general relativity – november 20

key concepts

- Some standard cosmological parameters are the Hubble constant

$$H = \frac{\dot{a}}{a}$$

which is not really a constant. Its present value is somewhere around 70 kilometers per second per megaparsec, a parsec is a bit over 3 lightyears. Another constant is the deceleration parameter

$$q = -\frac{a\ddot{a}}{\dot{a}^2}$$

which indicates whether the expansion of the universe is accelerating or not. Experimentally there is now a fair amount of evidence that $q < 0$, i.e. the rate of expansion is increasing. Other quantities are the critical density

$$\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$$

and the ratio of the current density to the critical density,

$$\Omega = \frac{\rho}{\rho_{\text{crit}}}.$$

These parameters are of interest because the second of the Friedman equations

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) \\ \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} \end{aligned} \tag{1}$$

can be rewritten as

$$\Omega - 1 = \frac{k}{H^2 a^2}.$$

Notice that this result is not valid in the presence of a cosmological constant. Thus we see that the geometry of the universe at fixed time is determined by whether or not the present energy-momentum density exceeds the critical density. If it does, the universe is closed.

- In view of the first equation of (1), if there is only conventional matter in the universe, i.e. nonnegative p and ρ, then $\ddot{a} \leq 0$. Using this, we can give an upper bound for the present age of the universe. The age of the universe is

$$\int_{a=0}^{a_{\text{now}}} dt = \int_{a=0}^{a_{\text{now}}} \frac{da}{\dot{a}} \leq \int_{a=0}^{a_{\text{now}}} \frac{da}{\dot{a}_{\text{now}}} = \frac{1}{H_{\text{now}}}.$$

This bound assumes only conventional matter, but the conclusion that at some finite point in the past the universe started out in a singular state with $a = 0$ is quite robust. This is of course also known as the big bang.

- Now consider solutions of (1). We will assume an equation of state

$$p = w\rho \tag{2}$$

so that

$$\rho = \rho_0 a^{-3(1+w)}. \tag{3}$$

One can easily verify that once we insert (2) and (3) in (1), the second equation in (1) implies the first equation in (1), so it will be sufficient to solve the second equation in (1) which becomes

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3} a^{-1-3w} - k. \tag{4}$$

For simplicity, we denote

$$C = \frac{8\pi G\rho_0}{3}. \tag{5}$$

- We first consider the case with matter, $w = 0$, and $k = -1$. From (4) we obtain

$$t = \int da \sqrt{\frac{a}{a + C}} \tag{6}$$

and with the substitution $a = C\sinh^2\phi$ one obtains an integral that one can do, notice that $\cosh^2\phi - \sinh^2\phi = 1$ and $2\sinh^2\phi = \cosh 2\phi - 1$. We finally get

$$\begin{aligned} a &= \frac{C}{2}(\cosh 2\phi - 1) \\ t &= \frac{C}{2}(\sinh 2\phi - 2\phi). \end{aligned} \tag{7}$$

- Similarly for matter with $k = +1$ we need a substitution $a = C\sin^2\phi$ resulting in

$$\begin{aligned} a &= \frac{C}{2}(1 - \cos 2\phi) \\ t &= \frac{C}{2}(2\phi - \sin 2\phi). \end{aligned} \tag{8}$$

- Next we consider radiation, $w = 1/3$. Now it is straightforward to solve for a, and for $k = -1$ we get

$$a = \sqrt{t(t + 2\sqrt{C})} \tag{9}$$

whereas for $k = 1$

$$a = \sqrt{t(2\sqrt{C} - t)}. \tag{10}$$

- Next, we look at $k = 0$ with w arbitrary, as long as $w > -1$. Solving for a is straightforward, we obtain

$$a = \left(\frac{3(1+w)}{2} Ct\right)^{\frac{2}{3(1+w)}}. \tag{11}$$

- We see a general pattern, for $k = -1$ we see a universe that expands forever ((7) and (9)), for $k = 1$ we see a universe that has a finite life time before it collapses into a big crunch ((8) and (10)), and for $k = 0$ the universe also expands forever, albeit at a slower rate than for $k = -1$.

- It remains to do the case with a cosmological constant. That is the case $w = -1$, and as we explained last week $8\pi G\rho_0$ should be replaced by $\Lambda/3$, so that (4) becomes

$$\dot{a}^2 = \frac{\Lambda}{3}a^2 - k. \tag{12}$$

The solutions of this differential equation are summarized below

$$\begin{aligned}
k = -1 \quad & \Lambda < 0 \quad a = \sqrt{3}|\Lambda| \sin\left(\sqrt{\frac{|\Lambda|}{3}}t\right) \\
k = -1 \quad & \Lambda = 0 \quad a = \pm(t - t_0) \\
k = -1 \quad & \Lambda > 0 \quad a = \sqrt{3}\Lambda \sinh\left(\sqrt{\frac{\Lambda}{3}}t\right) \\
k = 0 \quad & \Lambda < 0 \quad \text{no solution} \\
k = 0 \quad & \Lambda = 0 \quad a = \text{const} \\
k = 0 \quad & \Lambda > 0 \quad a = \exp\left(\pm\frac{\Lambda}{3}t\right) \\
k = +1 \quad & \Lambda < 0 \quad \text{no solution} \\
k = +1 \quad & \Lambda = 0 \quad \text{no solution} \\
k = +1 \quad & \Lambda > 0 \quad a = \sqrt{3}\Lambda \cosh\left(\sqrt{\frac{\Lambda}{3}}t\right).
\end{aligned}$$

The solution with $\lambda < 0$ is anti-de Sitter space, all solutions with $\Lambda > 0$ cover part of de Sitter space, and the solution with $\Lambda = 0, k = -1$ is called the Milne universe, whereas the solution with $\Lambda = k = 0$ is simply Minkowski space.

- From (3) we see that if the universe contains a mix of radiation and matter and a cosmological constant, than at very early times the universe was dominated by radiation, at some intermediate time it may have been dominated by matter, and at very late times it will be dominated by a cosmological constant. Experimental evidence points towards a value for k which is 0, so that means that at very early times the universe grew as $a \sim \sqrt{t}$ (11), at intermediate times it grew as $a \sim t^{2/3}$ and at very late times it grows as $a \sim \exp(\Lambda t/3)$. It cannot be dominated at very late times by a negative cosmological constant.

- Notice that $q = -\frac{a\ddot{a}}{a^2}$ can be written as $q = (1 + 3w)\Omega/2$, so that $q > 0$ for matter domination, but $q < 0$ for cosmological constant domination.

- How do we measure cosmological parameters? All we have is some images of faraway galaxies and stars. One thing we can usually figure out is the

redshift

$$z = \frac{\omega_{\text{emitted}}}{\omega_{\text{received}}} - 1 \tag{13}$$

which is equal to

$$z = \frac{a_{\text{receiver}}}{a_{\text{emitter}}} - 1 \tag{14}$$

(see exercises). This can be done for example by finding spectral lines whose profile can be recognized. We can also measure the amount of flux F received on earth. If we know the total absolute luminosity L of the object, for instance because we recognize what it is and we have a similar object nearby ("standard candles"), or because we have a reasonable theoretical model for it, we can compute the luminosity distance

$$d_L^2 = \frac{L}{4\pi F}. \tag{15}$$

Actually, it is usually quite difficult to determine L, so this results in a fair amount of theoretical uncertainty. Light that is being emitted from a distant object is redshifted and loses energy, but besides the redshift there is also the fact that photons arise less frequent due to the expansion, and all this is responsible for the fact that

$$d_L = ar(1+z) \tag{16}$$

where a is the present size of the universe, and r the coordinate distance to the distant object. By making a Taylor series for the scale factor a, we can find an expression of the form

$$d_L = H_0^{-1}(z + \frac{1}{2}(1-q_0)z^2 + \ldots) \tag{17}$$

with H_0, q_0 the present Hubble constant and deceleration parameter. Thus if have a table of values of (z, d_L), we can use this equation to fit H_0 and q_0.

- Measurements of this type have been done for some time, but things changed in 1999 with measurements of distant type Ib supernovae. It was then discovered that $\ddot{a} > 0$, which implies that the universe cannot just contain ordinary matter: it has to contain unconventional matter (dark energy) or a cosmological constant. This is not to be confused with dark matter, which is conventional matter that we do not directly observe but which has to exist due to indirect measurements (e.g. rotation curves of galaxies.) Besides type Ib supernovae, there have been two other recent sets of experimental data that have become available, data from the WMAP satellite which measured the cosmic microwave background in great detail, and a survey of a quarter of a million galaxies, the Sloan digital sky survey. For a detailed discussion of recent data, see astro-ph/0310723. The model that emerges is one where $k = 0$ and $\Omega = 1$. However, only ~ 0.05 of this is due to visible matter, about ~ 0.25 is due to cold dark matter it seems, and the rest, 0.70, is due to a cosmological constant or dark energy. The present universe expands at an exponential rate!!!

exercises

- Verify that (13) is equal to (14): consider signals emitted from $(t_0, r_0, \theta_0, \phi_0)$ that are received by an observer at $(t_1, r_1, \theta_1, \phi_1)$, and compute the red-shift of the signal.

- Consider the metric with a cosmological constant $\Lambda > 0$ and $k = 0$,

$$ds^2 = -d\tau^2 + e^{2H\tau}(dx^2 + dy^2 + dz^2), \tag{18}$$

  with H the Hubble parameter. A light signal is emitted at $\tau = 0$ from $x = y = z = 0$. Show that at $\tau = \infty$, the light signal travels only a finite distance in the metric $ds^2 = dx^2 + dy^2 + dz^2$, and compute this distance. Thus, light cannot access all of the space-time. This phenomenon is called the "cosmological horizon."