general relativity – october 2

key concepts

- In special relativity, the derivative of a tensor was again a tensor. This is no longer true in general relativity, if A^μ is a tensor, $\partial_\nu A^\mu$ is not. Of course we still would like to be able to take derivatives somehow, and this is where the concept of a connection comes in. Schematically, a connection is denoted by ∇, and it has to satify:
(i) If T is a tensor, so is ∇T.
(ii) ∇ is linear: $\nabla(T+S) = \nabla T + \nabla S$.
(iii) ∇ satisfies the Leibnitz rule: $\nabla(TS) = (\nabla T)S + T(\nabla S)$.
(iv) ∇ commutes with contraction: $\delta^\lambda_\nu \nabla T_{\mu\nu}{}^\lambda{}_\rho = \nabla(T_{\mu\nu}{}^\nu{}_\rho)$. In other words, the Kronecker delta is annihilated by ∇, $\nabla\delta^\lambda_\nu = 0$.
(v) ∇ reduces to ordinary derivatives on scalars, $\nabla_\mu \phi = \partial_\mu \phi$.

- The Leibnitz rule implies, as one can show using a Taylor expansion, that ∇_μ must be equal to ∂_μ up to a linear term. Therefore

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda} A^\lambda$$

for some set of coefficients $\Gamma^\nu_{\mu\lambda}$. These coefficients are called the connection coefficients.

- If we require that $\nabla_\mu A^\nu$ is a tensor, we find a specific transformation rule for Γ:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}. \tag{1}$$

See exercises.

- This does not completely fix $\Gamma^\nu_{\mu\lambda}$. Notice that $\Gamma^\nu_{\mu\lambda}$ is not a tensor. But if $\Gamma^{(1)}$ and $\Gamma^{(2)}$ both satisfy (1), then the difference $\Gamma^{(1)} - \Gamma^{(2)}$ is a tensor, as the inhomogeneous term in (1) drops out.

- The other requirements on ∇ determine how ∇ acts on general tensors. For example,

$$\nabla_\mu X_{\alpha\beta}{}^\gamma = \partial_\mu X_{\alpha\beta}{}^\gamma + \Gamma^\gamma_{\mu\lambda} X_{\alpha\beta}{}^\lambda - \Gamma^\lambda_{\mu\alpha} X_{\lambda\beta}{}^\gamma - \Gamma^\lambda_{\mu\beta} X_{\alpha\lambda}{}^\gamma$$

and similar for other tensors.

- We have only given an implicit characterization of Γ, but would like an explicit expression. This can be found once we impose two more requirements:
(i) $\Gamma^\nu_{\mu\lambda} = \Gamma^\nu_{\lambda\mu}$, in other words, the connection is symmetric. The antisymmetric part of a connection is a tensor which is called the torsion of a connection, and though it is interesting to think about possible applications of the presence of torsion, we will in the remainder assume the

torsion vanishes.
(ii) The connection should be compatible with the metric (so that the norm of a vector will be preserved under transport - this is explained later), which means that

$$\nabla_\alpha g_{\mu\nu} = 0.$$

- By writing out what $\nabla_\alpha g_{\mu\nu} = 0$ means, and by permuting the indices in this equation and by adding and subtracting the resulting equations one finds that

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}).$$

  This connection is called then Christoffel connections, the components are sometimes called the Christoffel symbols. The connection also goes under the names Levi-Civita connection and Riemannian connection.

- A smooth manifold with metric (and Christoffel connection) constitute a Riemannian manifold.

- How to compare vectors at different points? In flat space there is a natural way to do this, but no longer in curved space, as tangent spaces at different points cannot be naturally identified. To compare vectors at points p_1 and p_2, we need a path from p_1 to p_2, and we need to drag the vector from p_1 to p_2 in a "natural way." This natural way of dragging things along is called parallel transport, but in curved space the result depends on the choice of path.

- Take a path $x^\mu(\lambda)$ from p_1 to p_2. In flat space we drag a vector or tensor along by taking it to be constant. Therefore, we require that $\frac{d}{d\lambda}T = 0$. In curved space this equation does not make sense, since the λ derivative of a tensor is no longer a tensor, and therefore this equation would depend on an explicit choice of coordinates. To come up with a well-defined equation, we need the covariant derivative:

$$\frac{D}{D\lambda}T \equiv \frac{\partial x^\mu(\lambda)}{\partial\lambda}\nabla_\mu T = 0.$$

  If this equation is satisfied for a tensor defined everywhere along the path, we say that the tensor is parallel transported along the path.

- For example, for a vector A^ν we get

$$\frac{D}{D\lambda}A^\nu = \frac{\partial x^\mu(\lambda)}{\partial\lambda}(\partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda}A^\lambda) = \frac{\partial A^\nu}{\partial\lambda} + \frac{\partial x^\mu(\lambda)}{\partial\lambda}\Gamma^\nu_{\mu\lambda}A^\lambda.$$

- Parallel transport is a first order differential equation, so if we specify the value of the tensor at one point on the path, parallel transport can be used to find its value everywhere else.

- If we parallel transport two vectors A^{mu} and B^ν along a path, the inner product is preserved,

$$\frac{d}{d\lambda}(g_{\mu\nu}A^\mu B^\nu) = 0.$$

In particular, the length of a vector and the inner product between two vectors do not change, which is what we want.

- If we have a vector at some point p and we parallel transport it around a closed loop, the result will be a different vector at the point p. In view of the previous comment, the two vectors will be related by an orthonormal matrix which depends only on the path. If we consider all possible closed loops, we obtain a set of matrices that form a subgroup of the full orthonormal group, and this subgroup is called the holonomy group at p. For flat space, the holonomy group is the identity matrix only, since parallel transport does not do anything nontrivial.

- Another important concept: geodesics. Generalizes the notion of straight lines and inertial observers to curved space. In flat space, a straight line is given by
$$\frac{d^2x^\mu}{d\lambda^2} = 0$$
but this is not a tensor equation so it does not generalize to curved space. In curved space, there is an obvious version which is a tensor and uses the covariant derivative,
$$\frac{D^2x^\mu}{D\lambda^2} \equiv \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0. \tag{2}$$
This important equation is the geodesic equation, a curve $x^\mu(\lambda)$ that solves it is called a geodesic. It is a second order differential equation, so we need to specify both an initial point $x^\mu(0)$ as well as an initial "velocity" $\frac{dx^\mu}{d\lambda}|_\lambda = 0$ to specify a geodesic, just as in flat space.

- One characterization of a geodesic: it is a curve that parallel transports its own tangent vector. This is clear from the definition of tangent vector to a curve and that of parallel transport.

- Other characterization: as paths that extremize the length of the path. This can be verified using the variational principle just as in flat space, see exercises.

- In general the geodesic equation has some parameter λ in it. The meaning of λ depends on the type of geodesic. A geodesic is characterized by the number $e = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$ which is constant along the geodesic (parellel transport preserves norms.) If $e < 0$, the geodesic is called timelike, and λ is linearly related to the proper time along the geodesic. If $e = 0$, the geodesic is called null, and λ is called an affine parameter (not related to proper time). If $e > 0$, the geodesic is space-like, and λ is related to i times the proper time, which is imaginary for space-like separations.

- A sometimes useful trick to find geodesics is to observe that the geodesic equation can also be obtained by applying the variational principle to
$$\int d\lambda g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}.$$

- How to get Newtonian gravity? Use the geodesic equation as the equation for free fall, and consider almost flat space: Assume
  (i) all movement is slow
  (ii) the field are weak
  (iii) the field is static.

- Slow movement means $\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau}$. Therefore the geodesic equation becomes
$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0.$$
  Static means that all time derivatives vanish. Weak field means that $g_{\mu\nu} = \eta_{\mu\nu} = h_{\mu\nu}$ with very small $h_{\mu\nu}$, in other words we work to first order in h. Then $\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00}$. The $\mu = 0$ component of the geodesic equation states that t and τ are linearly related, and the other components therefore simplify to
$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial_i h_{00}.$$
  Comparing to the standard equation
$$\frac{d^2x^i}{dt^2} = -\partial_i \Phi$$
  with Φ the potential, $\Phi = -GM/r$, shows that things match if $h_{00} = -2\Phi$. Thus curving space appears to a freely falling observer as if there is a gravitational field acting. This is of course just an approximation, the full story is quite a bit more involved.

exercises

- Show that the requirement that $\nabla_\mu A^\nu$ is a tensor implies the transformation rule (1).

- Compute the eight Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ for the two-sphere of unit radius.

- Consider the unit two-sphere, and consider the vector A^μ_0 which is the unit vector in the θ-direction, at the point p_0 which is $\theta = \pi/2$, $\phi = 0$ in polar coordinates. What happens to the vector if we parallel transport it once around the equator, i.e. along the path $(\theta(\lambda), \phi(\lambda)) = (\pi/2, \lambda)$ with $0 \le \lambda \le 2\pi$? Next consider a curve which consists of four separate pieces,
$$\begin{aligned}
\gamma_1(\lambda) &= (\pi/2, \lambda) \text{ for } 0 \le \lambda \le \lambda_1 \\
\gamma_2(\lambda) &= (\pi/2 - \lambda, \lambda_1) \text{ for } 0 \le \lambda \le \lambda_2 \\
\gamma_3(\lambda) &= (\pi/2 - \lambda_2, \lambda_1 - \lambda) \text{ for } 0 \le \lambda \le \lambda_1 \\
\gamma_4(\lambda) &= (\pi/2 - \lambda_2 + \lambda, 0) \text{ for } 0 \le \lambda \le \lambda_2
\end{aligned}$$

with $0 \leq \lambda_1 \leq 2\pi$ and $0 \leq \lambda_2 \leq \pi/2$. Find out what happens to the vector A_0^μ once we parallel transport it around this closed path. What is the area of the two-sphere enclosed by this closed curve? Is this area somehow related to the result of the parallel transport?

- Consider the length of a path which is everywhere timelike,

$$\Delta\tau = \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}.$$

Work out the variation of this expression, keeping in mind that now $g_{\mu\nu}$ is also a function of x. After computing the variation, change the parameter λ to proper time τ, so that

$$d\tau = \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda.$$

Show that demanding that the variation vanishes leads to the geodesic equation (2) with λ replaced by τ.