general relativity – March 10, 2011

material discussed in class

Roughly 8.4,8.5, and a little bit of 8.7 in the book.

exercises

- Show that the Friedmann equations

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho+3p) \\ \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \end{aligned} \tag{1}$$

  imply the equation we derived from the conservation of energy and momentum,

$$\nabla_\mu T^{\mu i} = \dot{\rho} + 3\frac{\dot{a}}{a}(p+\rho) = 0. \tag{2}$$

  why should they imply this?

- Study the solutions of the Friedmann equations for universes filled with either pure matter, pure radation, and nothing but a cosmological constant. Separately treat the cases $k=-1$, $k=0$ and $k=+1$, and draw pictures of the time evolution of the size $a(t)$ of the universe.

  It is possible to give closed expressions for $a(t)$ in all cases, except when the universe contains only matter and $k=\pm 1$. For $k=-1$, show that the solutions take the form $a=(C/2)(\cosh\phi-1)$, $t=(C/2)(\sinh\phi-\phi)$ for some constant C: these equations implicitly determine a as a function of t.

  For a matter dominated universe with $k=+1$, similarly show that the solutions take the form $a=(C/2)(1-\cos\phi)$, $t=(C/2)(\phi-\sin\phi)$.

- Consider the FRW metric with $k=0$,

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2 d\Omega_2^2).$$

  Assume that a star is located at $r=0$, and that we are located at $r=R$. A light ray is emitted at $t=t_1$ from the star, and reaches us at $t=t_0$. Show that

$$R = \int_{t_1}^{t_0} \frac{dt}{a(t)}. \tag{3}$$

  Recall that we computed the redshift of the light that we receive, and it was equal to

$$z = \frac{a(t_0)}{a(t_1)} - 1. \tag{4}$$

Also recall that the luminosity distance was defined by $d_L^2 = \frac{L}{4\pi F}$, where L is the total luminosity of the star, and F the flux measured by us. In a static universe, d_L would be the actual distance to the star, but in an expanding universe this is no longer the case. In an expanding universe the flux is diluted because photons are redshifted, and in addition by the fact that we receive fewer photons per second. Verify that therefore the correct relation between R, F and L reads

$$\frac{F}{L} = \frac{1}{4\pi R^2 a(t_0)^2 (1+z)^2}. \tag{5}$$

From this we deduce that the luminosity distance (which we determine experimentally by measuring F and using for L the known approximate value for the objects we consider, we only know L reasonably well for so-called standard candles) is equal to

$$d_L = R a(t_0)(1+z). \tag{6}$$

Now Taylor expand (3) and (4) around $t = t_0$. In other words, replace $a(t)$ by $a(t_0) + (t_1 - t_0)\dot{a}(t_0) + \frac{1}{2}(t_1 - t_0)^2 \ddot{a}(t_0) + \ldots$ and drop the higher order terms.

Then use (3) and (4) to write R and z as a power series in $t_1 - t_0$ up to second order. Eliminate $t_1 - t_0$ from these power series to find a relation between R and z, valid up to second order in z, and finally use (6) to show that

$$d_L = H_0^{-1}\left[z + \frac{1}{2}(1 - q_0)z^2 + \ldots\right], \tag{7}$$

where $H_0 = \frac{\dot{a}}{a}|_{t=t_0}$ is the present Hubble parameter, and $q_0 = -\frac{a\ddot{a}}{\dot{a}^2}|_{t=t_0}$ is the present value of the deceleration parameter.