general relativity – September 15, 2009

material discussed in class

Roughly equal to sections 1.9, 2.1 in Carroll

exercises

- Recall that the delta-function $\delta(x)$ is defined as

$$\int dx \delta(x) f(x) = f(0) \tag{1}$$

for any function $f(x)$. Also recall that for a monotonously increasing function $g(x)$ with $g(a) = 0$,

$$\delta(g(x)) = \frac{1}{g'(x)} \delta(x - a). \tag{2}$$

Define the charge and current of a collection of point particles which have charge q_n and move along trajectories $x_n^i(t)$ $(i = 1, 2, 3)$ as follows

$$\begin{aligned} Q(t,x) &= \sum_n q_n \delta^{(3)}(x - x_n(t)) \\ J^i(t,x) &= \sum_n q_n \delta^{(3)}(x - x_n(t)) \frac{dx_n^i(t)}{dt}. \end{aligned}$$

Here, x is shorthand for x^i, $i = 1, 2, 3$, and similarly $x_n(t)$ is shorthand for $x_n^i(t)$, $i = 1, 2, 3$. Furthermore, $\delta^{(3)}(x - x_n(t))$ is the three-dimensional delta function

$$\delta^{(3)}(x - x_n(t)) \equiv \delta(x^1 - x_n^1(t)) \delta(x^2 - x_n^2(t)) \delta(x^3 - x_n^3(t)).$$

Why are these the correct definitions of the charge density and current of a collection of point particles? These combine nicely into a four-vector $J^\mu = (Q, J^i)$ which indeed transforms as a vector under Lorentz transformations. Verify this. Hint: first show that if the trajectories are parametrized more generally as $x_n^\mu(\lambda)$, we can rewrite J^μ as follows:

$$J^\mu(x^\nu) = \sum_n \int d\lambda \; q_n \frac{dx_n^\mu(\lambda)}{d\lambda} \delta^{(4)}(x^\nu - x_n^\nu(\lambda)).$$

- Show that J^μ is a conserved current, $\partial_\mu J^\mu = 0$.
- Show that $Q = \int d^3x J^0(x)$ is conserved, i.e. time-independent, using only the fact that $\partial_\mu J^\mu = 0$, and not the detailed form of J^μ.

- A relativistic particle with four-momentum p^μ is moving in a square box of size $L \times L \times L$. Every time is collides with a wall it is reflected back, and if we average these forces over a long time a net pressure will be exerted on the walls of the box. Show that this pressure equals

$$P = \frac{\sum_{i=1}^{3} p^i p^i}{3p^0}. \tag{3}$$

  Consider e.g. first only the collisions with the walls perpendicular to the x-axis, compute how many collisions with these walls take place per unit time, and use the relation $F\Delta t = \Delta p$ to compute the total force. After including also the effects of the other walls, the result (3) will then follow.

- [OPTIONAL] Carroll problem 1.9