Calculation of threepoint functions of chiral primaries in a symmetric orbifold

The uses of Hilbert schemes

Master's Thesis Theoretical Physics

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Abstract

The AdS/CFT conjecture in string theory suggests that quantum gravity in Anti de Sitter space, as a low energy limit of some string theory configuration, is equivalent to a conformal field theory on the boundary. One of the simplest examples is the D1-D5 system that has a super-gravity limit in $AdS3$. It is argued that the boundary theory has a point in its moduli space that is a 2 dimensional CFT with as target space the symmetric orbifold of a 4 dimensional hyperkähler space. To compare both theories one has to study objects that do not depend on the coupling constants. On the CFT side these objects can be described from a conformal field theory point of view or by dimensional reduction to a certain cohomology of the target space. In this thesis the link between these two methods is studied. An explicit description of the cohomology ring is given and the benefits of each description is discussed.

Contents

Chapter 1

Introduction

The interplay between theoretical physics and mathematics has allways been fascinating. One could say that most areas in mathematics have started out of need for a description of physical phenomena, often diverging away from physics only much later making contact again. A contact from which again whole new branches of mathematics have emerged. Classical examples are development of Riemannian geometry in the nineteenth century, swiftly followed by the discovery of general relativity, which again has strongly influenced again the growth of differentiable geometry.

This interplay has been particular strong in describing fermionic behaviour, where one of the most important mathematical discoveries of last century, the Atiyah-Singer index theorem, has been strongly influenced by the discovery by Dirac of the equation that bears his name, describing the motion of spin 1/2 particles.

The discovery of supersymmetry in the early 1970s made a lot of more esoteric mathematics suddenly common good among theoretical physicists. Among the surprising connections one can count the beautiful interplay between complex structures and supersymmetric models. The search for a theory unifying gravity and quantum field theory, both areas in physics that have been highly successfull on their own, resulted in seeing string theory as the most likely candidate we have today for this unification. String theory itself was created first as a model for just strong interactions, but discarded because of certain inconsistencies.

String theory as we know it today again needed a lot of new mathematical concepts, partly not well known to the physical community, among them algebraic geometric, topological and representation theoretical theories. On the other hand many new insights in areas as algebraic topology and algebraic geometry have emerged often first based on "physical intuition" that only later have been made exact. Well known examples are new stringy types of cohomology, like orbifold cohomology and quantum cohomology, Gromov-Witten invariants, counting curves on a manifold, and mirror symmetry relating two seemingly different manifolds to each other.

In this thesis we will touch upon some of these new concepts and show how some esoteric mathematical objects that were discovered by Alexander Grothendieck, one of the greatest mathematicians from the 20th century, suddenly are seen to be of practical use. This is done in the context of certain

calculations on some singular spaces called orbifolds. The reason behind these calculations is the conjectured duality between two previously unconnected theories. One a theory describing gravity. The other a quantum field theory on a flat space time totally unaware of any gravitational influence. This conjecture, based on observations in string theory by Maldacena at the end of last century, has made a big impact on theoretical research done in high energy physics of the last decade and hopefully will help to unravel the puzzle of how to build this long searched for unifying theory.

The theory on the boundary is in the case that we will consider, a certain conformal field theory (CFT) for which a relevant set of correlators have been calculated in a paper by Mathur and Lunin [1]. They managed to get analytic expressions for a certain asymptotic region.

We will start doing a prcise analysis of their method. We will then analyse the geometric meaning of these correlators from a topological point of view and then try to formulate them in an algebraic topological language. Using some recent results from algebraic geometry we are then able to give an algorithm that makes it in principle possible to calculate an important part of these correlators exactly.

We will do a comparison of both methods and will try to match the results.

Chapter 2

The setting

One of the most remarkable, conjectured, dualities in string theory is the Maldacena conjecture relating string theory in an AdS background to a conformal field theory on the boundary. Several tests of this conjecture have been done, the biggest obstacle being that the duality is a strong - weak coupling duality, the weak string coupling, resulting in the supergravity limit, being dual to a strongly coupled CFT.

Although the precise formulation of the conjecture is outside of the scope of this thesis, we will briefly touch upon some of the aspects, the main reference being [2]. String theory contains, next to closed and open strings also extended objects that can be seen as hypersurfaces on which open strings can end. They follow from imposing Dirichlet boundary conditions on some of the spatial open string dimensions and so these hypersurfaces go by the name of D-branes. The open strings living on the branes make them into dynamical objects. The low energy description can be given as a decoupled system of closed strings in the bulk, corresponding to supergravity in a Minkowski spacetime and a super Yang-Mills theory describing the open string modes on the branes. One can also look at the effective action of the dynamics of the D-brane excitations as they interact with the closed strings. Far away from the branes this corresponds again to the free supergravity solution. Close to the branes (one speaks of the near horizon limit) the geometry is that of Anti de Sitter space and the theory is described by a string theory in AdS.

Comparing both descriptions lead to the conjecture that string theory on AdS should be equivalent to $\mathcal{N} = 4$ SYM on the boundary of this AdS. One special case, relevant to this thesis, is a system of Q_5 D5 branes wrapped around a 4-dimensional manifold M that is taken to be $K3$ or T^4 (see the appendix C for definitions) and Q_1 D1 branes stacked upon the D5 branes along the non compactified direction. When we look at the near horizon-limit, the geometry is that of $AdS3 \times S^3$. The boundary SYM theory has a point in its parameter space that is given by a 2 dimensional CFT with as target space the symmetric orbifold $S^N M$. The symmetric orbifold is not the point dual to the low energy supergravity theory though and this has to be taken into account when comparing both theories. The number of copies of M in $S^N M$ is $Q_1 Q_5 + 1$ for $M = K3$. It is one less for $T4$.

To get a better handle on the theories on both side of the conjecture, it is helpfull to consider a limit in which N goes to ∞ while keeping the combination

g^2N fixed. Here g is the Yang-Mills coupling constant. This limit is called the 't Hooft limit. In the CFT this corresponds to a $1/N$ expansion, in lowest order restricting to planar Feynman diagrams. On the supergravity side this limit corresponds to a more tractable expansion in the string coupling constant.

One of the unexpected results [3] is that the three point functions of chiral operators at small coupling in 4 dimensional $\mathcal{N} = 4$ SYM (supersymmetric Yang Mills) ,in the large N limit, are equal to the the three point functions in the weak coupling supergravity limit in the dual $AdS_5 \times S^5$. Combining this with the AdS/CFT conjecture this would mean that the 3 points functions are "protected" when going from strong to weak coupling pointing to a kind of nonrenormalisation. This is a concept common in supersymmetric theories, meaning here that these 3 point functions do not receive any corrections from higher loops.

The question one could pose is if this protection of 3 point functions also occurs in the $D1 - D5$ system where the supergravity description is in $AdS_3 \times S^3 \times M$ with M a 4 dimensional manifold. It is believed that a sigma model with as target space the symmetric orbifold M^N/S_N is contained in the moduli space of the $D1 - D5$ system.

Chapter 3

Supersymmetry

In this chapter we will, after a short motivating historical introduction, define supersymmetry algebra's, first in 4 dimensions, as an extension of the Poincaré algebra. We will then review 2-dimensional conformal symmetry and extend it to $\mathcal{N} = 4$ superconformal symmetry. The appearance of some special states that are annihilated by part of the supersymmetry generators is treated concisely as preparation for the theories described in the main body of the thesis.

3.1 Extensions of Poincaré Symmetry

The discovery of an ever increasing amount of elementary particles in the second half of last century made it clear that some symmetry principles where needed to group the different particles, just like the Poincaré symmetry "connects" the 4 components of the vector potential into one spin 1 photon. A very attractive idea was the possibility to combine the extra continuous symmetries with the Poincaré group in a non trivial way. The first attempt actually dates already from 1937 [4] when Wigner tried to combine SU(2) isospin symmetry with the Poincaré group resulting in an SU(4) symmetry for the combined spin and isospin. Later SU(6) models looked promising, but all nontrivial extensions of the Poincaré group lead to models that lacked realistic scattering. It was finally the Coleman-Mandula theorem [5] that proved that only trivial extensions are possible, with the notable exception of conformal symmetry for massless particles. The proof of this theorem hinges mostly on 2 basic concepts : the almost everwhere analytic behaviour of the S-matrix and the discreteness of the spectrum of particles (the finiteness axiom). If some generators, that form the extension, would not commute with the Poincaré generators then it would be possible to continuously transform from one one particle state to another (with another mass) contradicting the finiteness axiom.

3.2 Introducing SUSY

With the discovery of supersymmetric theories, first in string theory [6] and later in 4 dimensions [7] it became clear that by enlarging the allowed symmetries with anticommuting generators a non trivial extension of the Poincaré

algebra is possible. The theorem by Haag, Łopuszański and Sohnius [8] is the supersymmetric version of the Coleman-Mandula theorem, classifying all possible extensions of the Poincaré group but now including anticommuting generators resulting in supersymmetric algebra's for massive theories and superconformal algebra's in the massless case. The proof is based on Coleman-Mandula and is for the rest purely algebraic. A sketchy overview of the constructions in 4 dimensions follows.

Lets start with the Poincaré group as the matrix group that leaves the Minkowski metric, $\eta_{\mu\nu} = -+++$, invariant (the Lorentz group) combined with 4 dimensional translations . Expanding around the group unit we readily find the Poincaré algebra:

$$\begin{aligned}
&[P_\mu, P_\nu] = 0 \\
&[P_\mu, M_{\nu\rho}] = i\left(\eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu\right) \\
&[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho}\right)
\end{aligned}$$

The generator of translations, P_μ, is the momentum 4-vector. Since the concept of a particle makes sense only if it is stable over (at least some) time, it is natural to split the antisymmetric tensor $M_{\mu\nu}$ in a conserved (commuting with the generator of time translations P_0) angular momentum $J_i := 1/2\varepsilon_{ijk}M^{jk}$ and a boost vector $K_i := M_{0i}$. Working out the commutation relations for the complex combinations $A_i^{\pm} := J_i \pm iK_i$, we find 2 decoupled copies of the $\mathfrak{su}(2)$ algebra. In this way we can classify the representations by giving the "spin"of each $SU(2)$ representation. A scalar is then noted as $(0,0)$, a Dirac fermion as $(1/2,0)\oplus(0,1/2)$ and a vector as $(1/2,1/2)$. It is custom to write the second spin index with a dot, indicating it transforms as the hermitian conjugate representation, so for example spin $1/2$ fermions are written as Q_α or $\bar{Q}_{\dot\alpha}$. From the definition of $A_i^{\pm}$ follows that the constructed finite dimensional representations are not unitary representations of the Poincaré group, which is not a big surprise since one can show that a non compact lie algebra has no non-trivial finite dimensional hermitian representations.

From the Coleman-Mandula theorem we know that the only bosonic operators that are *not* in the centre of extensions of the Poincaré algebra are the vector P_μ and the $(1,0)\oplus(0,1)$ tensor $M_{\mu\nu}$, we see that the anti-commutator of two fermionic generators can add up to at most spin 1, showing that these generators have spin $1/2$. After some lengthy but rather straightforward algebra, the most general extension for a *massive* theory can be shown to be [9]:

$$[Q_{\alpha i}, M_{\mu\nu}] = \frac{1}{2}\left(\sigma_{\mu\nu}\right)_\alpha{}^\beta Q_{\beta i} \tag{3.1}$$

$$\left[\bar{Q}^i{}_{\dot\alpha}, M_{\mu\nu}\right] = -\frac{1}{2}\bar{Q}^i{}_{\dot\beta}\left(\bar{\sigma}_{\mu\nu}\right)^{\dot\beta}{}_{\dot\alpha} \tag{3.2}$$

$$\{Q_{\alpha i}, \bar{Q}^j{}_{\dot\beta}\} = 2\delta_i^j\left(\sigma^\mu\right)_{\alpha\dot\beta}P_\mu \tag{3.3}$$

$$[Q_{\alpha i}, B_r] = \left(b_r\right)_i{}^j Q_{\alpha j} \tag{3.4}$$

$$\left[\bar{Q}^i{}_{\dot\alpha}, B_r\right] = -\bar{Q}^j{}_{\dot\alpha}\left(b_r\right)_j{}^i \tag{3.5}$$

$$[B_r, B_s] = ic_{rs}{}^t B_t \tag{3.6}$$

$$[Q_{\alpha i}, P_\mu] = \left[\bar{Q}^i{}_{\dot\alpha}, P_\mu\right] = 0 \tag{3.7}$$

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\varepsilon_{\alpha\beta}Z_{ij} \tag{3.8}$$

$$\{\bar{Q}^i{}_{\dot{\alpha}}, \bar{Q}^j{}_{\dot{\beta}}\} = 2\varepsilon_{\dot{\alpha}\dot{\beta}} Z^{ij} \tag{3.9}$$

All other commutators being zero. The Q's are the spin 1/2 supercharges; the B's are internal symmetries; Z_{ij} are central charges, commuting with all generators and related to the B generators through $Z_{ij} = a^r_{ij} B_r$ with the a^r antisymmetric matrices ; $Z^{ij} := (Z_{ij})^\dagger$; ε the antisymmetric 2×2 tensor (for conventions see the end of the thesis), with which we can lower and raise spinor indices; $\sigma^{\mu\nu}$ are spin matrices constructed from the Pauli matrices as follows :

$$\sigma^\mu = \left(\mathbb{1}.\sigma^i\right) \quad , \quad \bar{\sigma}^\mu = \left(-\mathbb{1}.\sigma^i\right)$$
$$\sigma^{\mu\nu} = \frac{1}{2} i \left(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu\right)$$

One immediate result is, since the supercharges commute with P^2, that the masses of particles in a supermultiplet are the same, so that supersymmetry must be broken to yield a realistic model. The roman index on the supercharges runs over the supersymmetry generations, $\mathcal{N}$. When $\mathcal{N} = 1$ one speaks of unextended supersymmetry.

If there are no central charges the algebra is invariant under a $U(\mathcal{N})$ symmetry transforming the $\mathcal{N}$ supercharges into eachother. This symmetry goes under the name of **R-symmetry**. R-symmetry

Since the central charges commute with the supercharges and with P^μ it is possible to choose particle representations in a supermultiplet that are also eigenstates of the central charges. We can now rotate in the internal supercharge space . Let $Z = HU$ be the polar decomposition of Z, U unitary and H positive hermitian. Define the supercharges

$$T_{\alpha i} := Q_{\alpha i} - U_{ij} \bar{Q}^j_{\dot{\alpha}}$$

Now forming the anticommutator and sandwitching between a one particle state, $|m\rangle$ of mass m, $|m\rangle$, we find

$$\begin{aligned}
0 \leq & \sum_{\alpha,i} 2 \parallel T_{\alpha i}|m\rangle \parallel^2 = \langle m|\{T_{\alpha i}, \bar{T}^{\dot{\alpha} i}\}|m\rangle \\
= & \langle m|\{Q_{\alpha i}, \bar{Q}^{\dot{\alpha} i}\}|m\rangle - U_{ij}\langle m|\{\bar{Q}^j_{\dot{\alpha}}, \bar{Q}^{\dot{\alpha} i}\}|m\rangle \\
& - \left(U^\dagger\right)^{ji} \langle m|\{Q_{\alpha i}, Q^\alpha_j\}|m\rangle + U_{ij} \left(U^\dagger\right)^{ki} \langle m|\{\bar{Q}^j_{\dot{\alpha}}, Q^\alpha_k\}|m\rangle \\
= & 2\mathcal{N} \left(\sigma^\mu\right)^{\dot{\alpha}}_\alpha \langle m|P_\mu|m\rangle - 2\delta^{\dot{\alpha}}_{\dot{\alpha}} \langle m|U_{ij} Z^{ij}|m\rangle \\
& - 2\delta^\alpha_\alpha \langle m| \left(U^\dagger\right)^{ji} Z_{ij}|m\rangle + 2U_{ij} \left(U^\dagger\right)^{ki} \delta^k_j \left(\sigma^\mu\right)^{\dot{\alpha}}_\alpha \langle m|P_\mu|m\rangle \\
= & 8\mathcal{N} p_0 - 8\langle m|\mathrm{Tr}H|m\rangle
\end{aligned} \tag{3.10}$$

Where the anti-commutation relations (3.3) and (3.8) have been used. Since H is a positive operator we get a lower bound on the mass of the one particle state:

$$m \geq \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} h_i$$

Where the sum is over the eigenvalues of H. When the mass is the lowerbound the state is called a **BPS state** , a terminology that has its origin in monopole solutions in gauge theories. From the construction we see that a state in a BPS state

BPS multiplet is characterised by its annihilation by the $T_{\alpha i}$. The multiplet corresponding to BPS states is then also called a short multiplet. It is now custom to call all multiplets that get annihilated by a supercharge BPS states.

3.3 Adding conformal symmetry

The Coleman Mandula theorem for massless particles allows for a nontrivial extension by the generators of the conformal group.

As a side note, this is a somewhat misleading statement since even in 4 dimensional Minkowski space to give a rigorous definition of the conformal group one has to define it on the conformal compactification of $\mathbb{R}^{3,1}$ In 2 dimensions the confusion is much bigger, as we will see. First though we will continue with the treatment of the $\mathbb{R}^{3,1}$ case, because the main features new to superconformal algebras are easy to to calculate. We define a conformal transformation as an orientation preserving diffeomorphism $\phi : U \supset \mathbb{R}^{3,1} \to O \supset \mathbb{R}^{3,1}$ such that the pull back of the metric equals the original metric up to a factor (the conformal factor):

$$(\phi^* g)\,(p)(X,Y) = \lambda(p)^2 g(p)(X,Y) \tag{3.11}$$

where $X, Y \in T_p\mathbb{R}^{3,1}$ and the pull back is defined through the push forward of the vectors in the tangent space:

$$(\phi^* g)\,(p)(X,Y) := g(\phi(p))(\phi_* X, \phi_* Y) = g(\phi(p))(\phi' \circ X, \phi' \circ Y) \tag{3.12}$$

Here the prime denotes the differential of the map. This is easy to define through local coordinates as follows: suppose $\{x^i\}$ are some local coordinates at a point p in $\mathbb{R}^{3,1}$ and $\{\phi^i\}$ local coordinates around the image of p under ϕ. A tangent vector at p has in local coordinates the form: $X = X^\mu \partial/\partial x^\mu$ under the diffeomorphism it gets mapped to a vector at $\phi(p)$. In local coordinates we have for the image:

$$\frac{\partial \phi^\nu}{\partial x^\mu} X^\mu \frac{\partial}{\partial \phi^\nu}$$

In physics literature, in case only one chart is needed, it is custom to express everything in a local coordinate basis. Then we would say that the vector X^μ gets mapped to the vector $\dfrac{\partial \phi^\nu}{\partial x^\mu} X^\mu$.

The conformal transformations as defined do not form a group since even composition is not necessarily defined. We will first classify them locally. Locally we can define the conformal transformations through one parameter groups resulting in the extra generators (next to the Poincaré generators that, as isometry generators, are clearly also conformal) the dilatation generator, D and the generators of the special conformal transformations, K_μ. In local coordinates they can be written as:

$$D = -i x^\mu \frac{\partial}{\partial x^\mu}$$
$$K_\mu = -2i x_\mu x^\nu \frac{\partial}{\partial x^\nu} + x^\nu x_\nu \frac{\partial}{\partial x^\mu}$$

Exponentiating the special conformal generator generates the one parameter group of transformations:

$$\phi_b^\mu(x,t) = \frac{x^{\mu - b^\mu t}}{1 - 2b^\nu x_\nu t + b^\nu b_\nu x^\rho x_\rho t^2}$$

with b some fixed point. From this expression it is clear that for *any* nonzero value of t the domain of the transformation is a subset of $\mathbb{R}^{3,1}$. Thats why one has to conformally compactify $\mathbb{R}^{3,1}$, by embedding it in 5 dimensional projective space, to properly define the conformal group. Since we won't need global properties for the discussion that follows we refer for the details to [10]. The full conformal algebra is now:

$$\begin{aligned}
&[P_\mu, P_\nu] = [K_\mu, K_\nu] = 0 \\
&[P_\mu, M_{\nu\rho}] = i\left(\eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu\right) \\
&[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho}\right) \\
&[D, P_\mu] = iP_\mu \\
&[D, K_\mu] = -iK_\mu \\
&[K_\mu, P_\nu] = 2i\left(\eta_{\mu\nu}D - M_{\mu\nu}\right) \\
&[K_\mu, M_{\nu\rho}] = i\left(\eta_{\mu\nu}K_\rho - \eta_{\mu\rho}K_\nu\right)
\end{aligned}$$

We can combine the D, K and P into 2 extra indices of the "rotation" generator showing that the conformal group is isomorphic to $SO(4,2)$. When classifying the particle content we then should search for projective representations or, equivalently, representations of its (4 times) cover.

The conformal algebra can again be extended with supersymmetry charges analogous to the Poincaré algebra. leading to the general 4 dimensional superconformal algebra. An important tool in finding the maximal extension is looking at the commutator of D with the generators, the "dimension", Δ of the generator A is defined by $[D, A] = i\Delta A$, so that M, P, K and D have dimensionality of $0, 1, -1$ and 0. Combining the supersymmetry algebra with the conformal algebra it is not hard to show that the dimensionality of the supercharges is $\pm 1/2$. Central charges do not appear, since they would have dimensionality 0, commuting with the D but 1 when using (3.8) and the Jacobi identity. An important feature follows from the Jacobi identity for D, k and Q:

$$[D, [K^\mu, Q_{i\alpha}]] = [[D, K^\mu], Q_{i\alpha}] + [K^\mu, [D, Q_{i\alpha}]] = -\frac{i}{2}[K^\mu, Q_{i\alpha}]$$

so apparently when one includes supercharges with dimension $1/2$, charges with dimension $-1/2$ should also appear, lets call them Q and $Q^{(1)}$. Finally the anticommutator of the Q's can be calculated combining the (anti)commutation relations found :

$$\{Q_{i\alpha}, Q^{(1)}_{j\beta}\} = \varepsilon_{\alpha\beta}\delta_{i,j}D - \delta_{i,j}(\sigma_{\mu\nu})_{\alpha\beta}M^{\mu\nu} + i\varepsilon_{\alpha\beta}O_{ij}$$

Where O_{ij} is a hermitian (and traceless for $\mathcal{N} = 4$) matrix. The essential novelty is that the generalised R-symmetry (internal symmetry) is now interwoven with the spacetime symmetries. Comparing the anticommutator with 3.8 we

see that the BPS condition will now, because of the appearance of the dilatation and angular momentum generator, result in BPS states with a conformal dimension that is quantised when angular momentum is. This means also that the dimension of these BPS states will be constant, over a range of the coupling constant for which the correlators are analytic as a function of the coupling constant. This can be used to relate properties of BPS states in the (unknown) strong coupling region to those in the weak coupling region, where perturbative approximations can be used.

The supersymmetric extensions to other signatures of the metric and dimensions larger than 2 have been classified by Nahm [11].

3.4 Conformal symmetry in 2 dimensions

In 2 dimensions it is dubious at least to speak of a conformal group in the way it was discussed in the last sections. As is immediately clear from the conformal Killing equation in 2 dimensions. Writing out (3.11) in local coordinates we find:

$$(\phi^*\eta)_{\mu\nu}X^\mu Y^\nu = \eta_{ij}\partial_\mu\phi^i\partial_\nu\phi^j X^\mu Y^\nu = \lambda^2(x,y)\eta_{\mu\nu}X^\mu Y^\nu$$

For $\mathbb{R}^{2,0}$ the metric is $\eta_{\mu\nu} = \delta_{\mu,\nu}$ and by interpreting x,y (resp. ϕ^x, ϕ^y) as the real, imaginary components of complex numbers it follows that the conformal transformations correspond to the locally invertible holomorphic or antiholomorphic functions. It is impossible though to compactify $\mathbb{R}^{2,0}$ in a way to make these conformal transformations into a group like we did in $\mathbb{R}^{3,1}$. The best we can get is to use at most linear conformal factors resulting in the finite dimensional group of Moebius transformations. So defined the conformal group is isomorphic to $SO(3,1)$, but this captures only a small part of the symmetry. In the 2 dimensional Minkowski space, $\mathbb{R}^{1,1}$, it is possible though to have a infinite dimensional conformal group, again since we will define the theory only locally we can do without and refer to the literature for details [10].

The 2 dimensional analogue of the local conformal algebra can be taken to be 2 copies of the Witt algebra as generators of the holomorphic and antiholomorphic functions.The quantisation procedures shifts our interest into projective representations of the algebra, which translates back into representations of central extensions of the Witt algebra, the Virasoro algebra (restricting ourselves to the holomorphic part)

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

Where the L_n represent the conformal transformation generators $-z^{n+1}\partial_z$ and thus are the modes of the generator of conformal transformations:

$$T(z) = \sum_{n\in Z} z^{-n-2}L_n$$

Alternatively the Virasoro algebra can be expressed through the symmetry generating currents (energy mommentum density) OPE as:

$$T(z)T(w) \sim \frac{\partial T(w)}{z-w} + \frac{2T(w)}{(z-w)^2} + \frac{c}{2(z-w)^4}$$

The central charge of the theory depends on the field content. In case the central charge is nonzero we don't have conformal invariance anymore in our theory on the quantum level and any conformal scaling of the metric needs to be compensated for, as we do in the next chapter. In case of worldsheet conformal invariance in string theory, where conformal invariance is a gauge invariance and the quantised theory does not allow an anomaly, this restricts the allowed theories.

The basic states, $|\phi\rangle$ in the theory are eigenstates of L_0 for which $L_n|\phi\rangle = 0$ for $n > 0$. These fields are called **primary**, the eigenvalue under L_0 is their **conformal weight**.

primary

conformal weight

The Virasoro algebra contains a finite dimensional subalgebra generated by L_{-1}, L_0 and L_1 corresponding to the group of global conformal transformations. In contrast to higher dimensional conformal theories, there are many possible extensions of the Virasoro algebra possible. Many of those have a nonlinear algebra and are not easy to interpret geometrically. The $\mathcal{N} = 1$ theory is the superstring theory symmetry (heterotic if it is the symmetry of just the left or right movers).

3.5 $\mathcal{N} = 4$ Superconformal Algebra

One of the possible extensions is the $(\mathcal{N}, \bar{\mathcal{N}}) = (4, 4)$ algebra that is expected to be a symmetry of the conformal field theory on the boundary of AdS^3 where a global $SU(2) \times SU(2)$ symmetry corresponds to the $SO(4)$ symmetry of the S^3 in the $AdS_3 \times S^3 \times M \quad D1 - D5$ system.

The $\mathcal{N} = 4$ superconformal current algebra extension is [12] (holomorphic part only)

$$i, j, k \in \{1, 2, 3\} \quad a, b \in \{1, 2\}$$

$$J^i(z)J^j(w) \sim \frac{i\varepsilon^{ijk}J^k(w)}{z-w} + \frac{c\delta^{i,j}}{12(z-w)^2} \tag{3.13}$$

$$T(z)J^i(w) \sim \frac{\partial J^i(w)}{z-w} + \frac{J^i}{(z-w)^2} \tag{3.14}$$

$$G^a(z)\tilde{G}_b(w) \sim \frac{2T(w)\delta^a_b}{z-w} - \frac{2(\sigma^i)^a{}_b\partial J^i(w)}{z-w} - \frac{4(\sigma^i)^a{}_b J^i(w)}{(z-w)^2} + \frac{2c\delta^a_b}{2(z-w)^3} \tag{3.15}$$

$$T(z)G^a(w) \sim \frac{\partial G^a(w)}{z-w} + \frac{3G^a(w)}{2(z-w)^2} \tag{3.16}$$

$$T(z)\tilde{G}^a(w) \sim \frac{\partial \tilde{G}^a(w)}{z-w} + \frac{3\tilde{G}^a(w)}{2(z-w)^2} \tag{3.17}$$

$$J^i(z)G^a(w) \sim -\frac{(\sigma^i)^a{}_b G^b(w)}{2(z-w)} \tag{3.18}$$

$$J^i(z)\tilde{G}_a(w) \sim \frac{\tilde{G}_b(w)(\sigma^i)^b{}_a}{2(z-w)} \tag{3.19}$$

J is the $SU(2)$ R-symmetry current , G_a and $\tilde{G}^a$ are 4 supersymmetry currents corresponding to a complex $SU(2)$ doublet, with $\tilde{G}^a = (G_a)^\dagger$, an other $SU(2)$ coming from the antiholomorphic version.

The mode expansion depends on the boundary conditions we impose upon the currents. Restricting ourselves to periodic bosonic currents we can choose as boundary conditions for the super currents:

$$G^1(e^{2\pi i}z) = \exp[-2\pi i\eta]G^1(z) \qquad G^2(e^{2\pi i}z) = \exp[2\pi i\eta]G^2(z) \tag{3.20}$$

$$\tilde{G}^1(e^{2\pi i}z) = \exp[2\pi i\eta]\tilde{G}^1(z) \qquad \tilde{G}^2(e^{2\pi i}z) = \exp[-2\pi i\eta]\tilde{G}^2(z) \tag{3.21}$$

$$J^{\pm}(e^{2\pi i}z) = \exp[\mp 2\pi i\eta]J^{\pm}(z) \tag{3.22}$$

Where $J^{\pm}(z) := J^x(z) \pm iJ^y(z)$. These boundary conditions correspond to mode expansions of the form:

$$G^1(z) = \sum_{n\in\mathbb{Z}} G^1_{n+\frac{\eta}{2}+\frac{1}{2}} z^{-n-\frac{\eta}{2}-\frac{3}{2}} \qquad G^2(z) = \sum_{n\in\mathbb{Z}} G^2_{n+\frac{\eta}{2}+\frac{1}{2}} z^{-n+\frac{\eta}{2}-\frac{3}{2}} \tag{3.23}$$

For $\eta = 0$ we get the Neveu-Schwarz algebra for the modes($n, m \in \mathbb{Z}; r, s \in \mathbb{Z} + 1/2$):

$$\begin{aligned}
&[J^i_m, J^j_n] = i\varepsilon^{ijk}J^k_{m+n} + \frac{c}{12}m\delta_{i,j}\delta_{m+n,0}\\
&[L_m, J^i_n] = -nJ^i_{n+m}\\
&\{G^a_r, \tilde{G}_{b,s}\} = 2\delta^a_b L_{r+s} - 2(r-s)(\sigma^i)^a{}_b J^i_{r+s} + \frac{c(4r^2-1)}{12}\delta^a_b\delta_{r+s,0}\\
&\{G^a_r, G^b_s\} = \{\tilde{G}_{a,r}, \tilde{G}_{b,s}\} = 0\\
&[L_m, G^a_r] = \left(\frac{m}{2} - r\right) G^a_{m+r}\\
&[L_m, \tilde{G}_{a,r}] = \left(\frac{m}{2} - r\right) \tilde{G}_{a,m+r}\\
&[J^i_m, G^a_r] = -\frac{1}{2}(\sigma^i)^a{}_b G^b_{m+r}\\
&[J^i_m, \tilde{G}_{a,r}] = \frac{1}{2}\tilde{G}_{b,m+r}(\sigma^i)^b{}_a
\end{aligned}$$

The algebras for different values of η are isomorphic . This is also called the spectral flow connecting the Ramond algebra, for $\eta = 1$, with the Neveu-Schwarz algebra. The modes for the supercurrents are related through:

$$\tilde{G}_{a,r} = (G^a_{-r})^\dagger \tag{3.24}$$

It is interesting to note that there is a finite subalgebra consisting of just the $L_0, L_{\pm 1}$ and J^i_0, combined with $\pm 1/2$ modes of the supercurrent and that this algebra does not contain a central charge, so that it is a symmetry for the quantised field theory. This is the lie algebra that generates (together with the antiholomorphic part), after conformal compactification, the group of global $(\mathcal{N}, \bar{\mathcal{N}}) = (4, 4)$ superconformal symmetry with the global $SU(2) \times SU(2)$ R-symmetry mentioned above.

The smallest possible representation is for $c = 6$, which can be realised by 4 bosons and 4 fermions. This is related to the lack of existence of a highest weight state for smaller c.

Superconformal primary fields are fields that get annihilated by the positive modes, not only of the energy momentum density as in bosonic conformal field theory, but also of the supercurrent and the R-symmetry current. The

states that get annihilated by $\tilde{G}_{1,-1/2}$ or $G^2_{-1/2}$ are called left **chiral**, those that get annihilated by $\tilde{G}_{2,-1/2}$ or $G^1_{-1/2}$ left **anti-chiral** and the same for the anti-holomorphic sector, but then as "right" variants. From the algebra we immediately find the BPS conditions : chiral anti-chiral

$$0 \leq \{G^a_{1/2}, \tilde{G}_{a,-1/2}\}|\phi\rangle = (2L_0 - 2(2\delta_{a,1} - 1)J^3_0)|\phi\rangle \tag{3.25}$$

$$0 \leq \{G^a_{-1/2}, \tilde{G}_{a,1/2}\}|\phi\rangle = (2L_0 + 2(2\delta_{a,1} - 1)J^3_0)|\phi\rangle \tag{3.26}$$

Apparently the primary states that are (anti)chiral, the so called **chiral primaries**, fulfil the lower bound, identifying them as BPS states with R-symmetry charge equal to their conformal weight in the chiral case and minus their conformal weight in the anti-chiral case. On the other hand we find for any BPS state that it is a chiral primary by using the conjugacy relation (3.24) : chiral primaries

$$0 = \langle\phi|\{G^a_{\pm 1/2}, \tilde{G}_{a,\mp 1/2}\}|\phi\rangle = \| \tilde{G}_{a,\mp 1/2}|\phi\rangle \|^2 + \| G^a_{\pm 1/2}|\phi\rangle \|^2 \tag{3.27}$$

Note that there is an ambiguity in the choice of direction in which we measure the charge.

3.6 Bosonisation

The OPE of the exponential of a scalar field, ϕ, is equal to the OPE of a complex fermion, $\psi = (\psi_1 + i\psi_2)/\sqrt{2}$ where the ψ_i are Majorana fermions. Writing out the holomorphic part (see [13]):

$$\begin{aligned} :e^{i\phi(z)}::e^{-i\phi(w)}: &\sim e^{i(\phi(z)-\phi(w)}\frac{1}{(z-w)} \\ :e^{i\phi(z)}::e^{i\phi(w)}: &\sim z-w \\ :e^{-i\phi(z)}::e^{-i\phi(w)}: &\sim z-w \end{aligned}$$

Comparison with the fermion OPE

$$:\psi(z)::\bar{\psi}(w):=\frac{1}{2}:(\psi_1(z)+i\psi_2(z))::(\psi_1(w)-i\psi_2(w)):\sim \frac{1}{z-w}+\psi_1(w)\psi_2(w)$$

makes it natural to make the identification: $\psi(z) \cong e^{i\phi(z)}$.
The fermion current is then

$$:\bar{\psi}(z)\psi(z):=i\partial\phi(z) \tag{3.28}$$

The operator $S^{\pm} := e^{\pm i/2\phi}$ is a vertex operator that maps a NS vacuum to an R vacuum, commonly called a **spin field**. We could also let s vary continuously,in $e^{is\phi}$ recovering the spectral flow from the previous section for the vacuum. In a string picture it would correspond to a continuous stretching of the string along the boundary. The spectral flow shows that the $\mathcal{N} = 4$ superconformal algebras for the Ramond sector and the NS sector are isomorphic. Note though that the spin fields represent a spectral flow only on the vacuum. spin field

The state resulting from this spectral flow from a NS vacuum has conformal dimension $s^2/2 = 1/8$ (as bosonic vertex operator with momentum 1/2). For the Ramond groundstate of $\mathcal{N} = 4$ this means, in the minimal 4 boson case,

that the conformal dimension gets raised by $2 \cdot 1/8 = 1/4$ since there are 2 pairs of fermions. In general the spectral flow raises the dimension of the Ramond vacuum by $c/24$. Notice that we have 2 ground states for a generic s value. Only the NS sector has one, degenerate, vacuum. When we continue the spectral flow till $s = \pm 1$ we reach an excited NS state.

Chapter 4

Correlation functions

In this chapter we will describe a calculation of 3 point functions of chiral primaries, as originally done by Mathur and Lunin in 2001 [1]. After a short introduction to orbifolds, we will go in some detail through their calculations. We will see how part of their results follow from the $SU(2) \otimes SU(2)$ R-symmetry resulting in some pure spin algebra piece and a part from geometric origin. The advantage of this method is that it is almost independent of the target manifold that is the base of the symmetric orbifold, although, as we will see in the next chapter, the $\mathcal{N} = 4$ supersymmetry automatically implies a hyperkähler structure and so fixes the target manifold already for a big part. Furthermore this method makes it possible to include other primaries from the chiral multiplet, although it is unlikely that they are as rigid as the chiral primaries.

Disadvantages are that not all chiral primaries can be obtained with this method and that it is in principle a large N approximation. The final result is a slight, but not essential, generalisation of the original.

4.1 The symmetric orbifold

An **orbifold** is defined as the quotient of a manifold by a finite group G. This means that we have to identify points on the manifold that transform into eachother by the group action, that we assume to be continuous. We might have fixed points of this action, meaning in other words that the action is not free. Because of these fixed points the orbifold is not necessarily a manifold. orbifold

A simple example is the real line with inversion through the origin (fig. 4.1(a)). The group is the group of the elements, $\mathbb{Z}_2 = \{1, s\}$ with $s(x) = -x$. The origin is a fixed point.

When we start from the circle S^1 (this can be seen as the interval $[-1, 1]$ with the endpoints identified), then we can construct the orbifold $S^1/\mathbb{Z}_2$ (fig. 4.1 (b)). The **symmetric orbifold** M^N/S_N, is defined as the quotient of the Cartesian product of N copies of a manifold, M, by the symmetric group of N objects, S_N (fig. 4.1 (c)). When we define a 2 dimensional conformal field theory with with an orbifold as target space we can construct this theory by means of projecting the states, defined on the original manifold as target space, onto the orbifold. Since in our case the field theory is defined on a cylinder (the boundary of AdS_3), say with parameters σ_0 and σ_1, the bosonic fields should symmetric orbifold

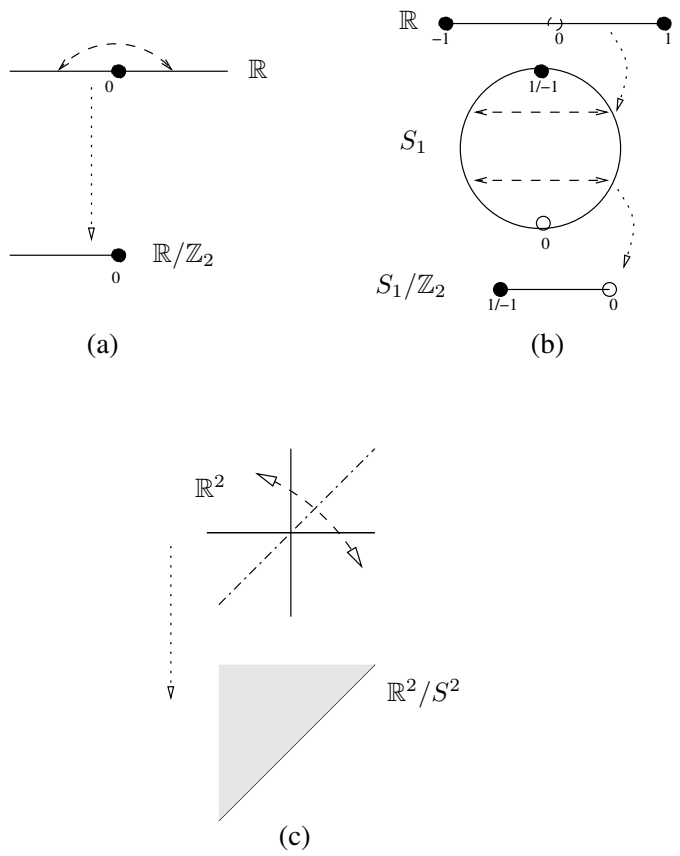


Figure 4.1: The orbifolds $\mathbb{R}/Z_2$ (a), S_1/Z_2 (b) and $\mathbb{R}^2/S^2$ (c)

be periodic around the cylinder modulo a twist by the group action:

$$\phi(\sigma_0, \sigma_1 + 2\pi) = \hat{g}\phi(\sigma_0, \sigma_1)$$

These twisted states form a separate sector in the Hilbert space. The demand of modular invariance (i.e. the invariance of the complex structure under diffeomorphisms) can be fulfilled by inclusion of all possible twisted sectors. An other argument to include all the sectors is to realise that a loop in the untwisted sector as parametrised by by a slice of the cylinder can, when reaching the fixed point split into loops that are in twisted sectors. This is true in the case of an Abelian group, although it is possible that we can have modular invariant twisting by just a subset of the group. For example had we just used the original theory, without any twisting and without projection onto invariant states, the theory would have been modular invariant.

In the case of a non-Abelian group we have to be more carefull how to define the twisted sectors. Let $g \in G$ and ϕ a g-twisted field, then, because of the

identification, we should have:

$$\begin{aligned} h\phi(\sigma_0, \sigma_1 + 2\pi) &= h \circ g\phi(\sigma_0, \sigma_1) \\ &= (h\phi)(\sigma_0, \sigma_1 + 2\pi) \\ &= (h \circ g \circ h^{-1}) \circ h\phi(\sigma_0, \sigma_1) \text{ for any } h \in G \end{aligned}$$

So the group action transforms a g-twisted state into one twisted by an element in the same **conjugacy class** as g. The right prescription for forming an orbifold, in this case, is to project a g twisted sector by the **centraliser** of g (that is the set of group elements that commute with g), and then average over the conjugacy class of which g is a representant. It is not complicated to show that the resulting theory is indeed modular invariant and invariant under the group transformations [14].

conjugacy class

centraliser

When studying the symmetric orbifold the finite group is the symmetric group. We refer to appendix A for the structure of this group and its classes. For the moment we will just concentrate on one cycle from the class it belongs to. When we calculate correlation functions of operators from several, equal or non equal, sectors the sum over the centralisers and representatives will give us the final answer. We will see in the following that in the large N limit these double sums will result in a series expansion in $1/N$.

We will describe our theory in the complex plane with $z = e^{\sigma_0 + i\sigma_1}$. When we consider the bosonic field as a map from $\mathbb{C}$ to M^N, then by imposing twisted boundary conditions on the fields when going around a point $z_0 \in \mathbb{C}$, we make this point into a **branching point**, also called a **ramification point** for this map. This means that the map is multivalued in any neighbourhood of this point. Suppose Σ is a Riemann surface and $f : \mathbb{C} \to \Sigma$ is a map such that every point in Σ has a neighbourhood whose preimage is a disjoint union of open sets in $\mathbb{C}$, each of which is homeomorphic to that neighbourhood under f, then f is called a **covering map**. The nice thing is, that we can use covering maps to resolve the orbifold singularities as follows. We know that for an n-cycle twist, after encircling the twist insertion n times, the bosonic fields will again have the values they had originally. We can make a covering map that has the same periodicity, that is periodic after n loops in the z plane, by cutting out a small disk around the insertion and choosing a map to the complex plane that goes close to insertion as

branching point

ramification point

covering map

$$z = at^n \tag{4.1}$$

When we go around 0 in the t-plane once, we go around 0 in the z-plane n times . Clearly the map is a covering map on the z plane with the small disk removed. The bosonic fields, as coordinates on M^N, get unwounded to fields that are single valued on the t-plane. Such a map that lifts a non simply connected surface to a simply connected cover is also called a **monodromy map**. We still need to specify boundary conditions around the disk that was cut out. As far as the bosonic theory is concerned we will just glue in a flat disk in the t-plane, choosing boundary conditions that reflect the single valuedness of the fields (i.e. periodic under the map $t \mapsto \exp(2\pi i)t$). We can now consider the theory as a sigma model on the cover with as target space just one copy of M, since the unwounded fields are equivalent. and we can make the choice to start with the first copy of M in the cycle.

monodromy map

For fermions we have to be more carefull. As argument for a choice of boundary conditions is the following [2]: since AdS_3 is contractible a fermion on

the boundary going around the cylinder is equivalent to a fermion in the centre of AdS_3 turning around its axis, so it should be, as a fermion, antiperiodic. In other words the fermion is in the **Neveu-Schwartz** (NS) sector :

Neveu-Schwartz

$$\psi(\sigma_0, \sigma_1 + 2\pi) = -\psi(\sigma_0, \sigma_1)$$

When changing coordinates to the z plane, we have to take into account that fermions have conformal dimension $1/2$, so that in the z-plane we have , with $w \stackrel{\text{def}}{=} \sigma_0 + i\sigma_1$

$$\psi_z(z) = \psi_w(w)(\frac{dw}{dz})^{\frac{1}{2}} = \psi_w(w)z^{-\frac{1}{2}}$$

making $\psi(z)$ periodic. The choice of a Ramond vacuum on the other hand would correspond to a minimal mass black hole [2]. We still have to check what boundary conditions we have to impose on the cover that correspond to this. We will do that in the next section.

4.2 Raising the dimension

A twist operator $\omega_n(z)$, is defined as an operator that creates the lowest dimensional state, the vacuum, of a twisted sector from the untwisted one. It does change the dimension of the vacuum. This can be calculated using the energy-momentum tensor and a local mode expansion of the fields [15], but it can be deduced in the following way: we go to a cylinder around the insertion point. The vacuum contribution *before* we insert the twist operator is the standard Casimir energy of $-c/24$, with N copies of M we have $c = 6N$ and so the energy is $-N/4$. After the twist of n strings we have replaced the contribution of these to the energy with the Casimir energy of a cylinder that has a circumference of n times the original leading to a contribution of $-1/4n^2$ per copy that participates in the twisting, amounting to a total of $-1/4n$. The difference of these energies gives us the dimension of the twist operator:

$$\Delta_n = \frac{1}{4}\left(n - \frac{1}{n}\right) \tag{4.2}$$

The vacuum in the NS sector has no R-symmetry charge. We will shortly see that, depending on the cycle length, we might have to impose Ramond boundary conditions along the disk cut out on the cover resulting in a charge of $\pm 1/2$. In both cases though, the conformal dimension is bigger than the R-symmetry charge so that they do not represent chiral operators (see the discussion on chiral operators in section (3.5)).

The idea is now to construct chiral operators from twist operators by applying operators that raise the dimension slower than the charge until they are equal. We start by defining ladder current operators in the standard way. Starting from the current algebra (3.13): we define :

$$J^{\pm}(z) \stackrel{\text{def}}{=} J^1(z) \pm iJ^2(z)$$

with OPE's :

$$J^3(z)J^{\pm}(w) \sim \pm\frac{J^{\pm}(w)}{z - w} \tag{4.3}$$

We now define a kind of fractional mode operators on the z plane in the sector twisted by the cycle $(1, 2, \cdots, n)$:

$$J^{+,z}_{-m/n} \stackrel{\text{def}}{=} \oint_C \frac{dz}{2\pi i} \sum_{k=1}^{n} J^{+,z}_k(z) e^{-2\pi i m(k-1)/n} z^{-m/n} \tag{4.4}$$

Where C is a loop around the origin, the z superscript on the first J just a reminder on what surface the operator is defined and the k subscripts indicating on which 4-manifold from the n that are twisted the current operator acts.The exponential is chosen such that the integrand is single valued along the contour:

$$\sum_{k=1}^{n} J^{+,z}_k(ze^{2\pi i}) e^{-2\pi i m(k-1)/n} (ze^{2\pi i})^{-m/n} = \sum_{k=1}^{n} J^{+,z}_{k+1}(z) e^{-2\pi i m((k+1)-1)/n} z^{-m/n} \tag{4.5}$$

where $J^{+}_{n+1}(z) \stackrel{\text{def}}{=} J^{+}_1(z)$. Changing the summation index gives the desired result.

Since the $J^{+,z}_k(z)$ have conformal dimension 1, as can be seen from the OPE with the energy momentum tensor, $J^{+,z}_{-m/n}$ raises the conformal dimension by m/n, while it raises the charge corresponding to the left SU(2) ,with current $J^i(z) \stackrel{\text{def}}{=} \sum_{k=1}^{N} J^i_k(z)$, of the orbifold by one:

$$\begin{aligned}
&\left[\oint_C \frac{dz}{2\pi i} J^3(z), J^{+,z}_{-m/n}\right] \\
&\qquad = \oint_C \frac{dw}{2\pi i} \oint_{C'} \frac{dz}{2\pi i} \sum_{k,l=1}^{n} J^{3,z}_l(z) J^{+,z}_k(w) e^{-2\pi i m(k-1)/n} z^{-m/n} \\
&\qquad = \oint_C \frac{dw}{2\pi i} \oint_{C'} \frac{dz}{2\pi i} \sum_{k=1}^{n} \frac{J^{+,z}_k(w)}{z-w} e^{-2\pi i m(k-1)/n} z^{-m/n} \\
&\qquad = J^{+,z}_{-m/n}
\end{aligned} \tag{4.6}$$

Where the contours in the commutator have been deformed and merged in the usual way([16]) to change the commutator in a radial ordered product. C' is a contour around w and the OPE (4.3) has been used. So , with $|j\rangle$ a $J^3(z)$ eigenstate with eigenvalue j, we find the usual $\mathfrak{su}(2)$ relation:

$$J^3(z) J^{+,z}_{-m/n} |j\rangle = (j+1) J^{+,z}_{-m/n} |j\rangle$$

Next we are going to lift the fractional modes (4.4) to the t-plane. We take the contour, C, a circle around $z = 0$ and split it up in n connected arcs ,α_k so that each arc lifts to a full contour under the lifting map (4.1).

$$J^{+,z}_{-m/n} = \int_{\alpha_1} \frac{dt}{2\pi i} nat^{n-1} \sum_{k=1}^{n} J^{+,z}_k(at^n) e^{-2\pi i m(k-1)/n} a^{-m/n} t^{-m} \tag{4.7}$$

twist J_k back to J_1

$$= a^{-m/n} \sum_{k=1}^{n} \int_{\alpha_1} \frac{dt}{2\pi i} nat^{n-1} J^{+,z}_1(a(te^{2\pi i m(k-1)/n})^n) e^{-2\pi i m(k-1)/n} t^{-m} \tag{4.8}$$

changing variable $t \to te^{\frac{2\pi i(k-1)}{n}}$

$$= a^{-m/n} \sum_{k=1}^{n} \int_{\alpha_k} \frac{dt}{2\pi i} \left(\frac{dt}{dz}\right)^{-1} J_1^{+,z}(at^n) t^{-m} \tag{4.9}$$

since the conformal dimension of $J_k^{+,z}$ is 1

$$= a^{-m/n} \sum_{k=1}^{n} \int_{\alpha_k} \frac{dt}{2\pi i} J_1^{+,t}(t) t^{-m} \tag{4.10}$$

$$= a^{-m/n} J_{-m}^{+,t} \tag{4.11}$$

We now have constructed an operator on the t-plane, $J_{-m}^{+,t}$, that raises the dimension with m/n and the charge with 1. The t superscript will be omitted in the following unless possible ambiguity demands it. To define the theory on the t-plane we need to define the vacuum to which we apply the current modes, but the lift around a twist insertion might change the boundary conditions for fermions. Remember that the fermions were periodic in the z plane so taking into account the fermionic conformal dimension we find :

$$\psi(t) = (an)^{\frac{1}{2}} t^{\frac{n-1}{2}} \psi(z)$$

Apparently we need periodic boundary conditions around the twist insertion for n odd, but anti-periodic ones for n even, this slightly changes the construction of the chiral primaries.

4.3 Chiral Primaries

In the odd cycle case we can start from the NS vacuum in the t-plane , where we have, because of the twist, a conformal dimension on the z-plane, Δ_n, given by (4.2). The fastest change towards a chiral primary we get by applying the lowest possible J^+ mode to the twisted vacuum. J_0^+ gives a state of zero norm as can be seen from the affine $\mathfrak{su}(2)$ algebra together with the fact that the NS vacuum has no charge. So the first possible operator is J_{-1}^+ resulting in a state with charge $q = 1$ and dimension $\Delta = \Delta_n + 1/n$. By calculating the norm of the resulting state, using the affine algebra, we see that the next lowest possible mode operator is J_{-3}^+ :$(m \geq 1)$

$$\begin{aligned}
\langle 0|J_1^- J_m^- J_{-m}^+ J_{-1}^+|0\rangle &= \langle 0|J_1^- \left(J_m^+ J_{-m}^- - 2J_0^3 + \frac{c}{6}m\right) J_{-1}^+|0\rangle \\
&= \langle 0|J_1^- \left(J_{-m}^+ \left(\frac{c}{6}m\delta_{m,1} - 2J_{m-1}^3\right) + \left(-2 + \frac{c}{6}m\right) J_{-1}^+\right)|0\rangle \\
&= \langle 0| \left(\frac{c}{6}\delta_{m,1} - 2J_{1-m}^3\right) \left(\frac{c}{6}m\delta_{m,1} - 2J_{m-1}^3\right) + \left(\frac{c}{6}m - 2\right) \frac{c}{6}|0\rangle \\
m = 1 \ &= \left(\frac{c}{6}\right)^2 + \frac{c}{6}\left(\frac{c}{6} - 2\right) = 0 \text{ for } c = 6 \\
m \geq 2 \ &\left(m\frac{c}{6} - 2\right)\frac{c}{6} = 0 \text{ for } m = 2 \text{ and } c = 6
\end{aligned}$$

The same result we can get after bosonisation (see section 3.6). Continuing in this manner we arrive at the chiral primaries:

$$\sigma_n^- \stackrel{\text{def}}{=} a^{-\frac{q^2}{n}} J^+_{-(n-2)}.....J^+_{-1}|0\rangle \text{ with } q = \Delta = \frac{n-1}{2} =: \Delta_n^- \qquad (4.12)$$

$$\text{and} \qquad \sigma_n^+ \stackrel{\text{def}}{=} Aa^{-\frac{q^2}{n}} J^+_{-n}.....J^+_{-1}|0\rangle \text{ with } q = \Delta = \frac{n+1}{2} =: \Delta_n^+ \qquad (4.13)$$

The factor in front of the primaries is a normalisation required to make the definitions independent of the slope a of the cover map, as can be seen from recursively applying (4.11), and so making it possible to define the primary from the t-plane. From the construction we also get an excited chiral primary in the untwisted sector:

$$\sigma_1^+ = J^+_{-1}|0\rangle \text{ with } q = \Delta = 1 \qquad (4.14)$$

Note that, somewhat confusingly, the primaries are given with respect to the t-plane, but the dimension is calculated as seen from the z-plane. Of course the dimension does not change under the conformal map from the z-plane with punctures to the t-plane.

To differentiate in the following between the primaries as defined on the t-plane and the primaries as defined on the z-plane we will use τ to denote the latter:

$$\tau_n^\pm(z) = \left(\frac{dz}{dt}\right)^{-(n\pm1)/2} \sigma_n^\pm(t) \qquad (4.15)$$

For even cycle length we need a Ramond vacuum around the lift of the twist insertion point. We can use the spin fields from the appendix to create a Ramond vacuum . Here we have 4 Majorana fermions or 2 complex fermions, ψ_1 and ψ_2. They transform under the $SU(2)$ R-symmetry, so we have a continuum of directions for the spectral flow to choose from, the natural choice being along the J^3 direction. Combining the 2 complex fermions into a $SU(2)$ doublet and bosonising following the conventions of [1]:

$$\Psi \stackrel{\text{def}}{=} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} e^{i\phi_5} \\ e^{i\phi_6} \end{pmatrix}$$

The SU(2) current can now be written as :

$$J^i(z) = \bar{\Psi}(z)\sigma^i\Psi(z)$$

Where σ^i are the Pauli matrices. Written out in bosonic fields we find as current components:

$$J^3(z) = \frac{i}{2}(\partial\phi_5 - \partial\phi_6)$$
$$J^\pm = exp(\pm i(\phi_5 - \phi_6))$$

Our choice for the spinfields is then

$$S^\pm \stackrel{\text{def}}{=} exp(\pm\frac{i}{2}(\phi_5 - \phi_6))$$

Targetting for a highest weight state we start from the Ramond vacuum $|0^+\rangle_R = S^+|0\rangle_{NS}$. From this bosonised representation follows that the charge of this vacuum 1/2 is. The conformal dimension is (see section (3.6)) 1/4 . The other vacuum would have had charge $-1/2$. We are now in the position to construct the chiral primaries analogous to the odd cycle length case to arrive at the two chiral primaries:

$$\sigma_n^- \stackrel{\text{def}}{=} a^{-\frac{q^2}{n}} J^+_{-(n-2)}.....J^+_{-2}J^+_0 S^+|0\rangle \text{ with } q = \Delta = \frac{n-1}{2} =: \Delta_n^- \tag{4.16}$$

$$\sigma_n^+ \stackrel{\text{def}}{=} a^{-\frac{q^2}{n}} J^+_{-n}.....J^+_{-2}J^+_0 S^+|0\rangle \text{ with } q = \Delta = \frac{n+1}{2} =: \Delta_n^+ \tag{4.17}$$

where we have used the fact that we now also need to transform back the dimension change from the spinfield operator (since it was defined on the t-plane), so that we have to add $1/n \times 1/4$ to the twisted vacuum dimension, since the vacuum energy gets raised as if it is one string out of n. We will slightly generalise the primaries used in the calculations because charge conservation otherwise would give us pretty boring results. A general primary used will be of the form $(J_0^-)^k \sigma_n^\pm$. Where $k \leq n \pm 1$, so we stay in the same $SU(2)$ multiplet.

Combining with the antiholomorphic fields we get a total of 4 chiral primaries to start from per cycle . They will be denoted by σ_n^{--}, σ_n^{-+}, σ_n^{+-} and σ_n^{++}.

4.4 separating the geometry from the algebra

Our general objective is to calculate 3 point functions of the chiral primaries defined above. The singularity at the twist insertions can be smoothed out by first cutting out a small disk and then going to the cover using the monodromy map. This can be done consistently for each value of n by using the spinfield for even cycles. Before the lift to the cover we need to regularise the theory by compactifying the z-plane. This is done by cutting off the z-plane at $|z| = 1/\delta$ and gluing on a half sphere by switching to coordinates

$$\tilde{z} = \frac{1}{\delta^2 z} \text{ and metric } ds^2 = d\tilde{z}d\bar{\tilde{z}} \tag{4.18}$$

for $z > 1/\delta$ The construction of the lifting map was done in [17] as a rational polynomial in terms of the quotient of Jacobi polynomials (appendix B). The global conformal symmetry of the theory on the sphere (section (3.5)) can be used to situate the insertions at $z = 0, a$ and ∞. The covering map lifts the insertions to $t = 0, 1$ and ∞. When calculating the path integral on the t-plane, we should use in the Sigma model action the metric induced from the z-plane by the lifting. The idea is to conformally scale this metric to a fiducial metric , $\hat{g}$, that we choose to be the same again as the z plane in the case the cover has genus 0, with a cutoff at $|t| = \delta'$. Although the action was classically conformal invariant, this is not the case in the quantum theory. The conformal anomaly can be accounted for by introducing an extra field, the Liouville field, to compensate the conformal scaling of the metric. The n-points function generating partition sum gets an extra factor after a conformal transformation. When 2 metrics are conformally related through $ds^2 = e^\phi d\hat{s}^2$, the partition functions are related

through:

$$Z^{(s)} = e^{S_L} Z^{(\hat{s})}$$

Where S_L is the Liouville action defined through:

$$S_L = \frac{c}{96\pi} \int d^2t \sqrt{-g^{(\hat{s})}} \left(\partial_\mu \phi \partial_\nu \phi g^{(\hat{s})\mu\nu} + 2R^{(\hat{s})}\phi \right)$$

Note that the original metric is a fixed given metric inherited from the way we have taken the limit to the boundary of the bulk AdS geometry, and it has no gauge freedom in the metric, as in the worldsheet CFT, thus every conformal transformation should be accounted for. As an example we will investigate the effect on the partition function coming from the compactification of the z plane. On the glued on half of the sphere, that is the region $z > 1/\delta$, we have for the metric a relative δ dependence of:

$$e^{\phi_\delta} \sim \frac{1}{\delta^4}$$

In the Liouville action only the curvature term contributes since the scaling is constant. For a sphere the curvature is easy to calculate by embedding the sphere in $\mathbb{R}^3$ and changing to polarcoordinates. The metric is then:

$$g = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 sin^2(x_1) \end{pmatrix}$$

and the curvature scalar is $1/r^2$. For the relative Liouville action we then find :

$$S_L^\phi = -4\ln(\delta) \frac{1}{2} \frac{c}{48} \int_0^{2\pi} \int_0^{\pi} dx_2 dx_1 r^2 |sin(x_1)| . \frac{1}{r^2} = \ln(\frac{1}{\delta^{\frac{c}{3}}})$$

And so the δ dependence of the partition function is :

$$Z_\delta \sim (\delta)^{-\frac{c}{3}}$$

The whole process of compactification on the z plane and going from the metric on the t plane, induced by the covering map, to a flat metric was carried out in [17]. Of course the final results do not depend on δ nor on the size of the compactification sphere in the cover Σ. Also the dependence on the size of the disc that is cut out around the twist insertion does not enter the final results. The induced metric on Σ, on the other hand, does depend on the specific twist insertions in the n-point function. And so the Liouville term coming from the change to a flat metric on Σ, does enter the final results in an essential way. These calculations were done for the bosonic orbifold. The cover map that was used in the construction can be found in appendix B. Since these results depend on the geometry of the twists, they will be part of the n-point functions in the superconformal theory.

We will now concentrate on the superconformal theory. A general n-points

function for n, on one cycle based, chiral operators is given by :

$$\begin{aligned}\langle\tau_{[l_1]}(z_1)\cdots\tau_{[l_n]}(z_n)\rangle &= \prod_{i=1}^{n}\left(\frac{\lambda_{[l_i]}}{|[l_i]||C_{[l_i]}|}\sum_{g_i\in[l_i]}\sum_{h_i\in C_{g_i}}\right)\langle\hat{h}_1\tau_{g_1}(z_1)\cdots\hat{h}_n\tau_{g_n}(z_n)\rangle \\ &= \frac{1}{(N!)^n}\prod_{i=1}^{n}\left(\lambda_{[l_i]}l_i(N-l_i)!\sum_{g_i\in[l_i]}\right)\langle\tau_{g_1}(z_1)\cdots\tau_{g_n}(z_n)\rangle\end{aligned} \tag{4.19}$$

Of course the projection onto the centraliser invariant subspace could have been omitted from the start since we constructed our operators to be invariant. l_i is the cycle length, $[l_i]$ the conjugacy class and the $\lambda' s$ are normalisation constants, to be defined shortly, that we take to be constant within a class. The $\hat{h}$'s are the action of the permutation on the fields. The orders of the cycle class, $|[l_i]|$, and the centraliser, $|C_{[l_i]}|$ were determined in (A.9) resp. (A.8).

We will calculate first one term in the sum of (4.19) after which it will be shown that the contributions of the individual terms in the projection on the invariant space depend strongly on the topological properties of the cover. The n-pointsfunction for one particular choice of twist insertions takes the following form

$$\langle\tau_{g_1}(z_1)..\tau_{g_n}(z_n)\rangle = \frac{1}{Z_0^N}\int\prod_{m=1}^{s}\mathcal{D}\left[X_m,\psi_m\right]e^{-\sum_{j=1}^{N}S(X_j,\psi_j)}\tau_{g_1}(z_1)..\tau_{g_n}(z_n) \tag{4.20}$$

Here Z_0 is the partition sum for one copy of M:

$$Z_0 = \int\mathcal{D}\left[X,\psi\right]e^{-S(X,\psi)} \tag{4.21}$$

Notice that this is just a calculational convention. Z_0^N is not the partition sum of the untwisted sector, that would be $Z_0^N/N!$. We now assume that just s copies of the N get twisted by the insertions, resulting in

$$\langle\tau_{g_1}(z_1)..\tau_{g_n}(z_n)\rangle = \frac{1}{Z_0^s}\int\prod_{m=1}^{s}\mathcal{D}\left[X_m,\psi_m\right]e^{-S(X_1..X_s,\psi_1..\psi_s)}\tau_{g_1}(z_1)..\tau_{g_n}(z_n) \tag{4.22}$$

In general the sum over the permutations in the conjugacy classes results in cycles with different amounts of overlap. We will see though in the next section that the dominant terms in a $1/N$ expansion are given by those combinations that result in a sphere as cover. The lift of the path integral in (4.22) to the cover Σ with the induced metric results in:

$$\langle\tau_1(z_1)..\tau_n(z_n)\rangle = \frac{1}{Z_0^s}\int_{\Sigma,\text{induced}}\mathcal{D}\left[X,\psi\right]e^{-S(X,\psi)}\sigma_1(t_1)..\sigma_n(t_n)$$

the σs were defined in (4.15). We are now assuming a genus 0 cover and change the metric to the flat one:

$$=\frac{e^{S_L}}{Z_0^s}\left(\int_\Sigma\mathcal{D}\left[X,\psi\right]e^{-S(X,\psi)}\right)\frac{\int_\Sigma\mathcal{D}\left[X,\psi\right]e^{-S(X,\psi)}\sigma_1(t_1)..\sigma_n(t_n)}{\int_\Sigma\mathcal{D}\left[X,\psi\right]e^{-S(X,\psi)}}$$

The Liouville term contains the data from the change of metric.

$$=\frac{e^{S_L}}{Z_0^{s-1}}\langle\sigma_1(t_1)..\sigma_n(t_n)\rangle \tag{4.23}$$

In the last step we have used the definition of Z_0. Note that Σ is identical to the original space time up to the value of the cutoff on which the final results do not depend.

4.5 Large N

It will now be shown that the main contribution comes from the sphere as cover. First we have to define some normalisation of the cycle operators :

$$\langle\tau_{c_n}(0)\tau^\dagger_{c_n}(1)\rangle = 1$$

Demanding the same normalisation from the full cycle based chiral operators we find , using the group summation as in (4.19):

$$1 = \langle\tau_{[c_n]}(0)\tau^\dagger_{[c_n]}(1)\rangle = \frac{(\lambda_n n(N-n)!)^2}{(N!)^2}\sum_{g_1,g_2\in[c_n]}\langle\tau_{g_1}(0)\tau^\dagger_{g_2}(1)\rangle$$

but the correlator is only nonzero for 2 matching twists

$$\begin{aligned}&= \frac{(\lambda_n n(N-n)!)^2}{(N!)^2}\sum_{g\in[c_n]}\\ &= \frac{(\lambda_n n(N-n)!)^2}{(N!)^2}|[c_n]|\end{aligned}$$

Resulting in:

$$\lambda_n = \left[\frac{n\,(N-n)!}{N!}\right]^{-1/2} \tag{4.24}$$

for 3 point functions the combinatorics is more complicated, since now the last insertion cycle should match the permutation from the first 2 combined. So the first 2 combined should end up into one cycle. This means that they should have overlap. When we lift to the t-plane we note that the lifting map (4.1) has a ramification index l_i (the number of sheets that come together at the branch point). The Riemann-Hurwitz formula [18] then let us calculate the **genus** (the number of handles) from the cover: genus

$$g = \frac{1}{2}\sum_i(l_i-1) - s + 1 \tag{4.25}$$

Where s is the generic number of sheets, being the total number of copies of M, s, that get twisted. An n-points function that has s twisted copies will have a factor, corresponding to the different ways these s copies can be picked from the total of N, of

$$\left(\begin{array}{c} N \\ s \end{array}\right) \sim N^s \text{ for large } N$$

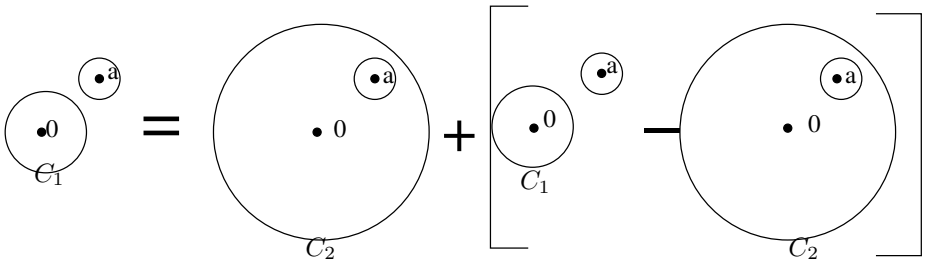


Figure 4.2: Change of contour

The normalisation by the 2 point functions contributes a factor of

$$\lambda_{l_i} \sim N^{-l_i/2} \text{ for large } N$$

So that we find as combinatoric factor for some given combination of cycles as weight:

$$N^{s-(\sum_i l_i)/2} = N^{-g+(2-n)/2}$$

We conclude that terms in the n-point functions leading to covers with g get suppressed by a factor of N^{-g}

4.6 Vertex representation of the currents

Having shown that the main contribution to the 3 points function in the large N limit comes from the genus 0 covers we will concentrate on those from now on. This has the big advantage that, since the fundamental group is trivial, there are no topological obstructions to deform the contours around insertions. As a consequence we can move the R-symmetry ladder operators through eachother, picking up correlators of the $\mathcal{N} = 4$ algebra. An example will illustrate this (see figure 4.6):

Suppose we have in our correlator insertions of the form

$$\oint_{C_1} ds \oint_a dt J^-(t)J^+(s)s^{-1}$$

then we can write this as (see figure 4.6):

$$\begin{aligned}
&\oint_{C_2} ds \oint_a dt J^+(s)s^{-1}J^-(t) \\
&\qquad + \big(\oint_{C_1} ds \oint_a dt J^-(t)J^+(s)s^{-1} - \oint_{C_2} ds \oint_a dt J^+(s)s^{-1}J^-(t)\big) \\
&\quad = \oint_{C_2} ds \oint_a dt J^+(s)s^{-1}J^-(t) + [J_0^-, J_{-1}^+]
\end{aligned}$$

Where the change of order of operators is dictated by the radial ordering (time ordering). In this way we can rearrange the current operators until J_0^- is next to the vacuum, resulting in a vanishing correlation function. We can now conclude that the correlation functions only depend on the algebra and so we can take a suitable representation for it. Following ideas from Kac and Frenkel

[19] we construct a bosonic vertex representation. We start from a 2 complex free boson, 2 complex free fermion representation (and so $c = 6$):

$$
\begin{aligned}
J^i(z) &= \frac{1}{2}\mathbf{\Psi}^\dagger \sigma^i \mathbf{\Psi} \qquad (4.26)\\
T(z) &= \partial X_i^\dagger \partial X_i + \frac{1}{2}(\mathbf{\Psi}^\dagger \cdot \partial \mathbf{\Psi} - \partial \mathbf{\Psi}^\dagger \cdot \mathbf{\Psi})\\
G^1(z) &= \sqrt{2}\psi_2^\dagger \partial X_1 - \sqrt{2}\psi_1 \partial X_2 \qquad & G^2(z) &= \sqrt{2}\psi_1^\dagger \partial X_1 + \sqrt{2}\psi_2 \partial X_2\\
\tilde{G}_1(z) &= \sqrt{2}\psi_2 \partial X_1^\dagger - \sqrt{2}\psi_1^\dagger \partial X_2^\dagger & \tilde{G}_2(z) &= \sqrt{2}\psi_1 \partial X_1^\dagger + \sqrt{2}\psi_2^\dagger \partial X_2^\dagger
\end{aligned}
$$

Here $\Psi \stackrel{\text{def}}{=} (\psi_1, \psi_2)$ are the as vector combined fermions, σ^i the Pauli matrices. We can check that the currents indeed satisfy the $\mathcal{N} = 4$ currentalgebra (3.13). For example we have:

$$
\begin{aligned}
&J^i(z)J^j(w)\\
&\quad = \frac{1}{4} : \mathbf{\Psi}^\dagger(z)\sigma^i \mathbf{\Psi}(z) :: \mathbf{\Psi}(w)^\dagger \sigma^j \mathbf{\Psi}(w) :\\
&\quad = \frac{1}{4}(\sigma^i)_{\alpha\beta}(\sigma^j)_{\gamma\delta}\left(\overbrace{\mathbf{\Psi}_\alpha^\dagger(z)\overbrace{\mathbf{\Psi}_\beta(z)\mathbf{\Psi}_\gamma^\dagger(w)}\mathbf{\Psi}_\delta(w)} + \overbrace{\mathbf{\Psi}_\alpha^\dagger(z)\overbrace{\mathbf{\Psi}_\beta(z)\mathbf{\Psi}_\gamma^\dagger(w)}\mathbf{\Psi}_\delta(w)}\right)\\
&\quad = \frac{1}{4}(\sigma^i)_{\alpha\beta}(\sigma^j)_{\gamma\delta}\left(\frac{\delta_{\alpha,\delta}}{z-w} + : \mathbf{\Psi}_\alpha^\dagger \mathbf{\Psi}_\delta(w) : + \mathcal{O}(z-w)\right)\\
&\qquad \left(\frac{\delta_{\beta,\gamma}}{z-w} - : \mathbf{\Psi}_\gamma^\dagger \mathbf{\Psi}_\beta(w) : + \mathcal{O}(z-w)\right) + \mathcal{O}(z-w)\\
&\quad \sim \frac{1}{4}(\delta_{i,j} + i\varepsilon^{ijk}\sigma^k)_{\alpha\beta}\left(2\frac{\mathbf{\Psi}_\alpha^\dagger \mathbf{\Psi}_\beta(w)}{z-w} + \frac{\delta_{\alpha,\beta}}{(z-w)^2}\right)\\
&\quad = \frac{i\varepsilon^{ijk}J^k(w)}{z-w} + \frac{\delta_{i,j}}{2(z-w)^2} \qquad (4.27)
\end{aligned}
$$

Bosonising the 2 fermions as outlined in section (3.6), $\psi_1 =: exp(i\phi_1)$ respectively $\psi_2 =: exp(i\phi_2)$, leads to the following bosonic representation of the $SU(2)$ currents and spin fields:

$$J^3(z) = \frac{i}{2}\mathbf{e} \cdot \partial \mathbf{\Phi}(z) \qquad (4.28)$$

$$J^\pm(z) = \exp[\pm i\mathbf{e} \cdot \mathbf{\Phi}(z)] \qquad (4.29)$$

$$S^\pm(z) = \exp[\pm \frac{i}{2}\mathbf{e} \cdot \mathbf{\Phi}(z)] \qquad (4.30)$$

Here $\mathbf{\Phi}$ is the 2 component boson (ϕ_1, ϕ_2) and $\mathbf{e}$ is given by:

$$\mathbf{e} \stackrel{\text{def}}{=} (1, -1)$$

4.7 The correlators

We normalise the chiral operators through:

$$\check{\tau}_{n,m}^s(z) \stackrel{\text{def}}{=} \frac{\tau_{n,m}^s(z)}{\langle \tau_{n,m}^s(0)(\tau_{n,m}^s)^\dagger(1)\rangle^{1/2}} \qquad (4.31)$$

sign index

We use the notation where n is the cycle length, m the number of ladder down, J^-, operators and the **sign index** $s = \pm 1$, denotes the basic chiral operator choice. Global conformal invariance dictates the 2 and 3 point functions to be of the form [13] (leaving out the antiholomorphic part):

$$\langle \check{\tau}^{s_1}_{n_1,m_1}(z_1)(\check{\tau}^{s_2}_{n_2,m_2})^\dagger(z_2)\rangle = \frac{A_{\{1\}\{2\}}}{z_{12}^{2\Delta_1}}\delta_{\Delta_1,\Delta_2} \tag{4.32}$$

$$\langle \check{\tau}^{s_1}_{n_1,m_1}(z_1)\check{\tau}^{s_2}_{n_2,m_2}(z_2)(\check{\tau}^{s_3}_{n_3,m_3})^\dagger(z_3)\rangle = \frac{A_{\{1\}\{2\}\{3\}}}{z_{12}^{\Delta_1+\Delta_2-\Delta_3}z_{23}^{\Delta_2+\Delta_3-\Delta_1}z_{13}^{\Delta_3+\Delta_1-\Delta_2}} \tag{4.33}$$

The $\{i\}$ subscripts are an abbreviation for the 3 parameters defining the operator and $z_{ij} \stackrel{\text{def}}{=} |z_i - z_j|$. The dimension depends on the cycle length and the sign index through (4.2) $\Delta_i = (n_i + s_i)/2$. Putting the insertions at $0, a$ and ∞ we can define normalised correlation functions as:

$$\begin{aligned}&\frac{\langle \check{\tau}^{s_1}_{n_1,m_1}(0)\check{\tau}^{s_2}_{n_2,m_2}(a)(\check{\tau}^{s_3}_{n_3,m_3})^\dagger(\infty)\rangle}{\langle \check{\tau}^{s_3}_{n_3,m_3}(0)(\check{\tau}^{s_3}_{n_3,m_3})^\dagger(\infty)\rangle}\\ &= \lim_{|z|\to\infty}\frac{C_{\{1\}\{2\}\{3\}}|z|^{2\Delta_3}}{|a|^{\Delta_1+\Delta_2-\Delta_3}(|a-z|)^{\Delta_2+\Delta_3-\Delta_1}|z|^{\Delta_3+\Delta_1-\Delta_2}} = \frac{C_{\{1\}\{2\}\{3\}}}{|a|^{\Delta_1+\Delta_2-\Delta_3}}\end{aligned} \tag{4.34}$$

With $C_{\{1\}\{2\}\{3\}} = A_{\{1\}\{2\}\{3\}}/A_{\{3\}\{3\}}$ Using (4.23) we can separate the topological part :

$$\begin{aligned}&\frac{\langle \check{\tau}^{s_1}_{n_1,m_1}(0)\check{\tau}^{s_2}_{n_2,m_2}(a)(\check{\tau}^{s_3}_{n_3,m_3})^\dagger(\infty)\rangle}{\langle \check{\tau}^{s_3}_{n_3,m_3}(0)(\check{\tau}^{s_3}_{n_3,m_3})^\dagger(\infty)\rangle}\\ &\qquad = |a|^{\Delta_3-\Delta_1-\Delta_2}\frac{\langle \check{\tau}^{s_1}_{n_1,m_1}(0)\check{\tau}^{s_2}_{n_2,m_2}(1)(\check{\tau}^{s_3}_{n_3,m_3})^\dagger(\infty)\rangle}{\langle \check{\tau}^{s_3}_{n_3,m_3}(0)(\check{\tau}^{s_3}_{n_3,m_3})^\dagger(\infty)\rangle}\end{aligned}$$

$$\begin{aligned}&= \frac{e^{S_L([n_1,0],[n_2,a],[n_3^\dagger,\infty])-\frac{1}{2}S_L([n_1,0],[n_1^\dagger,1])-\frac{1}{2}S_L([n_2,0],[n_2^\dagger,1])+\frac{1}{2}S_L([n_3,0],[n_3^\dagger,1])}}{Z_0^{s-1-\frac{1}{2}\sum_i(n_i-1)}}\\ &\quad\times e^{-S_L([n_3,0],[n_3^\dagger,\infty])}\frac{\langle \check{\sigma}^{s_1}_{n_1,m_1}(0)\check{\sigma}^{s_2}_{n_2,m_2}(1)(\check{\sigma}^{s_3}_{n_3,m_3})^\dagger(\infty)\rangle}{\langle \check{\sigma}^{s_3}_{n_3,m_3}(0)(\check{\sigma}^{s_3}_{n_3,m_3})^\dagger(\infty)\rangle}\end{aligned}$$

The Liouville action $S_L([n_1, z_1], \cdots)$ has as parameters the cycle and the insertion point of the twists involved. Since the cover is a sphere , using 4.25, we find that the Z_0 factor cancels from the denominator. The exponential of the Liouville action was calculated in [17] and can be expressed, extracting its a dependence, in a "bosonic fusion coefficient" $\mathbf{C}_{n_1,n_2,n_3}$. Its formula can be found in the appendix of [1]. The final result can then be written as:

$$\begin{aligned}&\frac{\langle \check{\tau}^{s_1}_{n_1,m_1}(0)\check{\tau}^{s_2}_{n_2,m_2}(a)(\check{\tau}^{s_3}_{n_3,m_3})^\dagger(\infty)\rangle}{\langle \check{\tau}^{s_3}_{n_3,m_3}(0)(\check{\tau}^{s_3}_{n_3,m_3})^\dagger(\infty)\rangle}\\ &\qquad = (\mathbf{C}_{n_1,n_2,n_3})^6|a|^{-\Delta_1-\Delta_2+\Delta_3}\frac{\langle \check{\sigma}^{s_1}_{n_1,m_1}(0)\check{\sigma}^{s_2}_{n_2,m_2}(1)(\check{\sigma}^{s_3}_{n_3,m_3})^\dagger(\infty)\rangle}{\langle \check{\sigma}^{s_3}_{n_3,m_3}(0)(\check{\sigma}^{s_3}_{n_3,m_3})^\dagger(\infty)\rangle}\end{aligned} \tag{4.35}$$

And analogous for the antiholomorphic part. Here $\sigma^s_{n,m}$ is defined as in (4.31) above:

$$\check{\sigma}^s_{n,m}(t) \stackrel{\text{def}}{=} \frac{\sigma^s_{n,m}(t)}{\langle \sigma^s_{n,m}(0)(\sigma^s_{n,m})^\dagger(1)\rangle^{1/2}} \tag{4.36}$$

The exponent of $\mathbf{C}_{n_1,n_2,n_3}$ is the central charge of the theory. We have extracted all a dependence and so contrary to the calculations in [1] we will set $a = 1$ in what follows.

4.8 Vertex representation of the chiral primaries

Using the bosonic representation for the $J^\pm$ (4.29) and the spinfield (4.30) we see that the chiral operators (4.12) should be of the form:

$$\check{\sigma}^s_{n,0}(t) = A\alpha^{-\frac{q^2}{n}} : \exp[iq\mathbf{e}\cdot\mathbf{\Phi}(t)] :$$

Where, in light of the analogous expression for J^3, q is the R-charge $(n+s)/2$ of the operator. The slope of the insertion is given by α. The normalisation constant we can find using the cover map. For one n-twist insertion at $z = 0$ and one reverse twist at $z = 1$. We use the map:

$$z = \frac{t^n}{t^n - (t-1)^n} \tag{4.37}$$

The slope of the map is at both insertions 1 and so we find that when we choose $A = 1$ we have the right normalisation of $\langle \sigma^s_{n,0}(0)(\sigma^s_{n,0})^\dagger(1)\rangle = 1$.

Applying J_0^- recursively leads to other operators in the BPS multiplet. since the weight of J_0^- is one we get in the t-plane after applying J_0^- once :

$$\sigma^s_{n,1}(t) = \alpha^{-\frac{q^2}{n}} \oint du J_0^-(u) : \exp[iq\mathbf{e}\cdot\mathbf{\Phi}(t)] : \tag{4.38}$$

$$= \alpha^{-\frac{q^2}{n}} \oint \frac{du}{2\pi i} : \exp[-i\mathbf{e}\cdot\mathbf{\Phi}(u)] :: \exp[iq\mathbf{e}\cdot\mathbf{\Phi}(t)] : \tag{4.39}$$

Wick contracting (see for example [16])

$$= \alpha^{-\frac{q^2}{n}} \oint \frac{du}{2\pi i}(u-t)^{-q\mathbf{e}\cdot\mathbf{e}} : \exp[-i\mathbf{e}\cdot\mathbf{\Phi}(u)] + iq\mathbf{e}\cdot\mathbf{\Phi}(t)] : \tag{4.40}$$

$$\tag{4.41}$$

Now expand the exponential, retaining the residue:

$$\begin{aligned}
&\sigma^s_{n,1}(t) \\
&= \alpha^{-\frac{q^2}{n}} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\oint \frac{du}{2\pi i}(u-t)^{-2q+j} \lim_{r\to t} \left(\partial_r^j : \exp[-i\mathbf{e}\cdot\mathbf{\Phi}(r) + iq\mathbf{e}\cdot\mathbf{\Phi}(t)] :\right)\right) \\
&= \alpha^{-\frac{q^2}{n}} \frac{1}{(2q-1)!} \lim_{r\to t} \left(\partial_r^{(2q-1)} : \exp[-i\mathbf{e}\cdot\mathbf{\Phi}(r) + iq\mathbf{e}\cdot\mathbf{\Phi}(t)] :\right)
\end{aligned} \tag{4.42}$$

the other operators we get by repeating this procedure:

$$\begin{aligned}
\sigma^s_{n,2}(t) &= \frac{\alpha^{-\frac{q^2}{n}}}{(2q-1)!}\lim_{r_1\to t}\oint\frac{du}{2\pi i}:\exp[-i\mathbf{e}\cdot\mathbf{\Phi}(u)]:\partial^{(2q-1)}_{r_1} \\
&\qquad :\exp[-i\mathbf{e}\cdot\mathbf{\Phi}(r_1)+iq\mathbf{e}\cdot\mathbf{\Phi}(t)]: \\
&= \frac{\alpha^{-\frac{q^2}{n}}}{(2q-1)!}\lim_{r_1\to t}\partial^{(2q-1)}_{r_1}\oint\frac{du}{2\pi i}\frac{(u-r_1)^{\mathbf{e}\cdot\mathbf{e}}}{(u-t)^{q\mathbf{e}\cdot\mathbf{e}}} \\
&\qquad :\exp[-i\mathbf{e}\cdot\mathbf{\Phi}(u)-i\mathbf{e}\cdot\mathbf{\Phi}(r_1)+iq\mathbf{e}\cdot\mathbf{\Phi}(t)]: \\
&= \frac{\alpha^{-\frac{q^2}{n}}}{(2q-1)!}\lim_{r_1\to t}\partial^{(2q-1)}_{r_1}\sum_{k=0}^{\infty}\oint\frac{du}{2\pi i}\frac{1}{k!(u-t)^{2q-k}}\lim_{r_2\to t}\partial^k_{r_2} \\
&\qquad (r_2-r_1)^2:\exp[-i\mathbf{e}\cdot\mathbf{\Phi}(r_2)-i\mathbf{e}\cdot\mathbf{\Phi}(r_1)+iq\mathbf{e}\cdot\mathbf{\Phi}(t)]: \\
&= \frac{\alpha^{-\frac{q^2}{n}}}{(2q-1)!}\lim_{r_1\to t}\partial^{(2q-1)}_{r_1}\frac{1}{(2q-1)!}\lim_{r_2\to t}\partial^{(2q-1)}_{r_2}(r_2-r_1)^2 \\
&\qquad :\exp[-i\mathbf{e}\cdot\mathbf{\Phi}(r_2)-i\mathbf{e}\cdot\mathbf{\Phi}(r_1)+iq\mathbf{e}\cdot\mathbf{\Phi}(t)]: \qquad (4.43)
\end{aligned}$$

and we find using induction:

$$\begin{aligned}
\sigma^s_{n,m}(t) = \alpha^{-\frac{q^2}{n}}&\left(\prod_{j=1}^{m}\frac{\lim_{r_j\to t}}{(2q-1)!}\partial^{(2q-1)}_{r_j}\right)\left(\prod_{k<j}(r_j-r_k)^2\right) \\
&:\exp[-i\sum_{l=1}^{m}\mathbf{e}\cdot\mathbf{\Phi}(r_l)+iq\mathbf{e}\cdot\mathbf{\Phi}(t)]: \qquad (4.44)
\end{aligned}$$

The normalised operators we can find using the SU(2) commutation relations, writing $\check{\sigma}^s_{n,0}(t)|0\rangle = |q,t\rangle, q=(n+s)/2$:

$$\langle(\sigma^s_{n,m})^\dagger(0)\sigma^s_{n,m}(1)\rangle = \langle q,0|(J^+_0)^m(J^-_0)^m|q,1\rangle \qquad (4.45)$$

$|q,t\rangle$ is a highest weight state so $J^+_0|q,t\rangle = 0$

$$= m(2q+1-m)\langle q,0|(J^+_0)^{m-1}(J^-_0)^{m-1}|q,1\rangle \qquad (4.46)$$

by induction

$$= \frac{m!(2q)!}{(2q-m)!}\langle q,0|q,1\rangle = \frac{m!(2q)!}{(2q-m)!} \qquad (4.47)$$

Since $\check{\sigma}^s_{n,0}$ is a normalised operator. With this result the normalised chiral

operator is given by:

$$\check{\sigma}^{s}_{n,m}(t) = \alpha^{-\frac{(n+s)^2}{4n}} \left(\frac{(n+s-m)!}{m!(n+s)!}\right)^{1/2} \left(\prod_{j=1}^{m} \frac{\lim_{r_j \to t}}{(n+s-1)!} \partial_{r_j}^{(n+s-1)}\right) \left(\prod_{k<j} (r_j - r_k)^2\right) \exp[-i\sum_{l=1}^{m} \mathbf{e}\cdot\mathbf{\Phi}(r_l) + i\frac{(n+s)}{2}\mathbf{e}\cdot\mathbf{\Phi}(t)] : \quad (4.48)$$

4.9 Calculation of the three point functions

To calculate the three point functions we will need the explicit covering maps as constructed in [17]. Since the normalised chiral operators, with insertions at $0, 1$ and $/infty$ depend explicitly on the slope of the cone close around the insertion we will need the asymptotic shape of the covering map (B.1) when approaching the ramification points.

$$\begin{aligned} t \to 0 : &\qquad z \approx \frac{s!(s-n_3)!(s-n_2)!}{n_1!(s-n_1)!(n_1-1)!} t^{n_1} \\ t \to 1 : &\qquad z \approx 1 + \frac{s!(s-n_3)!(s-n_1)!}{n_2!(s-n_2)!(n_2-1)!} (t-1)^{n_2} \\ t \to \infty : &\qquad z \approx \frac{(s-n_3)!(n_3-1)!n_3!}{s!(s-n_1)!(s-n_2)!} t^{n_3} \end{aligned} \quad (4.49)$$

Where $s = (n_1+n_2+n_3-1)/2$ is the number of copies of M that participates in the twisting. We will now choose the insertions at a and ∞ to be chiral. Thus we are calculating

$$\frac{\langle \check{\tau}^{s_1}_{n_1,m_1}(0)\check{\tau}^{s_2}_{n_2,0}(a)(\tau^{s_3}_{n_3,0})^\dagger(\infty)\rangle}{\langle \check{\tau}^{s_3}_{n_3,0}(0)(\tau^{s_3}_{n_3,0})^\dagger(\infty)\rangle} = (\mathbf{C}_{n_1,n_2,n_3})^6 a^{-\Delta_1-\Delta_2+\Delta_3} \frac{\langle \check{\sigma}^{s_1}_{n_1,m_1}(0)\check{\sigma}^{s_2}_{n_2,0}(1)(\sigma^{s_3}_{n_3,0})^\dagger(\infty)\rangle}{\langle \check{\sigma}^{s_3}_{n_3,0}(0)(\sigma^{s_3}_{n_3,0})^\dagger(\infty)\rangle} \quad (4.50)$$

The operator inserted at ∞ is not explicitly normalised. It first of all doesn't need to be because the denominator cancels the constant, and the point $z = \infty$, corresponding to $\tilde{z} = 0$ (4.18), is in another chart then the reference points.

We now use the vertex representation as defined in the previous section to

calculate the operator product of the first 2 insertions:

$$\breve{\sigma}^{s_1}_{n_1,m_1}(0)\breve{\sigma}^{s_2}_{n_2,0}(1) = a_1^{-\frac{(n_1+s_1)^2}{4n_1}} a_2^{-\frac{(n_2+s_2)^2}{4n_2}} \left(\frac{(n_1+s_1-m_1)!}{m_1!(n_1+s_1)!}\right)^{1/2}$$
$$\left(\prod_{j=1}^{m_1} \frac{\lim_{r_j\to 0}}{(n_1+s_1-1)!}\partial_{r_j}^{(n_1+s_1-1)}\right)\left(\prod_{k<j}(r_j-r_k)^2\right)$$
$$:\exp[-i\sum_{l=1}^{m}\mathbf{e}\cdot\mathbf{\Phi}(r_l)+i\frac{(n_1+s_1)}{2}\mathbf{e}\cdot\mathbf{\Phi}(0)]::\exp[i\frac{n_2+s_2}{2}\mathbf{e}\cdot\mathbf{\Phi}(1)]:$$
$$= a_1^{-\frac{(n_1+s_1)^2}{4n_1}} a_2^{-\frac{(n_2+s_2)^2}{4n_2}} \left(\frac{(n_1+s_1-m_1)!}{m_1!(n_1+s_1)!}\right)^{1/2} \left(\prod_{j=1}^{m_1} \frac{\lim_{r_j\to 0}}{(n_1+s_1-1)!}\partial_{r_j}^{(n_1+s_1-1)}\right)$$
$$\left(\prod_{k<j}(r_j-r_k)^2\right)\left(\prod_{j=1}^{m}(1-r_j)^{-n_2-s_2}\right)$$
$$:\exp[-i\sum_{l=1}^{m}\mathbf{e}\cdot\mathbf{\Phi}(r_l)+i\frac{(n_1+s_1)}{2}\mathbf{e}\cdot\mathbf{\Phi}(0)+i\frac{n_2+s_2}{2}\mathbf{e}\cdot\mathbf{\Phi}(1)]:$$
$$= a_1^{-\frac{(n_1+s_1)^2}{4n_1}} a_2^{-\frac{(n_2+s_2)^2}{4n_2}} \left(\frac{(n_1+s_1-m_1)!}{m_1!(n_1+s_1)!}\right)^{1/2} \left(\prod_{j=1}^{m_1} \frac{\lim_{r_j\to 0}}{(n_1+s_1-1)!}\partial_{r_j}^{(n_1+s_1-1)}\right)$$
$$\left(\prod_{k<j}(r_j-r_k)^2\right)\left(\prod_{j=1}^{m}(1-r_j)^{-n_2-s_2}\right)$$
$$:\exp[i\left(\frac{(n_1+s_1)}{2}+\frac{n_2+s_2}{2}-m\right)\mathbf{e}\cdot\mathbf{\Phi}(0)]$$
$$\left(1+i\left(\frac{n_2+s_2}{2}-\sum_{l=1}^{m}r_l\right)\mathbf{e}\cdot\partial\mathbf{\Phi}(0)+\text{ higher derivatives }\right): \quad (4.51)$$

In the end when calculating the 3 points function we are contracting with an exponential operator at $t=\infty$. $\partial\mathbf{\Phi}$ has conformal weight 1 and so we find for the operator on the chart for large t ($t>1/\delta'$) where we switch to coordinates $t'=1/t$:

$$\Phi'(t') = \left(\frac{\partial t'}{\partial t}\right)^{-1}\Phi(t) = t^2\Phi(t)$$

and so for $\Phi'(t')$ to be analytic at $t'=0$, Φ should go at least as t^{-2}. With this

argument we can neglect the higher derivative terms resulting in:

$$\begin{aligned}
&\frac{\langle \check{\sigma}^{s_1}_{n_1,m_1}(0)\check{\sigma}^{s_2}_{n_2,0}(1)(\sigma^{s_3}_{n_3,0})^\dagger(\infty)\rangle}{\langle \check{\sigma}^{s_3}_{n_3,0}(0)(\sigma^{s_3}_{n_3,0})^\dagger(\infty)\rangle} \\
&\qquad = a_1^{-\frac{\Delta_1^2}{n_1}} a_2^{-\frac{\Delta_2^2}{n_2}} a_3^{\frac{\Delta_3^2}{n_3}} \left(\frac{(2\Delta_1-m_1)!}{m_1!(2\Delta_1)!}\right)^{1/2} \left(\prod_{j=1}^{m_1} \frac{\lim_{r_j\to 0}}{(2\Delta_1-1)!}\partial_{r_j}^{(2\Delta_1-1)}\right) \\
&\qquad\quad \left(\prod_{k<j}(r_j-r_k)^2\right)\left(\prod_{j=1}^{m_1}(1-r_j)^{-2\Delta_2}\right) \\
&\qquad\quad \langle : \exp[i\,(\Delta_1+\Delta_2-m_1)\,\mathbf{e}\cdot\mathbf{\Phi}(0)] :: \exp[-i\Delta_3 \mathbf{e}\cdot\mathbf{\Phi}(\infty)] :\rangle \\
&\qquad = a_1^{-\frac{\Delta_1^2}{n_1}} a_2^{-\frac{\Delta_2^2}{n_2}} a_3^{\frac{\Delta_3^2}{n_3}} \left(\frac{(2\Delta_1-m_1)!}{m_1!(2\Delta_1)!}\right)^{1/2} \left(\prod_{j=1}^{m_1} \frac{\lim_{r_j\to 0}}{(2\Delta_1-1)!}\partial_{r_j}^{(2\Delta_1-1)}\right) \\
&\qquad\quad \left(\prod_{k<j}(r_j-r_k)^2\right)\left(\prod_{j=1}^{m}(1-r_j)^{-2\Delta_2}\right)\delta_{q_1+q_2,q_3}
\end{aligned} \tag{4.52}$$

Remember that the R-charge is related to the dimension through $q_i = \Delta_i - m_i$. The delta function just tells us that charge is conserved. There are no restrictions on dimensions in this part of the correlator, those will come from the demand that the cover has genus 0. Using the slopes from the cover map from (4.49) and the bosonic fusion coefficients from [17] the final form of the 3 point functions was, based on numerical calculations, conjectured to be:

$$\frac{\langle \check{\tau}^{s_1}_{n_1,m_1}(0)\check{\tau}^{s_2}_{n_2,m_2}(a)(\check{\tau}^{s_3}_{n_3,m_3})^\dagger(\infty)\rangle}{\langle \check{\tau}^{s_3}_{n_3,m_3}(0)(\check{\tau}^{s_3}_{n_3,m_3})^\dagger(\infty)\rangle} = |a|^{\Delta_3-\Delta_1-\Delta_2}\hat{C}^{s_1,s_2,s_3}_{n_1,n_2,n_3}\begin{pmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ q_1 & q_2 & -q_3\end{pmatrix} \tag{4.53}$$

$\begin{pmatrix}\Delta_1 & \Delta_2 & \Delta_3\\ q_1 & q_2 & -q_3\end{pmatrix}$is the $3j$ symbol [20] taking care of the SU(2) part of the correlator. The "reduced" fusion coefficient is defined through:

$$\begin{aligned}
\hat{C}^{s_1,s_2,s_3}_{n_1,n_2,n_3} &\stackrel{\text{def}}{=} \left(\frac{(\sum_i s_i n_i+1)^2}{4n_1n_2n_3}\right)^{1/2} \\
&\left(\frac{(\sum_i \Delta_i+1)!(-\Delta_1+\Delta_2+\Delta_3)!(\Delta_1-\Delta_2+\Delta_3)!(\Delta_1+\Delta_2-\Delta_3)!}{(2\Delta_1)!(2\Delta_2)!(2\Delta_3)!}\right)^{1/2}
\end{aligned} \tag{4.54}$$

These results hold in the case of a genus 0 cover. We will now show that the contributions of covers of this type dominate over higher genus surfaces in the sum over the conjugacy classes that we still have to do. We know that the first 2 cycles should combine to one cycle, namely the last one. The cycles should have some overlap or else the result will be two cycles. We can take the second cycle to have part of the first cycle in reverse order, and part containing new elements. To be specific:

$$\begin{aligned}
\omega_1 &= (1,\cdots,k,k+1,\cdots,n_1) \\
\omega_2 &= (k,k-1,\cdots,1,n_1+1,\cdots,n_1+n_2-k)
\end{aligned} \tag{4.55}$$

they combine into

$$\omega_3 = (k, k+1, \cdots, n_1 + n_2 - k)$$

resulting in a third cycle with length $n_3 = n_1 + n_2 - 2k + 1$.The total number of copies of M that get twisted is then $s = n_1 + n_2 - k$ and the genus of the corresponding cover is $g = (n_1 - 1 + n_2 - 1 + n_3 - 1)/2 - s + 1 = 0$. Any other choice of the second cycle will either result in more then one cycle or a longer cycle resulting in a higher genus.

The fraction of all possible choices from the 3 classes resulting in a genus 0 cover is determined as follows:

1. the first cycle we can choose freely from its class
2. we have n_1 choices for the start of the overlap k
3. we have $\begin{pmatrix} N - n_1 \\ n_2 - k \end{pmatrix} (n_2 - k)!$ choices for the remaining $n_2 - k$ members of the 2nd cycle
4. the 3rd cycle is fixed with the choice for the first two
5. we have to divide through the class size of the 2nd and 3rd cycle class

The final factor is thus

$$\frac{n_1 \begin{pmatrix} N - n_1 \\ n_2 - k \end{pmatrix} (n_2 - k)!}{\begin{pmatrix} N \\ n_2 \end{pmatrix} (n_2 - 1)! \begin{pmatrix} N \\ n_3 \end{pmatrix} (n_3 - 1)!} = \frac{n_1 n_2 n_3 (N - n_1)!(N - n_2)!(N - n_3)!}{N! N! (N - s)!} \tag{4.56}$$

Combining this with (4.24) results for large N in:

$$\frac{\langle \check{\tau}^{s_1}_{[n_1],m_1}(0) \check{\tau}^{s_2}_{[n_2],m_2}(a) (\check{\tau}^{s_3}_{[n_3],m_3})^\dagger(\infty) \rangle}{\langle \check{\tau}^{s_3}_{[n_3],m_3}(0) (\check{\tau}^{s_3}_{[n_3],m_3})^\dagger(\infty) \rangle} \simeq$$
$$(\frac{n_1 n_2 n_3}{N})^{1/4} \left(\frac{\langle \check{\tau}^{s_1}_{n_1,m_1}(0) \check{\tau}^{s_2}_{n_2,m_2}(a) (\check{\tau}^{s_3}_{n_3,m_3})^\dagger(\infty) \rangle}{\langle \check{\tau}^{s_3}_{n_3,m_3}(0) (\check{\tau}^{s_3}_{n_3,m_3})^\dagger(\infty) \rangle} \right)_{g=0} \tag{4.57}$$

Where n_1, n_2 and n_3 are chosen so that they combine to a twist, whose cover is a sphere and simlar for the anti-holomorphic sector (that is the reason for the exponent of 1/4th.).

Chapter 5

Chiral rings, cohomology and some algebra

The chiral primaries that we studied in the previous chapter were defined purely algebraic. To get a picture of the geometry behind these objects we need some understanding of the σ-model. This will lead us first to the beautiful interplay between differentiable structures and supersymmetry and the relation between the cohomology of the target space and the chiral primaries, this will make it possible to extract three point functions of chiral primaries from the ring structure of the cohomology ring. The first section is a brief incomplete review of these concepts. Our next task is the construction of the cohomology ring. We will first describe this from the orbifold description where we will see how some of the structure characteristics can be compared to those of the cohomology ring of a resolution of the singularities of the orbifold. Before we end this chapter, we will define a symmetric algebra out of the cohomology ring of the original $K3$ manifold. In that construction already some concepts of the previous chapter will show up and the resulting construction will turn out to be essential for the ring construction in the next chapter.

5.1 The chiral rings

The chiral primaries can be divided into four groups according to the choice of chiral or anti-chiral two from the holomorphic, "left", combined with 2 from the anti-holomorphic "right" sector. We know from invariance under scaling that the OPE of the product of 2 chiral primaries should be of the form (see eg. [10]):

$$\phi_i(z)\phi_j(w) \sim \sum_k C_{ijk}(z-w)^{h_k-h_i-h_j}\phi_k(w) \tag{5.1}$$

Where the sum is over all primary fields, secondary fields allways being less singular. We know that the charges of ϕ_i and ϕ_j, being chiral primary operators, are equal to their conformal weights. The charge of their product is the sum of the respective charges since

$$[J_0^3, \phi_i\phi_2] = [J_0^3, \phi_i]\phi_j + \phi_1[J_0^3, \phi_j] = (q_i + q_j)\phi_i\phi_j$$

and so we find as exponent in the OPE (5.1) $h_k - q_k$. But now we can make use of the BPS condition (3.25) to conclude that the OPE has no singular terms, and although the primaries in the OPE are not necessarily chiral, in the limit $z \to w$ the only terms that survive are the chiral ones. The chiral primaries apparently close under multiplication thus forming a ring [1], the **chiral ring**, under pointwise multiplication and addition. Analogous we find that the anti-chiral primaries form a ring. Combining the left and right movers we thus find 2 sets of 2 rings that we denote by (c,c) and (a,c) and their conjugate rings (a,a) and (c,a). Of course there is actually for every ring a two dimensional space of charges that annihilates it because we have $\mathcal{N}=4$ supersymmetry. Next to these rings we can use 0 mode ladder operators as we did in the last chapter to create operators within the chiral multiplet. In the Ramond sector it would have boiled down to a rotation within the chiral ring.

chiral ring

An instructive way to build the supersymmetric σ-model action is using the superspace formalism where the supersymmetry is inherent in the construction process (see for example [21]). For a Riemann surface Σ with a flat metric the resulting action is given by:

$$S = \int d\sigma d\tau (-g_{ij}(\phi)\partial^\mu \phi^i \partial_\mu \phi^j + i g_{ij}\bar{\psi}^i D_+ \psi^j + i g_{ij}\psi^j D_- \bar{\psi}^j + R(\phi)_{ijkl}\psi^i\psi^j\bar{\psi}^k\bar{\psi}^l) \tag{5.2}$$

The fields ϕ are the coordinates in a target manifold, whose metric is given by g. The ψ's are left chiral fermions. The barred fermions are right chiral. $D_\pm$ are covariant derivatives with respect to g:

$$D_\pm \psi^i = (\partial_\tau \pm \partial_\sigma)\psi^i + \Gamma(\phi)^i_{jk}(\partial_\tau \pm \partial_\sigma)\phi^j \psi^k \tag{5.3}$$

R_{ijkl} is the Riemann curvature. Since the metric depends on the bosonic fields, the action is called nonlinear. Note the resemblance to Yang-Mills theory if we look upon the target space as an internal gauge manifold. For us the flat metric on the $2d$ plane will be good enough. In case the 2 dimensional space time is curved the covariant derivative has to be modified with an extra term that defines how spinors are to be compared at different points in two dimensional spacetime, the so called spin connection.

The action is invariant under the following supersymmetry, that can be defined on any target space with a Riemannian metric:

$$\begin{aligned} \delta\phi^i &= \epsilon\bar{\psi}^i - \bar{\epsilon}\psi^i \\ \delta\psi^i &= i\bar{\epsilon}(\partial_\sigma + \partial_\tau)\phi^i + \epsilon\Gamma^i_{jk}\psi^j\bar{\psi}^k \\ \delta\bar{\psi}^i &= i\bar{\epsilon}(\partial_\sigma - \partial_\tau)\phi^i + \bar{\epsilon}\Gamma^i_{jk}\psi^j\bar{\psi}^k \end{aligned} \tag{5.4}$$

This symmetry also commutes with coordinate transformations from the target space. Here ϵ is an infinitesimal grassman number (an anticommuting number). When we try to define a second supersymmetry we find that we need a complex structure. In fact the target space has to be Kähler [22]. A third supersymmetry corresponds to a change from a complex structure to a quaternionic structure resulting in a hyperkähler manifold automatically admitting a fourth

[1] it is a ring since not every element has an inverse and multiplication is not per se commutative

supersymmetry. This is an important aspect for our further discussions since we will lift our symmetric orbifold to a smooth manifold that we now know has to be hyperkähler to be able to describe an $\mathcal{N} = 4$ supersymmetric theory.

Supersymmetry is not the only symmetry that restricts our choice of target manifold. We also want our theory to be conformally invariant. For this to be true in the quantum theory we should have a vanishing Ricci tensor on the target manifold [21]. Equivalent to the disappearing Ricci tensor is the the triviality of the **canonical bundle** of the target space. This is defined for an analytic $2n$-dimensional manifold as the line bundle of holomorphic top forms. That is of differential forms that in a local basis in complex coordinates can be written as $adz_1 \wedge \cdot dz_n$, with $a \in \mathbb{C}$. These demands restrict the possible allowed target manifolds to be so called Calabi-Yau manifolds (see appendix C.

canonical bundle

Let's now consider the chiral ring by spectral flow from the Ramond sector perspective. The chiral ring corresponds here to the groundstates. The argument is that the ground states can have no nonzero momentum and thus we just need to retain the zero modes of the fields, effectively reducing the theory to one dimension [23]. To see where this leads to we are going to look briefly at a one dimensional $N = 2$ model first

When we take the target space to be Kähler, the resulting ($\mathcal{N} = 2$) Lagrangian is [21]

$$L = g_{i\bar{j}} \frac{d\phi^i}{dt} \frac{d\bar{\phi}^{\bar{j}}}{dt} + i g_{i\bar{j}} \bar{\psi}^{\bar{j}} D_t \psi^i + i g_{i\bar{j}} \bar{\psi}^i D_t \psi^{\bar{j}} + R_{i\bar{j}k\bar{l}} \bar{\psi}^i \psi^k \psi^{\bar{j}} \bar{\psi}^{\bar{l}} \tag{5.5}$$

This action has the following supersymmetries:

$$\delta\phi^i = \epsilon_+ \bar{\psi}^i - \epsilon_- \psi^i \qquad \delta\bar{\phi}^{\bar{i}} = \bar{\epsilon}_+ \psi^{\bar{i}} - \bar{\epsilon}_- \bar{\psi}^{\bar{i}} \tag{5.6}$$

$$\delta\psi^i = i\bar{\epsilon}_- \frac{d\phi^i}{dt} - \epsilon_+ \Gamma^i_{jk} \bar{\psi}^j \psi^k \qquad \delta\bar{\psi}^{\bar{i}} = -i\epsilon_- \frac{d\bar{\phi}^{\bar{i}}}{dt} - \bar{\epsilon}_+ \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \bar{\psi}^{\bar{j}} \psi^{\bar{k}} \tag{5.7}$$

$$\delta\bar{\psi}^i = -i\bar{\epsilon}_+ \frac{d\phi^i}{dt} - \epsilon_- \Gamma^i_{jk} \bar{\psi}^j \psi^k \qquad \delta\psi^{\bar{i}} = i\epsilon_+ \frac{d\bar{\phi}^{\bar{i}}}{dt} - \bar{\epsilon}_- \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \bar{\psi}^{\bar{j}} \psi^{\bar{k}} \tag{5.8}$$

We are going to use canonical quantisation for this system. but first we need to extract the Noether charges for the supersymmetries. They can be calculated to be:

$$Q_+ = g_{i\bar{j}} \psi^i \bar{\phi}^{\bar{j}} \qquad Q_- = g_{i\bar{j}} \bar{\psi}^i \bar{\phi}^{\bar{j}} \qquad \bar{Q}_+ = g_{i\bar{j}} \bar{\psi}^{\bar{j}} \phi^i \qquad \bar{Q}_- = g_{i\bar{j}} \psi^{\bar{j}} \phi^i \tag{5.9}$$

Apart from supersymmetry there are symmetries that rotate the phase of the fermions resulting in two fermion numbers:

$$F_V = g_{i\bar{j}} (\bar{\psi}^{\bar{j}} \psi^i - \bar{\psi}^i \psi^{\bar{j}}) \qquad F_A = g_{i\bar{j}} (\bar{\psi}^{\bar{j}} \psi^i + \bar{\psi}^i \psi^{\bar{j}}) \tag{5.10}$$

For the quantisation we need the conjugate momenta for the fields:

$$\pi_{\phi^i} = g_{i\bar{j}} \frac{\bar{\phi}^{\bar{j}}}{dt} \qquad \pi_{\bar{\phi}^{\bar{j}}} = g_{i\bar{j}} \frac{\phi^j}{dt}$$
$$\pi_{\psi^i} = i g_{i\bar{j}} \bar{\psi}^{\bar{j}} \qquad \pi_{\psi^{\bar{j}}} = i g_{i\bar{j}} \bar{\psi}^i$$

From canonical quantisation we find the following anticommutation relations for the fermions:

$$i g_{k\bar{j}} \{\psi^i, \bar{\psi}^{\bar{j}}\} = i\delta^i_k \Rightarrow \{\psi^i, \bar{\psi}^{\bar{j}}\} = g^{\bar{j}} i \tag{5.11}$$

$$i g_{j\bar{k}} \{\psi^{\bar{i}}, \bar{\psi}^j\} = i\delta^{\bar{i}}_{\bar{k}} \Rightarrow \{\psi^i, \bar{\psi}^{\bar{j}}\} = g^{\bar{i}j} \tag{5.12}$$

We can now choose the operator odering for the Hamiltonian H such that the charges satisfy the following (anti-)commutation relations:

$$\{Q_+, \bar{Q}_+\} = \{Q_-, \bar{Q}_-\} = H \tag{5.13}$$

$$[F_V, Q_\pm] = -Q_\pm \quad [F_V, \bar{Q}_\pm] = -\bar{Q}_\pm \quad [F_A, Q_\pm] = \mp Q_\pm \quad [F_A, \bar{Q}_\pm] = \pm \bar{Q}_\pm$$

The anti-commutation relations of the conjugate fermions makes the algebra constructed out of products of these fields into an exterior product algebra, like the algebra of differential forms on a manifold. This analogue can be extended to a representation of the whole algebra, constructed above, with the identification:

algebra	$\bar{\psi}^i$	$\bar{\psi}^{\bar{i}}$	ψ^i	$\psi^{\bar{i}}$	Q_-	$\bar{Q}_+$	Q_+	$\bar{Q}_-$
geometry	dz^i	$d\bar{z}^{\bar{i}}$	$g^{\bar{j}i}\imath_{\bar{j}}$	$g^{\bar{i}j}\imath_j$	$-i\partial$	$-i\bar{\partial}$	$i\bar{\partial}^\dagger$	$i\partial^\dagger$

The ∂s are the Dolbeault operators (see appendix C) and the $\imath_i$ is the inner product operator of the basis vector $\partial/\partial z^i$. The Hamiltonian translates in this language to the Laplacian so that we can identify the groundstates with harmonic forms. And thus with cohomology classes.

A state made out of the product of $\bar{\psi}^i$s and $\bar{\psi}^{\bar{i}}$s is an eigenvector of F_V and F_A with eigenvalue $-p+q$ and $p+q$. with p and q counting the number of $\bar{\psi}^i$s and $\bar{\psi}^{\bar{i}}$s. This follows directly from the commutation relations (5.13).

When we compare these results finally with the zero mode algebra of the $\mathcal{N}=4$ theory in the Ramond sector:

$$\{G^a_0, G^b_0\} = \{\tilde{G}_{a,0}, \tilde{G}_{b,0}\} = 0 \qquad \{G^a_0, \tilde{G}_{b,0}\} = (2L_0 - \frac{c}{12})\delta^a_b \tag{5.14}$$

$$[J^3_0, G^1_0] = -\frac{1}{2}G^1_0 \quad [J^3_0, G^2_0] = \frac{1}{2}G^2_0 \quad [J^3_0, \tilde{G}_{1,0}] = \frac{1}{2}\tilde{G}_{1,0} \quad [J^3_0, \tilde{G}_{2,0}] = -\frac{1}{2}\tilde{G}_{2,0}$$

Comparing this with (5.13) leads to the identification of G^1_0 or G^2_0 with Q_- The Q_+ can be identified with charges in the antiholomorphic part of the algebra. The 3-direction of the R-symmetry current, here just in the holomorphic part of the algebra, measures the holomorphic degree, up to a constant, and so we have the identification $J^3_0 + c/12 \leftrightarrow \pm(F_A - F_V)/4$, the sign depending on which of the two choices we make for Q_-. The construction on the antiholomorphic part of the algebra goes analogous

We have established a correspondence between the ring of groundstates of the Ramond sector and the Dolbeault cohomology ring. With spectral flow we have now also the identification of the (a,c) or (c,c) ring with the cohomology ring. In fact we have we have a whole sphere of directions in which we can flow to from the Ramond sector to the NS sector. In the geometry this is related to the choice of complex structures we have on a hyperkähler manifold. In the next chapter we will come back to the $SU(2)$ symmetries behind this purely from the cohomology point of view.

We now take the central charge of the N-fold symmetric orbifold: $c = 6N$ The spectral flow from the Ramond sector in the J^3 direction with $s = \pm 1$ takes the following form on the zero modes of the supercharges and R symmetry:

$$\begin{aligned} &J^3_0 \to J^3_0 + \frac{N}{2}s \qquad J^\pm_0 \to J^\pm_{\mp s} \\ &G^1_0 \to G^1_{-s/2} \qquad G^2_0 \to G^2_{s/2} \qquad \tilde{G}_{1,0} \to \tilde{G}_{1,s/2} \qquad \tilde{G}_{2,0} \to \tilde{G}_{2,-s/2} \end{aligned} \tag{5.15}$$

We see that depending on s we either arrive in the chiral or in the anti-chiral ring. The R-symmetry charge is for operators in the (anti-)chiral ring half of the Dolbeault degree in cohomology language. Notice also that the zero-mode ladder operator, that moved us around in the Ramond groundtate $SU(2)$ multiplet does not map to the chiral ring anymore, but to operators in the chiral multiplet that are outside of the chiral ring. The $J^{\pm}_{\mp s}$ on the other hand do map to the chiral ring and it are just those operators that made it possible to move inbetween the two chiral primaries within one cycle class in the last chapter.

The restriction we have that the cohomology ring $H^{pq}(X)$ is identical zero for p or q larger than $2N$ for our $4N$-dimensional targetspace[2] we recover from the $\mathcal{N}=4$ algebra by examining the $\pm 3/2$ modes from the supercurrents, for example in the *chiral* ring:

$$\{G^1_{3/2}, \tilde{G}_{1,-3/2}\} = \{G^2_{-3/2}, \tilde{G}_{2,3/2}\} = 2L_0 - 6J^3_0 + 4N \tag{5.16}$$

for chiral primaries we find:

$$0 \leq \| \, G^1_{3/2}|\phi\rangle \, \|^2 = 2h - 6q + 4N = -4q + 4N$$

and so we find the sought for restriction that the Dolbeault degree $2q$ is bounded by $2N$.

The only missing link is how we can recover the three-point functions from the cohomology. In principle the answer was already given in the the OPE of the chiral primaries (5.1). The structure constants of the cohomology ring are the fusion coefficients of the chiral primaries from the previous chapter. As we mentioned above, this holds only for true chirals. The primaries that were generated by adding J^-_0 operators do not appear in the cohomology ring, They would if the ring would have been in the Ramond sector groundstate, but under spectral flow the J^-_0 changes in $J^-_{\pm 1}$, the sign depending on the diretion of the flow. On the other hand is the cohomology ring much richer then the two chiral primaries per sector we found using the CFT-construction.

In the symmetric orbifold (or in general any orbifold) the simple description outlined above can not be the whole picture. The theory also contains twisted sectors that are not reflected in the topology of the target manifold and won't appear in normal cohomology. It is possible to take into account the twisted sectors in a manner related to the way we have calculated correlation functions in chapter (4) by defining an **orbifold cohomology**[24]. This is in fact based on ideas coming from string theory [14] where an orbifold cohomology, for an orbifold X/G, is defined as: (this is in essence equivalent to (4.19):

orbifold cohomology

- For each conjugacy class $[g]$ in G we pick a member, g, and consider the subspace X^g of fixed points under the action of g.
- Let $H^*_{[g]}(X)$ be the part of the cohomology ring of X^g invariant under the centraliser C_g of g.
- The orbifold cohomology is now defined as

$$H^*(X/G) \stackrel{\text{def}}{=} \oplus_{[g]} H^*_{[g]}(X) \tag{5.17}$$

[2] It is "almost everywhere" $4N$ dimensional and the smooth resolution is that everywhere

We could also try to resolve the singularity of the orbifold and calculate the cohomology of the resulting nonsingular space. From the previous considerations concerning the target manifold, namely the Ricci flatness, or equivalently the disappearing of the first Chern class, we should make the same demand from the resulting smooth resolution. In other words we should demand that the canonical bundle is again trivial. There is of course the problem that in the way we have defined the canonical bundle, such an object does not exist on general varieties and even less if they are singular. There exists though a construction called a canonical divisor [25] that corresponds to the canonical bundle in case the variety is an analytic manifold. We will just abuse the word canonical bundle in the text. The symmetric orbifold of $K3$ has a trivial canonical bundle in thus loose sense and we would like to have a resolution which has the same. Such a resolution of singularities in which the canonical line bundle stays essentially the same is called a **crepant resolution**. It is generally believed that orbifold cohomology is isomorphic to the cohomology of its crepant resolution in the hyperkähler case. For a $K3$ surface this has recently been shown to be true by construction of an explicit ring isomorphism of the two rings [26].

crepant resolution

In the case of the n-fold symmetric orbifold of a complex surface a crepant resolution is the, to be defined, Hilbert scheme of n points of that surface and it is hyperkähler when the original surface is. This space can be seen as the moduli space of n points from that surface. The description of this space and construction of the cohomology ring will be of prime interest, its treatment will follow in the next chapter. We will start though doing some calculations concerning the orbifold cohomology and will see how this ring can be described in terms of a Fock space of string modes. The appearance of this algebra of oscillator modes has been a source of inspiration for the research done in that last 10 years in Hilbert scheme cohomology. We will then show how one can algebraicly construct an equivalent structure starting from the cohomology ring of the original surface. In the next chapter these two ideas will be used to give a full ring description of the cohomology ring of the crepant resolution of the orbifold and thus resulting in principle in the possibility to calculate 3-point functions of all the chiral primaries.

5.2 Orbifold cohomology of S^nK3

The calculation of a generating function for the Euler characteristic of the symmetric orbifold of a projective surface, in orbifold cohomological sense, was carried out by Hirzebruch and Höfer [27] and compared with the calculation of the Euler characteristic of its crepant resolution as carried out by Göttsche [28]. The orbifold calculation can be done using a Fock space description with a stringy interpretation of the twisted sectors [29]. The appearance of a Heisenberg algebra in this picture was formalised in the Hilbert scheme cohomology picture independently by Nakajima [30] and Grojnowski [31]. In this section we will briefly describe the calculation of the orbifold Euler characteristic using the Fock space language and show how the symmetric ring of the previous section is isomorphic to this Fock space.

Let's first consider the cohomology ring of the untwisted sector. This is the sector corresponding to the trivial conjugacy class [Id], whose centraliser is the whole group. The invariant subspace is just the whole n-fold symmetric product

of the surface. Using the Künneth formula [32] we find:

$$H^*_{[\mathrm{Id}]}(\times_n X) = \left(H^*(\times_n X)\right)^{S_n} = \left(H^*(X)^{\otimes n}\right)^{S_n} \tag{5.18}$$

In the first step we used the fact that the symmetrisation operator commutes with the exterior derivative. A way to think about this expression is as an n-particle state of (anti) commuting bosons and fermions, corresponding to the even respectively odd part of the cohomology ring. Although we will finally only need a bosonic Fock space, since $H^n(K3) = 0$ for n odd, let's for the time being keep it general. We take $\{a_i\}$ to form a basis of $H^*(X)$. We define creation operators, $\tilde{\mathfrak{p}}_{-1}(a_i)$, that (anti-)commute with each other. We can then write 5.18 symbolically as

$$H^*_{[\mathrm{Id}]}(\times_n X) = \{\prod_{i=1}^{n} \tilde{\mathfrak{p}}_{-1}(a_{j_i})|0\rangle\}$$

Now for a sector twisted by a permutation given in terms of a partition of n the invariant subspace is given by the cartesian product of one copy of X for each cycle in the partition. The centraliser of the permutation permutes cycles of equal length. This should look familiar by now! It is thus appropriate to introduce one set of creation operators for each cycle length separately. We will write them tentatively as $\tilde{\mathfrak{p}}_{-l}(a_i)$.

We can even make the picture complete by adding appropriate annihilation operators $\tilde{\mathfrak{p}}_l(a_i)$ while imposing the usual string oscillator (anti) commutation relations:

$$[\tilde{\mathfrak{p}}_l(a), \tilde{\mathfrak{p}}_m(b)]_\pm = l\delta_{l,-m}T(ab) \tag{5.19}$$

where the brackets define an anti commutator when both basis vectors have odd degree, This algebra is called an infinite dimensional Heisenberg algebra. the bilinear form $T(ab)$ is defined, up to a sign, as the integral over the cup product. We won't need the "annihilation operators" in this section, but we will come back to $T(ab)$. We next construct a zero mode Virasoro operator as a derivation on this algebra:

$$[L_0, \tilde{\mathfrak{p}}_{-1}(a)] = lL_0 \tag{5.20}$$

It counts the cycle length of the cycle created by $\tilde{\mathfrak{p}}_{-1}(a)$. The direct sum of the orbifold cohomology rings of all symmetric orbifolds of X corresponds then to the "Fock space of particles", $\mathcal{V}(H)$ generated by a basis of states created from the vacuum by the $\tilde{\mathfrak{p}}_{-l}(a_i)$ for $l > 0$. A basis-state is uniquely defined by giving a partition per cohomology class. The demand that a state $\prod \tilde{\mathfrak{p}}_{-l_i}(a_i)|0\rangle$ represents a class in $H^*(S^nX)$ translates then to a demand that its conformal weight equals n. $\mathcal{V}(H)$ can be written as a direct sum of components of weight n states :

$$\mathcal{V}(H) = \oplus_n \mathcal{V}(H)_n$$

We would like to know the **Euler characteristic** for S^nX in the orbifold sense, meaning that we take the sum over the dimension of the *orbifold* cohomology groups (the orbifold Betti numbers). It is easier though to calculate the generating function, since by summing over all n we loose the constraint that

Euler characteristic

all states need to have a fixed weight and just count states in the whole Fock space. This is in fact a common aspect of constructions related to the symmetric group: usually calculations are far easier to carry out when considering the sum over all n. In principle we are calculating string partition functions! The combinatorial calculations were already done by Euler some centuries ago though so let us make use of them:

$$\sum_{n=1}^{\infty} q^n \chi(S^n X) = \sum_{n=1}^{\infty} q^n \sum_m (-1)^m \dim H^m(S^n X)$$

$|\mathbf{p}(n)|$ is the number of partitions of n. For fermions the degree depends on the weight and we only should consider partitions $\tilde{\mathbf{p}}(n)$where no integer occurs twice

$$\begin{aligned}
&= \prod_{i\,\text{even}} \left(\sum_{n_i=0}^{\infty} q^{n_i} |\mathbf{p}(n_i)| \right)^{b_i(X)} \prod_{i\,\text{odd}} \left(\sum_{n_i=0}^{\infty} (-q)^{n_i} |\tilde{\mathbf{p}}(n_i)| \right)^{b_i(X)} \\
&= \prod_{i\,\text{even}} \left(\sum_{n_i=0}^{\infty} q^{n_i} |\mathbf{p}(n_i)| \right)^{b_i(X)} \\
&\qquad \prod_{i\,\text{odd}} \left(\sum_{n_i\,\text{even}}^{\infty} q^{n_i} |\tilde{\mathbf{p}}(n_i)| - \sum_{n_i\,\text{odd}}^{\infty} q^{n_i} |\tilde{\mathbf{p}}(n_i)| \right)^{b_i(X)}
\end{aligned}$$

This are the Euler functions [13]

$$= \prod_{n=1}^{\infty} \frac{\prod_{i\,\text{odd}} (1-q^n)^{b_i(X)}}{\prod_{i\,\text{even}} (1-q^n)^{b_i(X)}} \tag{5.21}$$

Poincaré polynomial
Hodge polynomial

The $b_i(X)$ are of course the Betti numbers of the surface X. Göttsche calculated actually the generating function for the **Poincaré polynomial** of $X^{[n]}$, defined as $P_t(X) \stackrel{\text{def}}{=} \sum_i t^i H^i(X)$. Later this calculation was refined to the **Hodge polynomial**, defined as $h(X^{[n]} : xy) \stackrel{\text{def}}{=} \sum_{p,q} h^{p,q}(X^{[n]}) x^p y^q$, among others by Cheah [33]. His result was:

$$\sum_{n=0}^{\infty} q^n h(X^{[n]} : xy) = \prod_{k=1}^{\infty} \left(\frac{\prod_{r+s\ \text{odd}} (1 + x^{r+k-1} y^{s+k-1} q^k)^{h^{r,s}(X)}}{\prod_{p+q\ \text{even}} (1 - x^{r+k-1} y^{s+k-1} q^k)^{h^{r,s}(X)}} \right) \tag{5.22}$$

From which it is easy to recover Göttsches result by realizing that $b_i(X) = \sum_k h^{k,i-k}(X)$ and thus

$$\begin{aligned}
\sum_{n=0}^{\infty} q^n P_t(X^{[n]}) &= \sum_{n=0}^{\infty} q^n h(X^{[n]} : tt) \\
&= \prod_{k=1}^{\infty} \left(\frac{\prod_{i\,\text{odd}} (1 + t^{2k+i-2} q^k)^{b_i(X)}}{\prod_{i\,\text{even}} (1 - t^{2k+i-2} q^k)^{b_i(X)}} \right)
\end{aligned} \tag{5.23}$$

From which the Euler characteristic follows by taking $t = -1$, leading indeed to the same result as the orbifold calculations. In the case of $K3$ the generating

function for the Euler characteristic takes the form:

$$\sum_{n=1}^{\infty} q^n \chi(S^n K3) = \prod_{n=1}^{\infty} (1-q^n)^{-24} \tag{5.24}$$

From (5.2) we could in principle read off all the Hodge numbers for any weight, or from (5.23) all the Betti numbers.

Although the dimensions of the cohomology rings are the same this doesn't mean that they have the same ring structure. When Vafa and Witten described the construction of the orbifold Euler characteristic, making use of stringy concepts as the infinite Heisenberg algebra it was natural to ask how theses structures could appear in the description of the cohomology ring of the corresponding Hilbert schemes. The next chapter will finally introduce the main concepts of these schemes, describe the appearance of an Heisenberg algebra and finally show how the cohomology ring of $X^{[n]}$ can be calculated from $H^*(X)$ using the symmetric algebra construction of section (5.3).

5.3 An algebraic intermezzo

Although the cohomology ring of the Hilbert scheme was known [34] to be isomorphic, as a vector space, to the Fock-space appearing in the orbifold cohomology calculation from a physical point of view (section 5.2), an explicit ring isomorphism, needed to calculate the structure constants of the cohomology ring, was not explicitly known. For compact surfaces with a trivial canonical bundle the explit ring structure was described in [35].

We first will build a so called graded Frobenius algebra on the cohomology ring of the surface, that will be isomorphic to the Fock space in the orbifold cohomology description of the last section. To start we will introduce the concept of a Frobenius algebra and investigate some of its properties, stressing those aspects that will be essential in forthcoming calculations. The notation of [35] will be used.

5.3.1 Frobenius Algebras

The cohomology ring of an even dimensional manifold is the prime exampe of special kind of algebra. So the real dimension of the manifold X, from which we are going to describe the cohomology ring, will be even and we will write it as $2d$. For the case that we will consider mostly, $K3$, $d = 2$. We start by defining an inner product on the cohomology ring. Although we should consider cohomology with rational coefficients as a consequence of the algebraic description of the varieties, in practice it is sufficient to work with de Rham cohomology (real coefficients) and we will be sloppy with the use of manifolds versus varieties and integration of forms, versus evaluation on cycles. So for example when we write $\int a$ for a class $a \in H^{2d}(X)$, then we mean evaluation over the **fundamental cycle** representing the manifold X, or from a more practical point of view, integration over the manifold, of a top form representing the class . fundamental cycle

Since X is compact we don't have to differentiate between normal cohomology and cohomology with compact support and will freely move, using Poincaré duality, between elements in $H_*(X)$, as cycles or closed submanifolds, and classes in $H^*(X)$.

Integration over the manifold provides us with a map from the cohomology ring to $\mathbb{Q}$, from which we can construct a non degenerate bilinear map on the cohomology ring as follows:

$$T(\alpha\beta) := -\int(\alpha \wedge \beta) \text{ for } \alpha, \beta \in H^*(X)$$

where the integral is zero for components not in the top class, the minus is a matter of convention. Note that the statement that T is nondegenerate is equivalent to Poincaré duality. It is notationally handy to work with a shifted cohomology ring, $\mathcal{H}^*(X) \stackrel{\text{def}}{=} H^*(X)[d]$ where the degree of a class is the cohomological degree $-d$, making the grading symmetrical around zero. The wedge product now has degree d so that we can add the degrees when multiplying 2 elements. When there is no confusion possible we will shorten the notation even further by defining $\mathcal{H} \stackrel{\text{def}}{=} \oplus_{n=-d}^{d}\mathcal{H}^n(X)$. $\mathcal{H}$ is an example of a Frobenius algebra:

Frobenius algebra

Definition 5.3.1. *A (graded) **Frobenius algebra** is a (graded) associative algebra A over a field k with a unit and a bilinear form $T : A \times A \to k$ such that*

1. *T is nondegenerate: $T(a, b) = 0, \forall b \in A \Leftrightarrow a = 0$*
2. *T is associative: $T(ab, c) = Ta, bc)$*

degree

We will call d also the **degree** of the algebra. We write $|a| = k$ for the degree of an element $a \in \mathcal{H}^k$ The unit has degree $-d$ and is here naturally the element $1 \in \mathbb{Q} \cong H^0(X)$

On the n-fold tensor product of $\mathcal{H}$ we define the multiplication as.

$$\Delta^{(n)*}(\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n, \beta_1 \otimes \beta_2 \otimes \cdots \otimes \beta_n) \stackrel{\text{def}}{=} \text{sign}(\{\alpha\}\{\beta\})\alpha_1 \wedge \beta_1 \otimes \otimes \cdots \otimes \alpha_n \wedge \beta_n$$

signature

Where $\text{sign}(\{\alpha\}\{\beta\})$ is the reorderings **signature** needed to pair the components, given by a factor $(-1)^{|a||b|}$ when interchanging two elements a and b. We won't be taking the signature into account in the following, since the cohomology ring of $K3$ is even graded. The whole construction is nonetheless easily extended to any Frobenius algebra, by just using the signature whenever an operation exchanges two elements. The reason for the notation used for the product operator will become clear in the following.

We get again a graded Frobenius algebra, now of degree dn using as map to $\mathbb{Q}$:

$$T(\alpha_1 \otimes \cdots \otimes \alpha_n) := T(\alpha_1) \cdots T(\alpha_1) \tag{5.25}$$

This construction of a tensor product can be generalised to a tensor product with respect to any indexset, I with finite cardinality n, by symmetrising over all bijections to $[n]$ in a two step process:

1. Take the direct sum over the n-fold tensor product by all bijections, defining the algebra product per component. For example if the index set is $\{a, b\}$ a typical product of two algebra elements looks like:

$$(\alpha_a \otimes \alpha_b \oplus \alpha_b \otimes \alpha_a) \cdot (\beta_a \otimes \beta_b \oplus \beta_b \otimes \beta_a) =$$
$$\alpha_a \cdot \beta_a \otimes \alpha_b \cdot \beta_b \oplus \alpha_b \cdot \beta_b \otimes \alpha_a \cdot \beta_a$$

2. Make the tensor product symmetric by dividing out by the action of S_n on the indexset $[n]$. In the above example by the equivalence:

$$\alpha_a \otimes \alpha_b \oplus \alpha_b \otimes \alpha_a \sim \alpha_b \otimes \alpha_a \oplus \alpha_a \otimes \alpha_b$$

The final general tensor product can be written as:

$$\mathcal{H}^{\otimes I} \stackrel{\text{def}}{=} \big(\oplus_{f:[n]\xrightarrow{\sim} I} \big(\mathcal{H}_{f(1)} \otimes \cdots \otimes \mathcal{H}_{f(n)}\big)\big)\big/ S_n \tag{5.26}$$

The usefulness of this definition lies in the way any isomorphism $f : [n] \to I$ induces a canonical isomorphism between $\mathcal{H}^{\otimes[n]}$ and $\mathcal{H}^{\otimes I}$

The multiplication in $\mathcal{H}$ can be seen as a pullback from the diagonal embedding as follows:

$$\begin{aligned} \Delta^{(2)} : X &\to X \times X \qquad (5.27)\\ p &\mapsto (p,p)\\ \text{wedge product : } \Delta^* : \mathcal{H}(X) \otimes \mathcal{H}(X) &\simeq \mathcal{H}(X \times X) \to \mathcal{H}(X) \qquad (5.28) \end{aligned}$$

Where in the last line again the Künneth formula ([32]) has been applied. Dualising using T we also have a **coproduct**, $\Delta^{(2)}_* : \mathcal{H}(X) \to \mathcal{H}(X) \otimes \mathcal{H}(X)$ also of degree d explicitly we have : coproduct

$$\big(-\int (\Delta^{(2)}_* \alpha)_1 \wedge \beta_1\big)\big(-\int (\Delta^{(2)}_* \alpha)_2 \wedge \beta_2\big) = -\int \alpha \wedge \beta_1 \wedge \beta_2 \tag{5.29}$$

It is an immediate consequence of the (anti)symmetry of the product that the coproduct is symmetric, that is to say that given an element $a \in \mathcal{H}$, there exists a $b \in \mathcal{H}$, perhaps zero, such that $\Delta^{(2)}_*(a) = b \otimes b$, although b is not necessarily of pure degree. This is one of the properties of the coproduct that will be of use in forthcoming calculations so lets state and proof it in a proper way:

Theorem 5.3.1. *Let $\mathcal{H}$ be a Frobenius algebra, its coproduct has the following properties*

1. *symmetry $a \in \mathcal{H}$ there is a (unique) $b \in \mathcal{H}$ such that $\Delta^{(2)}_* a = b \otimes b$*
2. *commutes with scalar multiplication $\Delta^{(2)}_* \lambda a = \lambda \Delta^{(2)}_* a$ for all $\lambda \in \mathbb{Q}, a \in \mathcal{H}$*

Proof. 1. Suppose e_i is a basis of $\mathcal{H}$. It suffices to proof the statement for $a = e_i$. Since $\Delta^{(2)}_* e_i \in \mathcal{H} \otimes \mathcal{H}$ we can write $\Delta^{(2)}_* e_i = f_i^{jk} e_j \otimes e_k$. By definition we have

$$\begin{aligned} T\big(\Delta^{(2)}_* e_i \cdot a_1 \otimes a_2\big) &= f_i^{jk} T\big(e_j \otimes e_k \cdot a_1 \otimes a_2\big)\\ &= f_i^{jk} T\big(e_j \cdot a_1\big) T\big(e_k \cdot a_2\big) = T\big(e_i \cdot a_1 \cdot a_2\big)\\ &= T\big(e_i \cdot a_2 \cdot a_1\big) = T\big(\Delta^{(2)}_* e_i \cdot a_2 \otimes a_1\big)\\ &= f_i^{kj} T\big(e_j \cdot a_1\big) T\big(e_k \cdot a_2\big) \end{aligned}$$

and comparing the second line with the last we conclude that $f_i jk = f_i^{kj}$.

2. Using again the definitions:

$$\begin{aligned} T\big(\Delta_*^{(2)}(\lambda a)\cdot(b_1\otimes b_2)\big) &= \lambda T\big(a\cdot b_1\cdot b_2\big) \\ &= \lambda T\big(\Delta_*^{(2)}(a)\cdot(b_1\otimes b_2)\big) \end{aligned}$$

and the stated follows from the non degeneracy of T.

□

Euler class

Using the coproduct we define the **Euler class**[3] of the Frobenius algebra as:

$$e := \Delta^{(2)*}\Delta_*^{(2)}(1)$$

Since, both multiplication and co-multiplication, have degree d, we see that $|e| = d$.

A surjective map ϕ from $[n]$ onto $[m]$ induces a natural "pullback", ϕ^* from $\mathcal{H}^*(X)^{\otimes n}$ to $\mathcal{H}^*(X)^{\otimes m}$ of degree $d(n-m)$, by multiplying the elements that get mapped to the same index, and because of the nondegeneracy of T a dual "pushforward". ϕ_* in the opposite direction and of the same degree. They are pullbacks and pushforwards from the appropriate diagonal embeddings. In the case $m = 1$ we will use the notation $\Delta_*^{(n)}$ and $\Delta^{(n)*}$. This n- fold coproduct has again the same symmetry properties as in the 2-fold case mentioned above. These maps have a natural generalisation to surjective maps between finite index sets I and J, and their induced pullbacks and pushforwards between the generalised tensor products as introduced above. This is a direct consequence of the canonical isomorphism between $\mathcal{H}^{\otimes[n]}$ and $\mathcal{H}^{\otimes I}$. It is easy to see that these maps both have a degree $d(|I| - |J|)$

The extension of the Euler class to an n-fold tensor product can be given formally, using the constant map $\phi : [n] \to [1]$, as

$$\begin{aligned} e^{[n]} &:= e^{n-1} \\ &= \otimes_{i\in[1]} e^{-1} \end{aligned} \tag{5.30}$$

With $|\phi^{-1}(i)|$ we get the number of elements that get mapped to i. Since e has degree d It is easy to check that this is indeed the right expression in the sense that $e^{[n]} = \phi^*\phi_*(1)$ at least formally since in fact $e^{[n]} = 0$ for n larger then 1. The use of this definition is in the possibility to extend it to general index sets $\phi : I \to J$ as

$$e^{I,J} := \otimes_{i\in J} e^{\phi^{-1}(i)-1}. \tag{5.31}$$

with the property : $e^{I,J} = \phi^*\phi_*(1_J)$. An even more general definition can be made using a function $\nu : I \to \mathbb{N}_0$ and the definition:

$$e^{\nu} := \otimes_{i\in I} e^{\nu(i)} \tag{5.32}$$

[3]The name Euler class is more than suggestive: the normal bundle of $\Delta^{(2)}(X)$ in $X \times X$ is isomorphic to the tangent bundle, TX. Since

$$\int_{X\times X}(\alpha_1\otimes\alpha_2)\Delta_*^{(2)}(1) = \int_X \alpha_1 \int_X \alpha_2$$

we see that $\Delta_*^{(2)}(1)$ corresponds in fact with the Thom class of the normal bundle and the Euler class in cohomology is the pullback of the Thom class by the zero section inclusion [32].

To calculate the degree of e^ν we use the fact that $|e| = d$ and that multiplication also has degree d:

$$\begin{aligned} |e^\nu| &= \sum_{i\in I} |e^{\nu(i)}| \\ &= \sum_{i\in I} (d(\nu(i)-1) + d\nu(i)) \end{aligned}$$

the first term comes from the multiplication, the second term from the degree of e

$$= 2d\sum_{i\in I}\nu(i) + d|I| \tag{5.33}$$

The reader might object that this manipulation is ill defined since any power of e larger than 1 gives zero, but the cohomology ring is a ring over $\mathbb{Q}$ and so we still can formally set $|e^n| = dn$ We will need this definition to define a multiplication in the twisted tensor product. There ν will be the equivalent of the genus of the cover from chapter 4 with respect to three matching twists.

5.3.2 The symmetric ring

As we know now from the previous section we can equip $\mathcal{H}^{\otimes n}$ with a bilinear form that makes it again a graded Frobenius algebra. The first step we will take is to define an analogue to the Fockspace approach of Vafa and Witten in terms of this algebra. It is clear that the symmetric group will have an important part in this construction. We start by defining the analogue of the genus of the cover (4.25): the **graph defect** of two permutations π and $\sigma \in S_n$ is defined as a map from the orbitspace of the subgroup generated by both π and σ to $\mathbb{N}$: graph defect

$$g(\pi,\sigma)(o) \stackrel{\text{def}}{=} \frac{1}{2}(l(o) + 2 - |\langle\pi\rangle\backslash o| - |\langle\sigma\rangle\backslash o| - |\langle\pi\sigma\rangle\backslash o|) \tag{5.34}$$

where $o \in \langle\pi,\sigma\rangle\backslash[n]$, $l(o)$ is the orbitlength as defined in the appendix (A) and with $|\langle\pi\rangle\backslash o|$ we mean the order (number of elements), of the resulting orbitspace. As an example we consider S_3. From the multiplication and orbit tables for S_3, table A and A.2 on page 65, we can read off the needed values to find (up to conjugacy) :

That this graphdefect is a nonnegative integer can be proven straight away geometrically. For us it is nice to use the intuition from our previous calculations in section 4.5) Let σ, π and $\sigma\pi$ be the cycles corresponding to the 3 insertions. We suppose that the two cycles do overlap and that o is the orbit representing the copies of m that get permuted, the other orbits all being trivial (having length 1). Note that $l(o)$ corresponds to s, the total number of copies of M that gets permuted, and $l(o) - |\langle\sigma\rangle\backslash o| + 1$ is equal to the ramification index for an insertion corresponding to the cycle σ. We can conclude that the two equations indeed match. This is a nice correspondence since it was argued in that first calculation that only genus zero contributions mattered in the large N limit.

From the graded Frobenius algebras $\mathcal{H}^{\otimes n}$ we build the following **twisted tensor product**: twisted tensor product

$$\mathcal{H}\{S_n\} \stackrel{\text{def}}{=} \bigoplus_{\pi\in S_n} (\mathcal{H}^{\otimes\langle\pi\rangle\backslash[n]}, \pi) \tag{5.35}$$

$$
\begin{aligned}
g(\mathrm{Id},\sigma)(o) &= \frac{1}{2}(l(o)+2-l(o)-1-1) = 0 \text{ for } \sigma \in S_3, o \in \langle\sigma\rangle\backslash[3] \\
g((12),(12))(\{1,2\}) &= \frac{1}{2}(2+2-1-1-2) = 0 \\
g((12),(12))(\{3\}) &= \frac{1}{2}(1+2-1-1-1) = 0 \\
g((12),(13))(\{1,2,3\}) &= \frac{1}{2}(3+2-2-2-1) = 0 \\
g((12),(123))(\{1,2,3\}) &= \frac{1}{2}(3+2-2-1-2) = 0 \\
g((123),(132))(\{1,2,3\}) &= \frac{1}{2}(3+2-1-1-3) = 0 \\
g((123),(123))(\{1,2,3\}) &= \frac{1}{2}(3+2-1-1-1) = 1
\end{aligned}
$$

Table 5.1: S_3 graph defects

The tensor product in the above is in the generalised sense as defined in (5.26) indexed by elements in the orbit space. We will use the convention to order the tensor product in decreasing size of the orbit length. We will call n the **weight** and $|(a,\pi)| \stackrel{\text{def}}{=} |a|$ the **degree** of an element of $\mathcal{H}\{S_n\}$. The rationale behind calling n the weight will become clear in the next section.

weight

degree

We next will give this vector space a ring structure by adding a multiplication. When $G \subset H$ are both subgroups of S_n then there is a natural surjection $f(G,H) : G\backslash[n] \to H\backslash[n]$ and so we can define pullbacks and pushforwards as introduced in the previous section. We use the notation:

$$f^{G,H} \stackrel{\text{def}}{=} f(G,H)^* : \mathcal{H}^{\otimes G\backslash[n]} \to \mathcal{H}^{\otimes H\backslash[n]} \text{ for the pullback} \tag{5.36}$$

$$f_{H,G} \stackrel{\text{def}}{=} f(G,H)_* : \mathcal{H}^{\otimes H\backslash[n]} \to \mathcal{H}^{\otimes G\backslash[n]} \text{ for the pushforward} \tag{5.37}$$

The multiplication in $\mathcal{H}\{S_n\}$ is now defined as:

$$(a,\pi)\cdot(b,\rho) \stackrel{\text{def}}{=} \left(f_{\langle\pi,\rho\rangle,\langle\pi\rho\rangle}(f^{\langle\pi\rangle,\langle\pi,\rho\rangle}(a) f^{\langle\rho\rangle,\langle\pi,\rho\rangle}(b) e^{g(\pi,\rho)}), \pi\rho\right) \tag{5.38}$$

The claim is that with this product the twisted tensor product is a ring. Per construction the pullbacks $f^{\langle\pi\rangle,\langle\pi,\rho\rangle}$ and $f^{\langle\rho\rangle,\langle\pi,\rho\rangle}$ both map to $\mathcal{H}^{\otimes\langle\pi,\rho\rangle\backslash[n]}$. Since $\langle\pi,\rho\rangle$ is the smallest subgroup containing both permutations, $\mathcal{H}^{\otimes\langle\pi,\rho\rangle\backslash[n]}$ is the "largest" tensor product where both elements can be pulled back to. The product of the images, multiplied by a weight that more or less measures the triviality of the overlap of the permutations involved, gets pushed forward to $\mathcal{H}^{\otimes\langle\pi\rho\rangle\backslash[n]}$. The Euler factor takes care of the associativity of the product. See for the rather technical proof of the associativity [35]. Note that for $\langle\pi\rangle = \langle\rho\rangle = \mathrm{Id}$ we recover the untwisted n-fold tensor product multiplication. To calculate the degree of the multiplication we have to take into account the degrees of all maps

and elements involved:

$$\begin{aligned}
|(a,\pi)\cdot(b,\rho)| =& |a|+|b| && \text{degree of arguments}\\
&+2(|\langle\pi\rho\rangle\backslash[n]|-|\langle\pi,\rho\rangle\backslash[n]|) && f_{\langle\pi,\rho\rangle,\langle\pi\rho\rangle}\\
&+2(|\langle\pi\rangle\backslash[n]|-|\langle\pi,\rho\rangle\backslash[n]|) && f^{\langle\pi\rangle,\langle\pi,\rho\rangle}\\
&+2(|\langle\rho\rangle\backslash[n]|-|\langle\pi,\rho\rangle\backslash[n]|) && f^{\langle\rho\rangle,\langle\pi,\rho\rangle}\\
&+4\sum_{o\in\langle\pi,\rho\rangle\backslash[n]} g(\pi,\rho)(o)-2|\langle\pi,\rho\rangle\backslash[n]| && e^{g(\pi,\rho)} \text{ from (5.33)}\\
&+2|\langle\pi,\rho\rangle\backslash[n]| && \text{2 multiplications}
\end{aligned}$$

using 5.34 we find that:

$$\sum_{o\in\langle\pi,\rho\rangle\backslash[n]} g(\pi,\rho)(o)=\frac{1}{2}(n+|\langle\pi,\rho\rangle\backslash[n]|-|\langle\pi\rangle\backslash[n]|-|\langle\rho\rangle\backslash[n]|-|\langle\pi\rho\rangle\backslash[n]|$$

and so:

$$|(a,\pi)\cdot(b,\rho)|=|a|+|b|+2n \tag{5.39}$$

showing that the multiplication has degree $2n$.

On $\mathcal{H}\{S_n\}$ we have a natural S_n action as follows: let $(a,\sigma)\in(\mathcal{H}^{\otimes\langle\sigma\rangle\backslash[n]},\sigma)$. For each $\pi\in S_n$ we define $f(\pi):\langle\sigma\rangle\backslash[n]\to\langle\pi\sigma\pi^{-1}\rangle\backslash[n]$ in the natural way,. Since this is a bijection we can define an isomorphism through the pullback $f(\pi)^*$, like we previously defined, and an automorphism $\hat{\pi}:\mathcal{H}\{S_n\}\to\mathcal{H}\{S_n\}$ through:

$$\hat{\pi}(a,\sigma)\stackrel{\text{def}}{=}(f(\pi)^*a,\pi\sigma\pi^{-1}) \tag{5.40}$$

We finally define $\mathcal{H}^{[n]}$ to be the invariant subring of $\mathcal{H}\{S_n\}$ under this action. That this is indeed a subring and thus closes under multiplication is a straightforward consequence of the definitions for details we refer again to [35]. As an example we will construct the different rings for S_3. Let's start with $\mathcal{H}\{S_n\}$. The orbitspace structure for S_3 is given in table A.2 on page 67. From there it is easy to read off the ringstructure:

$$\begin{aligned}
\mathcal{H}\{S_3\}=&(\mathcal{H}^{\otimes 3},\mathrm{Id})\oplus(\mathcal{H}^{\otimes 2},(12))\oplus(\mathcal{H}^{\otimes 2},(13))\oplus(\mathcal{H}^{\otimes 2},(23))\\
&\oplus(\mathcal{H},(123))\oplus(\mathcal{H},(132))
\end{aligned} \tag{5.41}$$

To work out multiplication in $\mathcal{H}\{S_3\}$ we use the results from table A and table 5.1. Two examples are worked out below.

$$\begin{aligned}
(a_1\otimes a_2,(12))\cdot(b_1\otimes b_2,(12)) &= \left(f_{\langle(12)\rangle,\langle\mathrm{Id}\rangle}((a_1\otimes a_2)(b_1\otimes b_2)(e^0\times e^0),\mathrm{Id}\right)\\
&= \left(f_{\langle(12)\rangle,\langle\mathrm{Id}\rangle}\left(a_1b_1\otimes a_2b_2\right),\mathrm{Id}\right)\\
&= \left(\Delta_*^{(2)}(a_1b_1)\otimes a_2b_2,\mathrm{Id}\right)\\
(a,(123))\cdot(b,(123)) &= \left(f_{\langle(123)\rangle,\langle(132)\rangle}(abe^1),(132)\right)\\
&= (abe,(132))
\end{aligned}$$

The multiplication table for $\mathcal{H}\{S_3\}$ can be found on page 81.

Note again that since e is a topform $e^n = 0$ for $n > 1$ and so for example in $\mathcal{H}\{S_5\}$:

$$\begin{aligned}\big(a,(1234)\big)\cdot\big(a,(1234)\big) &= \big(f_{\langle(1234)\rangle,\langle(1234)\rangle}(abe^2),(14523)\big)\\ &= \big(0,(14523)\big)\end{aligned}$$

From the multiplication table we can see that the multiplication is commutative up to conjugacy and so we see that indeed $\mathcal{H}^{[n]}$ is a subring.

Now for $\mathcal{H}^{[3]}$. The elements in this ring are of the form:

$$\begin{aligned}&\big(a\otimes b\otimes c,\mathrm{Id}\big)\oplus\cdots\oplus \text{ all permutations of } a,\ b \text{ and } c\\ &\big(a\otimes b,(12)\big)\oplus\big(a\otimes b,(13)\big)\oplus\big(b\otimes a,(23)\big)\\ &\big(a,(123)\big)\oplus\big(a,(123)\big)\end{aligned}$$

So that we find

$$\mathcal{H}^{[3]} \simeq S^3\mathcal{H}\oplus(\mathcal{H}\otimes\mathcal{H})\oplus\mathcal{H} \tag{5.42}$$

The general recipe to build the symmetric ring $\mathcal{H}^{[n]}$ is the following:

- draw all different Young diagrams corresponding to the partitions of n
- each row (cycle) corresponds to a copy of the shifted cohomology ring
- take the tensor product of the rows, symmetrising over rows of equal length.
- $\mathcal{H}^{[n]}$ is the direct sum of the resulting tensor products from all Young diagrams

The following construction of $\mathcal{H}^{[4]}$ illustrates the process:

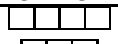
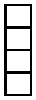
Young diagram	Tensor product
	$\mathcal{H}$
	$\mathcal{H}\otimes\mathcal{H}$
	$S^2\mathcal{H}$
	$\mathcal{H}\otimes S^2\mathcal{H}$
	$S^4\mathcal{H}$
$\mathcal{H}^{[4]}$	$\simeq\ S^4\mathcal{H}\oplus\mathcal{H}\otimes S^2\mathcal{H}\oplus S^2\mathcal{H}\oplus\mathcal{H}\otimes\mathcal{H}\oplus\mathcal{H}$

Table 5.2: construction of $\mathcal{H}^{[4]}$

We see now that there is a vector space isomorphism, through the Young diagrams that define the basis from this symmetric algebra to the Fock space representing the symmetric orbifold cohomology ring for all n:

$$\Phi : \mathcal{H}^{[n]} \to \mathcal{V}(H)_n \tag{5.43}$$

since symmetrising over rows of equal length corresponds to symmetrisation over equal mode oscillator operators.

Chapter 6

The cohomology ring of $X^{[n]}$

After introducing the concept of the Hilbert scheme of points, we will discuss the construction of the Heisenberg algebra analogue from the orbifold calculations, but now seen from the Hilbert scheme perspective. Build upon these concepts an explicit ring isomorphism can be constructed between the symmetric Frobenius ring and the cohomology ring of $K3^{[n]}$. This can finally be used to give an explicit prescription how to calculate 3 point functions of chiral primaries in the symmetric orbifold using as ingredient the cohomology ring structure of the original surface. We will use a shifted cohomology ring also on the Hilbertscheme side of the story and so define:

$$\begin{aligned} \mathbb{H}_n &\stackrel{\text{def}}{=} H^*(X^{[n]})[2n] \\ \mathbb{H} &\stackrel{\text{def}}{=} \oplus_n \mathbb{H}_n \end{aligned} \tag{6.1}$$

6.1 The Hilbert Scheme of points

Although the language of schemes will be used in this chapter, for which the appendix (D) gives the key concepts, the scheme that will be of interest to us is in fact a smooth variety as far as the original surface is a smooth variety. All schemes will be projective.

The symmetric orbifold is of course perfectly regular almost everywhere, that is in all points where the coordinates of the different copies of the defining surface differ from each other. There where 2 or more "coordinates" coincide the space will be singular though since in a sense the tangent space in those points becomes smaller. Technically the scheme we will describe now is a blow up of these singularities.

When talking about a **moduli space** of schemes we need to have some notion of what we demand of schemes to be in the same set of schemes. We first need the notion of a family of schemes. The analogue in the realm of differentiable manifolds is a fibre bundle where the fibres are the manifolds in the "family" parameterised by the base manifold. A **family** of schemes is defined as morphism between two schemes, $\phi : A \to B$. The fibres above any point in B are the schemes that make up the family: $\{\phi^{-1}(p)|p \in B\}$. This is still a very vague notion since those fibres can be totally different from each other.! On the other hand it is not possible to translate the notion of local triviality because

moduli space

family

of the coarseness of the Zariski topology. The right way to proceed is to start from the algebraic concept of flatness of modules (see appendix D.2). In the case of nonsingular varieties it happens to be that the fibres close to each other are similar enough such that any family is flat (see [36]).

This flatness is alltogether a pretty abstract notion, it can be shown though that it is the unique extension that reduces to the idea that any fibre can be seen as the limit of fibres close by "whenever this makes sense" [37]. The most important geometrical consequence of flatness, for our discussion, is that some numerical properties of projective schemes, related to the dimension of the scheme and degree of the defining polynomials are constant throughout a flat family. Hereto we have to introduce the notion of the Hilbert polynomial.

Hilbert polynomial

Roughly speaking the **Hilbert polynomial**, P_X of the scheme X, is the unique polynomial mapping integers into integers such that $P_X(n)$ equals the dimension of the homogeneous part of degree n of its coordinate ring (remember our schemes are projective) for sufficiently large n. The degree of the polynomial is the dimension of the scheme. In the case the scheme consists of a finite number of points, the degree is indeed 0, implying that the the polynomial is constant. The uniqueness of the polynomial is an old result by Hilbert.

The importance of the Hilbert polynomial is that a family of schemes is flat iff the members have the same polynomial. Knowing this we can now, given any (projective) scheme X, try to construct a flat family of closed subschemes of X having a given Hilbert polynomial, in particular we could consider the flat family of zero dimensional subschemes of given length.

Now this is easily stated, but where does this scheme structure come from that we put on the collection of subschemes. The answer is actually a rather deep result from Grothendieck. For some details and references to the literature see appendix D.2. What is important for us is that we can identify the collection of zero dimensional subschemes of fixed length n with a projective scheme that is

Hilbert scheme of points

called the **Hilbert scheme of points** in X. We will use the common notation $X^{[n]}$ for this scheme. The length corresponds to the number of points in the fibre.

As was mentioned in the last section, $X^{[n]}$ is a crepant resolution of the symmetric orbifold S^nX, of course under the assumption that X does not contain singularities. This is true in the case X is 2 dimensional (here we mean 2 complex dimensions !) . The smoothness of $X^{[n]}$ is a result from Fogarty (1968). That the resolution is crepant was proved by Beauville in 1983. For a proof of these theorems the main reference is [34], where most of the basics concerning these Hilbert schemes is explained in great detail. A preliminary version of this book might still be available in electronic format on the internet.

In case X is one dimensional (eg. a Riemann surface) the Hilbert scheme is in fact equivalent to the symmetric product, for higher dimensional varieties the Hilbert scheme of points is not smooth, so the relation between the Hilbert scheme and the symmetric product is really special for complex surfaces. The crepancy of the resolution furthermore implies that $X^{[n]}$ is hyperkähler whenever X is.

Hilbert-Chow map

The resolution map from $X^{[n]}$ to S^nX is called the **Hilbert-Chow map** and it maps a point in $X^{[n]}$ to the sum over the points in the subscheme it represents (in other words over the cycles in the subscheme) weighted by their multiplicity (by the length of the cycle).

To get a better understanding of the geometry of the Hilbert scheme we will

first look at the Hilbert scheme of points in $\mathbb{C}$. A subscheme of $\mathbb{C}$ of length n is described by a polynomial of degree n with complex coefficients:

$$\sum_{i=0}^{n} a_i z^i \text{ and } a_n \neq 0$$

since $a_n \neq 0$ we can normalise it to 1 and so can describe the scheme as a point in $\mathbb{C}^n$. Now it happens to be that also $S^n\mathbb{C} \simeq \mathbb{C}^n$ this can be seen by noticing that a point $(z_1, \cdots, z_n) \in S^n\mathbb{C}$ with $z_1 \leq z_2 \leq \cdots \leq z_n$ can be mapped to $(z_1 + \cdots + z_n, \sum_{i<j} z_i z_j, \cdots, z_1 \cdots z_n)$ and that this map is a bijection since the symmetric functions are linearly independent.

Now lets look at $\mathbb{C}^2$. In this case a generic point in $(\mathbb{C}^2)^{[n]}$ would consist of n different points in $\mathbb{C}^2$ and corresponds to a nonsingular point in $S^n\mathbb{C}^2$. At the moment two of the n coordinates coincide the Hilbert-Chow map would map this point to one singular point in $S^n\mathbb{C}^2$, but as a subscheme there is a direction in $\mathbb{C}^2$ attached that distinguishes from what direction the points approached each other. In a sense is the reduction in dimension of the tangent space by the addition of one more point to the cluster automatically compensated by the addition of the direction that point came from. This concept is described in slightly more detail in appendix D. In higher dimensions this compensation is not enough anymore and the resulting scheme can be shown to be singular for n larger than three.

6.2 Heisenberg algebra of Nakajima

In this section the appearance of the Heisenberg algebra of string oscillators in the cohomology ring of the Hilbert scheme of points will be reviewed. It is not the intention to be very precise, first of all since we will only need the final results and secondly because a precise treatment could easily fill a book, without enhancing the needed understanding. For the reader that wants to get a more in depth treatment several review articles [38],[39] next to Nakajima's book [34] are highly recommended.

As we know from the orbifold cohomology calculations from section (5.2), it is advantageous to treat all Hilbert schemes at once. It is only in this context that one can expect to be able to define a Heisenberg algebra structure. In the orbifold case the we constructed oscillator modes $\mathfrak{p}_l(a_i)$ that map a state in the $H^*(S^nX)$ sector to one in the $H^*(S^{n+l}X)$ sector. To translate this concept to the Hilbert scheme picture we want an operator on the *full* collection of cohomology rings, $\mathbb{H}$ (see 6.1) based on elements from $H^*(X)$, that maps a class in $\mathbb{H}_n$ to a class in $\mathbb{H}_{n+l}$ for any n. To accomplish this Nakajima defined first incidence schemes as pairs of subschemes of $X^{[n]}$ and $X^{[n+l]}$ that differ in just one point :

$$Z_{l,n} \stackrel{\text{def}}{=} \{(I, x, I') \in X^{[n]} \times X \times X^{[n+l]} | I' \subset I, \text{supp}(I/I') = x\} \tag{6.2}$$

$$Z_l \stackrel{\text{def}}{=} \bigsqcup_n Z_{l,n} \tag{6.3}$$

Let us convince ourselves that this definition makes sense. A point I in $X^{[n]}$ consists of a prime ideal of a polynomial ring that has n zeros. I' has l zeros

more all at the same point and so as ideal is a subset of I, moreover since the difference is just x the quotient has its support there.

We define the projection maps pr_1, pr_2 and pr_3 as the maps projecting from $X^{[n]} \times X \times X^{[n+l]}$ to its respective component. We now define the searched for oscillator modes as endomorphisms on $\mathbb{H}$ whose action on a class in $\mathbb{H}_m$ is given by:

$$\mathfrak{p}_{-l}(a)(y) \stackrel{\text{def}}{=} PD^{-1}(\text{pr}_{3*}(PD(\text{pr}_2^*(a) \cup \text{pr}_1^*(y)) \cap [Z_l])) \tag{6.4}$$

PD is the Poincaré duality map. The structure of this definition is the following: pr_2^* and pr_1^* pull the classes a in $H^*(X)$ and y in $\mathbb{H}_n$ first back to a class in $H^*(X^{[n]} \times X \times X^{[n+l]})$. The corresponding cycle is then intersected with the incidence scheme, or actually the fundamental homology class of the incidence scheme. The resulting class is finally pushed forward to a class in $H_*(X^{[n+l]})$. This use of a subvariety, namely Z, of a Cartesian product as an operator on the homology ring is called a **correspondence**. There are a lot of intricacies left out, for example it can be extended to non compact surfaces, but in that case one needs a replacement for Poincaré duality, relating homology with compact cohomology. Also is homology not allways a well defined concept when working with schemes and more appropriate would it be to work with the Chow ring [36]. But the main idea of the construction should be clear and is pictured in the following diagram:

correspondence

$$\begin{array}{ccccc} H^*(X^{[l]} \times X \times X^{[n+l]}) & \xrightarrow{\cap[Z_l]} & H^*(Z_{l,n}) \\ \uparrow{\scriptstyle pr_2^*} & \uparrow{\scriptstyle pr_1^*} & \downarrow{\scriptstyle pr_{3*}} \\ y & a & \mathfrak{p}_{-l}(a)(y) \end{array}$$

The main result of [30] (up to a constant) was that these indeed are Heisenberg algebra operators in the sense that:

$$[\mathfrak{p}_l(a), \mathfrak{p}_m(b)]_\pm = l\delta_{l,-m} T(ab) 1_{\mathbb{H}} \tag{6.5}$$

Here the vacuum is naturally $1 \in \mathbb{Q} \simeq \mathbb{H}_0$. That $\mathbb{H}_0 \simeq \mathbb{Q}$ can be seen by realising that $X^{[0]}$ consists of 1 point, the empty set, and so the cohomology ring is indeed $\mathbb{Q}$.

Li et. al. constructed [40] a complete set of generators for $\mathbb{H}$, that Lehn and Sorger managed to express in the Heisenberg operators defined above [35] in case the canonical bundle is trivial. It is interesting to know that they used some vertex algebra techniques to proof this. We will take this as our definition of the generators:

$$a^{[n]} \stackrel{\text{def}}{=} \sum_{m=0}^{n} \frac{(\mathfrak{p}_{-1}(1))^{n-m}}{(n-m)!} \sum_{\alpha \in \mathbf{p}(m)} \frac{\prod_{l \geq 1} (\mathfrak{p}_{-l})^{\otimes |\alpha_l|} (\Delta_*^{(|\alpha|)}(c_\alpha a))}{m!} 1_{\mathbb{H}} \tag{6.6}$$

c_α is a partition dependend class in $H^*(X)$. that is, based on numerics, conjectured to be:

$$c_\alpha = \frac{(-1)^{||\alpha||-|\alpha|}}{\prod_l \alpha_l!}\left(1 + \frac{|||\alpha||| - 1}{24}e\right) \tag{6.7}$$

e being again the Euler class of X. We recognise in (6.6) already a lot of structure we have seen previously when treating the orbifold cohomology ring and also in the threepoint calculations by Lunin and Mathur. If we just forget for the moment the dependence on the cohomology class of X, then it looks as if the generators are labelled by the way we twist. So for a given weight n the untwisted sector would then be represented by $m = 0$ We will see in the next section that this correspondence does not hold.

From the existence of the Heisenberg algebra as well as the dimensions from the cohomology group versus the Fock space dimension it is not hard to deduce that both structures are isomorphic as vector spaces :

$$\Psi : \mathcal{V}(H) \to \mathbb{H} \tag{6.8}$$

$$\tilde{\mathfrak{p}}_{-m}(a) \mapsto \mathfrak{p}_{-m}(a) \tag{6.9}$$

And so by composition with (5.43) we obtain a vectorspace isomorphism $\Gamma \stackrel{\text{def}}{=} \Psi \circ \Phi$. For the calculations needed to obtain the 3 point functions within the chiral ring, a vectorspace isomorphism is not enough, we need to know how to multiply elements in the Hilbert scheme cohomology ring. Lehn and Sorger finally proved, that the isomorphism created is in fact a ring isomorphism and thus respects the ring product. The obtained ring generators can now be expressed through a set of generators from the symmetric algebra:

$$a^{[n]} = \sum_{m=0}^{n} \sum_{\alpha \in \mathbf{p}(m)} \binom{n}{m} \Gamma\left(P_n\left(\Delta_*^{(|\alpha|)}(c_\alpha a) \otimes 1^{\otimes(n-m)}, \pi_\alpha\right)\right) \tag{6.10}$$

Here with π_α we mean an element from S_n that consist of $(n-m)$ 1 cycles in addition to the cycles from the partition α. The P_n operator is the projection to the invariant subspace we have encountered in earlier chapters that we can write using the group action (5.40)as

$$P_n \stackrel{\text{def}}{=} \frac{1}{n!}\sum_{\rho \in S_n} \hat{\rho} \tag{6.11}$$

So the claim is that with a running through generators of $H^*(X)$, the resulting $a^{[n]}$ form a complete set of generators for the cohomology ring of $X^{[n]}$. The statement that Γ is not only a vectorspace isomorphism, which is comparatively easy to infer from the construction of the Heisenberg algebra, but that it is in fact a ring isomorphism is the result we need.

Unfortunately though the generators $a^{[n]}$, although of pure weight n, are not of pure degree. This is not totally surprising, knowing how hard it is to relate different bases of symmetric functions to each other. On the other hand the *existence* of a ring isomorphism makes it possible to define a basis in $\mathbb{H}_n$ through a basis in $H^{[n]}$ formally using Γ. So to compare the cohomology ring structure to the calculations from chapter 3 we could try to find a suitable set of elements in $\mathcal{H}^{[n]}$ that correspond to chiral primaries. We will see in the next section that this is not an easy task.

6.3 The product of cycle based classes

To relate the cohomology ring structure to the correlation functions as calculated by Mathur and Lunin we need to find the classes in the cohomology ring of $K3^{[N]}$ that correspond to the chiral primaries as constructed in (4.12),(4.16).

6.3.1 Symmetry and complex structure

We look now at the cohomology ring for a Hilbert scheme of a fixed number of points N, using capital N to conform to the notation of chapter 3. We want now to assign to a cycle of length n with $n < N$ a ring element in the symmetric algebra As we noted in section 5.1, the chiral ring contains an $SU(2) \otimes SU(2)$ symmetry, that was for example given by the triple J^+_{-1}, J^-_1, J^3_0 for the holomorphic chiral NS sector. Of course we also need a splitting of the cohomology in Dolbeault cohomology

We start by fixing a complex structure I and its associated Kähler form ω_I. This will give us a splitting up of $H^n(X)$ in a direct sum of Dolbeault cohomology groups:

$$H^n(X) = \bigoplus_{p+q=n} H^{p,q}(X) \tag{6.12}$$

Now corresponding to the Kähler form we can define, following [41] an operator that raises the grading of a p,q-form to $p+1, q+1$ as follows:

$$L_I(a) \stackrel{\text{def}}{=} a \wedge \omega_I$$

Next we introduce the operator Λ_I as the dual under the Hodge pairing:

$$\int \Lambda_I a \wedge *b = \int a \wedge *(L_I b)$$

and an operator H that just multiplies by the total degree of the form:

$$Ha \stackrel{\text{def}}{=} (p+q)a \text{ for } a \in H^{p,q}(X)$$

Lefschetz triple

These three operators form a **Lefschetz triple**, satisfying the algebra

$$[L_I, \Lambda_I] = H \qquad [H, L_I] = 2L_I \qquad [H, L_I] = -2L_I \tag{6.13}$$

This is an $\mathfrak{su}(2)$ algebra, by multiplying the generators by $1/2$ it is in the more familiar form of ladder and charge operators, where the charge is now the total degree of the form. This fixing of a complex structure defines how we complexify the real manifold, but since $K3$ is hyperkähler we have two complex structures J and K left that give also rise to Lefschetz triples (L_J, Λ_J, H) and (L_J, Λ_J, H).

Although each of the Kähler forms is a $(1,1)$ form with respect to its own complex structure, that is not the case with respect to one of the other complex structures as can be easily checked from the definitions and the hyperkähler relation $I \circ J = -K$. For example are ω_J and ω_K $(2,0) \oplus (0,2)$ forms. Of course they still raise the total de Rham degree by two. We can define though

holomorphic and antiholomorphic 2-forms *with respect to the complex structure* I as follows [42]:

$$\begin{aligned} J^+ &\stackrel{\text{def}}{=} \frac{1}{2}(L_J + iL_K) : H^{p,q}(X) \to H^{p+2,q}(X) \\ J^- &\stackrel{\text{def}}{=} \frac{1}{2}(\Lambda_J - i\Lambda_K) : H^{p,q}(X) \to H^{p-2,q}(X) \\ \bar{J}^+ &\stackrel{\text{def}}{=} \frac{1}{2}(L_J - iL_K) : H^{p,q}(X) \to H^{p,q+2}(X) \\ \bar{J}^- &\stackrel{\text{def}}{=} \frac{1}{2}(\Lambda_J + i\Lambda_K) : H^{p,q}(X) \to H^{p,q-2}(X) \end{aligned} \tag{6.14}$$

Thus providing operators that do raise and lower the Hodge index in just the (anti)-holomorphic sector. We are now set to build a set of classes in $X^{[N]}$ that have the same kind of $SU(2)$ structure we had in the orbifold construction.

6.3.2 comparing degrees

In the orbifold construction every cycle has 2×2 primaries attached, forming the corners of a Hodge diamond. The $J^{\pm}_{\mp 1}$ and $\bar{J}^{\pm}_{\mp 1}$ operators connect the four corners of these Hodge diamonds. The $(1,1)$ centre of the diamond won't appear in this construction. Recall that a cycle of length n corresponds to chiral primaries with Dolbeault degree $p = n \pm 1$ The groundstate corresponds to the 1-cycle twist. Combining the Dolbeault classes in de Rham cohomology, the result is that an n cycle corresponds with 4 classes, one with degree $2n-2$, two with degree $2n$ and one with degree $2n+2$. A top form in the orbifold we can reach only from the untwisted sector by applying the ladder up to the ground state of each copy of M. The symmetrisation over the full group S_N, which is the centraliser of the unit element, does not change the degree of the resulting primary. Nonetheless this topform will never appear in the construction of one cycle based chiral primaries.

If we compare this with the construction of the symmetric ring then we see that the topform can also only be reached from the unit element, since it has the maximum number of N orbits. If we try to map one of the four n cycle based primaries to an n cycle based symmetric ring element, as defined through the symmetrised version of the twisted ring definition (5.35), it seems to be impossible! The simple ring elements based on an N-cycle for example have a degree of at most 4, although the chiral primaries corresponding to that same cycle would be of degrees $2N-2, 2N$ or $2N+2$. These last values lie in the centre of the Hodge diamond of the $4N$-dimensional smooth resolution of the orbifold (table 6.1).

The clue is that one should not calculate the cohomology degree just from the tensor product. So the cohomology degree is *not* the sum of the Frobenius algebra degrees of the elements plus the dimension of the base algebra times the number of tensor components, as it would be seen purely as a tensored Frobenius algebra. The degree should be calculated relative to N. Let us write the cohomology degree of an element as $\| a \|$ then:

$$\| a_1 \otimes \cdots \otimes a_k \| = \sum_{i=1}^{k} |a_i| + 2N \tag{6.15}$$

$$= |a_1 \otimes \cdots \otimes a_k| + 2N \tag{6.16}$$

		$h^{2N,2N}$			
		.....			
.....		$\mathbf{h^{N+1,N+1}}$		.....	
.....	$h^{N+1,N}$		$h^{N,N+1}$	.....	
$\mathbf{h^{N+1,N-1}}$		$h^{N,N}$		$\mathbf{h^{N-1,N+1}}$	
.....	$h^{N,N-1}$		$h^{N-1,N}$	.....	
.....		$\mathbf{h^{N-1,N-1}}$	.....		
		.....			
		$h^{0,0}$			

Table 6.1: Hodge diamond for with N cycle chiral primaries

With this relation between cohomology degree and Frobenius degree we can easily build candidates that correspond to chiral primaries from the CFT. To match the Dolbeault degrees good candidates would be of the form:

$$a_n = P(a \otimes 1^{\otimes(N-n)}, (n)) \tag{6.17}$$

The operator P is again the symmetrisation. We take a to be a zero, two or topform, for example 1, the two-form that squares to the Euler form $\sqrt{e}$, and the Euler form e. Note that $\Delta_*^(2)(1) = \sqrt{e} \otimes \sqrt{e}$. The cohomology degree is then

$$\| a_n \| = 2n + |a| \tag{6.18}$$

In the case of a topform "$a = e$", we need a small modification to be able to calculate the cup product from the classes corresponding to the CFT chiral primaries. The reason is the following: when we multiply two cycle based ring elements n_1 and n_2 we will first need to contract the element corresponding to the orbit of the cycle with a number of the elements that correspond to the other one element orbits. This is allways the case since the orbit space from the group generated by both cycles is never larger than the original orbitspaces and only equal to them in the case the 2 cycles have a total overlap. That only happens in case the cycles are, per construction (4.55), each others inverse. After this we might have to expand again the resulting element since the final cycle can again have a larger orbitspace. We will now analyse what kind of contraction/expansion we can have. First of all the cohomology degrees should match :

$$2(n_1 + s_1) + 2(n_2 + s_2) = 2(n_3 + s_3) \tag{6.19}$$

Where $s_i \in -1, 0, 1$ is total sign index (holomorphic plus anti holomorphic) defined in section 4.7. We also know the relation between the cycle lengths with respect to the (positive) overlap k:

$$n_3 = n_1 + n_2 - 2k + 1 \tag{6.20}$$

Combining these two relations shows that the overlap can only be 1 or 2. The orbitlength of the non trivial orbit from the group generated by n_1 and n_2, or

equivalently the total number of copies of M that gets twisted, is n_1+n_2-k. An expansion through the co-product can only happen in the case the cyclelength n_3 is strictly smaller than this total orbit length and this happens only in the case the overlap $k=2$. That again corresponds to values of $s_1=s_2=-1$. The size of the expansion is just 1. In these cases the element corresponding to the cycle is 1 and an expansion can at most lead to $1 \to \Delta_*^{(2)}(1) = \sqrt{e} \otimes \sqrt{e}$. So we come to the following modification:

$$\begin{aligned} a_n^- &\stackrel{\text{def}}{=} P(1 \otimes 1^{\otimes(N-n)}, (n)) \\ a_n^0 &\stackrel{\text{def}}{=} P(\sqrt{e} \otimes 1^{\otimes(N-n)}, (n)) \\ a_n^{+,1} &\stackrel{\text{def}}{=} P(\sqrt{e} \otimes \sqrt{e} \otimes 1^{\otimes(N-n-1)}, (n)) \\ a_n^{+,2} &\stackrel{\text{def}}{=} P(e \otimes 1^{\otimes(N-n)}, (n)) \end{aligned} \tag{6.21}$$

This is all we need, since $a_n^{+,i}$ can only combine with $a_{n'}^-$ in a $k=1$ overlap.

6.3.3 examples

We will work out a couple of three-point functions for $N=3$. Let us start with the chiral primaries. Since we won't be able to calculate three-point functions on the CFT for combinations that have a non-zero genus, we won't be able to calculate the fusion coefficient for the fusion of 2 threecycles into a threecycle or into a one cycle (see the table of graphdefects 5.1). We use the following notation for the fusion coefficients as defined in (4.34) : $(n_1^{s^1}, n_2^{s_2} | n_3^{s_3})$ for the fusion of $\check{\tau}_{n_{1,2},0}^{s_{1,2}}$ into $\check{\tau}_{n_3,0}^{s_3}$. To calculate the fusion coefficients are easy to calculate. The $3j$symbols simplify to:

$$\begin{pmatrix} \frac{n_1+s_1}{2} & \frac{n_2+s_2}{2} & \frac{n_3+s_3}{2} \\ \frac{n_1+s_1}{2} & \frac{n_2+s_2}{2} & -\frac{n_3+s_3}{2} \end{pmatrix} = \frac{1}{\sqrt{n_3+s_3}}$$

The fusion coefficients are given for one sector (holomorphic or anti-holomorphic, while the normalisation is given for the whole three-point function (holomorphic and anti-holomorphic parts). The last two entries result in 4 different Dolbeault classes, since holomorphic and antiholomorphic parts are allowed to differ as long as the twist is the same. The full 3 point functions can be eas-

	fusion	normalisation
$(1^+, 2^-\|2^+)$	$1/2$	$2/\sqrt{3}$
$(1^+, 3^-\|3^+)$	$1/3$	$\sqrt{3}$
$(2^-, 2^-\|1^+)$	$-1/2$	$2/\sqrt{3}$
$(2^-, 2^-\|3^-)$	$-\sqrt{3}/2$	$\sqrt{2}$
$(2^+, 2^-\|3^+)$	$-1/\sqrt{3}$	$\sqrt{2}$

Table 6.2: fusion of chiral primaries

ily obtained from table 6.3.3. They can be combined in a way to reflect the structure constants of the de Rham cohomology as:

In the cohomology ring we have, following the definition from last section, the following classes: As an example we calculate one multiplication:

holomorphic	anti-holom.	cohomology	structure constant
$(1^+, 2^-\|2^+)$	$(1^+, 2^-\|2^+)$	$[4] \wedge [2]$	$1/(2\sqrt{3})$
$(1^+, 3^-\|3^+)$	$(1^+, 3^-\|3^+)$	$[4] \wedge [4]$	$1/(3\sqrt{3})$
$(2^-, 2^-\|1^+)$	$(2^-, 2^-\|1^+)$	$[2] \wedge [2]$	$1/(2\sqrt{3})$
$(2^-, 2^-\|3^-)$	$(2^-, 2^-\|3^-)$	$[2] \wedge [2]$	$3/\sqrt{2}$
$(2^+, 2^-\|3^+)$	$(2^+, 2^-\|3^+)$	$[6] \wedge [2]$	$\sqrt{2}/3$
$(2^+, 2^-\|3^+)$	$(2^-, 2^-\|3^-)$	$[4] \wedge [2]$	$2\sqrt{2}$

Table 6.3: fusion of chiral primaries

twist	name	element	degree
1	a_1^-	$P(1^{\otimes 3}, \mathrm{Id})$	[0]
	a_1^0	$P(\sqrt{e} \otimes 1^{\otimes 2}, \mathrm{Id})$	[2]
	$a_1^{+,1}$	$P(\sqrt{e} \otimes \sqrt{e} \otimes 1, \mathrm{Id})$	[4]
	$a_1^{+,2}$	$P(e \otimes 1 \otimes 1, \mathrm{Id})$	[4]
2	a_2^-	$P(1 \otimes 1, (2))$	[2]
	a_2^0	$P(\sqrt{e} \otimes 1, (2))$	[4]
	$a_2^{+,1}$	$P(\sqrt{e} \otimes \sqrt{e}, (2))$	[6]
	$a_2^{+,2}$	$P(e \otimes 1, (2))$	[6]
3	a_3^-	$P(1, (3))$	[4]
	a_3^0	$P(\sqrt{e}, (3)$	[6]
	a_3^+	$P(e, (3))$	[8]

Table 6.4: cycle based ring elements

$$a_2^- \wedge a_2^- = P(1 \otimes 1, (2)) \wedge P(1 \otimes 1, (2)) \tag{6.22}$$

$$= \frac{2}{3} P(1, (3)) + \frac{1}{3} P(\sqrt{e} \otimes \sqrt{e} \otimes 1, \mathrm{Id}) \tag{6.23}$$

$$= \frac{2}{3} a_3^- + \frac{1}{3} a_1^{+,1} \tag{6.24}$$

The fractions in front are easy to calculate from the symmetrisation and is analogous to the calculations as done in the CFT 4.56. The the cohomology ring multiplication table that corresponds to the fusion of the chiral primaries of table 6.3.3 is given below: As can be seen from table 6.3.3 the two + variants

$a_1^{+,1} \wedge a_2^-$	$2/3 a_2^{+,1} + 1/3 a_2^{+,2}$
$a_1^{+,2} \wedge a_2^-$	$a_2^{+,2}$
$a_1^{+,1} \wedge a_3^-$	a_3^+
$a_1^{+,2} \wedge a_3^-$	a_3^+
$a_2^- \wedge a_2^-$	$2/3 a_3^- + 1/3 a_1^{+,1}$
$a_2^+ \wedge a_2^-$	a_3^+
$a_2^0 \wedge a_2^-$	a_3^0

Table 6.5: cohomology ring multiplication

are needed. The $(+, 2)$ class is the one that corresponds best with the + chiral primary in the CFT calculations. The $(+, 1)$ could have been implemented in the CFT calculations by adding an extra J_{-1}^+ operator to one of the copies of

M that do not get twisted. This wouldn't make the calculations more difficult since it doesn't have an effect on the geometry of the CFT.

In principle it should be possible to use the Lefschetz operators to separate the de Rham classes into Dolbeault classes, starting with the lowest class in the sector: $P(1^{\otimes(N-n+1),(n)}$ to $P(J^+1 \otimes 1^{\otimes(N-n),(n)}$, $P(J^-1 \otimes 1^{\otimes(N-n),(n)}$ and $P(J^-J^+1 \otimes 1^{\otimes(N-n),(n)}$.

The normalisation is not as easy to generate from the cohomology. A possibility would be to use Hodge duality to normalise the classes. It is unlikely to yield the same results as the CFT calculations. Nonetheless it seems to be possible to extract the normalisation from the CFT calculations and then generalise the threepoint functions to higher graphdefect correlation functions using the cohomology construction. From the definition of the symmetric ring product (5.38) we see that only genus 1 combinations give a non-vanishing result, making the large N approximation by restricting to genus zero more robust.

Apart from the normalisation, that is in principle not an approximation for large N in the CFT, there doesn't seem to be any obstacle in using cohomology to calculate the correlators in thr large N limit. When restricting calculations in the cohomology ring also to genus zero in the large N limit, there is only a limited amount of combinations of classes allowed, as outlined above. This simplyfies the calculations rather much.

Chapter 7

Conclusions

We have seen two ways in which to calculate three point functions from chiral primaries. The original CFT calculations have the benefit that they also incorporate elements that are in the chiral multiplet but not in the chiral ring. This benefit is in the physical context not totally clear though, since these states are probably not protected in the same way chiral primaries are and so might be more sensitive to how far away we are from the exact boundary theory CFT. The calculations do reveal the R- symmetry in a much more natural context then in the proposed cohomology classes. As a drawback we have to point out that exact calculations, are most likely very hard to carry out, because of the difficulties in doing CFT calculations on higher genus surfaces. It is likely that higher genus contributions can be neglected in the large N limit, but the constraint on how high a genus is still relevant for the calculations does not follow immediately from the calculation. If we would solely concentrate on the chiral primaries then this limitation to genus zero might be more severe.

The cohomology calculations show that it is in principle possible to make exact calculations for all the chiral primaries in the theory. Even in the large N limit, the calculations are comparatively straightforward to carry out and are easy to generalise to all classes in the cohomology ring.

To do a direct translation between these two methods seems to demand still some work. Although we have succeeded in creating a set of classes that are to a large extend equivalent to the class of primaries that were constructed by the monodromy map. It might be interesting to develop a normalisation procedure for the cohomology ring that would be comparable to the normalisation for the chiral primaries. It is on the other hand not immediately clear to me if the normalisation is truly relevant in the AdS/CFT correspondence. One possible normalisation procedure might be using Hodge duality pairing : $\int a * a = 1$, but this type of normalisation certainly will not correspond to the chiral anti-chiral two point function normalisation as done on the CFT side. It is furthermore not clear to me if such a normalisation would be of any importance to the AdS/CFT correspondence. In other words is the Hodge Riemann pairing protected? This seems to be crucial for having any relevance for the physics.

Overall is the final cohomology ring much simpler then beforehand expected and is it clear that most of the decoration in the CFT calculations were caused either by the normalisation procedure or by the inclusion of non-chiral primaries from the chiral multiplet in the correlators.

The final goal is to make a comparison of the structure of the cohomology ring with the structure of geometry fluctuations in AdS3. Hereto it would help to have the R-symmetry made explicit. For this we should use some form of the Lefschetz symmetry. To make the comparison it is essential to work in the large N limit. An interesting test would be to compare the structure of the 3 point functions with those from the supergravity side. As far as I know a comparison has not been made thus far.

Appendix A

The Symmetric group

The **symmetric group**, S_n, consists of all possible permutations of n objects. A common notation for a permutation, π, is the following: symmetric group

$$\begin{pmatrix} i_1 & i_2 & \dots & i_n \\ \pi(i_1) & \pi(i_2) & \dots & \pi(i_n) \end{pmatrix} \tag{A.1}$$

Where an object with an index from the upper row gets permuted to a slot with index underneath it. A **cycle** of length l is a cyclic shift of l elements : $(i_1, i_2, \dots, i_l) \mapsto (i_l, i_1, \dots, i_{l-1})$. The notation for cycles can simplified as follows: cycle

$$(i_1, i_2, \dots, i_l) := \begin{pmatrix} i_1 & i_2 & \dots & i_l \\ i_l & i_1 & \dots & i_{(l-1)} \end{pmatrix} \tag{A.2}$$

We can build any permutation by the disjoint product of cycles , this can be seen by iteration: pick an object and then follow the chain of permutations until we reach the starting object again, this defines a cycle; if all elements are used we are done, else pick an object from the remaining set and restart the procedure until all objects are used.

As an example the product table for S_3 is shown below:

	Id	(12)(3)	(13)(2)	(23)(1)	(123)	(132)
Id	Id	(12)(3)	(13)(2)	(1)(23)	(123)	(132)
(12)(3)	(12)(3)	Id	(123)	(132)	(13)(2)	(1)(23)
(13)(2)	(13)(2)	(132)	Id	(123)	(1)(23)	(12)(3)
(1)(23)	(1)(23)	(123)	(132)	Id	(12)(3)	(13)(2)
(123)	(123)	(1)(23)	(12)(3)	(13)(2)	(132)	Id
(132)	(132)	(13)(2)	(1)(23)	(12)(3)	Id	(123)

Table A.1: S_3 multiplication : first the row then the column

The product of 2 overlapping cycles can result in one or several disjoint cycles , like the following examples show:

$$(34) \circ (123) = (1243)$$
$$(345) \circ (1234) = (124)(35)$$

Where the group action is from right to left. One cycles will be omitted from the notation from now on since they act trivially anyway. Taking the conjugate of an l-cycle results in a permuted l-cycle:

$$\begin{aligned}\begin{pmatrix} 1 & \dots & n \\ \pi(1) & \dots & \pi(n) \end{pmatrix} \circ \begin{pmatrix} i_1 & \dots & i_l \\ i_2 & \dots & i_1 \end{pmatrix} \circ \begin{pmatrix} \pi(1) & \dots & \pi(n) \\ 1 & \dots & n \end{pmatrix} \\ = \begin{pmatrix} 1 & \dots & n \\ \pi(1) & \dots & \pi(n) \end{pmatrix} \circ \begin{pmatrix} \pi(i_1) & \dots & \pi(i_l) \\ i_2 & \dots & i_1 \end{pmatrix} \\ = \begin{pmatrix} \pi(i_1) & \dots & \pi(i_l) \\ \pi(i_2) & \dots & \pi(i_1) \end{pmatrix} \\ = (\pi(i_1), \pi(i_2), \cdots, \pi(i_l))\end{aligned}$$

Furthermore are two l-cycles related by conjugacy:

$$\begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_l \end{pmatrix} \circ \begin{pmatrix} i_1 & \dots & i_l \\ i_2 & \dots & i_1 \end{pmatrix} \circ \begin{pmatrix} j_1 & \dots & j_l \\ i_1 & \dots & i_l \end{pmatrix} \tag{A.3}$$

$$= (j_1, j_2, \cdots, j_l) \tag{A.4}$$

Apparently we can label the conjugacy classes by their cycle structure, or equivalently by the way we can split up the total number of objects in smaller groups: in a **partition** of n: $\alpha = (1^{\alpha_1} 2^{\alpha_2} \cdots)$. We will use the notation $\mathbf{p}(n)$ for the set of partitions of n. Given a partition α, the following definitions will be usefull:

partition
partition length
partition norm

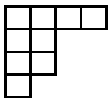
$$\text{the } \mathbf{length}\ |\alpha| \stackrel{\text{def}}{=} \sum_i \alpha_i \tag{A.5}$$

$$\text{the } \mathbf{norm}\ ||\alpha|| \stackrel{\text{def}}{=} \sum_i i\alpha_i = n \tag{A.6}$$

$$\text{the } \mathbf{permutation\ number}\ |||\alpha||| \stackrel{\text{def}}{=} \sum_i \binom{i+1}{2} \alpha_i \tag{A.7}$$

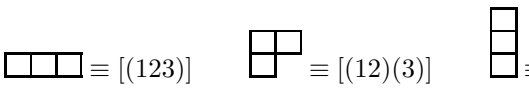
permutation number
Young diagrams

A graphical way to label the conjugacy classes is by means of **Young diagrams** where each row corresponds to a cycle. For example the class in S_9 given by one 4 cycle , two 2 cycles and a 1 cycle is given by:



$$\equiv [(1234)(56)(78)(9)]$$

The conjugacy classes from S_3 are:



degree

The **degree** $|\pi|$ of a permutation π is defined as the minimum number of 2 cycles needed to build it.
The centraliser of a permutation can be described, in terms of the disjoint cycles it decomposes into, as permutations that permute cycles of the same length and the cyclic permutations within a cycle [43] .For a cycle, c_l, of length l the the order, $|C_{c_l}|$, of the centraliser is:

$$|C_{c_l}| = l(n-l)! \tag{A.8}$$

Namely $n-1$ permutations of the remaining cycles of length 1 and l choices for the start of the cycle, or equivalently the order of $\mathbb{Z}_l$. The order, $|[c_l]|$, of the conjugacy class it belongs to is given by:

$$|[c_l]| = \binom{n}{l}(l-1)! \tag{A.9}$$

because we have to choose l cycle members out of n objects . These cycle members have $l!$ permutations each permutation being equivalent to $l-1$ other, because they define the same cycle. Note that the product of the conjugacy class order and the order of the centraliser is equal to the group order $n!$, as could have been expected.

The action of a subgroup $H \subset S_n$ on the set $[n] = 1, \cdots, n$ splits the set up in **orbits** of the group action, namely a set of numbers that are generated from one number by the H action . The resulting **orbitspace** we will write as $H\backslash[n]$. We will write $\langle \pi_1, \cdots, \pi_k \rangle$ for the subgroup generated by these k elements. The **orbitlength**, $l(x)$, of an orbit $x \in H\backslash[n]$ is the number of points in the orbit. Below as illustration the orbits of the subgroups up to conjugacy of S_3. The first row gives a choice of generators for the subgroup.

orbits

orbitspace

orbitlength

generator	subgroup	orbits	orbitlength $l(x)$
$\langle \mathrm{Id} \rangle$	Id	$\{1\}$	1
		$\{2\}$	1
		$\{3\}$	1
$\langle (12) \rangle$	$\{(12), \mathrm{Id}\}$	$\{1,2\}$	2
		$\{3\}$	1
$\langle (123) \rangle$	$\{(123), (132), \mathrm{Id}\}$	$\{1,2,3\}$	3
$\langle (12), (13) \rangle$	S_3	$\{1,2,3\}$	3

Table A.2: S_3 subgroup orbit table

Appendix B

Bosonic twists

In [17] correlation functions for a bosonic symmetric orbifold were studied. Since the calculations of the chiral primaries as explained in chapter 4 depend on some results from this paper, the needed results are collected in this appendix.
The covering map to a sphere for 3 "matching" twist insertions was found to be

$$z = at^n \frac{P_{s-n_1}^{(-n_1,-n_2)}(1-2t)}{P_{s-n_3}^{(-n_1,-n_2)}(1-2t)} \tag{B.1}$$

Here $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials. The $n_!, n_2$– and n_3-cycle based twist insertions are placed at $z = 0, a$ resp. ∞ and mapped to $t = 0, 1$ resp. ∞. The twists should match in a way that the genus of the resulting cover is zero. $s = (n_1 + n_2 + n_3 - 1)/2$ is the number of copies of M that participates in the twist.

Appendix C

Complex Manifolds

In this appendix , some of the basic facts on complex differential geometry will be given. References used for this appendix are [44] and [45]. Also the first part of [21] is a usefull reference. A complex manifold is defined analogous to a real differentiable manifold as a (Hausdorff, locally compact) topological space with an atlas of charts to $\mathbb{C}^n$ with holomorphic transition functions between intersecting charts. The dimension, given as the real dimension, has to be even.

The tangent space has a natural automorphism, I, defined through the multiplication by i. In the natural coordinates given by $z_i = x_i + iy_i$ this automorphism sends the vector $\partial/\partial x_i$ to $\partial/\partial y_i$ and $\partial/\partial y_i$ to $-\partial/\partial x_i$. Such an automorphism is called an **almost complex structure**. almost complex structure

An interesting question is: given an even dimensional manifold and an almost complex structure on it, does it represent a complex manifold? For this to be true the almost complex structure needs to match on overlapping charts. The criteria for this to be possible can be expressed in the disappearing of the **Nijenhuis tensor** defined through Nijenhuis tensor

$$N(v,w) := [v,w] + I[Iv,w] + I[v,Iw] - [Iv,Iw]$$

which is a measure for the torsion of the almost complex structure, v and w being arbitrary tangent vectors. In the case $N \equiv 0$ we call I a **complex structure**. complex structure

Let (X,J) be a manifold with complex structure. A Riemannian metric, g on X is called **hermitian** if $g(v,w) = g(Jv,Jw)$. In that case we can define the **hermitian form** as $\omega(v,w) := g(Jv,w)$ from which we can recover again the metric. A hermitian metric is called **Kähler** if its hermitian form is closed. The hermitian form is then called a **Kähler form**. Equivalent to the existence of a Kähler form is the disappearing of the Levi-Civita connection of the complex structure, implying that the holomorphic and anti-holomorphic parts of the tangent bundle don't mix under parallel transport. hermitian, hermitian form, Kähler, Kähler form

We can also construct a manifold over the quaternions with a **holonomy group**, the group of equivalence classes of vectors under parallel transport along loops, equal to $Sp(n)$, the symplectic group. This corresponds to a $4n$-dimensional real manifold. Seen from the perspective of this real manifold the quaternionic structure corresponds to the existence of 3 complex structures I, J and K that compose (remember they are automorphisms on the tangent bundle) as $I \circ J = -J \circ I = H$. Since $Sp(n) \subset SU(2n)$, using the to be discussed theorem by Yau, we can conclude that the manifold Kähler is. It is in fact Kähler holonomy group

hyperkähler

with respect to all 3 complex structures. Such a manifold is called **hyperkähler**. In fact there is a whole sphere of complex structures since any linear combination of I, J and K that stays within the unitsphere is automatically a complex structure.

Hodge star
Hodge pairing

Recall that on a real Riemannian manifold we can use the metric to define an inner product on differential forms using the **Hodge star** operation that sends a k-form to an $(n-k)$- form. The inner product, the **Hodge pairing**, is defined on the k-forms

$$\langle \alpha, \beta \rangle \stackrel{\text{def}}{=} \int \alpha \wedge *\beta \tag{C.1}$$

using this inner product we can define on a closed (so that we won't get boundary terms), compact Riemannian manifold the adjoint $d^\dagger$ of the exterior derivative with respect to the inner product:

$$\langle \alpha, d^\dagger \beta \rangle = \langle d\alpha, \beta \rangle$$

Laplacian

The **Laplacian** is defined as: $\triangle \stackrel{\text{def}}{=} dd^\dagger + d^\dagger d$. Note that since d makes an $r+1$ form out of an r-form, $d^\dagger$ *lowers* the degree of the form and the resulting Laplacian doesn't change the degree. When the metric is the Euclidean metric, the Laplacian is the standard calculus Laplacian.

exact
co-exact
harmonic
Hodge decomposition

Just like forms that can be written as the exterior derivative of another form are called **exact**, forms that can be written as $d^\dagger a$ are called **co-exact**. Forms that map to zero under the Laplacian are called **harmonic**. An important tool in the study of harmonic forms is the **Hodge decomposition** theorem that tells us that any form is uniquely writable as the sum of an exact, a co-exact and a harmonic form. An important consequence is that every class in de Rham cohomology corresponds to one unique harmonic form.

Dolbeault complex
Dolbeault cohomology

On a complex manifold we can decompose the exterior differential and its adjoint in differential operators $\bar\partial, \bar\partial^\dagger, \partial, \partial^\dagger$ working only on the (anti-)holomorphic part of forms. Just like the exterior differential the co-boundary operator is for the de Rham complex are the (anti)holomorphic exterior differential operators, co-boundary operators for the **Dolbeault complex** of (p,q)-forms Where the first index is the holomorphic-, the second one the antiholomorphic degree. The resulting cohomology is called the **Dolbeault cohomology**

One can now define two Laplacians each defining their harmonic forms:

$$\triangle_\partial \stackrel{\text{def}}{=} \partial\partial^\dagger + \partial^\dagger\partial \qquad\qquad \triangle_{\bar\partial} \stackrel{\text{def}}{=} \bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial$$

In the case of a Kähler manifold all three Laplacians are essentially the same :

$$2\triangle_\partial = 2\triangle_{\bar\partial} = \triangle$$

We have also in the complex case a Hodge decomposition from which we can conclude that every Dolbeault cohomology class corresponds to one harmonic form (for compact manifolds).

Hodge numbers
Betti numbers

The **Hodge numbers** $h^{p,q}$ are defined as the dimension of the (p,q) Dolbeault cohomology group, $H^{(p,q)}(X)$. In the case of a Kähler manifold, as a result of the equality of the Laplacians, the Hodge numbers are related to the **Betti numbers**, the dimensions of the de Rham cohomology groups, through

$$b^i = \sum_{p+q=i} h^{p,q}$$

Poincaré duality and the equality of the Laplacians combined with the Hodge decomposition makes the so called **Hodge diamond** symmetric with respect to the horizontal and the vertical axis. Hodge diamond

b^6				$h^{0,0}$			
b^5			$h^{1,0}$		$h^{1,0}$		
b^4		$h^{2,0}$		$h^{1,1}$		$h^{2,0}$	
b^3	$h^{3,0}$		$h^{2,1}$		$h^{2,1}$		$h^{3,0}$
b^2		$h^{2,0}$		$h^{1,1}$		$h^{2,0}$	
b^1			$h^{1,0}$		$h^{1,0}$		
b^0				$h^{0,0}$			

Table C.1: Hodge diamond for a 6 dimensional Kähler manifold

The renormalisability condition of the nonlinear σ-model translates into the demand that the target space is Ricci-flat, meaning that the Ricci form disappears. So there is a particular interest in Ricci flat manifolds. Now one of the basic theorems in complex differentiable geometry tells us that an $2n$-dimensional Kähler manifold Ricci flat is iff its **restricted holonomy** group, the holonomy group of contractable loops, is contained in $SU(n)$. In case the canonical bundle is trivial, Ricci flatness is equivalent to the full holonomy group being contained in $SU(n)$. Compact Kähler manifolds with holonomy equal to $SU(n)$ are called **Calabi-Yau** manifolds. Named after a conjecture by Calabi, proved by Yau, that given a Kähler manifold, there exist an unique metric, with Kähler form in the same class as the original, such that the Ricci form equals, up to a factor of 2π, the **first Chern class** $c_1(X)$ (This is a characteristic class that measures the number of zeroes of a typical section of the canonical bundle in). So when we have a Kähler manifold with trivial canonical bundle, then there exists a metric that makes the manifold Ricci flat. In 4 dimensions we have restricted holonomy Calabi-Yau first Chern class

$b^2 = 1$			1		
$b^3 = 0$		0		0	
$b^2 = 22$	1		20		1
$b^1 = 0$		0		0	
$b^0 = 1$			1		

Table C.2: Hodge diamond and Betti numbers for $K3$

accidentally that, since $Sp(1) = SU(2)$, a Calabi-Yau manifold also hyperkähler is. These manifolds are called $K3$. The four torus, T^4, has holonomy group that is a subgroup of $SU(2)$ and is not always considered to be Calabi-Yau. An important property of $K3$ surfaces (it has 2 *complex* dimensions) is that they are all diffeomorphic and so the Betti numbers are fixed, as are the Hodge numbers as can be seen from the Hodge diamond for $K3$ (table C.2

Appendix D

Algebraic geometry

This appendix collects the basic notions of algebraic geometry and schemes in particular as far as they are used in the main text. Algebraic geometry and in particular schemes are usually seen as a highly esoteric subject. And it is true that at first the amount of abstraction used and needed does not seem to have a lot of benefit in physics. Nonetheless in the last decades, especially due to the complexity of string theory, more and more ideas of algebraic geometry have penetrated the physics literature. At the same time a lot of new subjects in algebraic geometry have been developed out of string theory, with some remarkable results. A modest example shines through in the main text and although not many details to understand the theory behind these calculations are needed, I thought it to be of some use to give some hints of what these schemes actually are. Of course in an appendix we can only scratch the surface and many concepts necessarily stay vague. Hopefully the few concepts that become clear after reading the text will entice the reader to a more thorough study of this vast subject. The standard reference is Hartshorne's book [36] a more gentle approach covering the Hilbert scheme in more detail is [37].

D.1 Basics of schemes

Instead of defining spaces through charts to $\mathbb{R}^n$ algebraic varieties are locally defined as the zero locus of a set of polynomials over a closed field (closed meaning that you can always take roots, think $\mathbb{C}$). The reason for taking a closed field is that in that case the "dimension" of the locus is more or less the same everywhere, since the number of roots of a complex polynomial is equal to its degree. The polynomials form a ring since we can add and multiply polynomials.

Just like the, to many physicists more familiar, description of locally compact Hausdorff spaces through the C^*-algebra of continuous functions on that space, ideals in a polynomial ring are the main objects in algebraic geometry. Here by an **ideal** we mean a subset I of the polynomial ring R that is its own image under multiplication: $IR = I$. The points in the original space would correspond to **maximal ideals** in the ring, these are ideals that are not contained in any other "proper" ideal (the ring itself is not considered proper)[1]. As an example

ideal

maximal ideals

[1]This is quite analogous to the normed ring of continuous functions where the maximal

affine line

we look at the polynomial ring over $\mathbb{C}$, notation $\mathbb{C}[z]$, in one variable. This goes by the name of the **affine line**. A maximal ideal would be generated by the monomial $(a-z)$ with $a \in \mathbb{C}$, meaning that the ideal consists of all polynomials that contain a factor $(a-z)$. We now have a correspondence between maximal ideals and points in $\mathbb{C}$ by identifying a maximal ideal with the zero locus of all its polynomials.

But we would like to give more structure to a variety than just the set of points. One approach would be to include also the lines that are in this space and the surfaces and ... all subspaces of a fixed dimension. The right

prime ideals

choice is to look at **prime ideals** of the ring. These are ideals such that no element is decomposable into the product of elements that are not in that ideal.

Spec(R)

The collection of prime ideals of a ring R is denoted as $\mathrm{Spec}(R)$ and form the topological data of the scheme. To see what this means let us consider the ring of polynomials in *two* variables: $\mathbb{A}^2 = \mathbb{C}[z_1, z_2]$. A maximal ideal in this ring would be of the form $(a-z_1)(b-z_2)$ (here we use the convention to just write down the generators of the ideal between brackets). And indeed these correspond to all the points in $\mathbb{C}^2$. Of course these ideals are also prime, but there are more prime ideals that are not maximal, like the ideals $(a-z_1)$ or $(a-z_1-z_2$, both corresponding to lines in the plane. Also the zero polynomial, corresponding to the whole plane is considered to be a prime ideal. These kind

generic points

of ideals are called **generic points** of $\mathrm{Spec}(R)$, for the obvious reason that if we want to talk about some property for generic points (in the linguistic sense) on a line, we could investigate the generic point corresponding to that line.

To make a topological space out of $\mathrm{Spec}(R)$ the normal procedure would be to first define the set of functions and then define the topology in such a way as to make those functions continuous, because thats what topologies are for. Nonetheless there are no natural functions to a topological space available, although one could approach the construction of a topology also from a function perspective (see [37] for details). We will take another approach and start with defining a basis of *closed* sets. Let's consider a general subset $S \subset R$ and define the set $V(S) \subset \mathrm{Spec}(R)$ as the set of all prime ideals containing S. Notice first that if I is the ideal generated by S then $V(I) = V(S)$, so we just have to look at the ideals of R, furthermore the smaller the ideal the bigger the "set of zeroes" is. These sets have the property that the intersection of a collection of these subsets of $\mathrm{Spec}(R)$ again writable is as $V(J)$ for some ideal J. The same holds for a *finite* union. So the collection $\{V(I)\}$ could be a basis for the

Zariski topology

closed sets of a topology. This topology is called the **Zariski topology**. It is a rather unusual topology in that it is usually non-Hausdorff and even in an rather extreme way. An example will illustrate this:

One of the basic theorems in commutative algebra, Hilbert's nullstellensatz [46], tells us that ideals in $\mathbb{C}[z]$ are finitely generated, meaning that we need just a finite number of polynomials to generate the ideal. This implies that the closed sets in the Zariski topology consist of a finite number of points, next to the empty set and $\mathrm{Spec}(\mathbb{C}[z])$ itself. The open sets are then, as complements of closed ones, huge, covering all of the Spec up to a finite number of points, but then 2 different points are *never* separated in the Hausdorff sense. Now that's pretty extreme. Most of the use of the Zariski topology lies in the possibility to define a generalisation of the concept of vector bundles on schemes, as we will

ideals can be identified with the underlying space through the Gelfand transform.

see next. The topology should really be seen as a tool. When trying to visualise a scheme that is more or less a manifold, a point we will return to at the end of this section, it is better to stick to the normal topology.

One important structure ingredient is still missing from our definitions. In differential geometry we still had the transition functions between charts and most of the properties of manifolds depend on them, like homotopy, cohomology and so on. The topological invariant classes of algebraic topology, of which the Chern classes and the Euler class make their appearance in the main text, are encoded in the tangent bundle of the manifold. It is clear that we do need these kind of concepts to make the scheme into a geometric object, but the idea of vector bundles depends heavily on the existence of local charts which we lack in the scheme structure. The only concept of locality is through the open sets and so we will attach vector space like objects to every open set in an "appropriate way".

These construction are called **sheaves**. The objects we attach are general abelian groups, but they are allowed to have more structure of course, a vectorspace for example. To mimic the common vector bundles, there are some demands on how elements of this sheaf are related. First of all the restriction of a such a group defined on some open set to an open subset should exist and be unique in the sense that, if we restrict a sheaf first to an open set U and then restrict that restriction to the intersection of U with another openset V, the resulting sheaf should be the same as the restriction first to V and then to U. The other important property is that they glue together in a proper way, so if $F(U)$ is attached to the open set U (elements of such an abelian group $F(U)$ are called the **sections** of F) and U_1 ,U_2 are two open subsets of U that form a cover then these sections should add up to sections on U.

sheaves

sections

Now using this sheaf structure we can in essence generalise the concept of charts in an algebraic geometrical way. We define the regular functions at an element $\mathfrak{p} \in \mathrm{Spec}(R)$ as the quotient of two polynomials in R, where the denominator is not contained in $\mathfrak{p}$. This can be extended to open sets in $\mathrm{Spec}(R)$ by taking as ring the rational polynomials that are regular all over that set. Now the logic behind the Zariski topology shines through: since the closed sets are defined as the finite collection of prime ideals that represent zeros of polynomials, the open sets are those prime ideals representing points where a finite set of polynomials is *not* zero! The resulting sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ is called the **structure sheaf** of the scheme. You could see them as the equivalent of the transition functions between overlapping charts and thus more or less define the geometry of the scheme. These regular functions at a prime ideal form a so called **local ring**, which means that this ring contains just one maximal ideal. In this particular case it is called the **coordinate ring** at the point $\mathfrak{p}$, where the "at a point" is meant in a limiting way. This limit of sheaves to a point (it is called an inductive limit) results in what is like the counterpart of a fibre in vector bundle language and goes by the name of **stalk**. We will write it as $\mathcal{O}_{X,\mathfrak{p}}$.

structure sheaf

local ring

coordinate ring

stalk

The main idea in scheme theory is to turn this rather heavy machinery around, forget that the ring was originally a ring of polynomials, but just define it as a pair $(X, \mathcal{O}_X)$ where X is some topological space, $\mathcal{O}_X$ a sheaf of local rings and such that it looks locally as the Spec of a ring.

Usually the schemes that are used in this thesis can be seen just as subspaces of the original polynomial rings and are then the more familiar algebraic vari-

eties. When the scheme is a subspace of a projective space then that scheme is called projective. If it is an open subset of a projective space it is called quasi-projective.

Morphisms between schemes are defined, as to be expected, so that they respect all the structures. That means a continuous $f : X \to Y$ map for the topological part of the scheme and a map between the structure sheaves. But that map can normally only be defined in the opposite direction, by associating the sections on an open set $U \subset Y$ to the open set $f^{-1}(U) \subset X$ and taking the appropriate limit to get a sheaf over all of X. We use the notation $f^{\#}$ for the map of sheaves. This last map should now also be a homomorphism (in the sense of local rings).

subscheme

It is easy to see how one could define an open **subscheme**: since the intersection with the open sets of the scheme are again open in the subscheme, using the induced topology, we can just use the structure sheaf from the original scheme. For a closed subscheme it is a bit more subtle. The intersection are in this case in general not open and so have a priori no sections defined on them. The trick is to use the inclusion map and demand that the induced map from the structure sheaf of the scheme to that of the subscheme surjective is.

dimension

As a final notion we will briefly mention the definition of **dimension** of a topological space. In an Euclidean space the dimension can be described by counting how many times one can drop a dimension until the empty set is reached: for example a 3 dimensional space has 2 dimensional surfaces as closed subspaces , a surface has lines and a line points, that make a total of 3. This sound like a difficult way of describing something obvious, but it makes it clear how one could define the dimension of a general topological space : just see how to filter a topological space by an inclusion of closed irreducible[2] subspaces : $Z_0 \subset \cdots \subset Z_n \subset X$ the dimension of X is then the supremum of n for all possible filtrations. In algebraic geometry there is an in this context equivalent

Krull dimension

definition but now using the inclusion of prime ideals. This is called the **Krull dimension**. From the above it looks reasonable to suspect that both definitions coincide since the prime ideals either represent the points, or the generic points representing just the closed irreducible subspaces.

D.1.1 Fat points, an example

As an illustration of some of the hardcore concepts defined in the previous section and especially to show how "fat points" can get separated in the scheme context, an important aspect in the course of this thesis, we will consider a the subscheme of the affine line generated by the ring $\mathbb{C}[\epsilon]/(\epsilon^2)$. The Spec of this ring consists of just one point , namely the prime ideal generated by (ϵ). The reason is that any ideal generated by $(\epsilon - a)$ for nonzero a can be written as $(\epsilon - a + b\epsilon^2) = (\epsilon - a_1)(\epsilon - a_2)$ (since $\epsilon^2 \sim 0!$) so it is not prime. The structure sheaf we get from the inclusion map, as a closed subscheme, as outlined above. The local ring on the only open subset, and it is at the same time the stalk above the only point, has non-zero regular functions of the form az, higher degree polynomials getting mapped to zero or one of this form by the inclusion. One could say that the single point of the closed subscheme has a

[2] irreducible meaning that it can't be written as the union of two closed nonintersecting subspaces

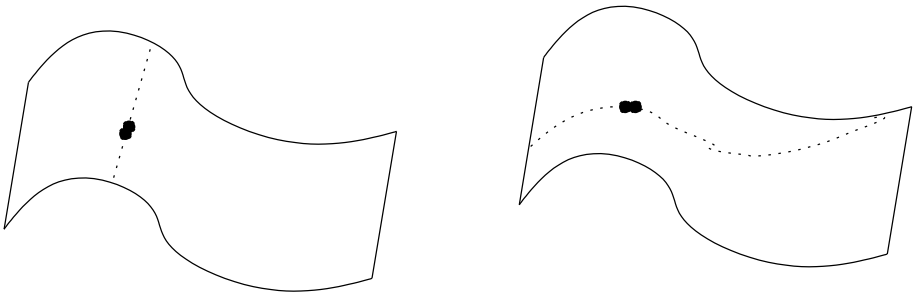


Figure D.1: 2 inclusions of a double point subscheme in a plane

one dimensional tangent space attached to it, that somehow distinguishes how two points collapsed into one single point. The degree of the polynomial that defined the subscheme in this case indicates how big the resulting tangent space is. When we construct the same subscheme in the affine plane, we would see that there are many ways to embed it, corresponding to the different directions two points can collide from in the plane (fig. D.1.1). This is the same mechanism we encounter in the Hilbert scheme of points in the plane, where subschemes containing points with a multiplicity higher than one remembers from what direction those points collided. Another way to look at it is to realise that we have to count the ways we can embed such a scheme in the original scheme.

D.2 Flat families and functors

In this second section on algebraic geometry we will focus on the concepts of flat families and how to represent families by a scheme. This will give some understanding of how the Hilbert schemes are formally constructed.

We will start with a short description of **categories**. Although pervading many branches of mathematical physics nowadays, it seems appropriate to give the basic definitions. In short the idea behind categories is define different structures in mathematics in such a broad sense that it is possible to relate seemingly unrelated structures to each other. A structure in a category does not consist only of elements, but also maps in between elements. There are two building blocks in a category $\mathcal{C}$: the objects, $\mathrm{Ob}(\mathcal{C})$ and the arrows, $\mathrm{Hom}(\mathcal{C})$, connecting objects. categories

Examples are the objects of "finite sets", topological spaces, Lie groups, commutative algebras,... The arrows are the morphisms in the category, for example they are continuous functions in the category of topological spaces, group diffeomorphisms for lie groups and so on. Besides the obkects and the arrows, there is still the composition map for the arrows, a source and a target map that map arrows to their respective source and target object and an identity map that maps an object to the identity map for this object. Note that not all objects need to be connected to each other by an arrow and so composition is only defined if the source of one arrow is the target of the next one.

A **functor** is a map between two categories and has naturally two parts: one that maps the objects and a second one for the arrows. If the functor reverses the direction of the arrows it is called **contravariant** else it is **covariant**. A functor contravariant covariant

typical example of a functor is the forgetfull functor to the category of sets that forgets about all the structure that category had. An example of a contravariant functor is the functor that maps from the category of commutative rings to the category of affine schemes that maps R to Spec(R) (see [36]).

flatness

The concept of **flatness** of a module M, over a ring R, is a rather technical concept that says that when this module is tensored with any 2 modules over the same ring that are related by an injective morphism [3] , the induced morphism between the tensor product is again injective. It can be shown that modules over a field, like $\mathbb{R}$ or $\mathbb{C}$ are allways flat. For a more thorough treatment of flatness the book of Atiyah and Macdonald [46] could be consulted.

When $f : R \to S$ is a map between rings, S is called flat over R if it is flat as an R-module. Now we finally make again contact with schemes by defining family $\phi : A \to B$ of schemes to be flat when the the local ring $\mathcal{O}_{A,p}$ is flat over $\mathcal{O}_{B,\phi(p)}$.

Now we are ready to tie some concepts together that makes it possible to speak of a Hilbert scheme of points. We are going to construct a functor from the category of schemes to the category of sets. Let X be a scheme define a

functor of points

contravariant functor $\mathcal{P}_X$ called the **functor of points** as follows:

- on objects to the morphisms between schemes

$$\mathcal{P}_{(X,1)} : Obj_{\mathrm{Scm}} \to Obj_{\mathrm{Set}}$$
$$Y \mapsto Hom_{\mathrm{Scm}}(Y, X)$$

- on arrows by composition $g \in Hom_{\mathrm{Scm}}(Z, X) \mapsto g \circ f \in Hom_{\mathrm{Scm}}(Y, X)$

$$\mathcal{P}_{(X,2)} : Hom_{\mathrm{Scm}} \to Hom_{\mathrm{Set}}$$
$$Hom_{\mathrm{Scm}}(Y, Z) \ni f \mapsto \circ f \in Hom_{\mathrm{Set}}(Hom_{\mathrm{Scm}}(Z, X), Hom_{\mathrm{Scm}}(Y, X))$$

the morphisms in $Hom_{\mathrm{Scm}}(Y, X)$ go by the name of Y valued points of X In the case we have a contravariant functor from Schemes to Sets and there exists a scheme X such that that functor can be written as its functor of points, then

representable

the functor is called **representable** by X. This is exactly the situation we have in the case of Hilbert schemes: We look for a given scheme B at the set of flat families $A_B \to B$ with as fibres subschemes of some projective scheme X with a constant Hilbert polynomial (these are zero dimensional subschemes of given length as explained in (6.1)). Grothendieck has shown that the functor that assigns this set to the scheme B representable is as a functor of points of the scheme $X^{[n]}$. For a deeper understanding of these ideas the book of Eisenbud and Harris [37] is a good point to start.

To close this appendix some words about the way schemes are related to analytic manifolds [36]. Since usually in physics we assume all functions to be sufficiently differentiable, a question that comes to mind is under what circumstances schemes , in all its abstraction, can be seen as common, everyday manifolds, and conversely, is every complex manifold a scheme? In all generality the answer is no in both directions. Riemann surfaces can always be seen as schemes and all projective schemes and projective analytic manifolds can be shown to be equivalent. In other cases one needs some rationality conditions before an analytic manifold can be made into a scheme.

[3] such a morphism is called a monomorphism. In categorical language it means that composition is injective

Tables

	$\big(b_1 \otimes b_2 \otimes b_3, \mathrm{Id}\big)$	$\big(b_1 \otimes b_2, (12)\big)$	$\big(b_1 \otimes b_2, (13)\big)$
$\big(a_1 \otimes a_2 \otimes a_3, \mathrm{Id}\big)$	$\big(a_1b_1 \otimes a_2b_2 \otimes a_3b_3, \mathrm{Id}\big)$	$\big(a_1a_2b_1 \otimes a_3b_2, (12)\big)$	$\big(a_1a_2b_1 \otimes a_3b_2, (13)\big)$
$\big(a_1 \otimes a_2, (12)\big)$	$\big(a_1b_1b_2 \otimes a_2b_3, (12)\big)$	$\big(\Delta_*^{(2)}(a_1b_1) \otimes a_2b_2, \mathrm{Id}\big)$	$\big(a_1a_2b_1b_2, (132)\big)$
$\big(a_1 \otimes a_2, (13)\big)$	$\big(a_1b_1b_2 \otimes a_2b_3, (13)\big)$	$\big(a_1a_2b_1b_2, (123)\big)$	$\big(\Delta_*^{(2)}(a_1b_1) \otimes a_2b_2, \mathrm{Id}\big)$
$\big(a_1 \otimes a_2, (23)\big)$	$\big(a_1b_1 \otimes a_2b_2b_3, (23)\big)$	$\big(a_1a_2b_1b_2, (123)\big)$	$\big(a_1a_2b_1b_2, (132)\big)$
$\big(a, (123)\big)$	$\big(ab_1b_2b_3, (123)\big)$	$\big(\Delta_*^{(2)}(ab_1b_2), (13)\big)$	$\big(\Delta_*^{(2)}(ab_1b_2), (23)\big)$
$\big(a, (132)\big)$	$\big(ab_1b_2b_3, (132)\big)$	$\big(\Delta_*^{(2)}(ab_1b_2), (23)\big)$	$\big(\Delta_*^{(2)}(ab_1b_2), (12)\big)$

	$\big(b_1 \otimes b_2, (23)\big)$	$\big(b, (123)\big)$	$\big(b, (132)\big)$
$\big(a_1 \otimes a_2 \otimes a_3, \mathrm{Id}\big)$	$\big(a_1b_1 \otimes a_2a_3b_2, (23)\big)$	$\big(a_1a_2a_3b, (123)\big)$	$\big(a_1a_2a_3b, (132)\big)$
$\big(a_1 \otimes a_2, (12)\big)$	$\big(a_1a_2b_1b_2, (132)\big)$	$\big(\Delta_*^{(2)}(a_1a_2b), (23)\big)$	$\big(\Delta_*^{(2)}(a_1a_2b), (13)\big)$
$\big(a_1 \otimes a_2, (13)\big)$	$\big(a_1a_2b_1b_2, (123)\big)$	$\big(\Delta_*^{(2)}(a_1a_2b), (12)\big)$	$\big(\Delta_*^{(2)}(a_1a_2b), (23)\big)$
$\big(a_1 \otimes a_2, (23)\big)$	$\big(a_1b_1 \otimes \Delta_*^{(2)}(a_2b_2), \mathrm{Id}\big)$	$\big(\Delta_*^{(2)}(a_1a_2b), (13)\big)$	$\big(\Delta_*^{(2)}(a_1a_2b), (12)\big)$
$\big(a, (123)\big)$	$\big(\Delta_*^{(2)}(ab_1b_2), (12)\big)$	$\big(abe, (132)\big)$	$\big(\Delta_*^{(3)}(ab), \mathrm{Id}\big)$
$\big(a, (132)\big)$	$\big(\Delta_*^{(2)}(ab_1b_2), (13)\big)$	$\big(\Delta_*^{(3)}(ab), \mathrm{Id}\big)$	$\big(abe, (123)\big)$

Table D.1: S_3 multiplication in $\mathcal{H}\{S_3\}$ for $K3$

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Conventions

The 4 dimensional Minkowski metric is $\eta^{\mu} = (-,+,+,+)$

The Pauli matrices are defined as:

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The total antisymmetric tensor for spinor algebras:

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \epsilon^{\beta}_{\alpha} = \delta^{\beta}_{\alpha}$$

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