymptotically Locally Flat (ALF)* space. Topologically it has

[3]Note that even if (5.10) would have been our complete space, the singularity at $r = 0$ would have prevented us from making this transformation.

the same structure as the one we described for the ALE case, but this time $Y = S^2 \times S^1$ and $X = \mathbb{R}^3 \times S^1$.

With this detailed analysis of the geometry associated with (5.1) we have given ourselves a solid basis to understand the physics of M-theory compactified on such a space. In the next subsection we will explore in detail what we have already briefly mentioned. Namely, the multi-instanton metric contains $(N-1)$ 2-spheres and describes the smooth geometry of a resolved A_{N-1} singularity, when all the $\vec{r}_i$ are distinct. When k of the instantons are brought to the same location, the space degenerates and will contain an A_{k-1} singularity.

5.2.2 Intersection numbers for two-cycles

In this subsection we will calculate the intersection numbers between the two-cycles we saw to be present in the gravitational instanton metric. In order to do this, we will first show that there are $(N-1)$ homologically inequivalent two-cycles, then we explain why these intersect transversally and finally we calculate the intersection numbers to see that the matrix made up by all these numbers (the *intersection matrix*) is exactly equal to the Cartan matrix of the $A_{N-1} = SU(N)$ Lie algebra.

Consider an oriented straight path c_{ij} through $\mathbb{R}^3$ from the point $\vec{r}_i$ to $\vec{r}_j$. Erect an oriented circle S^1 at each point $p \in c_{ij}$ of radius (5.11) (which vanishes at all $\vec{r}_i$) and call the two-cycle that we create in this way S_{ij}. If we take the orientation of S^1 to respect, say, the right hand rule with respect to the orientation of c_{ij}, we will have given S_{ij} a definite orientation. We have seen in section 2.1.4 that it is necessary for the intersecting cycles to be oriented in order to be able to calculate the intersection numbers.

Because the total number of paths between N distinct points is $\sum_{i=1}^{N-1}(N-i) = \frac{1}{2}N(N-1)$, we would naively expect there to be this number of two-cycles. That the actual number is smaller can be understood by remembering that because we are trying to find the topology of the space, we should count cycles in the same homology class only once. To see which cycles are homologically equivalent, we look at the configuration of three instantons as depicted in figure 5.2. All the possible oriented paths between the instantons are c_{12}, c_{23} and c_{13}. But if we deform c_{12} and c_{23} to the dashed paths $\tilde{c}_{12}$ and $\tilde{c}_{23}$ respectively, we see that the vertical paths cancel and we have effectively constructed the third path c_{13}. So in this particular case we have $N-1=2$ homologically inequivalent cycles.

Now add a point $\vec{r}_4$ to this configuration and connect it to either $\vec{r}_1$ or $\vec{r}_3$, with the orientation respecting the line $c_{12}c_{23}$. It is not hard to see that now the *all* the lines connecting $\vec{r}_4$ to any of the other points can be constructed by deforming adding up a combination of the paths already present. In a similar way we can add an arbitrary number of additional points to the

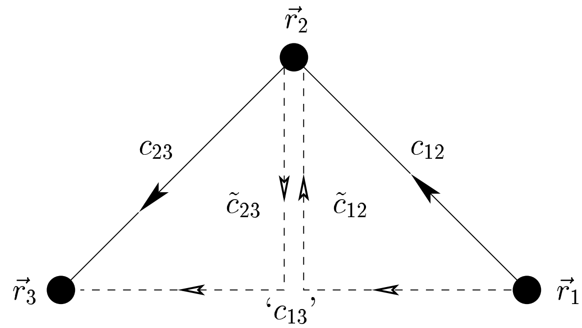

Figure 5.2: Deformation of oriented lines. c_{13} can be constructed by deforming c_{12} and c_{23} to $\tilde{c}_{12}$, $\tilde{c}_{23}$ in the way indicated.

configuration. Each of these points adds only one homologically inequivalent two-cycle. So the total number of two-cycles in both the ALE and Taub-NUT spaces is given by the minimal number of links in a chain connecting N points, which is $(N-1)$. Note that this configuration corresponds exactly to the *Dynkin diagram of the A_{N-1} algebra*. From this we also see that we can take the $S_{i(i+1)}$ with $1 \leq i \leq (N-1)$ to be a basis for the independent two-cycles.

Note that in [7] a way was presented to come to the same conclusion without knowing an explicit metric. In this article the same result is obtained by carefully carrying out the blow-up procedure for an $\mathbb{C}^2/\mathbb{Z}_N$ orbifold singularity. If we carry out this procedure, $(N-1)$ exceptional divisors $\mathbb{C}P^1$ will be generated that are linked in a chain in the same way as above. And because $\mathbb{C}P^1 \cong S^2$, this corresponds to the same geometry we found here. In his later lectures [8] and the original article, further details about this blow-up procedure can be found.

The only place the two-cycles intersect (after a deformation if necessary) is at the $\vec{r}_i$. The next thing to do is to show that here the two-cycles $S_{(i-1)i}$ and $S_{i(i+1)}$ indeed intersect transversally. To show this, we need to find bases for the tangent spaces $T_{r_i}S_{(i-1)i}$, $T_{r_i}S_{i(i+1)}$ that span the total four-dimensional tangent space at $\vec{r}_i$. A basis wherein one of the vectors in tangent to the S^1 is not suited, as the S^1 degenerates at the point $\vec{r}_i$. We already mentioned for both the ALE and Taub-NUT cases, close to the instantons the space looks like flat Euclidean space $\mathbb{R}^4$. So we expect to find that the tangent spaces of these two-spheres at $\vec{r}_i$ are just two planes in $\mathbb{R}^4$ that (for generic mutual orientations) intersect transversally. To see this explicitly, we go to a different (quaternionic) coordinate system that demonstrates the transversality. This idea is taken from [24], which was in turn inspired by [35]. For the definition of quaternions, see Appendix A. Here we will only consider the case of an ALE space, but it should be possible to do something similar for the Taub-NUT metric.

Because a quaternion consists of a one-dimensional real and a three-dimensional imaginary part, we expect to be able to identify these with the periodic and Euclidean coordinates of the ALE metric respectively. For reasons that will become clear shortly, we do not make this identification in the obvious 'one-on-one' way. First note that any quaternion $q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ can be written as $q = ce^{\mathbf{i}\sigma}$, where c is a pure imaginary quaternion and $\sigma \in (0, 2\pi]$. Now σ can be identified with x^4 by $\sigma = x^4/R$. The combination $q\mathbf{i}\bar{q}$ can be seen to be independent of σ and pure imaginary[4]. With this, we define

$$\frac{1}{2}q\mathbf{i}\bar{q} = \frac{1}{2}(w^2 + x^2 - y^2 - z^2)\mathbf{i} + (xy + wz)\mathbf{j} + (xz - wy)\mathbf{k} \quad (5.12)$$

$$=: x^1\mathbf{i} + (x^2 + x^3\mathbf{i})\mathbf{k} = x^1\mathbf{i} - x^3\mathbf{j} + x^2\mathbf{k}. \quad (5.13)$$

Near the instanton at $\vec{r}_i$, the ALE metric looks like $ds^2 = 1/r d\vec{r}^2 + r(dx^4 + \vec{A} \cdot d\vec{r})^2$. After carrying out the coordinate transformation defined above and noting that any quaternion can be written as $q = a + b\mathbf{j}$ $(a, b \in \mathbb{C})$, we find that the metric can be written as:

$$ds^2 = dq d\bar{q} = da d\bar{a} + db d\bar{b}. \quad (5.14)$$

This form again expresses the fact that this space locally looks like $\mathbb{C}^2 \cong \mathbb{R}^4$. The tangent spaces $T_{r_i}S_{(i-1)i}$ and $T_{r_i}S_{i(i+1)}$ are just two-dimensional planes in $\mathbb{R}^4$. If it happens to be that they coincide or intersect along a line, we can deform one of the two-cycles so that its tangent plane gets rotated in such a way that they only intersect in $\vec{r}_i$. This means that their intersection number is either $+1$ or -1, depending on whether we can define an oriented basis for $\mathbb{R}^4$ using basis vectors of these two tangent spaces. It turns out that this is possible, so the intersection number between all cycles $S_{(i-1)i}$ and $S_{i(i+1)}$ is positive. If we call the intersection number between the cycles $S_{i(i+1)}$ and $S_{j(j+1)}$ I_{ij}, we thus find $I_{(i-1)i} = 1$.

Calculating the self-intersection of, say, $S_{i(i+1)}$ ($\equiv S$ for now) takes a bit more work. To calculate this intersection, we make a new cycle by deforming the surface within its homology class and then calculate the intersection number of this cycle with the original cycle. Because the points $\vec{r}_i$, $\vec{r}_{i+1}$ stay fixed, these cycles necessarily intersect there, so we expect the self-intersection to be either $+2$ or -2. If we consider the surface as an embedding in $\mathbb{C}^2$, a deformation of the surface corresponds to a section in its normal bundle N_S. The self-intersection is then given by the number of zeros of this section. This can be calculated by integrating the first Chern class over the cycle [8],

$$\#(S \cdot S) = \int_S c_1(N_S). \quad (5.15)$$

[4] Also note that $q\mathbf{i}\bar{q} = c\mathbf{i}\bar{c}$.

We can then use the *adjunction formula* [8]

$$c_1(T_{E|_S}) = c_1(N_S) + c_1(T_S) \tag{5.16}$$

to find a calculable expression for (5.15). In the expression on the left hand side $E|_S$ is the embedding space restricted to S, which in our case is $\mathbb{C}^2$. Because the tangent bundle of $\mathbb{C}^2$ is a trivial bundle, its first Chern class $c_1(T_{\mathbb{C}^2}) = 0$. But this will also mean that the first Chern class of the restriction of this bundle to any subset will be zero. If we use this and then integrate (5.16) over S, we find

$$\#(S \cdot S) = -\int_S c_1(T_S). \tag{5.17}$$

But the *Gauss-Bonnet theorem* for a complex manifold M of complex dimension m states that [30]

$$\int c_m(M) = \chi(M), \tag{5.18}$$

where $\chi(M)$ is the Euler characteristic. For a complex orientable one-dimensional surface (a so-called *Riemann surface*) of genus g, the Euler characteristic is widely known to give $\chi(M) = 2 - 2g$. Since our two-sphere can be considered as a zero-genus complex surface, we finally find

$$\#(S \cdot S) = -2. \tag{5.19}$$

The fact that the number of zeros in a section is given by a negative number is a result of our chosen conventions. So for all $1 \leq i \leq N-1$, we have $I_{ii} = -2$.

It should be clear from our configuration of two-cycles that the cycles S_{ij} with $j \notin \{i-1, i, i+1\}$ do not intersect and hence have $I_{ij} = 0$. Taking all of the above into account, we finally see that the intersection matrix is the following symmetric $(N-1) \times (N-1)$ matrix:

$$I = -\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}. \tag{5.20}$$

This is exactly minus the Cartan matrix of the $SU(N)$ algebra. Using different conventions it could have been possible to get exactly the Cartan matrix. This, together with the fact that the blow-up of an A_{N-1} singularity gives us the Dynkin diagram of the $SU(N)$ algebra, is a first indication of the fact that there is a deep relation between the discrete subgroups of $SU(2)$ (which have an ADE classification) and the simply-laced Lie groups. This has been stated [8] to be one of the most miraculous interrelations in mathematics. It was discovered in 1980 [57] and is known as the *McKay correspondence*.

In the next subsection we will see that if we have M2-branes wrapping the two-spheres of this space, an enhanced $SU(N)$ gauge symmetry is generated by these branes if we take the singular limit.

5.2.3 Enhanced gauge symmetry from wrapping branes

Consider first a configuration of only two monopoles (so effectively Eguchi-Hanson space) and for simplicity choose a coordinate system in which $\vec{r}_1 = (x, 0, 0)$ and $\vec{r}_2 = (y, 0, 0)$ (with $y > x$). The area of cycle S_{12} corresponding to the straight path between $\vec{r}_1$ and $\vec{r}_2$ is simply given by the integral

$$A_{12} = \int_x^y \int_0^{2\pi a} \sqrt{V(\vec{r})} dx^1 dx^4 = 2\pi R(y - x), \tag{5.21}$$

where the factor $\sqrt{V(\vec{r})}$ comes from the scaling of $\mathbb{R}^3$ in (5.1) and a is given by (5.11). The obvious generalization to any straight path is

$$A_{ij} = 2\pi R \int_{c_{ij}} |d\vec{r}| = 2\pi R |\vec{r}_i - \vec{r}_j| \tag{5.22}$$

and we expect that the area of any other two-cycle will be proportional to the length of the curve c_{ij}. So the two-cycle corresponding to the straight path is actually the cycle with minimal area.

As we have discussed in chapter 3, eleven-dimensional supergravity classically admits an M2-brane solution. If such an M2-brane is wrapped around a two-cycle in a compactification manifold, it will look from the lower-dimensional perspective like a point-particle. The mass of a wrapped M2-brane is proportional to its area and tension T_{M2}, so in our case we simply have

$$m_{ij} = 2\pi R T_{M2} |\vec{r}_i - \vec{r}_j|. \tag{5.23}$$

Because of their tension the M2-branes will seek to wrap the cycle with minimal area. This configuration will have minimal mass and will in fact be a BPS state [12] as described in chapter 3. Such a state will be invariant under only a part of the supersymmetry algebra, which means that it will live in a short representation of this algebra. The cycle about which it is wrapped is actually a calibrated submanifold and what we have just said is precisely the reason why these are often also called *supersymmetric cycles* (see section 2.1.3). One of the implications of a wrapped M2-brane being a BPS state is that its (classical) mass 5.23 will not get quantum corrections.

For a two-sphere $b_2(S^2) = 1$, which implies that we have exactly one harmonic two-form ω on our two-cycle. Furthermore, $b_1(S^2) = 0$, so there is no harmonic one-form and $b_3(S^2)$ is identically zero. If we now make the Kaluza-Klein ansatz for the three-form field, $C(x,y) = A(x) \wedge \omega(y) + \cdots$, we see that get one $U(1)$ gauge field in the effective seven-dimensional field theory. In section 5.2.1 we showed that the moduli space of the Eguchi-Hanson space is three-dimensional. Just as in chapter 4, these moduli will give us three scalar fields in the effective theory. An Abelian vector multiplet in seven dimensions contains precisely one gauge field and three scalars [5], so our effective field content is exactly that of an Abelian gauge multiplet. We have implicitly wrapped the M2-branes with a certain orientation. If we had done this with the opposite orientation, the seven-dimensional gauge field would have gotten the opposite charge under the $U(1)$ gauge group. These particles look rather like the $W^{\pm}$-bosons, except that they do not generate a $SU(2)$ symmetry (yet).

If we now bring the two monopoles close together, the Eguchi-Hanson space will degenerate and develop an A_1 singularity. The wrapped M2-branes will in this singular limit become massless. In the lower dimensional effective theory we will thus find extra massless oppositely charged vector multiplets, which turn out to have exactly the right quantum numbers to generate an $A_1 = SU(2)$ gauge symmetry.

In a similar way, if we take an N-centered gravitational instanton metric to begin with, we find at a generic point in the moduli space $(N-1)$ Abelian vector multiplets, generating a $U(1)^N$ gauge symmetry. If we now move to a special point in the moduli space, where all of the instantons coincide, the space will develop an A_{N-1} singularity. The extra massless degrees of freedom will in that case generate an enhanced gauge symmetry $SU(N)$.

One might object that since the metric (5.1) describes a non-compact space, we have a continuous particle spectrum in the effective field theory. This would also imply that the states we describe here would not be the only massless states we find. Although that might be true, we already indicated that we only use this as a local description and hope to embed these types of singularities in a compact space. What we are doing here is describe the states that would be massless if we were to do this.

The importance of ADE singularities in symmetry enhancement in theories with wrapping membranes was first pointed out in the famous paper [76]. Later in [25] this statement was expanded upon and the relation between these singularities and so-called 'quiver-diagrams' was found. Then in [25] this mechanism was carried over into the Matrix-theory [10] description of M-theory and shown to be consistent with the previous description. For more on this, see for example [11].

5.3 Chiral Fermions from Isolated Singularities

In the previous section we learned how to get non-Abelian gauge groups from M-theory compactifications on singular G_2-manifolds. To find a realistic four-dimensional theory from these kinds of compactifications, we still need to find out how to incorporate charged chiral fermions - the second basic ingredient of the Standard Model. As we will find out in this section, we need to compactify on a manifold X which contains isolated singularities to be able to find them.

There are actually four ways to show that chiral fermions can be supported at isolated singularities. One is via the duality with the Heterotic string [4], the second is using anomaly cancellation ([77] and below), the third is by treating a specific conical metric [9] and finally we can utilize the duality with Type IIA string theory (see [22] or [71] for a review). This last method will be shortly discussed in the next chapter.

The first mechanism through which we can see that our low-energy effective theory can contain chiral fermions is by doing an analysis of the anomalies in the theory. A theory is said to contain anomalies if a certain (gauge) symmetry that was present in the classical theory disappears after quantization. As anomalies indicate inconsistencies in the theory, something has to added to the theory to exactly cancel the anomalies. Anomaly cancellation has been a recurring theme in both Field and String Theories. Suppose we have a space with isolated singularities that can locally be described as conical singularities. Because a full M-theory description is still lacking, we would like to try to find a supergravity solution on such a space. But because at a conical singularity the curvature blows up, supergravity is not valid on such a space. What we can try to do is cut out a small region around the singularity to see if we can model the effect of the singularity by describing supergravity on manifold with boundary. It turns out that this is possible. Now remember that eleven-dimensional supergravity has a gauge invariance under $C \to C + d\Lambda$. This gauge symmetry is broken if we put supergravity on a manifold with boundary. The anomaly we find can exactly be cancelled by adding chiral fermions at the location of the singularity. This idea was first put forward in [77] and our treatment of the subject is based on this article. For a very thorough treatment of the presence of anomalies in G_2-compactifications in general and some more details about this particular mechanism, see [59].

Suppose we have G_2-manifold X, which is smooth, except for isolated singularities that are located at points $P_\alpha \in X$, $\alpha = 1, \ldots, s$. Assume as well that near these singularities, the space can be modeled as a cone over some six-dimensional base-manifold Y_α (see Definition 2.12). If we now excise the singularity from the space by cutting out a small open neighborhood of P_α, then we end up with a manifold-with-boundary X', whose boundary is

$$\partial X' = -\cup_\alpha Y_\alpha. \tag{5.24}$$

The minus sign is included to give the boundary a convenient orientation [13]. Because the manifold X' is smooth, we can now use the standard Kaluza-Klein mechanism to determine its low energy effective theory. We have seen that if X would have been smooth, the gauge bosons in four dimensions come from the Kaluza-Klein ansatz

$$C(z) = \sum_{i=1}^{b_2(X)} A^i(x) \wedge \omega_i(y) + \cdots , \tag{5.25}$$

where the $\omega_i \in H^2(X;\mathbb{Z})$ are harmonic two-forms on X and the gauge group is $U(1)^{b_2(X)}$. However, things go a bit differently in the current situation as we now have a manifold with boundary. In [77] it was stated that in this case the ω_i are arbitrary harmonic two-forms on Y_α, independent of the radial coordinate r, and that the low-energy gauge group is now $H^2(X';U(1))$. Note that although strong circumstantial evidence for this statement is present, it was not explicitly proven.

We will consider here only anomalies that come purely from the gauge field C. If we include one-loop quantum corrections in the eleven-dimensional supergravity action (see e.g. section 14.1 of [29] for a short discussion), we find terms in which the metric couples to the C field. For the cancellation of these so-called mixed gauge-gravitational anomalies, we refer to [77]. The only term in the action (3.23) that leads to anomalies is the Chern-Simons term, as the Yang-Mills term is manifestly gauge-invariant even on a manifold with boundary. By integrating this term over $M' = \mathbb{R}^4 \times X'$ and dropping the exact factors in front of it, we see that under the gauge transformation $C \to C + d\Lambda$ the Chern-Simons term transforms like

$$\delta S \sim \int_{M'} d\Lambda \wedge G \wedge G. \tag{5.26}$$

Note that here we have made use of $d^2\Lambda = 0$. If we now remember that $dG = 0$ and then use Stokes' theorem, we see that this can be written as

$$\begin{aligned} \delta S &\sim \int_{M'} (d\Lambda \wedge G \wedge G + \Lambda \wedge dG \wedge G + \Lambda \wedge G \wedge dG) && (5.27)\\ &= \int_{M'} d(\Lambda \wedge G \wedge G) && (5.28)\\ &= -\sum_\alpha \int_{\mathbb{R}^4 \times Y_\alpha} \Lambda \wedge G \wedge G. && (5.29) \end{aligned}$$

In [77] it was then stated that if we write $F^i = dA^i$ for the four-dimensional field strength and make the Kaluza-Klein ansatz for C as above and for Λ as

$$\Lambda(z) = \sum_i \Lambda^i(x) w_i(y), \tag{5.30}$$

with Λ^i harmonic functions on $\mathbb{R}^4$, that then the contribution of the α-th singularity to the anomaly is given by

$$\delta_\alpha S \sim \int_{\mathbb{R}^4} \sum_{i,j,k} \Lambda^i \wedge F^j \wedge F^k \int_{Y_\alpha} w_i \wedge w_j \wedge w_k. \tag{5.31}$$

Since anomaly cancellation should be a local phenomenon (i.e. it should be satisfied at all P_α's individually), they then set forth that this anomaly should be cancelled by adding massless charged chiral multiplets Φ^σ of charges q_i^σ at all the P_α. These charges take their value in a set T_α that depends on the location P_α. By comparing the form (5.31) to the anomalies we find in four-dimensional gauge theories, it can be seen that anomaly cancellation is realized if we take the Y_α to satisfy

$$\int_{Y_\alpha} w_i \wedge w_j \wedge w_k = \sum_{\sigma \in T_\alpha} q_i^\sigma q_j^\sigma q_k^\sigma. \tag{5.32}$$

What this equation means is that if Y_α is such that the right hand side is not zero, chiral fermions Φ^σ with charges q_i^σ need to be added at P_α to cancel the anomalies. That the complete theory is then anomaly free can be seen by summing (5.32) over all α. Anomaly cancellation is guaranteed, because this gives

$$\begin{aligned} \sum_\alpha \sum_{\sigma \in T_\alpha} q_i^\sigma q_j^\sigma q_k^\sigma &= \sum_\alpha \int_{Y_\alpha} w_i \wedge w_j \wedge w_k && (5.33) \\ &= \int_{X'} d(w_i \wedge w_j \wedge w_k) && (5.34) \\ &= 0, && (5.35) \end{aligned}$$

because dw_i is zero for all i.

It is interesting to note that in [14] a minor flaw in this line of reasoning was discovered. The procedure outlined above we apply Stokes theorem to equation (5.28) to write it as a sum over contributions from the s boundaries. Although this is perfectly correct mathematically, it does not allow us to conclude that in the steps leading to (5.33) we are forced to considering *local* anomaly cancellation. So, the above line of argument only gives us a prescription for *global* anomaly cancellation. After a careful reconsideration [14], the conclusion remained the same, though.

5.4 Model building with singular G_2-manifolds

In this chapter we have established two things: that if the compactification manifold contains ADE singularities, the effective four-dimensional field theory will have non-Abelian ADE gauge symmetry and that chiral fermions are supported at isolated singularities. So, we now have ways to construct the two basic components of the Standard Model in framework of M-theory. But we have not discussed how one goes about constructing a (quasi) realistic model of particle physics within this setting. We will shortly describe a possible procedure here.

Because ADE singularities are codimension four, we need them to be supported on a three-dimensional manifold Q in X. We could for example take X to be some fibre bundle over Q, with fibres K3. We then take the K3 fibres to have points in their moduli space were they develop at (several) points ADE singularities of some specific type. We saw in section 2.3.4 that G_2-manifolds with A_1-singularities exist, but it has been shown [49] that they exist with the other types of ADE singularities as well. The most promising possibilities are those that correspond to one of the well-studied Grand Unified gauge groups, i.e. one of $A_4 = SU(5)$, $D_5 = SO(10)$ or E_6.

X should then develop isolated (conical) singularities at a number of points on Q, which can support chiral fermions that transform in appropriate representations of the ADE gauge group. The exact nature of the isolated singularity determines in what specific representation these chiral fermions live. It is not at all obvious that G_2-manifold with the needed singularities exist, but it can be motivated in (at least) two ways that this has to be the case. The first is by making use of the duality with the Heterotic String Theory [5]. If we compactify this theory on singular Calabi-Yau manifolds, we can construct many different models with chiral fermions. In the dual M-theory picture these compactifications can be shown to correspond to singular G_2-compactifications, which means that G_2-manifolds should exist that have the right kind of singularities to construct these chiral fermions. The second way to motivate this is by making use of the duality with the Type IIA String Theory [71]. In this model many different types of chiral fermions can be constructed by considering D6-branes intersecting so-called orientifold O6-planes[5] at slight angles. These types of models also lead to $\mathcal{N}=1$ supersymmetric theories in four dimensions with chiral fermions, which in the lift to M-theory (i.e. in taking the limit of the radius of the eleventh circular dimension to infinity) corresponds to singular G_2-compactifications. As was discussed at the end of chapter 2, compact G_2-manifolds with isolated singularities have not been constructed yet, but we know from these considerations that they have to exist. If we take X to contain for example A_4-singularities, we would like to find a manifold that has three conical singularities that give chiral fermions transforming in the $\mathbf{5}$ of $SU(5)$ and three leading to chiral fermions transforming in the $\bar{\mathbf{10}}$ of $SU(5)$. This would correspond to the field content of the Grand Unified

[5] Orientifold planes are basically D-branes with some additional identification under a reversal of orientation.

Theories constructed using this group.

If we then take Q to be non-simply connected, like for example $Q = S^3/\mathbb{Z}_n$, we can use Wilson lines (see e.g. [63] for a definition) to break the $SU(5)$ group to the Standard Model group $SU(3) \times SU(2) \times U(1)$. In [78] such models are described and a method is provided for so-called doublet-triplet splitting, i.e. a mechanism that prescribes how the pentaplet of $SU(5)$ splits into a doublet of leptons and a triplet for the quarks.

Chapter 6

Conclusions

After this long exposition, we are now about to comment on the merits of the path we have chosen for doing M-theory phenomenology and - more importantly - on the open problems that remain to be solved. But before we do so, let us briefly review how we built up the story in this thesis.

6.1 Summary

In chapter 1 we set the stage by explaining that the Standard Model - although a phenomenal experimental success - is believed by many to be only a low-energy limit of an as yet unknown theory. On a road passing through Grand Unified Theories, Supersymmetry, Supergravity and String Theory we were led to the idea that M-theory might be this theory. Although an explicit definition of the theory is still lacking, it has been conjectured that M-theory allows for a description as a Matrix Theory [10]. It is certain that the eleven-dimensional low-energy limit of M-theory should be eleven-dimensional supergravity. In chapter 3 we introduced this theory and explained the general Kaluza-Klein mechanism for dimensional reduction. We then showed that Kaluza-Klein compactification of eleven-dimensional supergravity on a circle yields the Type IIA Supergravity theory in ten dimensions. This ten-dimensional theory is itself the low-energy limit of Type IIA String Theory. M-theory is conjectured [76] to be the strong coupling limit of Type IIA String Theory, but it is hard to make sense of that theory at strong coupling, because we do not yet know how to describe it non-perturbatively.

It turns out that the holonomy group of the compactification manifold is important for counting the number of supersymmetries that survive in the lower-dimensional theory. In order to maintain hope for finding chiral fermions in the effective field theory, we want at most $\mathcal{N}=1$ supersymmetry to survive the compactification, because any $\mathcal{N}>1$ supersymmetric theory is always CPT-invariant. In compactifying eleven-dimensional supergravity on a seven-manifold, imposing $\mathcal{N}=1$ supersymmetry leads to G_2-holonomy for the compactification manifold X. As was explained in chapter 3, in standard Kaluza-Klein theory the four-dimensional effective theory has a non-Abelian gauge group if the isometry group of X is non-Abelian. But in chapter 2 we showed that the isometry group of a manifold of G_2-holonomy is always trivial. In chapter 4 we carried out this compactification and for the given reason we found that the effective field theory has only Abelian gauge symmetry, which originated from the three-form C-field. Furthermore, in 1983 it was shown [74] that these compactifications can not yield chiral fermions. These facts dimmed enthusiasm for doing research into eleven-dimensional supergravity compactifications considerably for some ten years.

However, in 1995 it was put forward [76] that degrees of freedom living on codimension four ADE singularities can be used to generate non-Abelian gauge symmetry. Manifolds of G_2-holonomy that contain ADE singularities are known [49]. Later, in [77] it was concluded that chiral fermions can be supported at isolated singularities in the compactification space. The asymptotically conical non-compact G_2-manifolds introduced in chapter 2 can develop such singularities. The Kaluza-Klein procedure does not work on non-compact manifolds, because it would lead to a continuous spectrum of Kaluza-Klein states. Therefore, the asymptotically conical G_2-manifolds can only be used as local descriptions of the physics close to the singularity. Furthermore, because spaces with isolated singularities contain curvature singularities, the supergravity approximation of M-theory is not valid on such spaces. Therefore, we cannot use standard Kaluza-Klein theory to do a compactification on a singular space. Without going into the question of which exact procedure we *can* use, we know that these singularities need to be embedded in a *compact* space in order to get a discrete spectrum in the four-dimensional effective theory. And as discussed at the end of chapter 2, such manifolds have not been constructed.

6.2 Discussion

Besides mathematical difficulties, there are from the perspective of physics of course also still enormous challenges to be met. First of all, until a complete description of M-theory is found, we are limited to using indirect methods for understanding phenomena beyond the supergravity approximation. One of the best methods is using the various dualities of M-theory with the five String Theories, as we have seen various times throughout this thesis. One of the other big open questions is how to break supersymmetry. The models we have discussed all have low-energy supersymmetry. Low energy here means compared to the Planck scale. We still need to find a good method to break supersymmetry at some intermediate scale (between the Planck and the Standard Model scale) that breaks supersymmetry.

After describing the generic features of G_2-compactifications, let us quickly discuss how these models hold up to basic phenomenological tests. The first question we address is that of *proton stability*. Some Grand Unified and Supersymmetric models predict that there is a small but finite chance that an individual proton spontaneously decays. To test these models, the stability of the proton has been measured and a lower bound on its average decay time has been established of about 10^{35} years. Because this has been measured to great accuracy, this has become one of the standard checks for validity of new theories. Research described in [32] showed that proton stability is guaranteed in M-theory compactifications on G_2-manifolds. The second question is that of *doublet-triplet splitting*. In theories that on some intermediate energy scale are described by a Grand Unified Theory with gauge group $SU(5)$, a mechanism has to be provided that prescribes how the pentaplet splits into a doublet of leptons and a triplet for the quarks. In [78] such a prescription was provided in the context of M-theory compactifications. Finally, first studies into cosmological implications of various String and M-theory models [50] seem to indicate that G_2-compactifications of M-theory may be one of the few models that lead to *observational cosmological effects*. If this statement proofs to be true, it would open up an exciting possibility for testing these types of models.

We now come back to the statement made in the last sentence of the previous section about the need for compact G_2-manifolds with isolated singularities. To this date no compact manifolds with G_2-holonomy and isolated singularities have been constructed. Furthermore, the fact that there is no analogue of Yau's theorem for G_2-holonomy metrics on a seven-dimensional manifold, complicates the construction of G_2-manifolds. Compared to the big stream of Calabi-Yau manifolds being constructed, G_2-manifolds seem to only slowly trickle into the literature. These are the two great mathematical challenges still to be resolved. One possible generalization of the models described here is considering compactifications with a G-flux turned on in a way that is consistent with the other assumptions we made about the vacuum. If we do this, it turns out that we are led to the concept of *weak G_2-holonomy*. One of the reasons that these are interesting models to consider is that complete *compact* weak G_2-holonomy metrics *with isolated singularities* have been constructed in [14]. Furthermore, compactifications with fluxes might provide us with a mechanism for moduli fixing and an explanation of the hierarchy of Yukawa couplings. For a taste of this subject, see [1], [3] and [2]. However, the presence of a non-zero G-flux complicates the story greatly and leads to a number of problems as well. The biggest problem is probably that these kinds of compactifications typically lead to a negative (possibly very big) cosmological constant and observations teach us that we live in a Universe with a positive cosmological constant.

6.3 Conclusion

In this thesis we have only been able to scratch the surface of a vast subject. In a sense, the place at which this thesis ends, should actually be the starting point for the more interesting research. The two directions that strike us as the most promising are intersecting D-brane models in Type IIA String Theory (as shortly described at the end of chapter 5) and compactifications with fluxes.

All in all, we can conclude that this subject knows a number of mathematical difficulties that have made progress in the field slower than maybe many people would have hoped. Slowly these mathematical challenges are being resolved, which opens up the arena for discovering fascinating physics from M-theory compactifications and at the same time enabling us to learn more about the structure of M-theory itself.

Appendix A

Notation and conventions

Throughout this thesis - unless stated otherwise - we use $D = d + k$ to indicate the dimension of the complete spacetime, d for the dimension of the macroscopic (observable) part of spacetime and k for the dimension of the internal (compact) space. The following conventions for indices and coordinates are used:

dim	Indices*	Range	Coordinates
D	$M, N, P \ldots$ / $A, B, C \ldots$	$0 \ldots D-1$	z^M
d	$\mu, \nu, \rho \ldots$ / $\alpha, \beta, \gamma \ldots$	$0 \ldots d-1$	x^μ
k	$m, n, p \ldots$ / $a, b, c \ldots$	$d \ldots D-1$	y^m

$*$: Spacetime/Tangent space (vielbein) Indices

We use the following convention for the signature of the metric:

$$\eta = \mathrm{diag}(-1, 1, \ldots, 1). \tag{A.1}$$

A curved metric is always written as g_{MN} and we define $g := \sqrt{|\det g_{MN}|}$. The (anti)symmetric versions of multi-index objects are defined as

$$T_{(M_1 \cdots M_p)} := \frac{1}{p!} \sum_\pi T_{M_{\pi(1)} \cdots M_{\pi(p)}}, \tag{A.2}$$

$$T_{[M_1 \cdots M_p]} := \frac{1}{p!} \sum_\pi \mathrm{sgn}(\pi) T_{M_{\pi(1)} \cdots M_{\pi(p)}}. \tag{A.3}$$

When proofs are given $\diamond$ **Proof:** indicates the beginning of the proof and $\Box$ signifies the end of the proof.

Quaternions

The quaternions are defined as the set

$$\mathbb{H} = \{q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \mid w, x, y, z \in \mathbb{R}\}, \tag{A.4}$$

where the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are defined by

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 = \mathbf{ijk}. \tag{A.5}$$

Note that conjugation $\bar{q}$ reverses the sign of all imaginary components: $\bar{q} = w - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$. A pure imaginary quaternion is one for which $w = 0$. Conjugation of a bilinear satisfies $\overline{pq} = \bar{q}\bar{p}$. Also note that any quaternion can be written as $q = a + bj$ with $a, b \in \mathbb{C}$.

Appendix B

Kähler Geometry and Calabi-Yau Manifolds

In section 2.1.2, we saw that having special holonomy could imply that the manifold had the so-called Kähler property. This is the case for manifolds with holonomy $U(n)$, $SU(n)$ and $Sp(n)$. Specifically, if the manifold has holonomy $SU(n)$, not only is it then Kähler, but it is then also Ricci-flat. Because chapter 2 is aimed at the introduction of G_2-manifolds, we have not gone into any details of the geometry of Kähler and Calabi-Yau manifold, but would like to devote this appendix to this for two reasons. Firstly, Kähler geometry shows beautifully that imposing a certain holonomy group ($U(n)$ in this case) can lead to a dramatic simplification in the geometry of the space and secondly, Calabi-Yau manifolds have played and still play a central role in string theory compactifications.

B.1 Kähler differential geometry

In this section we briefly list the defining qualities and properties of Kähler manifolds, but we omit the proofs. The corresponding proofs can be found in many places in the literature. See for example [37]. To be able to define a Kähler manifold, we first need to define

Definition B.1 (Hermitian metric) *Let g be the metric on a complex manifold X. The metric g is called* Hermitian *if in local coordinates, the components satisfy $g_{ij} = g_{\bar{\imath}\bar{\jmath}} = 0$.*

A hermitian metric can therefore always be expressed as

$$g = g_{i\bar{\jmath}} dz^i \otimes d\bar{z}^j + g_{\bar{\imath}j} d\bar{z}^i \otimes dz^j. \tag{B.1}$$

The following theorem obviously shows that this notion is very useful:

Theorem B.2 *A complex manifold always admits a Hermitian metric.*

With the definition of Hermiticity given, we can proceed to the definition of a Kähler manifold.

Definition B.3 (Kähler form) *Given a Hermitian metric g on X, we can locally define a form $J \in \Omega^{1,1}(X)$ by*

$$J = i g_{i\bar{\jmath}} dz^i \wedge d\bar{z}^j. \tag{B.2}$$

This (1,1)-form is called the Kähler form.

Definition B.4 (Kähler manifold) *A* Kähler manifold *is a Hermitian manifold whose Kähler form is closed, i.e. $dJ = 0$.*

Without showing it explicitly, we would like to mention that the Kähler form J we is actually a calibration and the corresponding calibrated submanifolds are the holomorphic curves in the Kähler manifold. Actually, all the powers of the Kähler form (with the multiplication given by the wedge product), $J^k/k!$ for all $1 \leq k \leq \dim_{\mathbb{C}}(X)$, are calibrations on X as well.

Perhaps surprisingly, the closure of the Kähler form results in a drastic simplification in the description of the geometry of X. First of all, it can be shown that any Kähler metric can locally (on a chart U_q) be written as

$$g_{i\bar{\jmath}} = \frac{\partial^2 K_q}{\partial z^i \partial \bar{z}^j}, \tag{B.3}$$

where $K_q \in \mathcal{F}(U_q)$ is known as the *Kähler potential.* Note that this potential can be deformed by so-called *Kähler transformations* without changing the metric:

$$K(z, \bar{z}) \to K(z, \bar{z}) + f(z) + g(\bar{z}) \tag{B.4}$$

with $f(z)$ and $g(\bar{z})$ strictly holomorphic and anti-holomorphic functions respectively.

With a little bit of algebra one can show that with the above form of the metric, numerous cancellations occur in the curvature tensors. Concretely, when written in complex coordinates, all the Christoffel symbols with mixed indices vanish. The only non-zero components are

$$\Gamma^i{}_{jk} = g^{i\bar{s}} \frac{\partial g_{k\bar{s}}}{\partial z^j}; \qquad \Gamma^{\bar{\imath}}{}_{\bar{\jmath}\bar{k}} = g^{\bar{\imath}s} \frac{\partial g_{\bar{k}s}}{\partial \bar{z}^{\bar{\jmath}}} \tag{B.5}$$

Because of this the curvature tensors simplify greatly as well. For the Riemann tensor the only non-vanishing components are

$$R_{i\bar{\jmath}k\bar{l}} = g_{i\bar{s}} \frac{\partial \Gamma^{\bar{s}}{}_{\bar{\jmath}\bar{l}}}{\partial z^k}, \tag{B.6}$$

and so for the Ricci tensor they become

$$\mathcal{R}_{\bar{\imath}j} = R^{\bar{k}}{}_{\bar{\imath}\bar{k}j} = -\frac{\partial \Gamma^{\bar{k}}{}_{\bar{\imath}\bar{k}}}{\partial z^j}. \tag{B.7}$$

Just as with a plain Riemannian manifold, the Riemann and Ricci tensors satisfy a number of symmetry conditions in the indices. We refer to the literature [61] for these properties.

It's not hard to come up with examples of Kähler manifolds. The simplest examples are: $\mathbb{C}^n$, the Riemann surface (orientable complex manifold of complex dimension 1) and $\mathbb{C}P^n$. An important singular space admitting a Kähler structure is the weighted projective space $\mathrm{W}\mathbb{C}P^n$. This space is like $\mathbb{C}P^n$, but with the identifications of the inhomogeneous coordinates weighted by different factors in different directions.

B.2 Calabi-Yau Manifolds

Like we stated in section 2.1 and above, a Calabi-Yau manifold is a complex, compact Kähler manifold of $\dim_{\mathbb{C}} X = n$ that has $SU(n)$ holonomy. As could already be seen in table 2.1 and the remarks accompanying it, an equivalent definition could be that a Calabi-Yau manifold is a complex, compact Kähler manifold which admits a Ricci-flat metric. Generally it is quite hard to determine whether X admits a Ricci-flat metric or not. Owing to Yau, who in 1977 proved a conjecture made by Calabi in 1957, we now have a much simpler condition. To understand it we first define

Definition B.5 (Chern class) *Define the* total Chern class *of X by*

$$c(X) = \det(\mathbb{1} + \mathcal{R}), \tag{B.8}$$

where $\mathcal{R}$ is the matrix-valued 2-form

$$\mathcal{R} = R^k{}_{li\bar{j}} dz^i \wedge dz^{\bar{j}}. \tag{B.9}$$

Then the k-th Chern class $c_k(X)$[1] is an element of $H^{2k}(X)$ defined from the expansion

$$c(X) = 1 + \sum_j c_j(X) = 1 + \mathrm{tr}\mathcal{R} + [\mathrm{tr}\mathcal{R} \wedge \mathcal{R} - 2(\mathrm{tr}\mathcal{R})^2] + \ldots. \tag{B.10}$$

Although they are constructed from the (local) curvature tensor the Chern classes are topological invariants. It's not hard to prove the following theorem.

Theorem B.6 *Let (M,g) be a Kähler manifold. If M admits a Ricci-flat metric h, then its first Chern class must vanish.*

Yau's theorem goes the other way: it states that if $c_1(X)$ vanishes, there exists a Ricci-flat metric on X.

[1] The notation $c_k(X)$ actually is a bit sloppy, as the Chern class should be defined using the tangent bundle T_X. We stick to the sloppy notation $c_k(X)$ instead of writing $c_k(T_X)$.

Theorem B.7 (Yau) *If X is a complex Kähler manifold with vanishing first Chern class and with Kähler form J, then there exists a unique Ricci-flat metric on X whose Kähler form J' is in the same cohomology class as J.*

So now we basically have another equivalent definition of a Calabi-Yau manifold. It is in general not hard to compute the first Chern class and, in particular, find examples with $c_1(X) = 0$. With this knowledge, people have been able to construct thousands of Calabi-Yau manifolds, while on none of these manifolds we know an explicit Ricci-flat metric. Often a lot can be learned about the physics without knowing the exact form of the metric.

It is important to note that for any Calabi-Yau manifold the Hodge numbers satisfy $h^{0,s} = h^{s,0} = 0$ for $1 < s < n$ and that $h^{n,0} = h^{0,n} = 1$. This together with the Poincaré duality leaves only a limited number of Hodge numbers to be determined.

In the next two subsections we will concretely describe Calabi-Yau manifolds in two and three complex dimensions, but before we do so we mention that on every CY_n manifold we can define a complex n-form θ, called the holomorphic volume form. The real part of this form (possibly even multiplied by a phase factor), $\mathrm{Re}(e^{i\gamma}\theta)$, is a calibration on X and the corresponding calibrated submanifolds are called *special Lagrangian submanifolds.*

B.2.1 The $K3$ Surface

We define a $K3$ manifold simply as a Calabi-Yau manifold of complex dimension two. We can also define a $K3$ *surface* in such a way that it can include singularities, but we refer to the literature [47] for that. Because of the following theorem (which we will not prove) things turn out to be quite simple for $K3$ surfaces as

Theorem B.8 *Any two $K3$-surfaces are diffeomorphic.*

So there is basically one topologically distinct $K3$ surface and if we have found one, we have found them all. The simplest example of the construction of a $K3$ surface is the so-called *Fermi quartic*

$$FQ := \{[z_0, \ldots, z_3] \in \mathbb{C}P^3 | z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}. \tag{B.11}$$

Another interesting example is the so-called *Kummer construction* of $K3$-manifolds. Let T^4 be a complex torus defined by $T^4 = \mathbb{C}^2/\Lambda$, where Λ is some lattice in $\mathbb{C}^2$. Then $T^4/\mathbb{Z}_2$ is a complex orbifold with 16 singularities, each locally isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$. Now let $\widetilde{T^4/\mathbb{Z}_2}$ be the smooth complex manifold resulting from blowing up all the singularities in the way described in section 2.2.3. Then $\widetilde{T^4/\mathbb{Z}_2}$ can be shown to admit a metric of $SU(2)$ holonomy, which means that it is a $K3$ manifold.

It can be shown that the Betti numbers of a $K3$ are $b^0 = b^4 = 1$, $b^1 = b^3 = 0$ and $b^2 = 22$. Its only Hodge number which was not yet determined at the end of the previous subsection is $h^{1,1} = 20$.

$K3$ surfaces are almost omnipresent in string theory dualities. Because we need for example the duality between M-theory on $K3$ and heterotic string theory on T^3 to be able to determine that non-Abelian gauge groups can arise from singularities, the $K3$ surface is very important to us.

B.2.2 CY_3 manifolds

Calabi-Yau 3-folds have received considerable attention over the years, because they provided the first spaces on which we could compactify string theory to yield quasi-realistic particle phenomenology. Specifically, as was first pointed out in [18], when we compactify heterotic string theory on a CY_3, we find an $\mathcal{N}=1$ supersymmetric gauge theory with gauge group E_8. This gauge group can be seen as a GUT-group to be broken in such a way to give reasonable particle phenomenology.

The most straight-forward construction of CY_3's is as an intersection of hypersurfaces. Consider a set of r polynomials of degree $k_1, \ldots, k_r$ in $\mathbb{C}P^{r+3}$. With techniques from algebraic geometry it can be shown that the total Chern class is

$$c = \frac{(1+J)^{r+4}}{(1+k_1 J)\cdots(1+k_r J)}, \tag{B.12}$$

where J is the Kähler form normalized in a certain way. If we expand the right hand side in a power series, then the ith Chern class is the term proportional to J^i (which is zero if $i > 3$). So c_1 vanishes if and only if

$$\sum_{i=1}^{r} k_i = r + 4. \tag{B.13}$$

Since a linear subspace of $\mathbb{C}P^n$ is just $\mathbb{C}P^{n-1}$, we are only interested in solutions with all $k_i \geq 2$. If this is the case, there are only five possibilities:

- ($r = 1$) a quintic equation in $\mathbb{C}P^4$
- ($r = 2$) a quartic and quadratic equation *or* a pair of cubic equations in $\mathbb{C}P^5$
- ($r = 3$) a cubic and two quadratic equations in $\mathbb{C}P^6$
- ($r = 4$) four quadratic equations in $\mathbb{C}P^7$

So the simplest example of a Calabi-Yau 3-fold is the quintic

$$Q := \{[z_0, \ldots, z_4] \in \mathbb{C}P^4 | z_0^5 + \cdots + z_4^5 = 0\}. \tag{B.14}$$

Note that the defining equation can be deformed without changing the first Chern class (and hence the possibility of finding a Ricci-flat metric of $SU(3)$ holonomy) by adding terms like

$$\sum_{i=0}^{4} z_i^5 + \lambda z_0 z_2^4 + \kappa z_0^2 z_1^3 + \cdots = 0. \tag{B.15}$$

All such manifolds are diffeomorphic, but they have different complex structures.

Just as in the case of $K3$, we can also construct Calabi-Yau's by means of singularity resolution. We can for example take the product of three complex tori $T_i^2 = \mathbb{C}/\Lambda$ defined and divide by a $\mathbb{Z}_3$ action of a transformation α_i: $T_1^2 \times T_2^2 \times T_3^2/\alpha_1\alpha_2\alpha_3 = T^6/\mathbb{Z}_3$. This space has 27 singularities. If we resolve these singularities in a certain way, we can construct a compact manifold $\widetilde{T^6/\mathbb{Z}_3}$ with vanishing first Chern class. A resolution which preserves the defining properties of a Calabi-Yau manifold is called a *crepant resolution*. For Calabi-Yau 3-orbifolds (and certain other classes of singularities), crepant resolutions always exist.

To only thing we still want to mention about the Calabi-Yau manifolds constructed in the way described above, is that they are simply connected and that spinors can sensibly be defined on them. In other words, they are spin manifolds. The fact that they are spin manifolds can most easily be seen by observing that the second Stiefel-Whitney class ω_2 vanishes. This is the case because ω_2 is the mod 2 reduction of c_1.

Appendix C

Harmonic Operators

C.1 Hodge-de Rham Operator

The *Hodge-de Rham operator* or *Laplacian*, $\triangle : \Omega^k(M) \to \Omega^k(M)$, is defined as

$$\triangle \equiv dd^\dagger + d^\dagger d = (d + d^\dagger)^2, \tag{C.1}$$

where d is the exterior derivative and $d^\dagger$ the adjoint exterior derivative. A k-form ω is called *harmonic* if $\triangle\omega = 0$. This is the case iff it is both closed and co-closed,

$$d\omega = d^\dagger\omega = 0. \tag{C.2}$$

The set of harmonic k-forms on a manifold M is denoted $Harm^k(M)$. The set $Harm^k(M)$ can be shown to be isomorphic to $H^k(M)$, the k-th cohomology group, on a compact orientable Riemannian manifold. This fact is known as *Hodge's theorem.* When acting 0-, 1-, 2- and 3-forms we get respectively

$$\begin{aligned}
\triangle\omega &= -\nabla^2\omega && \text{(C.3)}\\
\triangle\omega_m &= -\nabla^2\omega_m + R_m{}^n\omega_n && \text{(C.4)}\\
\triangle\omega_{mn} &= -\nabla^2\omega_{mn} - 2R_{mpnq}\omega^{pq} - 2R^p{}_{[m}\omega_{n]p} && \text{(C.5)}\\
\triangle\omega_{mnp} &= -\nabla^2\omega_{mnp} - 6R_{[mn}{}^{qr}\omega_{p]qr} + 3R_{[m}{}^r\omega_{np]r} && \text{(C.6)}
\end{aligned}$$

This Laplacian is basically the curved space version of the ordinary Euclidian Laplacian. Note that on 7-manifolds (which is of course a case we're very interested in), we don't need the expressions for the Laplacian acting on k-forms with $k > 3$. The reason is that we can use the Hodge star $*$, which commutes with the Laplacian, to map k-forms into (7-k)-forms. We finally state the important Hodge decomposition theorem.

Theorem C.1 *Every k-form $\omega \in \Omega^k(M)$ on a compact orientable Riemannian manifold M can be uniquely written as*

$$\omega = d\alpha + d^\dagger\beta + \omega_H, \tag{C.7}$$

with $\alpha \in \Omega^{k+1}(M)$, $\beta \in \Omega^{k-1}(M)$ *and* $\omega_H \in Harm^k(M)$.

We restrict ourselves to to these rather minimal facts. For more information, consult [61], [30] or [17].

C.2 Derivation of the Lichnerowicz operator

In this Appendix, we calculate the variation of the Ricci tensor to first order in the metric. Doing this will lead us to the definition of the so-called *Lichnerowicz operator*, which is a natural generalization of the Laplacian working on tensors on curved manifolds. After a few straight-forward steps, the variation of the Ricci tensor can be seen to be

$$\delta\mathcal{R}_{MN} = \frac{1}{2} g^{PQ} \left[\nabla_N \nabla_M \delta g_{PQ} - \nabla_P \nabla_N \delta g_{QM} - \nabla_P \nabla_M \delta g_{QN} + \nabla_Q \nabla_P \delta g_{MN}\right], \tag{C.8}$$

where the covariant derivatives are of course calculated using the background metric. See for example [73], page 290, for a derivation of this result. We now use the fact that the metric is covariantly constant, $\nabla_P g_{MN} = 0$, to pull g^{PQ} inside the derivatives:

$$2\delta\mathcal{R}_{MN} = \nabla_P \nabla^P \delta g_{MN} + \nabla_N \nabla_M \delta g^P{}_P - \nabla_P \nabla_N \delta g^P{}_M - \nabla_P \nabla_M \delta g^P{}_N \tag{C.9}$$

After this we need the commutator of two covariant derivatives to get the derivatives in the last two terms in the same order. In [73], page 140, the following formula for the commutator acting on a (1,1) tensor is derived:

$$[\nabla_M, \nabla_N]\, T^P{}_Q = T^S{}_Q R^P{}_{SMN} - T^P{}_S R^S{}_{QMN} \tag{C.10}$$

Using this gives us

$$\begin{aligned} 2\delta\mathcal{R}_{MN} &= \nabla^2 \delta g_{MN} + \nabla_N \nabla_M \delta g^P{}_P - \nabla_N \nabla_P \delta g^P{}_M - \nabla_M \nabla_P \delta g^P{}_N \\ &+ \delta g^S{}_M \mathring{R}^P{}_{SPN} - \delta g^P{}_S \mathring{R}^S{}_{MPN} \\ &+ \delta g^S{}_N \mathring{R}^P{}_{SPM} - \delta g^P{}_S \mathring{R}^S{}_{NPM}, \end{aligned} \tag{C.11}$$

where quantities with superscripted zeros are calculated using the vacuum metric.

Our equation of motion at this point has an invariance under infinitesimal coordinate transformations. Basically, if we make an infinitesimal tranformation $z^M \to z'^M = z^M - \xi^M(z)$, the metric changes to $\delta g'(z) = \delta g(z) + \mathcal{L}_\xi g(z)$, where $\mathcal{L}$ is the Lie-derivative. If we look at this another way, we have here a set of transformations which change the field δg, but correspond to the same physical situation; in other words, there is a gauge invariance. It turns out that we can always choose a so-called *harmonic* coordinate system, in which the metric has the property

$$\nabla^M \delta g_{MN} - \frac{1}{2} \nabla_N \delta g^M{}_M = 0, \tag{C.12}$$

which can alternatively be seen to be just a specific gauge for the metric. This gauge is known in the literature as the Lorentz gauge, Einstein gauge, Hilbert gauge, de Donder gauge or Fock gauge, depending on where you look. We have a hard time resisting the urge to add insult to injury and call this the harmonic gauge, but we opt instead to conform to the Lorentz gauge. It has to be noted that there's still a residual gauge freedom: we can change the coordinates by a harmonic function and not change the physics.

In the Lorentz gauge, the variation of the Ricci tensor simplifies greatly. Using (C.12), subsequently (C.10), the algebraic properties of the Riemann tensor,

$$R_{MNPQ} = R_{PQMN} = -R_{NMPQ} = -R_{MNQP}, \tag{C.13}$$

and the second Bianci identity,

$$R_{PQMN} + R_{PNQM} + R_{PMNQ} = 0, \tag{C.14}$$

it's a straight-forward calculation to obtain the variation of the Ricci tensor in its final form

$$2\delta\mathcal{R}_{MN} = \nabla^2 \delta g_{MN} - 2\mathring{R}_{MPNQ}\delta g^{PQ} + \mathring{\mathcal{R}}^{Q}{}_{(M}\delta g_{N)Q} \equiv \triangle_L \delta g_{MN}. \tag{C.15}$$

This last equation gives the definition of the Lichnerowicz operator $\triangle_L$. Notice the striking similarity with the expression of the Laplacian acting on a 2-form (C.5).

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