General Relativity Delft 2009

Take Home Set 1

Hand in by September 30, no later than 10.00 hours (either give to Giorgi or put in his mailbox)

Problem 1

The components of the electromagnetic field tensor read

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}. \tag{1}$$

If E_i and B_i are the electric and magnetic fields as measured by an observer at rest, what are the electric and magnetic fields as seen by an observer who moves with constant velocity v in the x-direction? Use the fact that $F^{\mu\nu}$ is a tensor.

Problem 2

A particle with charge q which moves through an electric and magnetic field is subject to the well-known Lorentz force. The relativistic formulation of the Lorentz force reads

$$m\frac{du^\mu}{d\tau} = qF^{\mu\nu}u_\nu \tag{2}$$

where u^μ was the four velocity defined as

$$\frac{dx^\mu}{d\tau} = u^\mu. \tag{3}$$

Is (2) a tensor equation? Why? Show that in the limit of small velocities, or equivalently in the limit where the speed of light is taken to infinity, this reduces to the usual expression of the Lorentz force.

Problem 3

A relativistic particle with rest mass m and charge q is at rest at $t = 0$. There is a constant electric field E in the x-direction which accelerates the particle. Compute $x^\mu(t)$. Verify that the particle approaches the speed of light as $t \to \infty$.

Problem 4

A classical Lagrangian $L(q_i, \dot{q}_i)$ has the property that the action $I = \int dt L(q_i, \dot{q}_i)$ is invariant under the infinitesimal symmetry

$$\delta q_i = \epsilon f_i(q_j). \tag{4}$$

Show that Q defined as

$$Q = \sum_i \frac{\partial L}{\partial \dot{q}_i} f_i(q_j) \tag{5}$$

satisfies $\dot{Q} = 0$ when using the equations of motion. In other words, Q is constant for every solution of the equations of motion. This is a simple example of Noether's theorem. Verify that for the 2d harmonic oscillator with

$$L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2}m\omega^2(q_1^2 + q_2^2) \tag{6}$$

the conserved charge associated to the infinitesimal rotation in the q_1, q_2-plane ($\delta q_1 = \epsilon q_2$, $\delta q_2 = -\epsilon q_1$) is proportional to angular momentum.

Problem 5

An n-dimensional manifold, by definition, can be covered by open sets each of which can be mapped in a one-to-one way to an open subset of $\mathbb{R}^n$. Sketch such a covering for (i) the two-sphere, (ii) the two-torus (like e.g. an inflated rubber tire or a doughnut), and (iii) a genus-two Riemann surface (this is a two-dimensional surface just like the two-torus, except that it has two holes instead on one). For completeness, we recall the definition of an open subset of $\mathbb{R}^n$. This is a subset $U \subset \mathbb{R}^n$ such that for all $x \in U$ there exists an $\epsilon > 0$ such that $D(x, \epsilon) = \{y \in \mathbb{R}^n \,||x - y| < \epsilon\}$ satisfies $D(x, \epsilon) \subset U$. In other words, for all points in U there is a small disc around the point which also fits completely in U.