general relativity – november 15

material discussed in class

Roughly 8.1,8.2,8.3 in the book, or pp 217-225 of the lecture notes.

exercises

- Show that the Friedman equations

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) \\ \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \end{aligned} \tag{1}$$

  imply the equation we derived from the conservation of energy and momentum,

$$\nabla_\mu T^{\mu i} = \dot{\rho} + 3\frac{\dot{a}}{a}(p + \rho) = 0. \tag{2}$$

  why should they imply this?

- It is quite easy to find the Einstein equations with a cosmological constant,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

  This is the same as

$$G_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{\Lambda}{8\pi G} g_{\mu\nu}).$$

  Thus we just need to generate an extra term proportional to $g_{\mu\nu}$ in $T_{\mu\nu}$. Because

$$T_{\mu\nu} = (p + \rho)u_\mu u_\nu + p g_{\mu\nu} \tag{3}$$

  this can be done by shifting

$$\begin{aligned} p &\rightarrow p - \frac{\Lambda}{8\pi G} \\ \rho &\rightarrow \rho + \frac{\Lambda}{8\pi G} \end{aligned} \tag{4}$$

  Thus the Einstein equations with a cosmological constant become

$$\begin{aligned} \frac{\ddot{a}}{a} &= \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3p) \\ \left(\frac{\dot{a}}{a}\right)^2 &= \frac{\Lambda}{3} + \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \end{aligned} \tag{5}$$

  Consider (5) and assume that ρ and p describe conventional matter (i.e. both are nonnegative). When are there static solutions to these equations? If a static solution has $p = 0$, what is then the relation between a and ρ? Can you figure out whether these static solutions are stable against small perturbations?

- Consider the metric in n dimensions

$$ds^2 = -dt^2 + \sum_{i=1}^{n-1} t^{2\alpha_i} dx_i^2. \tag{6}$$

Find the conditions of the $n-1$ constants α_i so that this metric is a solution of the vacuum Einstein equations. Are these reasonable candidates for a realistic cosmology?