Problem 1 (Carroll 2.1)

Just because a manifold is topologically nontrivial does not necessarily mean that it cannot be covered with a single chart. In contrast to the circle $S^1$, show that the infinite cylinder $\mathbb{R} \times S^1$ can be covered with just one chart, by explicitly constructing the map.

Problem 2 (Carroll 2.3)

The easiest way to define the two-torus $T^2$ is as a square with opposite sides identified. Show that $T^2$ is a manifold by constructing an appropriate atlas.

Problem 3 (based on Carroll 2.6)

Consider $\mathbb{R}^3$ with standard spherical polar coordinates $(r, \theta, \phi)$. A particle moves along a parametrized curve given by

$$x(\lambda) = \cos \lambda, \quad y(\lambda) = \sin \lambda, \quad z(\lambda) = \lambda.$$  

Express the path of the curve in the $(r, \theta, \phi)$ system. Calculate the components of the tangent vector to the curve in both the Cartesian and spherical polar coordinate system. Verify explicitly that the components of the curve indeed transform as a tensor.

Problem 4

Consider the two-dimensional space $-x^2 + y^2 + z^2 = 1$, viewed as a subspace of three-dimensional Minkowski space $ds^2 = -dx^2 + dy^2 + dz^2$. Parametrize this space using two coordinates, and find the metric on this space.

Problem 5

Define $g \equiv \det g \equiv \det g_{\mu \nu}$ to be the determinant of the metric $g_{\mu \nu}$ viewed as a square matrix. Show that

$$\sqrt{|g|} \epsilon_{\mu_1 \ldots \mu_n}$$  

is a tensor. Show that if $f(x^\mu)$ is a scalar function on a manifold $M$, the integral

$$\int dx^0 \ldots dx^{n-1} \sqrt{|g|} f(x^\mu)$$

provides an answer which does not depend on the choice of coordinates $x^\mu$. In other words, this is the right way to integrate a quantity on a manifold, an operation that we would obviously like to be independent of the choice of coordinates.