general relativity – september 4

key concepts

- pre-special relativity: Galilean invariance, $x' = x - vt$, $t' = t$, both distance and time separation between events have an invariant meaning.
- problems: non-invariance of Maxwell equations, the Michelson-Morley experiment
- way out: principle of equivalence, all inertial observers have the same laws of physics
- special relativity: the only meaningful separation between events is the proper time separation or space-time interval $-c^2(\Delta\tau)^2 = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$, this is preserved by Lorentz transformations

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}, \qquad x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}.$$

- infinitesimal version: $-c^2 d\tau^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$, the *metric* of minkowski space. Define $(x^0, x^1, x^2, x^3) \equiv (ct, x, y, z)$ then $-c^2 d\tau^2 \equiv ds^2 \equiv \eta_{\mu\nu} dx^\mu dx^\nu$ where as always repeated indices are summed over (Einstein summation convention) and

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- length of a curved path $(x^0(\lambda), x^1(\lambda), x^2(\lambda), x^3(\lambda))$ parametrized by λ, $\lambda_0 \leq \lambda \leq \lambda_1$ is

$$\ell = \int_{\lambda_0}^{\lambda_1} |ds| \equiv \int_{\lambda_0}^{\lambda_1} \sqrt{\left(\frac{\partial x^0(\lambda)}{\partial\lambda}\right)^2 - \left(\frac{\partial x^1(\lambda)}{\partial\lambda}\right)^2 - \left(\frac{\partial x^2(\lambda)}{\partial\lambda}\right)^2 - \left(\frac{\partial x^3(\lambda)}{\partial\lambda}\right)^2} d\lambda.$$

  The total time elapsed on the watch of someone who travels along this path in space-time is $c\ell$.

- Lorentz transformations are linear tranformations $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ that preserve the Minkowski metric, i.e.

$$\Lambda^\mu{}_\nu \Lambda^{\mu'}{}_{\nu'} \eta_{\mu\mu'} = \eta_{\nu\nu'}.$$

  The Lorentz group is composed of rotations of the three spatial coordinates and boosts, like the Lorentz transformation written above. Proper Lorentz transformations have $\det\Lambda = +1$ and $\Lambda^0{}_0 \geq 1$, examples of improper Lorentz transformations are parity (denoted P,flipping the sign of one spatial coordinate) and time reversal (denoted T, flipping the sign of the time coordinate).

- general relativity: equivalence principle, freely falling observers see locally the same physics. Experimental evidence: inertial mass = gravitational mass.
- general relativity: Mach's principle, to define intertial frames need to know the rest of the galaxy ($\rightarrow$ rotating bucket of water).
- general relativity: space-time will become curved, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, with $g_{\mu\nu} \neq \eta_{\mu\nu}$ causing the apparent attractive force of one mass onto another.

exercises

- Give the transformation rules for velocities under Lorentz transformations.
- Discuss the car and garage paradox: Consider a car and garage of equal proper length. The driver speeds towards the garage, and a doorman at the garage is instructed to slam the door shut as soon as the back end of the car enters the garage. According to the doorman, "the car Lorentz contracted and easily fitted into the garage then I slammed the door." According to the driver, "the garage Lorentz contracted and was too small for the car when I entered the garage." Who is correct and why? Assume the car crashes through the back wall of the garage without stopping or slowing down.
- Discuss the twin paradox: consider twins where one makes a trip in a spaceship, first flying away at a speed close to c and then returning again at a speed close to c. According to the person remaining behind, the other person is much younger upon return due to Lorentz contraction, but according to the person in the spaceship it is the other way around. What is precisely happening?