General Relativity

Take Home Set 1

Hand in on February 24, 2011, before 11.00 hours

Problem 1

a) Prove Carroll (3.33), i.e.

$$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{\sqrt{|g|}}\partial_\lambda \sqrt{|g|}.$$

b) Use this result to show that if a vector field V^μ decays rapidly at infinity, then

$$\int d^n x \sqrt{|g|}\, \nabla_\mu V^\mu = 0. \qquad (1)$$

This is the analogue of the statement in flat space that the integral of a total derivative vanishes.

Problem 2

Consider the following metric on $\mathbb{R}^2$

$$ds^2 = e^{\phi(x,y)}(dx^2 + dy^2). \qquad (2)$$

a) Compute the Christoffel symbols and Riemann curvature tensor for this metric.

b) Next we take the metric on the plane induced by the stereographic projection, i.e.

$$e^{\phi(x,y)} = \left(\frac{4}{4 + x^2 + y^2}\right)^2 .$$

Compute $\int\int_{\mathbb{R}^2} \sqrt{g}\, dxdy$ and $\int\int_{\mathbb{R}^2} \sqrt{g} R\, dxdy$, where R is the Ricci scalar.

Problem 3

a) In four dimensions ($n = 4$), the Einstein equations imply

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}). \qquad (3)$$

Here, $T = g^{\mu\nu} T_{\mu\nu}$. What is $R_{\mu\nu}$ in terms of $T_{\mu\nu}$ if $n \neq 4$ and $n \neq 2$?.

b) Show that in two dimensions the Einstein equations are equivalent to $T_{\mu\nu} = 0$. To do this, first write down the most general Riemann tensor in two dimensions that is compatible with the symmetries that the Riemann tensor should have. From this infer the most general form of the Ricci tensor.

Problem 4

a) Suppose that in (2) the function $\phi(x,y)$ is a function of x^2+y^2 only. Show that in that case

$$K = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

is a Killing vector of (2). Which symmetry does this Killing vector generate?

b) Find the most general Killing vector of two-dimensional Minkowski space-time $ds^2 = -dt^2 + dx^2$.

c) Show that any Killing vector obeys $\nabla_\mu \nabla_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu} K^\nu$ and $K^\lambda \nabla_\lambda R = 0$ (Carroll problem 3.3).

Problem 5 (Carroll problem 3.6)

A good approximation to the metric outside the surface of the Earth is provided by

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4}$$

where

$$\Phi = -\frac{GM}{r}$$

may be thought of as the familiar Newtonian gravitational potential. Here, G is Newton's constant and M is the mass of the earth. For this problem Φ may be assumed to be small, in other words you only need to work to first order in G.

a) Imagine a clock on the surface of the earth at distance R_0 from the Earth's center (i.e. R_0 is the radius of the Earth) and another clock on a tall building of height h, so that it is a distance $R_0 + h$ from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time t. Which clock moves faster?

b) Solve for a geodesic corresponding to a circular orbit around the equator of the Earth ($\theta = \pi/2$). What is $d\phi/dt$?

c) How much proper time elapses while a satellite at radius R_0 (skimming along the surface of the earth, neglecting air resistance) completes one orbit? Plug in the actual numbers for the radius of the Earth and so on (don't forget to restore the speed of light) to get an answer in seconds. How does this number compare to the proper time elapsed on the clock stationary on the surface?