general relativity – october 30

key concepts

- As a simple application of the Schwarzschild solution, consider a stick at fixed θ, ϕ stretching from $r = r_1$ to $r = r_2$. The length of such a stick is obtained by integrating the line element

$$l = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - \frac{2GM}{r}}}$$

  which approaches $r_2 - r_1$ in the limit where $r \gg GM$.

- Consider a source of light at $r = r_1$, and an observer of light at $r = r_2$, both at fixed and equal θ, ϕ. Light follows a null geodesic, in this case the relevant one is one with θ and ϕ fixed, and t and r some function of an affine parameter λ. If $t(\lambda), r(\lambda)$ is a null geodesic, so is $t(\lambda) + \Delta t, r(\lambda)$ because the metric is independent of constant shifts of t. Thus if two light waves are emitted with time separation Δt at the source, they are received at the observer with the same time difference Δt. However, the amount of time that is being measured is the amount of proper time, which is

$$\Delta t = \left(1 - \frac{2GM}{r_i}\right)^{-1/2} \Delta \tau_i.$$

  It follows that the ratio of frequencies or wavelengths is given by

$$\frac{\omega_2}{\omega_1} = \frac{\lambda_1}{\lambda_2} = \left(\frac{1 - 2GM/r_1}{1 - 2GM/r_2}\right)^{1/2}.$$

  This is the gravitational redshift which can be experimentally measured.

- In general relativity, planets do not form orbits which are strict ellipsoids. After each orbit, the point where the planet is farthest from the sun (the aphelion) and where it is closest to the sun (the perihelion) are shifted by a small angle with respect to the previous one. This is called the perihelion precession and has first been measured for mercury.

- To compute this effect, we return to the geodesic equation in the Schwarzschild metric for a massive particle,

$$-1 = -(1 - 2GM/r)\dot{t}^2 + (1 - 2GM/r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2. \qquad (1)$$

  Previously we considered $\dot{r}$, but now we are interested in $\frac{\partial r}{\partial \phi}$. Divide left and right hand side of (1) by $\dot{\phi}^2$, use the conserved quantities E and L, and use that

$$\frac{\dot{r}}{\dot{\phi}} = \frac{\partial r}{\partial \phi}$$

to obtain

$$\left(\frac{\partial r}{\partial \phi}\right)^2 = -\frac{r^4}{L^2}\left(1-\frac{2GM}{r}\right) - r^2\left(1-\frac{2GM}{r}\right) + \frac{E^2 r^4}{L^2}. \qquad (2)$$

Call the right hand side $f(r)$. Then $f(r) = 0$ has three solutions, the smallest two are denoted r_- and r_+, and these are the values of the perihelion and aphelion respectively. The third root of $f(r)$ does not correspond to a stable orbit but could e.g. represent the point of passage of a comet. The angle that the particle goes through between perihelion and aphelion is thus

$$\Delta\phi = \int_{r_-}^{r_+} \frac{dr}{\sqrt{f(r)}}$$

which can be evaluated numerically or be estimated analytically. The total shift between two perihelia is twice this angle, and for Mercury one can compute that this is 43 arcseconds per century. For Mercury one measures 5601 arcseconds per century, but the difference can be attributed to effects due to other planets and the precession in the earth system.

- As a check, in the Newton limit we have instead

$$\left(\frac{\partial r}{\partial \phi}\right)^2 = -\frac{r^4}{L^2}\left(1-\frac{2GM}{r}\right) - r^2 + \frac{E^2 r^4}{L^2}$$

and the relevant integral becomes

$$\Delta\phi = \int_{r_-}^{r_+} \frac{dr}{r\sqrt{(1-r/r_+)(r/r_- - 1)}} = \pi$$

which is indeed the answer for a perfect ellipse.

- Along similar lines one can study light deflection near the surface of the sun. For a massless particle the analogue calculation that led to (2) leads to

$$\left(\frac{\partial r}{\partial \phi}\right)^2 = -r^2\left(1-\frac{2GM}{r}\right) + \frac{E^2 r^4}{L^2}. \qquad (3)$$

We want that light is precisely deflected at the surface of the sun. This means that we want that $\partial r/\partial\phi$ vanishes for $r = r_{\rm sun}$, the radius of the sun. This fixes E^2/L^2 as

$$\frac{E^2}{L^2} = \frac{1}{r_{\rm sun}^2}\left(1-\frac{2GM}{r_{\rm sun}}\right).$$

The right hand side of (3) can be written as $f(r)$ with

$$f(r) = -r^2\left(1-\frac{2GM}{r}\right) + \frac{r^4}{r_{\rm sun}^2}\left(1-\frac{2GM}{r_{\rm sun}}\right)$$

and clearly $f(r_{\rm sun}) = 0$. The total deflection of light can again be written as an integral,

$$\Delta\phi = 2\int_{r_{\rm sun}}^{\infty} \frac{dr}{\sqrt{f(r)}} \qquad (4)$$

which for the sun is equal to 1.75 arcseconds. The integral can be estimated analytically, or evaluated analytically and numerically. We will not do that here.

exercises

- (The so-called fourth test of general relativity) Consider the sun, mercury and earth, and assume that the sun is at $r = 0$, the metric around it is the Schwarzschild metric, and assume that mercury is at (r_1, θ, ϕ) for all t, and that earth is at (r_2, θ, ϕ) for all t. In other words, we assume both mercury and earth to be static. Assume that we send a radar signal from earth to mercury, where it is reflected back to earth. After some proper time on earth we measure the reflected signal. The total proper time as measured on earth it takes for the signal to go back to mercury and back is denoted $\Delta\tau$. Compute this quantity, and show that it is not equal to twice the proper distance between earth and mercury divided by the speed of light (which we take to be $c = 1$ throughout.)

- Consider a massive particle, initially at rest at some fixed θ, ϕ, at some fixed value $r = r_0$. The particle will disappear into the black hole described by the Schwarzschild metric. At some point the particle will reach the horizon. Show that it takes infinite coordinate time, but only finite proper time for the massive particle to reach the horizon.

- Extra exercise, not given in class: compute some of the periods of the orbits of massive particles described in the lecture of last week.