general relativity – november 6

key concepts

- Gravitational waves are ripples in space-time propagating at the speed of light, and are caused by a matter distribution that moves in time, like e.g. a binary star system. A very rough analogy is a wave moving through a blanket when you try to get the dust out. Gravitational waves may be directly measured in the near future, various experiments are under way. Gravitational waves can carry away energy, and gravitational waves have been indirectly observed in a binary system in the sense that the rate at which the period of rotation changes is in perfect agreement with the prediction from the amount of energy carried away by gravitational radiation.

- Gravity is a non-linear theory, and gravitational waves arise in the linearized approximation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with very small h.

- It is useful to first describe the relevant results from Maxwell theory. Maxwell theory has a gauge field A_μ with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The field strength containing the electric and magnetic fields is invariant under a gauge transformation

$$\delta A_\mu = \partial_\mu \chi \tag{1}$$

  with arbitrary χ.

- The Maxwell equations are

$$\partial^\mu F_{\mu\nu} = -4\pi J_\nu.$$

- To fix the ambiguity expressed by the gauge invariance, we need a gauge condition. There are several possible gauge conditions, we will take the Lorentz of covariant gauge $\partial^\mu A_\mu = 0$. This removes not all ambiguity, gauge transformations of the form (1) preserve the gauge choice as long as $\partial^\mu \partial_\mu \chi = 0$.

- Using the gauge condition the Maxwell equation in vacuum reduces to $\partial^\mu \partial_\mu A_\nu = 0$. This is a simple plane-wave equation. It is easily solved in momentum space,

$$A_\nu = \epsilon_\nu \exp(i k_\mu x^\mu).$$

  The gauge condition implies $k^\mu \epsilon_\mu = 0$, the Maxwell equation implies $k^\mu k_\mu = 0$, and the residual gauge invariance is $\delta\epsilon_\nu = \lambda k_\nu$ for some constant λ. Notice that the residual gauge invariance indeed preserves the gauge condition in view of $k^\mu k_\mu = 0$.

- Now choose a frame with $k^\mu = (k, 0, 0, k)$. The gauge condition then implies that $\epsilon_\mu = (a, b, c, -a)$ for some a, b, c. The residual gauge invariance acts on this by shifting a, and we can therefore choose $a = 0$. What remains is two parameters b, c, corresponding to the two physical degrees of freedom of a photon. Suitable combinations of b, c correspond to the two physical polarizations of light.

- Next we solve the Maxwell equation in the presence of a source. We have to solve $\partial^\mu \partial_\mu A_\nu = -4\pi J_\nu$. The general solution involves a retarded Green's function,

$$A_\nu(x) = -4\pi \int d^4y G(x-y) J_\nu(y).$$

  The Green's function we need should be such that the solution is in the right gauge (i.e. satisfies $\partial^\nu A_\nu = 0$), and we want that the sources only affect information in the future. Therefore the appropriate Green's function is the retarded Green's function

$$G(x-y) = -\frac{1}{4\pi|\vec{x}-\vec{y}|}\theta(x^0-y^0)\delta(|\vec{x}-\vec{y}|-(x^0-y^0)).$$

  Here, $\theta(x)$ is the function which is 1 for $x > 0$ and 0 for $x < 0$. We also wrote $x = (x^0, \vec{x})$ and $y = (y^0, \vec{y})$. If we do the y^0 integral be get

$$A_\nu(t, \vec{x}) = \int d^3\vec{y} \frac{1}{|\vec{x}-\vec{y}|} J_\nu(t - |\vec{x}-\vec{y}|, \vec{y}). \tag{2}$$

  We see that indeed the value of A_ν only depends on the values of J_ν in the past.

- A moving source radiates energy. The amount of energy that is being radiated (the power) is proportional to $\ddot{D}^2$, with D the dipole moment.

- We want to repeat the same calculations as for Maxwell theory, but now in gravity.

- In gravity, we will work in the linearized approximation, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

- The gauge invariance of gravity is general coordinate invariance. To determine how that acts on $h_{\mu\nu}$, we perform a small change of coordinates $\delta x^\mu = \xi^\mu$. The transformation properties of the metric then imply

$$(\eta_{\mu\nu} + h_{\mu\nu} + \delta h_{\mu\nu})(dx^\mu + d\xi^\mu)(dx^\nu + d\xi^\nu) = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu$$

  which implies

$$\delta h_{\mu\nu} = -\partial_\mu \xi_\nu - \partial_\nu \xi_\mu. \tag{3}$$

- As gauge condition we will impose $g^{\mu\nu}\Gamma^\rho_{\mu\nu} = 0$. These are four conditions, corresponding to the fact that we also have four gauge symmetries given by the four components of ξ. It is perhaps obscure why we pick this gauge condition, but it will turn out that this one gives very simple answers that

are similar to those of Maxwell theory in the covariant gauge. The gauge condition can be rewritten as

$$\partial_\mu \bar{h}^\mu{}_\lambda = 0,$$

where

$$\bar{h}_{\mu\lambda} = h_{\mu\lambda} - \frac{1}{2} h \eta_{\mu\lambda}$$

and

$$h = \eta^{\mu\nu} h_{\mu\nu}.$$

It will turn out that $\bar{h}$ rather than h will be analogous to the Maxwell field.

- The Einstein equation $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}$ becomes when expanded to first order in h

$$\begin{aligned} \kappa T_{\mu\nu} &= \frac{1}{2}(\partial_\sigma \partial_\nu h^\sigma{}_\mu + \partial_\sigma \partial_\mu h^\sigma{}_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} \\ &\quad -\eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu} \Box h) \end{aligned} \tag{4}$$

where $\Box = \partial^\mu \partial_\mu$ is the d'Alembertian.

- When we apply the gauge condition, the linearized Einstein equation simplifies dramatically and becomes

$$\Box \bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu}. \tag{5}$$

- This equation looks similar to the Maxwell equation, and to solve it we again take a plane wave

$$\bar{h}_{\mu\nu} = C_{\mu\nu} \exp(i k_\rho x^\rho).$$

The gauge condition is $k^\mu C_{\mu\nu} = 0$, the equations of motion (5) imply $k^2 = 0$.

- We also need the residual gauge transformations. The gauge condition $\partial^\mu (h_{\mu\lambda} - \frac{1}{2} h \eta_{\mu\lambda}) = 0$ is preserved by a transformation (3) if $\Box \xi_\nu = 0$, so that $\xi_\nu = -i B_\nu \exp(i k_\rho x^\rho)$ with $k^2 = 0$. We put the $-i$ for convenience. We need to determine how the residual gauge transformations act on $C_{\mu\nu}$. First we notice that

$$\delta \bar{h}_{\mu\nu} = \delta(h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}) = -\partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu} \partial^\rho \xi_\rho$$

and therefore

$$\delta C_{\mu\nu} = -k_\mu B_\nu - k_\nu B_\mu + \eta_{\mu\nu} k^\rho B_\rho. \tag{6}$$

- We can now figure out what happens to $C_{\mu\nu}$ once we impose the gauge condition and residual gauge invariance. Again, we take a frame where $k_\mu = (k, 0, 0, k)$. The most general $C_{\mu\nu}$ is a symmetric 4×4 matrix,

$$C_{\mu\nu} = \begin{pmatrix} C_{00} & C_{01} & C_{02} & C_{03} \\ C_{01} & C_{11} & C_{12} & C_{13} \\ C_{02} & C_{12} & C_{22} & C_{23} \\ C_{03} & C_{13} & C_{23} & C_{33} \end{pmatrix}.$$

The gauge condition restricts this form to

$$C_{\mu\nu} = \begin{pmatrix} C_{00} & C_{01} & C_{02} & C_{00} \\ C_{01} & C_{11} & C_{12} & C_{01} \\ C_{02} & C_{12} & C_{22} & C_{02} \\ C_{00} & C_{01} & C_{02} & C_{00} \end{pmatrix}$$

which contains six independent components. The residual gauge invariance acts as

$$\delta C_{\mu\nu} = k \begin{pmatrix} -B_0 - B_3 & -B_1 & -B_2 & -B_0 - B_3 \\ -B_1 & B_3 - B_0 & 0 & -B_1 \\ -B_2 & 0 & B_3 - B_0 & -B_2 \\ -B_0 - B_3 & -B_1 & -B_2 & -B_3 - B_0 \end{pmatrix}$$

and this can be used, as one easily sees, to put C in the form

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

This has only two independent components, and (7) is called the transverse traceless gauge in which $\bar{h}_{\mu\nu} = h_{\mu\nu}$. It is transverse because it is transverse to the momentum (just as in Maxwell theory), it is traceless because the trace of C vanishes.

- A ring of test particles responds to a wave with $C_{11} \neq 0$ and $C_{12} = 0$ by changing shape in a +-pattern, to one with $C_{11} = 0$ and $C_{12} \neq 0$ in a $\times$-pattern. For some pictures see the notes of Sean Carroll. The pattern is always invariant under reflection in the origin which is sign of the fact that gravity waves have spin two.

- We can now immediately solve the Einstein equations in the presence of a source, by analogy with Maxwell theory

$$\bar{h}_{\mu\nu}(t, \vec{x}) = 4G \int d^3\vec{y} \frac{1}{|\vec{x} - \vec{y}|} T_{\mu\nu}(t - |\vec{x} - \vec{y}|, \vec{y}).$$

It is convenient to Fourier transform, which we take in the form $f(\omega) = (2\pi)^{-1/2} \int dt e^{i\omega t} f(t)$ and with inverse $f(t) = (2\pi)^{-1/2} \int d\omega e^{-i\omega t} f(\omega)$. We get

$$\begin{aligned} \bar{h}_{\mu\nu}(\omega, x) &= 4G \int d^3\vec{y} \frac{1}{|\vec{x} - \vec{y}|} e^{i\omega|\vec{x}-\vec{y}|} T_{\mu\nu}(\omega, \vec{y}) \\ &\approx 4G \frac{e^{i\omega R}}{R} \int d^3\vec{y} T_{\mu\nu}(\omega, \vec{y}) \end{aligned} \tag{8}$$

where in the last line we assume the source of the gravitational radiation is approximately localized at a large distance R from the point $\vec{x}$, which is the right setup if we discuss gravitational radiation as seen on say earth emitted by a distant source in the universe.

- Next we perform the following trick. We notice that (i, j are space indices only)

$$\int d^3\vec{y} T_{ij} = \int d^3\vec{y} \partial_k (y^i T_{kj}) + i\omega \int d^3\vec{y} y^i T^{0j}. \tag{9}$$

We used here the fact that energy-momentum is conserved, so that $\partial_k T^{kj} = -\partial_0 T^{0j} = -i\omega T^{0j}$ in Fourier transformed language. In (9) the first term on the right hand side is a total derivative, and for localized sources this term can be dropped. Hence, we have

$$\int d^3\vec{y} T_{ij} = i\omega \int d^3\vec{y} y^i T^{0j}.$$

Similarly, we could have derived

$$\int d^3\vec{y} T_{ij} = i\omega \int d^3\vec{y} y^j T^{0i}.$$

Combining these we get

$$\begin{aligned} \int d^3\vec{y} T_{ij} &= \frac{i\omega}{2} \int d^3\vec{y} (y^i T^{0j} + y^j T^{0i}) \\ &= \frac{i\omega}{2} \int d^3\vec{y} \partial_k (y^i y^j T^{0k}) + \frac{(i\omega)^2}{2} \int d^3\vec{y} y^i y^j T^{00} \end{aligned}$$

where in the last line we repeated the same trick. Combining everything and introducing the quadrupole moment q^{ij} of the matter distribution via

$$q^{ij} = 3 \int d^3\vec{y} y^i y^j T^{00} \tag{10}$$

we obtain after undoing the Fourier transforms

$$\bar{h}_{ij}(t) = \frac{2G}{3R} \frac{d^2 q_{ij}}{dt^2}|_{t-R} \tag{11}$$

where the right hand side is evaluated at an earlier time $t - R$. This time difference is simply the amount of time it takes for a gravitational wave to travel from the source to the observer a distance R away.

- To compute the amount of energy carried in this gravitational wave is a subtle question. The energy in the gravitational field cannot be defined via $T_{\mu\nu}$, as this is zero in vacuum. It must somehow be extracted from the Einstein tensor itself. A procedure that works is to take the gravitational wave, and to expand the Einstein equation to second order. The gravitational wave only solves the Einstein equation to first order, so to second order we will get some non vanishing result $t_{\mu\nu}$. In principle this has to be cancelled against a second order correction to $h_{\mu\nu}$, in which case $t_{\mu\nu}$ looks like the energy-momentum source for this second order correction to the metric. It turns out that $t_{\mu\nu}$ is a good candidate for the energy and momentum carried by the gravitational field, but a precise justification is quite tricky.

- One can now compute the power radiated away in the gravitational field

$$\frac{\partial E}{\partial t} = -\frac{G}{45}\left[\frac{d^3Q_{ij}}{dt^3}\frac{d^3Q^{ij}}{dt^3}\right]_{t-R}. \tag{12}$$

  Here,

$$Q_{ij} = q_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}q_{kl}.$$

  Notice that whereas in electromagnetism it was the second derivative of the dipole moment, here it is the third derivative of the quadrupole moment that enters, a sign of the spin two nature of gravitational waves.

exercises

- Consider a system of particles moving in a plane with common angular frequency ω. Assume all particles lie on a straight line and are rotating around a point on this line. Show that the average total emitted power is given by

$$\left\langle \frac{\partial E}{\partial t} \right\rangle = -\frac{32GI^2\omega^6}{5}$$

  where I is the moment of inertia of the collection of particles.

- Consider a bar with length one meter, with a mass of one kilogram at each endpoint. Assume this bar rotates once a second. Compute the power emitted due to gravitational radiation. Remember $c = 1$ in our units.