general relativity – october 9

key concepts

- How to measure curvature, deviation from flat space? $g_{\mu\nu} - \eta_{\mu\nu}$ does not work, is not a tensor. First derivatives of the metric are also not tensors, and covariant derivatives of the metric vanish. We apparently need at least second derivatives of the metric. The extrinsic curvature obtained by embedding a manifold in $\mathbb{R}^n$ and by looking at the curvature radius by fitting in a small ball that barely touches the manifold is an extrinsic property and not an intrinsic property.

- Idea: parallel transport knows about the deviation from flat space. Parallel transport a vector around a small loop which is a rectangle spanned by vectors A^μ and B^ν. Then one finds

  $$\delta V^\mu = A^\mu B^\nu R^\rho{}_{\sigma\mu\nu} V^\sigma$$

  plus terms of higher order in A and B, with some tensor $R^\rho{}_{\sigma\mu\nu}$. Details of this construction are a little bit tricky.

- Since parallel transport is given by covariant derivatives, another measure of curvature is obtained by looking at the extent to which covariant derivatives do not commute. We will take this as our definition of curvature, but it turns out to be equivalent to the above definition.

- We define therefore the curvature or Riemann tensor $R^\rho{}_{\sigma\mu\nu}$ by the equation

  $$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma. \tag{1}$$

  Clearly, R is a tensor.

- From (1) one finds the explicit expression

  $$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \tag{2}$$

  see exercises.

- One also finds that

  $$[\nabla_\mu, \nabla_\nu] V_\rho = -R^\sigma{}_{\rho\mu\nu} V_\sigma \tag{3}$$

  and one can derive similar expression for commutators acting on more general tensors, see exercises.

- Riemann normal coordinates: since a manifold looks locally like a piece of $\mathbb{R}^n$, one would like to find a set of coordinates in which space looks locally like flat space. One can show that one can always find a coordinate system near a point p such that $g_{\mu\nu}(p) = \eta_{\mu\nu}$ and such that $\partial_\lambda g_{\mu\nu} = 0$.

- Special coordinate systems are useful to prove tensor identities. If a tensor vanishes in some coordinate system, it has to vanish in any coordinate system, so one can pick the most convenient coordinate system to verify any tensor identity.

- Riemann normal coordinates provide a construction of a set of coordinates near a point p so that $g_{\mu\nu}(p) = \eta_{\mu\nu}$g and so that $\partial_\lambda g_{\mu\nu}(p) = 0$: take a vector v^μ at p, and find the unique geodesic $x^\mu(\lambda)$ such that $x^\mu(0) = p$ and such that $\frac{dx^\mu}{d\lambda}|_{\lambda=0} = v^\mu$. Now $v^\mu \to x^\mu(1)$ gives a map from the set of vectors at p to points in the manifold, and locally this map is one-to-one. In this way we can use v^μ to define a local coordinate system near p. These coordinates are called Riemann normal coordinates, and one can check that indeed $g_{\mu\nu}(p) = \eta_{\mu\nu}$ and $\partial_\lambda g_{\mu\nu}(p) = 0$.

- The map from vectors at p to points in the manifold is called the exponential map.

- Example: Consider a two-sphere. The tangent space at the north pole is naturally identified with the x, y-plane, once we embed the two-sphere in $\mathbb{R}^3$. Thus we will label a vector at the north pole by its two components (v, w). One finds that the exponential map is

$$(v, w) \to (\theta = \sqrt{v^2 + w^2}, \phi = \arctan(w/v)).$$

  The metric in terms of v, w becomes

$$ds^2 = \frac{(vdv + wdw)^2}{v^2 + w^2} + \sin^2\sqrt{v^2 + w^2}\frac{(vdw - wdv)^2}{(v^2 + w^2)^2}.$$

  By exanding the sin in a Taylor series, we find that the metric near the north pole $v = w = 0$ is indeed given by $dv^2 + dw^2$ plus corrections of order v^2, w^2, so that the first derivatives of the metric near $v = w = 0$ vanish as claimed.

- If the domain of the exponential map is not the entire tangent space at p for some point p in a manifold, we say that the manifold is geodesically incomplete, i.e. has an "edge." This is problematic for physics, because we cannot explain to an observer who attempts to move through the edge.

- Riemann normal coordinates are the best approximation to flat space that is available.

- Inertial observers move along (timelike) geodesics. Riemann normal coordinates provide a local inertial frame, that to first order always looks like flat space. Therefore, locally all physics looks the same, which is the famous equivalence principle.

- Properties of the Riemann tensor. Define $R_{\alpha\beta\gamma\delta} = g_{\alpha\epsilon}R^\epsilon{}_{\beta\gamma\delta}$, then

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}$$

$$
\begin{aligned}
R_{\alpha\beta\gamma\delta} &= -R_{\beta\alpha\gamma\delta} \\
R_{\alpha\beta\gamma\delta} &= R_{\gamma\delta\alpha\beta} \\
R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} &= 0 \\
\nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon} + \nabla_\gamma R_{\alpha\beta\delta\epsilon} &= 0
\end{aligned}
$$

The fourth identity is called the cyclic identity, the last one the Bianchi identity.

- The first four follow rather easily be explicit calculation in Riemann normal coordinates, where

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}(\partial_\beta\partial_\gamma g_{\alpha\delta} - \partial_\alpha\partial_\gamma g_{\beta\delta} - \partial_\beta\partial_\delta g_{\alpha\gamma} + \partial_\alpha\partial_\delta g_{\beta\gamma}).$$

  The Bianchi identity follows with a bit more work by acting with the Jacobi identity

$$[[\nabla_\alpha, \nabla_\beta], \nabla_\gamma] + [[\nabla_\beta, \nabla_\gamma], \nabla_\alpha] + [[\nabla_\gamma, \nabla_\alpha], \nabla_\beta] = 0$$

  on e.g. a vector.

- With some combinatorics one finds that the number of independent components of R is $d^2(d^2-1)/12$ in d dimensions. For $d = 1, 2, 3, 4$ this is $0, 1, 6, 20$. Indeed, there is no curved one-dimensional space.

- The Riemann tensor is basically the unique object that one can make from the metric that is a tensor and has at most two-derivatives.

- Two related tensors: the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are defined by

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}, \qquad R = g^{\mu\nu}R_{\mu\nu}.$$

- From the Bianchi identity it follows that

$$\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R.$$

  This identity implies that the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

  is covariantly conserved,

$$\nabla^\mu G_{\mu\nu} = 0.$$

  The Einstein tensor plays an important role in the field equations of general relativity, which state that $G_{\mu\nu}$ should be proportional to the energy-momentum tensor $T_{\mu\nu}$.

exercises (hand in october 16)

- Verify (2).

- Show that (3) is correct. What would be the answer for $[\nabla_\mu, \nabla_\nu] A^\alpha{}_{\beta\gamma}$? Don't compute this explicitly, but use properties of the covariant derivative to figure out what the answer should be.
- Compute $R_{\alpha\beta\gamma\delta}$ for the unit two-sphere.
- Express in two-dimensions $R_{\alpha\beta\gamma\delta}$ in terms of R and $g_{\mu\nu}$ (without explicit derivatives). This is possible because in two-dimensions the Riemann tensor has only one independent component which we can take equal to R.