General Relativity

Take Home Set 2

Hand in on November 1

Problem 1

Consider the following metric on $\mathbb{R}^2$

$$ds^2 = e^{\phi(x,y)}(dx^2 + dy^2).$$

a) Compute the Christoffel symbols and Riemann curvature tensor for this metric.

b) Next we take the metric on the plane induced by the stereographic projection, i.e.

$$e^{\phi(x,y)} = \left(\frac{4}{4 + x^2 + y^2}\right)^2.$$

Compute $\int\int_{\mathbb{R}^2} \sqrt{g}\, dxdy$ and $\int\int_{\mathbb{R}^2} \sqrt{g}R\, dxdy$.

c) Suppose we add to ϕ a function of x, y that decays very rapidly at infinity. Would either of the two above integrals change? Explain your answer.

d) Consider geodesics in the above metric. Show that if a geodesic is a straight line, that straight line points everywhere in the direction of the gradient of $\phi(x, y)$.

Problem 2

a) Show that in four dimensions, the Einstein equations imply

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}). \tag{1}$$

Here, $T = g^{\mu\nu}T_{\mu\nu}$.

b) What is $R_{\mu\nu}$ in terms of $T_{\mu\nu}$ if $d \neq 4$?

Problem 3 (Carroll 3.10)

Define the Weyl tensor $C_{\rho\sigma\mu\nu}$ in terms of the Riemann curvature tensor by

$$\begin{aligned} C_{\rho\sigma\mu\nu} &= R_{\rho\sigma\mu\nu} - \frac{2}{(n-2)}\left(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}\right) \\ &\quad + \frac{2}{(n-1)(n-2)} g_{\rho[\mu}g_{\nu]\sigma}R. \end{aligned} \tag{2}$$

Recall that the notation $[\mu\nu]$ indicates antisymmetrization in the two indices, as e.g. in $F_{[\mu\nu]} \equiv \frac{1}{2}(F_{\mu\nu} - F_{\nu\mu})$. Show that in dimension $n \geq 4$,

$$\nabla_\rho C^\rho{}_{\sigma\mu\nu} = 2\frac{(n-3)}{(n-2)}\left(\nabla_{[\mu}R_{\nu]\sigma} + \frac{1}{2(n-1)} g_{\sigma[\mu}\nabla_{\nu]}R\right). \tag{3}$$

(OPTIONAL) Problem 4 (Carroll 3.9)

Consider an arbitrary metric tensor $g^{(1)}_{\mu\nu}$ and another metric $g^{(2)}_{\mu\nu}$ that is related to $g^{(1)}$ by an overall rescaling, $g^{(2)}_{\mu\nu} = \omega^2(x) g^{(1)}_{\mu\nu}$. Notice that $\omega^2(x)$ does not carry any indices. In this case we say that the two metrics $g^{(1)}$ and $g^{(2)}$ are conformally equivalent. Show that the Weyl tensor for the metric $g^{(1)}$ is identical to that for $g^{(2)}$.