general relativity – september 18

key concepts

- Energy-momentum tensor: very important for general relativity. In electrodynamics, there is a single conserved charge, and a corresponding conserved current. Momentum is a conserved four-vector, and expect the corresponding current to have two indices. This is the energy momentum tensor $T^{\mu\nu}$. Its definition is similar to the definition of charge and current for electromagnetism, simply replace charge by four-momentum:

$$\begin{aligned} P^\mu = T^{0\mu} &= \sum_n p_n^\mu(t)\delta^{(3)}(x - x_n(t)) \\ T^{\mu\nu} &= \sum_n p_n^\mu(t)\delta^{(3)}(x - x_n(t))\frac{dx_n^\nu(t)}{dt}. \end{aligned}$$

  Since $p^\mu = (E/c^2)\frac{dx^\mu}{dt}$, can rewrite this as

$$T^{\mu\nu} = \sum_n \frac{p_n^\mu(t)p_n^\nu(t)}{E/c^2}\delta^{(3)}(x - x_n(t)).$$

  Clearly, T is a symmetric tensor. As before, the energy momentum tensor is conserved in the absence of external forces.

- Relativistic hydrodynamics: excellent approximation for many realistic cosmological systems.

- Perfect fluid: at each point with velocity $\vec{v}$, a comoving observer sees an isotropic fluid (identical in all directions).Typically good if the length scales of interest are much larger than the typical distance between collisions, so that fine structure is not important.

- Isotropy restricts form of $T^{\mu\nu}$ in comoving frame. Rotational invariance restricts T to be

$$\begin{aligned} T^{ij}_{\text{comoving}} &= p\delta^{ij} \\ T^{0i}_{\text{comoving}} &= 0 \\ T^{00}_{\text{comoving}} &= \rho \end{aligned}$$

- This can be transformed to a non-comoving frame. If the velocity is v^i, then one can do a Lorentz transformation to obtain

$$\begin{aligned} T^{ij} &= p\delta^{ij} + (p+\rho)\frac{v_i v_j}{1-v^2} \\ T^{0i} &= (p+\rho)\frac{v^i}{1-v^2} \\ T^{00} &= \frac{\rho + pv^2}{1-v^2}. \end{aligned} \tag{1}$$

Nice form is by taking $U^\mu = \frac{dx^\mu}{d\tau} = (\gamma, \gamma v^i)$. Then

$$T^{\mu\nu} = p\eta^{\mu\nu} + (p+\rho)U^\mu U^\nu. \tag{2}$$

- Conservation of energy-momentum: $\partial_\mu T^{\mu\nu} = 0$. For $\nu = 0$ this is the relativistic Euler equation for a perfect fluid.

- Consider particles of mass m, go to comoving frame, then

$$p = \frac{1}{3}\sum_{i=1}^{3} T^{ii} = \frac{1}{3}\sum_n \frac{\vec{p}_n^2}{E_n}\delta^{(3)}(x - x_n(t)) = \left\langle \frac{\vec{p}^2}{3E} \right\rangle.$$

  For a thermodynamic collection of classical particles in a box one can easily verify that the pressure due to collisions on the walls of the box is indeed given by the expectation value of $\frac{\vec{p}^2}{3E}$. This justifies the symbol p, p is to be identified with the pressure.

- Similarly we see that

$$\rho = T^{00} = \sum_n E_n \delta^{(3)}(x - x_n(t))$$

  is the energy density, which is why it is called ρ.

- Since $\vec{p}^2 \leq E^2$ and $E \geq 0$ for standard particles $0 \leq p \leq \rho/3$. For a cool gas of non-relativistic massive particles, $\vec{p}^2 \ll E^2$, and therefore $p \ll \rho$. To first order, $p = 0$. For relativistic particles or massless particles, $E^2 \sim \vec{p}^2 \gg m$, and $p \sim \rho/3$.

- In general, $p = w\rho$ is called the equation of state. In view of the above, for matter $w = 0$ and for radiation $w = 1/3$. Standard particles have $w \geq 0$. Negative w, which is the present state of the universe, is sometimes called "dark energy," and can for instance be obtained from a positive cosmological constant, but not from any form of matter that we know.

- Einstein: replace flat space by curved space. Free fall in curved space is perceived as gravitational attraction. Therefore need to develop machinery of curved spaces.

- Examples of curved spaces: $\mathbb{R}^n$, the n-sphere S^n (the set of points a unit distance from the origin in $\mathbb{R}^{n+1}$), the set $\{f(x,y,z) = 0 | (x,y,z) \in \mathbb{R}^3\}$ for suitable f, the n-torus (the product of n circles, can be obtained by identifying opposite sides in an n-cube, for $n = 2$ this looks like a doughnut).

- These spaces are all smooth and locally look more or less look like a piece of $\mathbb{R}^n$.

- Before defining a manifold first need to define a topological space. A topological space is a set X, plus a given collection of subsets of X which we call the open sets. Requirements are that the empty set and X are open, any union of open sets is itself an open set and any finite intersection of open sets is an open set.

- For example, a subset $U \subset \mathbb{R}^n$ is open if for every point $p \in U$ there is an $\epsilon > 0$ so that the set $\{x \in \mathbb{R}^n | |x - p| < \epsilon\}$ is contained in U.
- A topological space is a crude concept but can already be used to define some basic properties. For example, a map $f : M_1 \to M_2$ is continuous if and only if for all open subsets $U \subset M_2$ the inverse image $f^{-1}(U)$ is an open subset of M_1. This corresponds on $\mathbb{R}^n$ to the usual definition of continuous maps.
- A cover of a space is a collection of open sets whose union is the entire space. A space is compact if every cover has a finite subcover. This is again the intuitive definition of compactness for $\mathbb{R}^n$.
- If $M_1 = \mathbb{R}^n$ with coordinates x_i and $M_2 = \mathbb{R}^m$ with coordinates y_j, then a map $f : M_1 \to M_2$ is describes by giving a collection of functions $y_j = f_j(x_1, \ldots, x_n)$. The map is continuous if and only if all functions f_j are continuous.
- A map is injective if $p_1 \neq p_2$ implies $f(p_1) \neq f(p_2)$. A map $f : M_1 \to M_2$ is surjective if $f(M_1) = M_2$ and not some subset of M_2. A map is bijective if it is both injective and surjective, sometimes also called a one-to-one map because bijective maps are invertible.
- If a map $f : M_1 \to M_2$ is continuous, bijective, and has a continuous inverse, we call M_1 and M_2 homeomorphic. A sphere and a cube are homeomorphic, they only become different once we put more structure one them, e.g. a metric.
- Recently a famous open problem has been solved, the Poincare conjecture, which states that every compact three manifold without non-contractible loops is homeomorphic to S^3. Solving this problem is good for one million dollars, see `http://www.claymath.org/Millennium_Prize_Problems/Poincare_Conjecture/` .

exercises

- Show that $T^{\mu\nu}$ is a tensor.
- Show that $\partial_\mu T^{\mu\nu} = 0$ if the particles appearing in the definition are free particles. Also show that quite generally if $\partial_\mu T^{\mu\nu} = 0$ that then $P^\mu = \int d^3x T^{\mu 0}$ is a conserved quantity. Incidentally, if the particles are not free there is some force field acting on them, and this force field will also carry energy and momentum. If we take this additional energy and momentum into account, we always end up with an energy-momentum tensor that obeys $\partial_\mu T^{\mu\nu} = 0$.
- Derive (1). Assume $c = 1$. If you find this quite difficult, an easier question is to argue that (1) is correct using the expression (2).