general relativity – november 27

key concepts

- We consider black holes, i.e. we take the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad (1)$$

  seriously for all values of r. We defined once and for all the metric on the two-sphere as

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

  The metric in (1) appears to be singular at $r = 2MG$, but this is an artifact of these coordinates. To study the interior of the black hole we need better coordinates.

- Notice that for $r < 2GM$ the roles of r and t are exchanged: r becomes the timelike coordinate and the metric becomes time-dependent. Since a physical particle can only move either always forward or always move backward in time, we see that inside the horizon any physical particle has to always move towards decreasing r, i.e. towards the interior of the black hole. Escape from the black hole is not possible.

- At $r = 2MG$, the volume element of the corresponding transversal three-dimensional space (spanned by t, θ, ϕ) vanishes,

$$\sqrt{g^{(3)}} = \sqrt{(1 - 2MG/r) r^2 \sin^2\theta}|_{r=2MG} = 0.$$

  This is because t becomes a null coordinate. In such cases it is better to look at the two-dimensional surface which is transversal to both r and the null coordinate, in this case that is a two-sphere with metric $ds^2 = r^2 d\Omega^2$. This two-sphere at $r = 2MG$ is usually called the horizon, and it has area $A = 4\pi(2MG)^2$. Once something crosses the horizon there is no way back.

- In history, several coordinate systems have been introduced for the Schwarzschild solution: Eddington in 1924 (who found coordinates that are good past the horizon but he did not realize this at the time), Lemaitre in 1933 who realized that $r = 2MG$ is not singular, Synge in 1950 who found coordinates that cover the entire Schwarzschild space-time to be described later, Fronsdal rediscovered these results in 1959, Kruskal in 1960 en Szekeres in 1960 found different coordinates which are also good everywhere and most frequently used, Novikov in 1963 who used coordinates comoving with massive particles, etc.

- To describe various coordinate systems, we first consider a massless geodesic which is given by

$$t = \pm \int \frac{dr}{1 - \frac{2MG}{r}} = \pm[r + 2MG \log\left(\frac{r}{2MG} - 1\right)] + \text{const} \qquad (2)$$

and which motives the following definition of the "tortoise coordinate" r^*

$$r^* = r + 2MG \log\left(\frac{r}{2MG} - 1\right) \qquad (3)$$

whose name is undoubtedly inspired by the Achilles and the tortoise anecdote, here we never reach the horizon since at $r = 2MG$, $r^* \to \infty$.

- The metric in coordinates t, r^* instead of t, r is

$$ds^2 = \left(1 - \frac{r}{2MG}\right)(-dt^2 + (dr^*)^2) + r^2 d\Omega^2 \qquad (4)$$

where r is to be interpreted as a function of r^*. This metric does not blow up but the horizon is at $r^* = -\infty$ so these coordinates are not yet a great way to study the metric beyond the horizon.

- To really go beyond the horizon we introduce

$$\tilde{u} = t + r^*, \qquad \tilde{v} = t - r^*.$$

It is clear from (4) that infalling null geodesics follow a trajectory $\tilde{u} =$ const, whereas outgoing null geodesics follow a trajectory $\tilde{v} =$ const.

- In terms of $\tilde{u}, r$ the metric becomes

$$ds^2 = -\left(1 - \frac{r}{2MG}\right) d\tilde{u}^2 + 2d\tilde{u}dr + r^2 d\Omega^2. \qquad (5)$$

The metric is smooth at $r = 2MG$, the determinant of the metric remains finite there. In the $r, \tilde{u}$ plane, ingoing null geodesics are vectors in the direction $(-1, 0)$, and outgoing null geodesics are in the direction $((1 - 2MG/r), 2)$. Thus for $r < 2MG$ both in and outgoing null geodesics are in the direction of decreasing r; there is no escape once we cross the horizon.

- In terms of $\tilde{v}, r$ the metric becomes

$$ds^2 = -\left(1 - \frac{r}{2MG}\right) d\tilde{v}^2 - 2d\tilde{2}dr + r^2 d\Omega^2. \qquad (6)$$

The metric is smooth at $r = 2MG$, the determinant of the metric remains finite there. In the $r, \tilde{v}$ plane, ingoing null geodesics are vectors in the direction $(1, 0)$, and outgoing null geodesics are in the direction $(-(1 - 2MG/r), 2)$. Thus for $r < 2MG$ both in and outgoing null geodesics are in the direction of increasing r; there is no way to get behind the horizon, and everything behind the horizon will try to leave that region.

- The metrics in (5) and (6) are in so-called Eddington-Finkelstein coordinates; notice the different behavior behind the horizon in each coordinate system. In (5) we see toward the horizon in the future, in (6) we look at the horizon in the past. We end up, once we cross the horizon, in two quite distinct regions of spacetime, as the properties of the null geodesics illustrate.

- To combine everything we will use $\tilde{u}, \tilde{v}$ instead of t, r. In order to have the horizon at finite coordinate values, we need to do some redefinition, and we choose
$$u' = e^{\tilde{u}/4MG}, \qquad v' = e^{-\tilde{v}/4MG} \tag{7}$$
and to have a slightly more conventional set of coordinates we finally define
$$u = \frac{1}{2}(u' - v'), \qquad v = \frac{1}{2}(u' + v'). \tag{8}$$
One easily finds that the relation between t, r and v, u is
$$v + u = e^{(t+r)/4MG}\left(\frac{r}{2MG} - 1\right)^{1/2} \tag{9}$$
$$v - u = e^{(-t+r)/4MG}\left(\frac{r}{2MG} - 1\right)^{1/2} \tag{10}$$
$$v^2 - u^2 = e^{r/2MG}\left(\frac{r}{2MG} - 1\right). \tag{11}$$
The metric in coordinates u, v becomes
$$ds^2 = \frac{32G^3M^3}{r}e^{-r/2MG}(dv^2 - du^2) + r^2 d\Omega^2 \tag{12}$$
where again we should view r as a function of u, v. These coordinates are known as Kruskal-Szekeres coordinates.

- Some properties of (12):
  - Radial null curves are curves $v = \pm u + \text{const}$.
  - The horizon is at $r = 2MG$ so that $u = \pm v$.
  - Curves of constant r are curves $u^2 - v^2 = \text{const}$.
  - Curves of constant t are $\frac{u+v}{u-v} = \text{const}$, in other words $u/v = \text{const}$. To be precise,
$$t = 2MG \log\left(\frac{v+u}{v-u}\right).$$
  - There is a singularity at $r = 0$, which means $u^2 - v^2 = 1$. Space-time only exists for $u^2 - v^2 \leq 1$.

- If we draw the u, v plane then there are two hyperbolae $u = \pm\sqrt{1+v^2}$ beyond which space-time does not exist. The horizon consists of the two lines $u = v$ and $u = -v$. This decomposes space in four regions, one $v > |u|$ is asymptotically Minkowski space and called region I, one is $u > |v|$, which is the future region behind the horizon where everything

ends on the singularity; this is really the black hole regions and called region II. The region $u < -|v|$ is the region behind the horizon in the past, we cannot send anything into this region, but things can leave this region (in fact they have to). This is the white hole, the opposite of the black hole, and called region III. Finally, for $v < -|u|$, there is another asymptotic Minkowski spacetime region, a new region on the other side of the black hole, called region IV.

- Region IV is a new "universe" that looks like our own. However, we cannot send signals from our universe to this new universe. If we consider the geometry of slices with fixed u, for very negative u the slice consists of two disconnected universes, then for some finite time ($-1 < u < 1$) a connection between the two universes opens, and finally at late time there are two disconnected universes again. Thus it looks like there is a temporary bridge between the two universes. The slice at $u = 0$ is also known as the Einstein-Rosen bridge. However, there is no time to send anything through the hole, it closes before the signal reaches the other side. Still, attempts have been made to use such holes to glue together different parts of the universe that are very far apart, with the idea to make space travel much faster. However, for this to work we have to somewhere identify regions I and IV, and it is quite difficult to make this idea work in practise (but the last word has not been said.)

exercises

- Verify that any particle, whether massive or massless, whether it moves just in the radial direction or also in the θ, ϕ direction, always moves in a direction in the u, v plane which is upward and makes an angle of 45 degrees of less. This verifies e.g. the statement that there is no escape once someone crosses the horizon.

- Someone is at $\tau = 0$ at $r = 2MG$. Show that no matter what this person does, he or she will always hit the singularity in a proper time $\Delta\tau \leq \pi MG$.

- Consider two point masses of mass $\mu = 40$ kg each, one meter apart, which are lying on some radial axis emanating from a Schwarzschild black hole. Take for M the mass of the sun, and assume that the masses are a distance 100 km from the center of the black hole. Compute $\mu\ddot{r}$ for each point mass, and compute the difference. If the point masses are some caricature of a living human being, would you survive this force?