general relativity – september 25

key concepts

- So far defined topological spaces. Want to define "smooth manifolds." These are spaces that locally look like $\mathbb{R}^n$ and if we imagine them as embedded in some larger dimensional flat space they should be described by infinitely often differentiable functions. The definition of a smooth manifold makes these ideas more precise. The idea is to locally describe a space by a set of coordinates, a set of labels that describe each point in some open subset of the manifold in a unique fashion, and to require that when two different coordinate systems are available, they are related by infinitely often differentiable functions, C^∞ functions.

- To define a manifold, we need a topological space M and a suitable collection of open set $\{V_\alpha\}$ that covers M. We also need that each V_α is homeomorphic to an open subset of $\mathbb{R}^n$ called U_α, and the bijective map from U_α to V_α will be called f_α. Since the points in U_α are in one-to-one correspondence with points in V_α, we can use the standard coordinates of $\mathbb{R}^n$ on U_α to label points in V_α, and in this way we obtain local set of coordinates on M. In this way we have a space which locally looks like $\mathbb{R}^n$. Smoothness arises by demanding that whenever $V_\alpha \cap V_\beta \neq \emptyset$, the changes of coordinates on the overlap, which are given by the functions

$$\begin{aligned} f_\beta^{-1} \circ f_\alpha &: U_\alpha \to U_\beta \\ f_\alpha^{-1} \circ f_\beta &: U_\beta \to U_\alpha \end{aligned}$$

  should be C^∞.

- As a side remark, the maps f_α are called charts in the math literature, and one often requires that the collection of charts is maximal, in which case it is called an atlas.

- In a similar way we define C^∞ maps between different manifolds. Take M with charts $U_\alpha \stackrel{f_\alpha}{\to} V_\alpha$, and another manifold N with charts $W_\alpha \stackrel{g_\alpha}{\to} X_\alpha$. A map $f : M \to N$ acts locally as a map from some V_α to some other open set X_β. We require that in local coordinates, f should be C^∞, in other words we require that $g_\beta^{-1} \circ f \circ f_\alpha : U_\alpha \to W_\beta$ is C^∞.

- An example of a chart is the map

$$(\theta, \phi) \to (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{1}$$

  from $(0, \pi) \times (0, 2\pi)$ to the two-sphere minus a line segment. We need strictly speaking at least one other patch to cover S^2 completely, see exercises, but in practise we usually don't and work with the usual spherical coordinates and their ambiguity at the north and south pole.

- We usually require another technical condition, namely that spaces are Haussdorf. This means that for each pair of distinct points p_1, p_2 there should be two disjoint open sets U_1, U_2 with $p_1 \in U_1$ and $p_2 \in U_2$. This means that we can cleanly distinguish points.

- Smooth functions $M \to \mathbb{R}$ are defined as functions that are C^∞ when expressed in local coordinates, i.e. $f \circ \phi_\alpha : U_\alpha \to \mathbb{R}$ should be C^∞, and the collections of all such functions is called $C^\infty(M)$.

- If we have a function $f(x)$ on M, and change coordinates $x \to x(x')$, the function in new coordinates obviously reads $f(x(x'))$. We say that f is a scalar, in analogy with the definition of a scalar in special relativity.

- Vectors arise by considering the velocities of points moving on a manifold. Vectors are arrows, but if we imagine the manifold embedded in some larger $\mathbb{R}^n$ the arrow points away from the manifold in general. The set of all possible vectors at a point span a vector space called the tangent space at that point. Vectors indicate infinitesimal displacements.

- A vector field is a smooth collection of arrows, i.e. a choice of vector at each point on a manifold that varies smoothly.

- A more precise definition of a vector field is a linear map $\nabla_A : C^\infty(M) \to C^\infty(M)$ such that

$$\begin{aligned} \nabla_A(fg) &= (\nabla_A f)g + f(\nabla_A g) \\ \nabla_A(\alpha f + \beta g) &= \alpha \nabla_A f + \beta \nabla_A g. \end{aligned}$$

  Such a map is also called a derivation and the first requirement is called the Leibnitz rule, similar to the product rule for differentiation. ∇_A represents the derivative in the direction of some vector.

- In local coordinates,

$$\nabla_A f = A^\mu \frac{\partial}{\partial x^\mu}.$$

  This expression also shows how vector fields transform under a change of coordinates. We will call A^μ a vector, or sometimes the components of a vector field, strictly speaking the $\frac{\partial}{\partial x^\mu}$ form locally a basis of the vector fields.

- By requiring $A^\mu(x)\frac{\partial}{\partial x^\mu} = A^{\mu'}(x')\frac{\partial}{\partial x^{\mu'}}$ we find the transformation rule

$$A^{\mu'}(x') = A^\mu(x)\frac{\partial x^{\mu'}}{\partial x^\mu}. \tag{2}$$

  This reduces to the definition in special relativity for Lorentz transformations.

- In flat space, we had Lorentz transformations. In curved space, this is replaced by general coordinate transformations. Obviously, physics should

not depend on a choice of coordinates, since such a choice carries no intrinsic information about a space. Thus physics should be invariant under general coordinate transformations, the principle of "general covariance."

- General tensors are defined to transform in a similar way, e.g.

$$A^{\mu'\nu'}{}_{\rho'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x^{\rho'}} A^{\mu\nu}{}_{\rho}.$$

  An easy way to remember this is to write down

$$A^{\mu\nu}{}_{\rho} \frac{\partial}{\partial x^{\mu}} \frac{\partial}{\partial x^{\nu}} dx^{\rho} \tag{3}$$

  and to require that this is a scalar.

- The object $A_\mu dx^\mu$ is called a one-form.

- As before, tensors can be added, contracted and multiplied, but to define raising and lowering indices and differentiation we need a few more ingredients. Tensors are the buildings blocks of expressions that are manifestly invariant under general coordinate transformations and therefore quite important in physics.

- To describe the size and shape of a manifold, we need something called the metric. This is a symmetric tensor $g_{\mu\nu}$ with two lower indices. It yields the square of the distance between a point x^μ and an infinitesimally close point $x^\mu + dx^\mu$ via

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

  In flat space $g_{\mu\nu} = \delta_{\mu\nu}$, in Minkowski space $g_{\mu\nu} = \eta_{\mu\nu}$. $g_{\mu\nu}$ is a tensor because ds^2 should be a scalar. Distances should be an intrinsic property and not depend on a choice of coordinates.

- The metric provides notions of past and future, inertial observers, gravity, the speed of light, etc.

- We define $g^{\mu\nu}$ as the inverse of $g_{\mu\nu}$,

$$g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu.$$

- Indices are raised and lowered using $g^{\mu\nu}$ and $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$.

- If p is the number of positive eigenvalues of the metric viewed as a square matrix, and q the number of negative eigenvalues, we say that (p, q) is the signature of the space. q roughly counts the number of time directions.

- If the signature is $(d, 0)$ space is Euclidean, if the signature is $(d-1, 1)$ it is Lorentzian. Sometimes spaces of different signature can be related to each other via a suitable analytic continuation.

- Another useful tensor: denote by $\epsilon_{\mu_1 \ldots \mu_d}$ the object which is completely antisymmetric in all indices and which satisfies $\epsilon_{01\ldots d-1} = +1$. This object is no longer a tensor. However,

$$\sqrt{\det g}\epsilon_{\mu_1 \ldots \mu_d} \tag{4}$$

  with $\det g$ the determinant of $g_{\mu\nu}$ viewed as a square matrix is a tensor (for Euclidean space; for Lorentzian space we use $\sqrt{-\det g}$ instead.)

- Because of this, the general coordinate invariant integration measure is $\int dx^0 \ldots dx^{d-1}\sqrt{\det g}$. if we integrate a scalar with respect to this measure the result will be invariant under general coordinate transformations.

exercises

- Find another chart that together with the chart given in (1) covers the two-sphere completely.
- Draw an arrow on the two-sphere and give its components in both coordinate systems found in the previous exercise. Verify it transforms as given by (2).
- Find the metric on S^2.
- Consider the space $-x^2 + y^2 + z^2 = 1$ as a subspace of three-dimensional Minkowski space $ds^2 = -dx^2 + dy^2 + dz^2$. Parametrize this space using two coordinates and find the metric on this space.
- Show that (4) is a tensor, as long as we consider coordinate transformations where $\det(\frac{\partial x^{\mu'}}{\partial x^{\mu}}) > 0$.