General Relativity Delft 2009

Take Home Set 2

Hand in by October 28, no later than 10.00 hours (either give to Giorgi or put in his mailbox)

Problem 1

a) Prove Carroll (3.33), i.e.

$$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{\sqrt{|g|}}\partial_{\lambda}\sqrt{|g|}.$$

b) Use this result to show that if a vector field V^{μ} decays rapidly at infinity, then

$$\int d^{n}x\,\sqrt{|g|}\,\nabla_{\mu}V^{\mu} = 0. \tag{1}$$

This is the analogue of the statement in flat space that the integral of a total derivative vanishes.

Problem 2

a) When parallel transporting a vector (A^{θ}, A^{ϕ}) on the two-sphere with metric

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

we know that the norm of this vector has to remain constant. Therefore, without loss of generality, we can parametrize the vector that is being parallel transported as

$$(A^{\theta}, A^{\phi}) = \mathcal{N}(\cos\alpha(\lambda), \frac{\sin\alpha(\lambda)}{\sin\theta(\lambda)}) \tag{2}$$

where $\mathcal{N}$ is a constant. Why is this the right form of (A^{θ}, A^{ϕ})?

b) Show that the equations for parallel transport found in a previous homework problem reduce to the very simple equation

$$\frac{d\alpha(\lambda)}{d\lambda} + \cos\theta(\lambda)\frac{d\phi(\lambda)}{d\lambda} = 0. \tag{3}$$

This equation is much easier to solve! Again, the lesson is to always use conservation laws whenever they are available.

Problem 3 (based on Carroll problem 3.6)

A good approximation to the metric outside the surface of the Earth is provided by

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4}$$

where

$$\Phi = -\frac{GM}{r}$$

may be thought of as the familiar Newtonian gravitational potential. Here, G is Newton's constant and M is the mass of the earth. For this problem Φ may be assumed to be small, in other words you only need to work to first order in G.

a) Imagine a clock on the surface of the earth at distance R_0 from the Earth's center (i.e. R_0 is the radius of the Earth) and another clock on a tall building of height h, so that it is a distance $R_0 + h$ from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time t. Which clock moves faster?

b) Recall that the geodesic equations can be derived from the action

$$I = \int d\lambda\, g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \tag{5}$$

by using the variational principle. The equations of motion of (5) are precisely the geodesic equations. Find the equations of motion obtained by varying t and ϕ. Show that they are equivalent to

$$(1+2\Phi)\dot{t} = E, \qquad r^2\sin^2\theta\dot{\phi} = L \tag{6}$$

where E and L are some constants. (These constants correspond roughly speaking to the "energy" and "angular momentum" of a geodesic).

c) Why was the quantity

$$\epsilon = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{7}$$

constant along a geodesic?

d) We now restrict to geodesics that lie in the equatorial plane $\theta = \pi/2$. Use (6) and (7) to find an equation of the form

$$\left(\frac{dr}{dt}\right)^2 + V(r) = 0 \tag{8}$$

which is valid along the geodesic. Here, $V(r)$ is a function of r which depends on ϵ, L, E. Remember that the dots denoted a derivative with respect to λ, not with respect to t!

e) Equation (8) looks like the equation of a classical point particle that moves in the potential $V(r)$. Circular orbits where r is constant therefore correspond to values of r where $V(r) = 0$ which are also extrema of V. Assuming that $V(r) = 0$ at $r = R$ and that this is also an extremum of $V(r)$, find two equations for L, E, ϵ. From these, compute $d\phi/dt$ as a function of R.

f) How much proper time elapses while a satellite at radius R_0 (skimming along the surface of the earth, neglecting air resistance) completes one orbit? Plug in the actual numbers for the radius of the Earth and so on (don't forget to restore the speed of light) to get an answer in seconds. How does this number compare to the proper time elapsed on the clock stationary on the surface?