General Relativity
Take Home Set 2
Hand in on March 14, 2011, before 11.00 hours

Write down your answers in ink.

Problem 1

A geodesic congruence is a family of geodesics such that through each point in some region of a curved space passes only one curve of the family. The congruence can be written as $x^\mu(s,\lambda)$, with the parameter s labelling each geodesic and λ the affine parameter along each curve.

a) If $u = \frac{d}{d\lambda} = \frac{\partial x^\mu(s,\lambda)}{\partial \lambda}\partial_\mu$ is, for each s, the tangent to a geodesic, begin by writing down the equation obeyed by u and which defines the curves as geodesics.
Then, introduce the displacement vector

$$S^\mu = \frac{\partial x^\mu(s,\lambda)}{\partial s},$$

connecting any two nearby geodesics of the congruence and define their relative velocity

$$V^\mu = u^\nu \nabla_\nu S^\mu = \frac{\partial S^\mu}{\partial \lambda} + \Gamma^\mu_{\nu\sigma} u^\nu S^\sigma \ .$$

Finally, introduce the relative acceleration

$$A^\mu = u^\nu \nabla_\nu V^\mu = \frac{\partial V^\mu}{\partial \lambda} + \Gamma^\mu_{\nu\sigma} u^\nu V^\sigma \ ,$$

These are indeed a measure of the relative velocity and acceleration of nearby geodesics given by two different but almost identical values of s. Show that V^μ and A^μ are tensors. What is A^μ in flat space?

We now want to deduce the *geodesic deviation* equation

$$A^\mu = R^\mu{}_{\nu\rho\sigma} \frac{\partial x^\nu}{\partial \lambda} \frac{\partial x^\rho}{\partial \lambda} S^\sigma. \tag{1}$$

b) Consider a generic point in the congruence defined by some fixed s and λ, say $s = s_0$ and $\lambda = \lambda_0$. Explain why we can always find a coordinate system such that locally

$$\Gamma^\mu_{\nu\rho}(x^\sigma(s_0,\lambda_0)) = 0.$$

Note, however, that unless the space is flat, derivatives of $\Gamma^\mu_{\nu\rho}$ do not vanish at that point.

c) Given such a local coordinate system, show that (1) is indeed true at the point $x^\sigma = x^\sigma(s_0, \lambda_0)$.
(Hint: Notice that the partial derivatives $\partial/\partial\lambda$ and $\partial/\partial s$ commute and $x^\mu(\lambda, s)$ are geodesics).

e) Explain why this implies that (1) is valid everywhere and in every coordinate system.

Problem 2

We consider gravitational waves propagating in the x^3 direction. The metric is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu}$ small. The solution of the linearized Einstein equations is that $h_{\mu\nu}$ has to be a linear combination of solutions of the form

$$h_{\mu\nu} = C_{\mu\nu}(k)e^{ik_\mu x^\mu} \tag{2}$$

with

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

Since we are interested in waves propagating in the x^3 direction, we will only consider momenta of the form $k_\mu = (-k, 0, 0, k)$.
The purpose of this exercise is then to deduce the behaviour of a ring of particles in Minkowski space under the passage of a gravitational wave parallel to the ring's plane.

a) Begin by deducing in flat space the most general form of each geodeisc $x^\mu(s, \lambda)$ as defined in the previous problem. Notice, therefore, that

$$\hat{x}^\mu(s, \lambda) = (\lambda, r\cos s, r\sin s, 0) \tag{4}$$

describes a geodesic congruence.

The family (4) describes how an infinitely thin ring of particles (with radius r) lying in the x^1, x^2-plane behaves in flat space: the ring does not change shape or location. We now want to investigate what happens when the gravitational wave passes through the ring. Since we assume that the gravitational wave is very weak, we will assume that the family of geodesics describing the ring is perturbed by a small amount due to the gravitational wave,

$$x^\mu(s, \lambda) = \hat{x}^\mu(s, \lambda) + \delta x^\mu(s, \lambda). \tag{5}$$

b) Consider the geodesic deviation equation (1), take for the metric a gravitational wave $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and take for the family of geodesics (3). Expand the geodesic deviation equation (1) to first order in $h_{\mu\nu}$ and $\delta x^\mu(s, \lambda)$ (so products of these two quantities with each other are also ignored). Give the form of the geodesic deviation equation to this order and show that at this order only ${R^\mu}_{00\sigma}$ appears.

c) Verify that

$${R^\mu}_{00\sigma} = \frac{1}{2}\partial_0\partial_0 {h^\mu}_\sigma$$

to first order in $h_{\mu\sigma}$ for $h_{\mu\nu}$ as in (2) and (3).

d) Show that the geodesic deviation equation (1) in the form as derived under b) reduces to

$$\frac{\partial^3 \delta x^\mu}{\partial \lambda^2 \partial s} = \frac{1}{2} \frac{\partial \hat{x}^\mu}{\partial s} \frac{\partial^2}{\partial \lambda^2} h^\mu{}_\nu. \tag{6}$$

e) Equation (6) is in particular solved by

$$\delta x^\mu = \frac{1}{2} h^\mu{}_\nu \hat{x}^\nu. \tag{7}$$

Verify this. Of all possible solutions of (6), why is this the physically relevant solution?

f) By taking a linear combination of gravitational waves, we can make one with

$$h_{11} = -h_{22} = 2C_{11} \cos k\lambda, \qquad h_{12} = h_{21} = 0$$

and also one with

$$h_{12} = h_{21} = 2C_{12} \cos k\lambda, \qquad h_{11} = h_{22} = 0.$$

(Recall that $t = \lambda$ to this order or approximation). Use (7) to describe for each of these two gravitational waves what the shape of the ring will look like as a function of time if the corresponding gravitational wave passes through it.

Problem 3

Consider again the Schwarzschild solution. A massive particle, not necessarily geodesic, is at $t = 0$ at $r = 2MG$ and assume it will inevitably hit the singularity. Show that, regardless of the motion of this particle, it will always hit the singularity in a proper time $\Delta\tau \leq \pi MG$.

Problem 4

Take the Schwarzschild metric and replace M with a function $M(r)$ to get

$$ds^2 = -(1 - \frac{2GM(r)}{r})dt^2 + (1 - \frac{2GM(r)}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{8}$$

Suppose that this is a solution of the Einstein equation for some perfect-fluid energy momentum tensor in comoving coordinates. Compute G_{00} from (8) and from this $\rho(r)$.

Problem 5

In relativistic cosmology, one begins by assuming the 3-space to be isotropic and therefore spherically symmetric in a neighbourhood of each point. This condition alone is sufficient to fix the geometry of the 3-space as:

$$ds^2 = F(r)\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \ ,$$

for some function $F(r)$.

a) Compute the components R_{rr} and $R_{\phi\phi}$ of the Ricci tensor of such 3-space.
(Hint: For a fast computation, use the Euler-Lagrange equations to find the non-vanishing components of the Levi-Civita connection).

b) The Riemann tensor of any space of constant curvature is given by:

$$R_{abcd} = K\left(g_{ac}\,g_{bd} - g_{ad}\,g_{bc}\right) \ ,$$

for some constant K. By taking our 3-space to be of constant curvature, show that the line element of the 3-space is given by:

$$ds^2 = \frac{1}{|K|}\,\frac{d\rho^2 + \rho^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)}{\left(1 + \frac{1}{4}\,a\,\rho^2\right)^2} \ ,$$

where $a = +1, -1$ or 0.
(Hint: Show that $F(r) = \left(1 - K\,r^2\right)^{-1}$ and perform an appropriate transformation of coordinates).

c) In relativistic cosmology, we take the metric tensor of the 4-dimensional spacetime to be the Friedmann-Robertson-Walker (FRW) metric:

$$ds^2 = -dt^2 + R^2(t)\,\frac{dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)}{\left(1 + \frac{1}{4}\,k\,r^2\right)^2} \ , \tag{9}$$

for some function $R(t)$ and such that the geometry of each hypersurface t constant is as in b), with $k = \pm 1, 0$.
We also postulate the matter content of the Universe to behave as a perfect fluid in comoving coordinates (the spatial coordinates of each particle in the fluid are constant along its geodesic). Show that the Einstein equations

$$G_{ab} + \Lambda\,g_{ab} = 8\pi\,T_{ab}$$

lead to the Friedmann's equations:

$$\begin{aligned} 8\pi\,\rho &= -\Lambda + 3\frac{k + R'(t)^2}{R^2(r)} \\ -8\pi\,p &= -\Lambda + \frac{k + R'(t)^2 + 2\,R(t)\,R''(t)^2}{R^2(t)} \ . \end{aligned}$$

d) Since the volume V enclosing a set of particles in the perfect fluid is proportional to the volume factor $R^3(t)$, where $R(t)$ is regarded as the radius of the Universe, show that the Friedmann's equations imply the first law of thermodynamics $dE + p\,dV = 0$. Note that this is the same result as that given by the contracted second Bianchi identity expressing the conservation of energy.

Problem 6

We want now to use the FRW metric (9) to deduce the cosmological redshift equation. Suppose a galaxy at $r = r_0$ emits a light signal with frequency $\omega_0 = \frac{2\pi}{T_0}$ such that two successive wave crests are emitted at time t_0 and $t_0 + T_0$. Suppose such wave crests are received by an observer at $r = r_1$ at times t_1 and $t_1 + T_1$. This means such observer will measure the signal with frequency $\omega_1 = \frac{2\pi}{T_1}$.

a) Since the light sinal follows null geodesics, begin by deducing the coordinate velocity $\frac{dr}{dt}$ of each wave crest. Then show that the waves satisfy:

$$\int_{t_1}^{t_1+T_1} \frac{dt}{R(t)} = \int_{t_0}^{t_0+T_0} \frac{dt}{R(t)} \ .$$

(Hint: begin by showing $\int_{t_0+T_0}^{t_1+T_1} \frac{dt}{R(t)} = \int_{t_0}^{t_1} \frac{dt}{R(t)}$ using the fact that the coordinate r is comoving, i.e., the galaxy and observer follow geodesics of constant r.)

b) By assuming in a first approximation that the scale factor $R(t)$ does not vary over the intervals T_0 and T_1, find that the observer will receive the light signal with a redshift z given by:

$$1 + z = \frac{R(t_1)}{R(t_0)} \ ,$$

where $z = \frac{\lambda_1 - \lambda_0}{\lambda_0}$, with $\lambda_{0,1}$ respectively the wavelengths of the emitted and received light signal.

Table

$$\int \frac{dr}{\sqrt{2M/r - 1}} = -r\sqrt{2M/r - 1} - M \arctan\left(\frac{M - r}{r\sqrt{2M/r - 1}}\right)$$