general relativity – February 3, 2011

material discussed in class

Roughly equal to sections 1.4-1.8 in Carroll. It is recommended that you read these sections as well.

exercises

- Recall that $\Lambda^{\mu'}{}_{\mu}$ and $\Lambda^{\mu}{}_{\mu'}$ are not the same, but each others inverse, in other words
$$\Lambda^{\mu'}{}_{\mu}\Lambda^{\mu}{}_{\nu'} = \delta^{\mu'}_{\nu'}.$$
Use this to show that $\eta^{\mu\nu}$ is a tensor, i.e.
$$\eta^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu}\Lambda^{\nu'}{}_{\nu}\eta^{\mu\nu}.$$
Verify also that δ^{ν}_{μ} is a tensor.

- Define the charge and current of a collection of point particles which have charge q_n and move along trajectories $x^i_n(t)$ $(i = 1, 2, 3)$ as follows
$$\begin{aligned} Q(t,x) &= \sum_n q_n \delta^{(3)}(x - x_n(t)) \\ J^i(t,x) &= \sum_n q_n \delta^{(3)}(x - x_n(t)) \frac{dx^i_n(t)}{dt}. \end{aligned}$$
Here, x is shorthand for x^i, $i = 1, 2, 3$, and similarly $x_n(t)$ is shorthand for $x^i_n(t)$, $i = 1, 2, 3$. Furthermore, $\delta^{(3)}(x - x_n(t))$ is the three-dimensional delta function
$$\delta^{(3)}(x - x_n(t)) \equiv \delta(x^1 - x^1_n(t))\delta(x^2 - x^2_n(t))\delta(x^3 - x^3_n(t)).$$
Why are these the correct definitions of the charge density and current of a collection of point particles? These combine nicely into a four-vector $J^\mu = (Q, J^i)$ which indeed transforms as a vector under Lorentz transformations. Verify this. Hint: first show that if the trajectories are parametrized more generally as $x^\mu_n(\lambda)$, we can rewrite J^μ as follows:
$$J^\mu(x^\nu) = \sum_n \int d\lambda \; q_n \frac{dx^\mu_n(\lambda)}{d\lambda} \delta^{(4)}(x^\nu - x^\nu_n(\lambda)).$$

- Show that J^μ is a conserved current, $\partial_\mu J^\mu = 0$.

- Show that $Q = \int d^3x J^0(x)$ is conserved, i.e. time-independent, using only the fact that $\partial_\mu J^\mu = 0$, and not the detailed form of J^μ.

- Exercises 1.6-1.7 in Carroll.