general relativity – november 10, 2009

material discussed in class

Roughly 4.1, 4.2 in the book.

exercises

1) In four dimensions ($n=4$), the Einstein equations imply

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}). \tag{1}$$

Here, $T = g^{\mu\nu}T_{\mu\nu}$. What is $R_{\mu\nu}$ in terms of $T_{\mu\nu}$ if $n \neq 4$ and $n \neq 2$?.

2) Show that in two dimenstions the Einstein equations are equivalent to $T_{\mu\nu} = 0$. To do this, first write down the most general Riemann tensor in two dimensions that is compatible with the symmetries that the Riemann tensor should have. From this infer the most general form of the Ricci tensor.

3) Suppose we make a change of coordinates of the form

$$x^{\mu'} = x^\mu + \epsilon K^\mu(x^\nu) \tag{2}$$

where ϵ is a very small parameter. Express $g_{\mu'\nu'}(x^{\lambda'})$ in terms of K^μ and $g_{\mu\nu}(x^\lambda)$ to first order in ϵ.

4) Show that the requirement that $g_{\mu'\nu'}(x^{\lambda'}) = g_{\mu\nu}(x^\lambda)$ to first order in ϵ is equivalent to the Killing vector condition

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0. \tag{3}$$

5) (Carroll 3.10) Define the Weyl tensor $C_{\rho\sigma\mu\nu}$ in terms of the Riemann curvature tensor by

$$\begin{aligned} C_{\rho\sigma\mu\nu} &= R_{\rho\sigma\mu\nu} - \frac{2}{(n-2)}\left(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}\right) \\ &\quad + \frac{2}{(n-1)(n-2)} g_{\rho[\mu}g_{\nu]\sigma}R. \end{aligned} \tag{4}$$

Recall that the notation $[\mu\nu]$ indicates antisymmetrization in the two indices, as e.g. in $F_{[\mu\nu]} \equiv \frac{1}{2}(F_{\mu\nu} - F_{\nu\mu})$. Show that in dimension $n \geq 4$,

$$\nabla_\rho C^\rho{}_{\sigma\mu\nu} = 2\frac{(n-3)}{(n-2)}\left(\nabla_{[\mu}R_{\nu]\sigma} + \frac{1}{2(n-1)} g_{\sigma[\mu}\nabla_{\nu]}R\right). \tag{5}$$