general relativity – december 4

key concepts

- To study global properties of spacetimes, it is useful to consider various asymptotic regions of a spacetime.
  - I^+, future timelike infinity, given by $t \to +\infty$ and r fixed.
  - I^-, past timelike infinity, given by $t \to -\infty$ and r fixed.
  - I^0, spacelike infinity, given by $r \to \infty$ with t fixed.
  - $\imath^+$, future null infinity, pronounced "scri plus", the endpoints of future null geodesics.
  - $\imath^-$, past null infinity, pronounced "scri minus", the endpoints of past null geodesics.

  To draw a diagram where the various asymptotic regions can be seen, it is convenient to change coordinates in such a way that infinity takes on a finite coordinate value.

- For Minkowski space

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{1}$$

  we introduce

$$t + r = \tan \frac{\psi + \xi}{2} \tag{2}$$

$$t - r = \tan \frac{\psi - \xi}{2}. \tag{3}$$

  The tangent covers all real values if the argument runs from $-\pi/2$ to $\pi/2$, so we need

$$-\frac{\pi}{2} \leq \frac{\psi+\xi}{2} \leq \frac{\pi}{2} \tag{4}$$

$$-\frac{\pi}{2} \leq \frac{\psi-\xi}{2} \leq \frac{\pi}{2} \tag{5}$$

  but we also need $r \geq 0$ which implies $\xi \geq 0$. Thus if we plot in the ξ, ψ plane the coordinate region which covers all of Minkowski space, dropping the dependence on $d\Omega^2$, we find a triangle. The bottom point of the triangle $(\xi, \psi) = (0, -\pi)$ is I^-, the right point is $(\xi, \psi) = (\pi, 0)$ is I^0, and the top point $(\xi, \psi) = (0, \pi)$ is I^+. The line between I^- and I^0 is $\imath^-$, the line between I^+ and I^0 is $\imath^+$. The line between I^- and I^+ belongs to the interior of Minkowski space. This diagram is the Penrose diagram of Minkowski space.

- The metric of Minkowski space in the new coordinates reads

$$ds^2 = \frac{-d\psi^2 + d\xi^2}{4\cos^2(\psi+\xi)/2\cos^2(\psi-\xi)/2} + r^2 d\Omega^2 \quad (6)$$

$$= \frac{1}{4\cos^2(\psi+\xi)/2\cos^2(\psi-\xi)/2}\left[-d\psi^2 + d\xi^2 + \sin^2\xi d\Omega^2\right] \quad (7)$$

  This expression shows that Minkowski space is conformally equivalent to a subset of a very simple space

$$ds^2 = -d\psi^2 + d\xi^2 + \sin^2\xi d\Omega^2 \quad (8)$$

  which we recognize as the direct product of time and a three-sphere. We call two metrics $g^1_{\mu\nu}(x)$ and $g^2_{\mu\nu}(x)$ conformally equivalent if there is a smooth function $p(x)$ such that $g^1_{\mu\nu}(x) = p(x)^2 g^2_{\mu\nu}(x)$. Conformally equivalent metrics yield the same angles between vectors (though distances are clearly different).

- The metric in (8) is the Einstein static universe. As we saw before in an exercise, it is a solution of the Einstein equations with a cosmological constant, some p and ρ, and $k = +1$.

- In general Penrose diagrams are constructed by dropping any $d\Omega^2$ terms in the metric, and by choosing new coordinates so that space becomes an area of finite extent in the new coordinates, and also so that all null geodesics are straight lines that make angles of plus or minus 45 degrees with the two coordinate axes.

- To construct a Penrose diagram, we in general want to bring the metric in the form

$$ds^2 = A(\alpha,\beta)(-d\alpha^2 + d\beta^2) + B(\alpha,\beta)d\Omega^2 \quad (9)$$

  where α, β have a finite range. In this form it is clear that null geodesics are straight lines $\alpha = \pm\beta + \text{const}$.

- Let's now find the Penrose diagram for the Schwarzschild metric in Kruskal-Szekeres coordinates. This was

$$ds^2 = \frac{32G^3M^3}{r}e^{-r/2MG}(dv^2 - du^2) + r^2 d\Omega^2 \quad (10)$$

  w with

$$v + u = e^{(t+r)/4MG}\left(\frac{r}{2MG} - 1\right)^{1/2} \quad (11)$$

$$v - u = e^{(-t+r)/4MG}\left(\frac{r}{2MG} - 1\right)^{1/2} \quad (12)$$

$$v^2 - u^2 = e^{r/2MG}\left(\frac{r}{2MG} - 1\right). \quad (13)$$

We change

$$u + v = \tan \frac{\psi + \xi}{2} \tag{14}$$

$$u - v = \tan \frac{\psi - \xi}{2} \tag{15}$$

just like what we did in Minkowski space, and the metric becomes

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2MG} \frac{-d\psi^2 + d\xi^2}{4\cos^2(\psi+\xi)/2\cos^2(\psi-\xi)/2} + r^2 d\Omega^2 \tag{16}$$

The range of ψ and ξ is again

$$-\frac{\pi}{2} \leq \frac{\psi+\xi}{2} \leq \frac{\pi}{2} \tag{17}$$

$$-\frac{\pi}{2} \leq \frac{\psi-\xi}{2} \leq \frac{\pi}{2} \tag{18}$$

but whereas in Minkowski space we needed to take $r \geq 0$ into account, we don't have to do that here. Here we need to impose that $u^2 - v^2 \leq 1$ in order to avoid the singularity. This implies $\cos\xi \geq 0$, so that $-\pi/2 \leq \xi \leq \pi/2$. It is now straightforward to draw the Penrose diagram and recognize the regions I,II,III and IV that we saw before.

- The singularity in the Penrose diagram will on the lines $\xi = \pm\pi/2$. These are drawn by wiggly lines. Except for that the diagram is quite similar to the diagram we drew last week, except that everything now takes a finite value.

- The remainder of this lecture was a minisummary of some other issues, such as the Penrose diagram for black hole creation and evaporation, the Penrose diagram for the electrically charged Reissner-Nordstrøm black hole, ergospheres in the Kerr-Newman black hole, mining energy from black holes via a Penrose process, closed timelike curves and naked singularities, no hair theorems, black hole thermodynamics, information loss, etc. All the stuff that we didn't do.

exercises

- Consider the three cosmological solutions with $\Lambda > 0$ and $p = \rho = 0$ that we constructed two lectures ago. The three solutions corresponded to $k = -1, 0, 1$. Find a Penrose diagram for each of these metrics, and verify that two of the three are actually submanifolds of the third one. Which of these have something that is reminiscent of an horizon?