general relativity – november 13

key concepts

- In cosmology a standard point of view is to look at scales much larger than that of say galaxies, so that the universe can be described as a spatially homogeneous and isotropic fluid. Actually, galaxies have been observed to appear in clusters which in turn appear in superclusters, so that the fluid description is only relevant at very large scales. The homogeneity assumption is partially based on observation, the universe does not seem essentially different anywhere we look, but we only see a small part of the universe so to some extent this is a philosophical statement. Isotropy appears slightly more robust, due to the presence of the cosmic microwave background. This is a thermal background with a temperature of about 2.7K, which is the same everywhere we look up to one part in 10^5, as measured by the COBE satellite. The universe is not static, Hubble was the first to observe that the universe is expanding by observing that objects are redshifted more as they are farther away from us.

- For a precise definition of homogeneity and isotropy see e.g. Wald. Homogeneity roughly means that at fixed time, every point can be connected to every other point by a diffeomorphism that preserves the metric. Isotropy roughly means that at a given point P, any two spatial tangent vectors can be mapped into each other by a diffeomorphism that preserves the metric (local rotations).

- We write the following ansatz for the metric

$$ds^2 = -d\tau^2 + a^2(\tau)\gamma_{ij}(u)du^i du^j \tag{1}$$

  where $a^2(\tau)$ is the scale factor, and $\gamma_{ij}(u)du^i du^j$ is a maximally symmetric space (completely homogeneous and isotropic). The u^i are comoving coordinates. Earth for example is not quite comoving, which is why we see a dipole moment in the cosmic microwave background.

- Consider just the metric $ds^2 = \gamma_{ij}(u)du^i du^j$, and look at the map $\xi_{ij} \to R_{ij}{}^{kl}(p)\xi_{kl}$ with some fixed p. This is a linear map on the space of antisymmetric rank two tensors. There are three such tensors, so this is a three by three matrix. If this matrix has unequal eigenvalues, then one can distinguish a preferred two-form and thereby a preferred direction in space, which contradicts the isotropy assumption. Therefore the matrix has to have equal eigenvalues and must be proportional to the identity matrix.

- A slightly more fancy argument is to observe that the above matrix has to be one that commutes with the rotation group in three dimensions, and by Schur's lemma it must be proportional to the identity matrix.

- Thus, we find
$$R_{ij}{}^{kl} = k(p)(\delta_i^k \delta_j^l - \delta_i^l \delta_j^k)$$
with some constant $k(p)$. Homogeneity then implies k should not depend on p, and therefore
$$R_{ijkl} = k(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}). \tag{2}$$

- In three dimensions the number of independent components of R_{ijkl} is six, but the number of independent components of R_{ij} is also six, so if we know R_{ij} we can determine R_{ijkl}. Therefore, in three dimensions it is sufficient to impose
$$R_{ij} = 2k\gamma_{ij} \tag{3}$$
which follows from (2) by contracting with γ^{ik}. In higher dimensions (3) no longer has to imply (2), but in three dimensions it does.

- A metric that is maximally symmetric certainly has rotational symmetry and therefore locally is of the form
$$ds^2 = e^{2\beta(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4}$$
The Ricci tensor of this metric is
$$\begin{aligned} R_{11} &= \frac{2}{r}\partial_1\beta \\ R_{22} &= e^{-2\beta}(r\partial_1\beta - 1) + 1 \\ R_{33} &= R_{22}\sin^2\theta \end{aligned}$$
Solving (3) is straightforward and we get
$$\gamma_{ij}du^i du^j = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$
so that we finally obtain the Robertson-Walker metric
$$ds^2 = -d\tau^2 + a^2(\tau)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right). \tag{5}$$

- After a rescaling of r we can always achieve $k = -1$ (in which case the spatial metric is called open), $k = 0$ (spatial metric is flat) or $k = +1$ (spatial metric is closed).

- For $k = -1$ the spatial space describes the induced metric on a hyperboloid embedded in Lorentzian space (which nevertheless carries a Euclidean metric). This space is also known as Euclidean three-dimensional anti-de Sitter space. By putting $r = \sinh\xi$ it can be written in the equivalent form
$$ds^2 = d\xi^2 + \sinh^2\xi(d\theta^2 + \sin^2\theta d\phi^2).$$

- For $k = 0$ the spatial metric describes flat space.

- For $k = +1$ the spatial metric describes a three-sphere, which becomes clearer by putting $r = \sin\xi$ so that it becomes

$$ds^2 = d\xi^2 + \sin^2\xi(d\theta^2 + \sin^2\theta d\phi^2).$$

This is the metric on a three-sphere in generalized spherical coordinates.

- Apart from this, all the above spaces can still be subject to certain discrete identifications which can be used to make e.g. flat space into a three-torus.

- Turning to (5), we compute

$$\begin{aligned} R_{00} &= -3\frac{\ddot{a}}{a} \\ R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} \\ R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \\ R_{33} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k)\sin^2\theta \\ R &= \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + 2k) \end{aligned}$$

- We now want to solve the Einstein equations with a nontrivial energy-momentum tensor for a homogeneous isotropic fluid. In flat space this was $T_{\mu\nu} = (p + \rho)u_\mu u_\nu + p\eta_{\mu\nu}$. Here u_μ was the fluid velocity. The relevant answer in curved space can be found by going to a local inertial frame, writing down this flat space answer, and transforming back to the original coordinates. It is not so difficult to guess the answer: u_μ becomes the velocity vector (which has to be of unit length!), and $\eta_{\mu\nu}$ becomes $g_{\mu\nu}$. Hence

$$T_{\mu\nu} = (p + \rho)u_\mu u_\nu + pg_{\mu\nu} \tag{6}$$

where for the cosmology (5) we obviously need $u^\mu = (1, 0, 0, 0)$.

- Let's now consider the conservation of energy and momentum. We compute $\nabla_\mu T^{\mu\nu}$ with $T_{\mu\nu}$ given in (6). For this we need

$$\nabla_\lambda u^\mu = \Gamma^\mu_{\lambda 0}.$$

The Christoffel symbols can be computed using the definition of $\Gamma^\mu_{\lambda\nu}$, of using the action principle trick. Either way we find that the only non-vanishing components of $\Gamma^\mu_{\lambda 0}$ are

$$\Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j.$$

It is now straightforward to see that $\nabla_\mu T^{\mu i} = 0$ identically, and that

$$\nabla_\mu T^{\mu i} = \dot{\rho} + 3\frac{\dot{a}}{a}(p + \rho) = 0. \tag{7}$$

- If we take the equation of state $p = w\rho$ then (7) implies

$$\rho = \rho_0 a^{-3(1+w)}. \tag{8}$$

For radiation, $w = 1/3$, and $\rho \sim a^{-4}$. The density of photons decreases as a^{-3}, but they also lose energy because of redshift making for a total decrease of a^{-4}. For matter, $w = 0$ and $\rho \sim a^{-3}$ as expected. For dark energy or a cosmological constant, $w = -1$ and ρ is constant. This is the crucial property of a cosmological constant.

- Thus as a keeps on increasing, the energy-momentum due to a cosmological constant will eventually dominate. Going back in time, when a keeps on decreasing, radiation will eventually dominate.

- The Einstein equations $G_{\mu\nu} = \kappa T_{\mu\nu}$ become

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) \\ \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \end{aligned} \tag{9}$$

These are the Friedman equations, the corresponding universes are Friedman-Robertson-Walker universes.

- It is quite easy to find the Einstein equations with a cosmological constant,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

This is the same as

$$G_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu}).$$

Thus we just need to generate an extra term proportional to $g_{\mu\nu}$ in $T_{\mu\nu}$. Because of (6), this can be done by shifting

$$\begin{aligned} p &\rightarrow p - \frac{\Lambda}{8\pi G} \\ \rho &\rightarrow \rho + \frac{\Lambda}{8\pi G} \end{aligned} \tag{10}$$

Thus the Einstein equations with a cosmological constant become

$$\begin{aligned} \frac{\ddot{a}}{a} &= \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3p) \\ \left(\frac{\dot{a}}{a}\right)^2 &= \frac{\Lambda}{3} + \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \end{aligned} \tag{11}$$

exercises

- Show that (9) imply (7). Why should they?

- Consider (11) and assume that ρ and p describe conventional matter (i.e. both are nonnegative). When are there static solutions to these equations? If a static solution has $p = 0$, what is then the relation between a and ρ?