general relativity – september 11

key concepts

- In special relativity, inertial observers follow straight lines; these are the lines whose length is maximal, so that inertial observers age more quickly than noninertial observers.

- Can be deduced from variational principle. The length of the curve $x^\mu(\lambda)$ with $\lambda_0 \leq \lambda \leq \lambda_1$ is

$$\Delta\tau = \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}. \tag{1}$$

  We vary the path keeping the endpoints fixed, $\delta x^\mu(\lambda_0) = \delta x^\mu(\lambda_1) = 0$. The variation is

$$\begin{aligned}
\delta \int_{\lambda_0}^{\lambda_1} d\lambda \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} &= \\
\int_{\lambda_0}^{\lambda_1} d\lambda \frac{-\eta_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}{\sqrt{-\eta_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}}} &= \\
\int_{\lambda_0}^{\lambda_1} d\lambda \delta x^\mu \frac{d}{d\lambda}\left[\frac{\eta_{\mu\nu}\frac{dx^\nu}{d\lambda}}{\sqrt{-\eta_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}}}\right]
\end{aligned}$$

  The result in (1) does not depend on the choice of λ, it is reparametrization invariant under $\lambda \to \lambda'(\lambda)$. In particular, we can choose λ equal to the proper time τ, so that the denominator $\sqrt{-\eta_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda}}$ in the above equation can be dropped. The resulting equation is

$$\frac{d}{d\tau}\left[\eta_{\mu\nu}\frac{dx^\nu}{d\tau}\right] = 0$$

  in other words

$$\frac{d^2x^\nu}{d\lambda^2} = 0$$

  and indeed these are straight lines.

- If a source is moving at velocity v and moving away from an observer with velocity v_r (the component of v along the axis pointing from the observer to the source), the frequency of the signal received by the observer is corrected because of Lorentz contraction and because of because of the radial velocity v_r. If ν_{obs} and ν_{em} denote the emitted and observed frequencies, then

$$\frac{\nu_{obs}}{\nu_{em}} = \frac{\sqrt{1 - v^2/c^2}}{1 + v_r/c}.$$

If the movement is purely radial, i.e. $v = |v_r|$, then

$$\frac{\nu_{obs}}{\nu_{em}} = \sqrt{\frac{1 - v_r/c}{1 + v_r/c}}.$$

Thus if $v_r > 0$, the source is moving away, the light is redshifted, for $v_r < 0$ it is always blueshifted.

- Recall that momentum $p^\mu = m\frac{dx^\mu}{d\tau}$, with $p^0 = E/c = mc\gamma$, and $p^i = m\gamma v^i$. The relativistic force is $f^\mu = m\frac{d^2x^\mu}{d\tau^2}$. Momentum is a vector, $p^{\mu'} = \Lambda^{\mu'}{}_\mu p^\mu$, and satisfies $p^2 = \eta_{\alpha\beta}p^\alpha p^\beta = -m^2c^2$. For massive particles we can always choose an intertial frame so that $p^\mu = (p, 0, 0, 0)$. For massless particles we can always choose an inertial frame so that $p^\mu = (p, p, 0, 0)$.

- Tensors are objects that "transform nicely." V^μ is contravariant, V_μ covariant. Define $\eta^{\alpha\beta}$ as the inverse of $\eta_{\alpha\beta}$, i.e. $\eta^{\alpha\beta}\eta_{\beta\gamma} = \delta^\alpha_\gamma$. The tensor η is used to raise and lower indices, e.g. $V_{\alpha\beta} = \eta_{\alpha\gamma}V^\gamma{}_\beta$ etc. Lorentz transformations satisfy $\Lambda_\alpha{}^\gamma\Lambda^\alpha{}_\beta = \delta^\gamma_\beta$. These two Λ's are each others inverse. Notice the location of the upper and lower indices. $\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}$ is covariant, and the differential operator $\eta^\alpha\beta\partial_\alpha\partial_\beta$ (the d'Alembertian or Laplacian) is a scalar.

- Tensors can be added, multiplied, contracted and differentiated and remain tensors. Contraction is e.g. the tensor $C_\mu{}^\rho = A_{\mu\nu}{}^{\nu\rho}$ which is a tensor if A is.

- Tensors transform as e.g.

$$A^{\mu'}{}_{\nu'\rho'} = \Lambda^{\mu'}{}_\mu\Lambda_{\nu'}{}^\nu\Lambda_{\rho'}{}^\rho A^\mu{}_{\nu\rho}.$$

- Examples of tensors are $\eta_{\mu\nu}$ and the Levi-Civita tensor $\epsilon_{\mu\nu\rho\sigma}$. The latter is the unique completely antisymmetric tensor that satisfies $\epsilon_{0123} = 1$. It is not strictly a tensor because it only transforms properly under Lorentz transformations with $\det(\Lambda) = 1$ and sometimes called a pseudotensor.

- Consider a collection of point-particles moving along trajectories $x^i_n(t)$, each with charge q_i, take $c = 1$. Define the charge and current as

$$\begin{aligned} Q &= \sum_n q_n\delta^{(3)}(x - x_n(t)) \\ J^i &= \sum_n q_n\delta^{(3)}(x - x_n(t))\frac{dx^i_n(t)}{dt}. \end{aligned}$$

These combine nicely into a four-vector $J^\mu = (Q, J^i)$ which indeed transforms as a vector under Lorentz transformations. We can write

$$J^\mu = \sum_n q_n\delta^{(3)}(x - x_n(t))\frac{dx^\mu_n(t)}{dt}. \tag{2}$$

exercises

- In any coordinate system we can define an object δ^μ_ν which is 1 if $\mu = \nu$ and 0 otherwise. From this definition is not yet clear whether this object is a tensor. Verify that it is.
- Show that J^μ is a conserved current, $\partial_\mu J^\mu = 0$.
- Show that $Q = \int d^3x J^0(x)$ is conserved, i.e. time-independent, using only $\partial_\mu J^\mu = 0$.
- Show that Q is a scalar, using the explicit expression for a collection of point particles