general relativity – october 23

key concepts

- The most general metric in 4 dimensions with spherical symmetry is of the form

$$ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

- A tedious calculation shows that the nonzero components of the Ricci tensor are (with $(0,1,2,3)$ denoting (t,r,θ,ϕ))

$$\begin{aligned}
R_{00} &= (\partial_0^2\beta + (\partial_0\beta)^2 - (\partial_0\alpha)(\partial_0\beta)) \\
&\quad + e^{2(\alpha-\beta)}(\partial_1^2\alpha + (\partial_1\alpha)^2 - (\partial_1\alpha)(\partial_1\beta) + \frac{2}{r}\partial_1\alpha) \\
R_{11} &= -(\partial_1^2\alpha + (\partial_1\alpha)^2 - (\partial_1\alpha)(\partial_1\beta) - \frac{2}{r}\partial_1\beta) \\
&\quad + e^{2(\beta-\alpha)}(\partial_0^2\beta + (\partial_0\beta)^2 - (\partial_0\alpha)(\partial_0\beta)) \\
R_{01} = R_{10} &= \frac{2}{r}\partial_0\beta \\
R_{22} &= e^{-2\beta}(r(\partial_1\beta - \partial_1\alpha) - 1) + 1 \\
R_{33} &= R_{22}\sin^2\theta
\end{aligned}$$

- To solve notice that $R_{01} = 0$ implies β is a function of r only. Next, $\partial_0 R_{22} = 0$ implies that $\partial_0\partial_1\alpha = 0$, so $\alpha(r,t) = \alpha_1(r) + \alpha_2(t)$. We can get rid of $\alpha_2(t)$ by a suitable redefinition of t, in other words we can always find a coordinate system with $\alpha_2 = 0$. We have no shown that the metric is static: there are no cross terms $dtdr$, $dtd\theta$ or $dtd\phi$, and none of the components of the metric depends on t. This is a bit stronger than the notion of stationary, where the only requirement is that the metric does not depend on t, $\partial_0 g_{\mu\nu}$. The fact that the metric is static is a consequence of the Einstein equations plus the assumption that the metric has rotational invariance.

- Next, $0 = e^{2(\beta-\alpha)}R_{00} + R_{11}$ implies that $\alpha + \beta$ is a constant, which after a constant rescaling of t can assumed to be equal to zero. Thus, $\alpha = -\beta$.

- Next we solve $R_{22} = 0$. This yields $e^{2\alpha} = 1 + \frac{\lambda}{r}$ with some constant λ.

- To determine λ, we study the resulting metric at very large r where the weak field approximation is appropriate and Newton's law should emerge. We showed before that $g_{00} = -1 + 2\Phi$ with $\Phi = -GM/r$ in this weak field limit. Comparing we see that $\lambda = -2GM$, and we obtain the Schwarzschild metric

$$ds^2 = -(1 - \frac{2GM}{r})dt^2 + (1 - \frac{2GM}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

- Notice that the metric has an apparent singularity when $r = 2GM$. This distance is called the Schwarzschild radius R_s and is where the horizon of a black hole is located. A black hole exists when all matter is concentrated within the Schwarschild radius.

- The metric is asymptotically flat space, as expected.

- Is the metric singular? This statement is not coordinate independent, would like to have a coordinate independent version of the statement that the metric is singular. We call a space singular if any scalar combination of the curvature tensor and its covariant derivatives is singular at some point of the space. The horizon of the Schwarzschild metric turns out not to be a singularity, however at $r = 0$ there appears to be a singularity, since one can show that $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{12G^2M^2}{r^6}$ for the Schwarzschild metric, and this blows indeed up at $r = 0$.

- Most test of general relativity involve testobjects near massive bodies like earth or the sun. The dynamics of such objects is described by geodesics. The geodesic equations in the Schwarzschild metric can be found by either computing the Christoffel symbols, or by applying the variational principle to $\int d\lambda g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, where the dot refers to the λ-derivative, and λ parameterizes the geodesic.

- The geodesic equations are

$$\begin{aligned}
0 &= \ddot{t} + \frac{2GM}{r(r-2GM)}\dot{t}\dot{r} \\
0 &= \ddot{r} + \frac{GM}{r^3}(r-2GM)\dot{t}^2 - \frac{GM}{r(r-2GM)}\dot{r}^2 \\
&\quad -(r-2GM)(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \\
0 &= \ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r} - \sin\theta\cos\theta\dot{\phi}^2 \\
0 &= \ddot{\phi} + \frac{2}{r}\dot{\phi}\dot{r} + 2\cot\theta\dot{\theta}\dot{\phi}. \qquad (2)
\end{aligned}$$

- Geodesic equations are complicated, by luckily there is a lot of symmetry: rotational invariance, and time translation invariance. In addition,

$$\epsilon = -g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \qquad (3)$$

  is constant along a geodesic. We can after a rescaling of λ assume $\epsilon = -1, 0, +1$ for a timelike, null and spacelike geodesic respectively. For timelike geodesics λ is therefore proper time. Timelike geodesics describe the orbits of massive particles, Null geodesics describe the orbits of massless particles.

- Use the symmetries. The rotational symmetries can be used to put the movement in a equatorial plane, so that $\theta = \pi/2$ and $\dot{\theta} = 0$ along the orbit. This is consistent with the geodesic equations.

- After this two symmetries remain, rotation in the equatorial plane, which sends $\phi \to \phi + \text{const}$ and time translation invariance, which sends $t \to t + \text{const}$. Associated to such symmetries are conserved quantities, in this case angular momentum and energy. To find the form of a conserved quantity, we can use the fact that they are of the form $V_\mu \dot{x}^\mu$, with V_μ the Killing vector that generates the invariance. We will use an alternative method, the Noether method.

- Noether method states that if an action $S = \int dt L(q, \dot{q})$ has an infinitesimal symmetry $\delta q = \epsilon f(q)$, then there is a corresponding conserved charge. To find the charge, we consider the variation of the action under $\delta q = \epsilon(t) f(q)$. The variation is of the form

$$\delta S = \int dt \dot{\epsilon} Q(q, \dot{q})$$

  which indeed vanishes for constant ϵ, which was the original symmetry. The claim is that $Q(q, \dot{q})$ is a conserved quantity, $\dot{Q} = 0$ upon using the equations of motion. Q is the conserved quantity that corresponds to the symmetry.

- Apply the Noether method to the action $\int d\lambda g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ with symmetries $\phi \to \phi + \text{const}$ and $t \to t + \text{const}$. Result is two conserved charges,

$$E = (1 - \frac{2GM}{r})\dot{t} \qquad L = r^2 \sin^2\theta \dot{\phi}. \tag{4}$$

- Now use the conserved charges and the fact that we are in the equatorial plane to convert (3) into the equation

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\epsilon(1 - \frac{2GM}{r}) + \frac{1}{2r^2}L^2(1 - \frac{2GM\gamma}{r}) = \frac{1}{2}E^2. \tag{5}$$

  We introduced a parameter γ, which is equal to one. One can do the same calculations for orbits in Newton theory and one finds the same equation with $\gamma = 0$. So γ can be used to analyze general relativity and the Newton limit at the same time.

- The equation (5) looks like a particle moving in an external potential

$$V(r) = \frac{1}{2}\epsilon(1 - \frac{2GM}{r}) + \frac{1}{2r^2}L^2(1 - \frac{2GM\gamma}{r}). \tag{6}$$

  Plotting the potential provides insight in the nature of the geodesics. A series of plots can be found in the lecture notes of Sean Carroll, pages 176 and 178. The properties of the potential depend on L. The potential only provides information about r as a function of λ, to find ϕ and t one needs to use equations (4)

- For Newtonian gravity and $L \neq 0$, $V \to +\infty$ as $r \to 0$. Massive particles can approach the star/planet, and disappear to infinity like a comet. Or they can be caught in circular or ellipsoidal orbits. Massless particles travel in straight lines and never stay near the object.

- For general relativity and $L \neq 0$, $V(r) \to -\infty$ as $r \to 0$. This indicates that massive and massless particles very close to the center $r = 0$ cannot escape anymore. This is the signal of the horizon of the black hole. Massive particles can follow comet-like trajectories, circular orbits that can be stable or unstable, and more or less ellipsoidal orbits that are not quite ellipsoidal. Photons can also travel in unstable circular orbits.

- To analyze circular orbits, look at values of r_0 for which $V'(r_0) = 0$. If $V''(r_0) > 0$, the orbit is stable, if $V''(r_0) < 0$ it is unstable. The equation $V'(r_0) = 0$ implies

$$GM\epsilon r_0^2 - L^2 r_0 + 3\gamma GML^2 = 0.$$

- For massless particles, this means $r_0 = 3\gamma GM$. Thus, in Newton theory photons never travel on circular orbits, in general relativity they can, the orbits are always unstable though and $r_0 = 3GM$, which is $3/2$ times the Schwarzschild radius.

- For massive particles,

$$r_0 = \frac{L^2 \pm \sqrt{L^4 - 12(GM)^2\gamma L^2}}{2GM}.$$

  So in Newton theory there is always a stable circular orbit for $r_0 = \frac{L^2}{GM}$. For general relativity there are no circular orbits for $L^2 < 12(GM)^2$, there is one at $r_0 = 6GM$ for $L^2 = 12(GM)^2$, and there are two for $L^2 > 12(GM)^2$, an unstable one at $r_0 = r_-$ and a stable one at $r_0 = r_+$. The latter is the one that approaches the Newtonian one as L becomes large. The orbit at $r_0 = r_-$ approaches the photon orbit at large L, $r_- \to 3GM$ as $L \to \infty$.

exercises

- What is the Scharzschild radius for earth? Recall that we are working in units where $c = 1$.

- Replace in the Schwarzschild metric M by $M(r)$, so that the mass becomes some function of r, like for a star interior or so. Now the metric no longer solves the Einstein equations with $T_{\mu\nu} = 0$, but solves it with a suitable $T_{\mu\nu}$. Compute $\rho = T_{00}$ from the Einstein equations, and express $M(r)$ as a function of $\rho(r)$. Is the pressure zero or not?

- Verify that (4) are indeed constant along geodesics.