general relativity – October 6, 2009

material discussed in class

Roughly 3.1-3.3 in the book.

exercises

- The metric on a two-sphere of unit radius is given by

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

Compute the eight Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ using (i) the definition of the Christoffel symbols in terms of the metric and (ii) the variational principle applied to the action

$$I = \int d\lambda\, g_{\mu\nu}(x^\rho)\dot{x}^\mu\dot{x}^\nu.$$

- The equation for parallel transport of a vector reads

$$\frac{d}{d\lambda}A^\mu + \dot{x}^\nu\Gamma^\mu_{\nu\rho}(x^\alpha(\lambda))A^\rho = 0.$$

Work out this equation in detail for the case of parallel transport on a two-sphere, using the results of the previous exercise.

- Show that when we parallel transport a vector A^μ on a two-sphere, the value of

$$(A^\theta)^2 + \sin^2\theta(A^\phi)^2 \tag{1}$$

does not change.

- Consider once more the unit two-sphere, and consider the vector A^μ_0 which is the unit vector in the θ-direction, at the point p_0 which is $\theta = \pi/2$, $\phi = 0$ in polar coordinates. What happens to the vector if we parallel transport it once around the equator, i.e. along the path $(\theta(\lambda), \phi(\lambda)) = (\pi/2, \lambda)$ with $0 \le \lambda \le 2\pi$? Next consider a curve which consists of four separate pieces,

$$\begin{aligned}
\gamma_1(\lambda) &= (\pi/2, \lambda) \text{ for } 0 \le \lambda \le \lambda_1 \\
\gamma_2(\lambda) &= (\pi/2 - \lambda, \lambda_1) \text{ for } 0 \le \lambda \le \lambda_2 \\
\gamma_3(\lambda) &= (\pi/2 - \lambda_2, \lambda_1 - \lambda) \text{ for } 0 \le \lambda \le \lambda_1 \\
\gamma_4(\lambda) &= (\pi/2 - \lambda_2 + \lambda, 0) \text{ for } 0 \le \lambda \le \lambda_2
\end{aligned}$$

with $0 \le \lambda_1 \le 2\pi$ and $0 \le \lambda_2 \le \pi/2$. Find out what happens to the vector A^μ_0 once we parallel transport it around this closed path. What

is the area of the two-sphere enclosed by this closed curve? Is this area somehow related to the result of the parallel transport? (hint: To solve the relevant differential equations, you may find it useful to use the fact that (1) does not change during parallel transport, though you can also solve this problem without using this fact)