general relativity – october 16

key concepts

- Equivalence principle: gravitational mass equals inertial mass. Einstein: all laws of physics are identical for all freely falling observers in "small enough regions" of spacetime (remember the standard elevator story).

- Consider two accelerated objects, separated by a distance Δx, accelerating with acceleration a in the direction of their separation. If the first object emits light with wavelength λ, the second one will receive it with a wavelength $\lambda - \Delta\lambda$ with to first order in a and to first order in Δx: $\Delta\lambda/\lambda = \frac{a\Delta x}{c^2}$. This because it takes time $\Delta t = \Delta x/c$ to travel from the one object to the other (this assumes a is small), so that the second object gains velocity $a\Delta x/c$, and the standard redshift equation says then $\Delta\lambda/\lambda = a\Delta x/c^2$.

- According to the equivalence principle, this is the same as when an object height Δx above the surface of earth emits a signal to the surface of earth: gravitational field is the same as constant acceleration. In this case $\Delta\lambda/\lambda \sim \Delta E/E$, with ΔE the energy gain due to gravitational potential energy, $\Delta E = ma\Delta x$, and for m we take E/c^2, so that we again arrive at $\Delta\lambda/\lambda = a\Delta x/c^2$, but now with a different interpretation.

- This suggests that the rate of a clock depends on the height above earth. Therefore space around earth is apparently not flat. Similarly, by taking flat space and transforming to an accelerating frame (not a Lorentz transformation) we obtain a non-Minkowski metric which should be the same as the metric in a uniform gravitational field, according to the equivalence principle.

- The idea is now to "covariantize" the laws of physics of flat space. In other words, we try to find a tensor equation (so that it is independent on the choice of coordinates, as we want in physics) which in flat space reduces to the laws of flat space. In particular, if one chooses coordinates at a point p such that $g_{\mu\nu}(p) = \eta_{\mu\nu}$ and $\partial_\rho g_{\mu\nu}(p) = 0$, for example Riemann normal coordinates, then this is a local inertial frame, and one should demand that the laws of physics in curved space look like those in flat space, at least near the point p.

- In practise, this often boils down to replacing ordinary derivatives by covariant derivatives: $\partial_\mu \to \nabla_\mu$. Or in notation that is often used where one writes e.g. $\partial_\mu y_\nu$ as $y_{\nu,\mu}$ and $\nabla_\mu y_\nu = y_{\nu;\mu}$, this boils down to comma becomes semicolon.

- This works fine for e.g. conservation of energy-momentum tensor: $\partial_\mu T^{\mu\nu} = 0 \Rightarrow \nabla_\mu T^{\mu\nu} = 0$ and the Maxwell equations, $\partial_\mu F^{\mu\nu} = 0 \Rightarrow \nabla_\mu F^{\mu\nu} = 0$.

- But the procedure can be ambiguous. For example, should $\partial_\mu \partial_\nu A_\lambda$ be written as $\nabla_\mu \nabla_\nu A_\lambda$ or as $\nabla_\nu \nabla_\mu A_\lambda$? These are not the same, they differ by a curvature term. In general there is no reason to pick either one or the other. One needs either a symmetry principle, another physical principle, or experimental input. Compare this to quantum mechanics, where it is also not clear how one should write the classical expression pq in the quantum theory. The difference is here related to operator ordering.

- In general relativity, in the absence of other length scales, the ambiguities that appear are typically of the order $l_p^2 R$, with $l_p = \sqrt{G\hbar/c^3}$ the Planck length, the unique combination of $G, \hbar$ and c of dimension length, and R the curvature. Since $l_p \sim 10^{-35} m$ such effects only become important if space is extremely strongly curved.

- Now try to find the equations for gravity in curved space: try to lift the Poisson equation
$$\nabla^2 \Phi = 4\pi G \rho$$
, with ρ the matter density and Φ the gravitational potential (this is the classical equation for gravity), to curved space. Not completely obvious. But remember that $h_{00} \sim -2\Phi$ in the weak field, static, slow moving approximation that we used to obtain Newton from the geodesic equation. Also recall that $\rho = T_{00}$. Thus we can rewrite the classical equation as $\nabla^2 h_{00} \sim -8\pi G T_{00}$.

- This suggests, some expression with two derivatives of the metric is equal to $T_{\mu\nu}$. Basically the only tensor we can make from the metric with two derivatives is the curvature tensor. This suggests an equation $\alpha R_{\mu\nu} + \beta R g_{\mu\nu} = \kappa T_{\mu\nu}$. Since $\nabla_\mu T^{\mu\nu} = 0$, the ∇^μ of the left hand side has to vanish, and this fixes $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}$. After rescaling κ we can always set $\alpha = 1$. Comparing with the classical limit, we fix $\kappa = 8\pi G$ (see exercises). Thus we have
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}, \qquad \kappa = 8\pi G. \tag{1}$$

- These are very difficult nonlinear equations. They are nonlinear because gravity couples to itself, in contrast to e.g. Maxwell theory, where a photon is not charged under the electromagnetic field.

- The general solution to $R_{\mu\nu} = 0$ is unknown.

- We can also include a cosmological constant in the Einstein equation,
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{2}$$
Experiments put an upper bound on Λ. Λ was originally introduces to allow for static universes by Einstein, but later he removed it (biggest blunder of his life). However, recent measurements suggests that $\Lambda \neq 0$ though very small. More about this later.

- Now want to find a rotational invariant solution of the Einstein equations. Rotational invariance means that the metric should allow so-called three Killing vectors fields $V^{(i)}$, $i = 1, 2, 3$. Killing vectors are vector fields such that a small displacement in the direction of these vectors leaves the metric invariant. For example, rotations along the axis of a cylinder. The technical requirement for a vector to leave the metric invariant if

$$\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0 \tag{3}$$

but we will not derive this here. Rotational invariance means that we have three vector fields each obeying (3), and in addition these vector fields form the "angular momentum algebra" (the Lie algebra of $SO(3)$ in math terms), namely

$$[V^{(i)}, V^{(j)}] = \epsilon_{ijk} V^{(k)}. \tag{4}$$

The commutator of two vector fields is defined as

$$[X, Y]^\mu = X^\nu \nabla_\nu Y^\mu - Y^\nu \nabla_\nu X^\mu. \tag{5}$$

- Without proof we state that any rotational invariant metric needs to locally like a two-sphere of varying radius times another two-dimensional space. The metric therefore has to be of the form

$$ds^2 = g_{aa}(a,b)\, da^2 + 2g_{ab}(a,b)\, da\, db + g_{bb}(a,b)\, db^2 + f(a,b)(d\theta^2 + \sin^2\theta d\phi^2). \tag{6}$$

This still contains four unknown functions of two undetermined variables. We now use the freedom to change coordinates to find a simpler form of the metric. First we choose a replace b by a function $b(r, a)$ that depends on a new variable r, so that $f(a, b(r, a)) = r^2$. This can always (locally) be solved for $b(r, a)$, unless f does not depend on b at all. In that case, one can first exchange a and b and then do the above procedure. Both fail if f is a constant. For that case, one can verify by explicit calculation that (6) can never yield a solution.

- We now have the metric

$$ds^2 = g_{aa}(a,r)\, da^2 + 2g_{ar}(a,r)\, da\, dr + g_{rr}(a,r)\, dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{7}$$

Next we want to find a function $a(r, t)$ so that when substituted in the above metric, leaves us with no $dr\, dt$ cross term. This yields the partial differential equation

$$2g_{aa}(a(r,t),r)\frac{\partial a(t,r)}{\partial r}\frac{\partial a(t,r)}{\partial t} + 2g_{ar}(a(r,t),r)\frac{\partial a(t,r)}{\partial t} = 0 \tag{8}$$

or in other words

$$g_{aa}(a(r,t),r)\frac{\partial a(t,r)}{\partial r} + g_{ar}(a(r,t),r) = 0. \tag{9}$$

This differential equation will in general have a solution for $a(t, r)$, so we can indeed locally get rid of the $dtdr$ cross-term. Exceptions can be dealt with on a case-by-case basis, we will not do so here.

- So conclusion: every rotationally invariant metric can after a suitable change of coordinates be brought in the form

$$ds^2 = -g_{tt}(t,r)\,dt^2 + g_{rr}\,dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{10}$$

  The minus sign was put in front of dt^2 because we want a solution with Minkowski signature and we will choose t to be the coordinate associated with time.

exercises

- Show that in four dimensions, (1) implies

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}). \tag{11}$$

  Here, $T = g^{\mu\nu}T_{\mu\nu}$.

- What is $R_{\mu\nu}$ in terms of $T_{\mu\nu}$ if $d \neq 4$?

- Show that for weak fields, slow moving and static fields (see previous discussion of derivation of Newtons law from geodesic equation) that

$$R_{00} = -\frac{1}{2}\eta^{ij}\partial_i\partial_j h_{00}. \tag{12}$$

- Now verify that $\kappa = 8\pi G$, using the Einstein equation (1) and the approximation derived in the previous exercise.